Approximate Analytic Solutions of the Inverse Cubic Truly Nonlinear Oscillator by Iterative Method

By

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Dedicated to My Parents

Md. Abdus Salek Molla

and

Ayesha Khatun

Who have chosen underprivileged life to continue my smile.

Declaration

This is to certify that the thesis work entitled "Approximate Analytic Solutions of the Inverse Cubic Truly Nonlinear Oscillator by Iterative Method" has been carried out by Md. Bayezid Bostami in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

Signature of Supervisor

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Approval

This is to certify that the thesis work submitted by Md. Bayezid Bostami entitled "Approximate Analytic Solutions of the Inverse Cubic Truly Nonlinear Oscillator by Iterative Method" has been approved by the board of examiners for the partial fulfillment of the requirements for the degree of Master of Science in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh in January 2016.

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Abstract

An analytical technique has been developed based on an iteration method to determine higher-order approximate periodic solutions for nonlinear oscillatory differential equations. Usually, a set of nonlinear algebraic equations is solved with this method. However, analytical solutions of these algebraic equations are not always possible, especially in the case of large oscillations**.** A new technique based on the Mickens iterative method has been presented to obtain approximate analytic solutions of the Inverse Cubic Truly Nonlinear Oscillator. In this thesis, we have adopted the method of Fourier series and utilized truncated terms in each steps of iteration. The solutions obtained by this method nicely matched with the exact frequency. Also the obtained solutions are much more accurate than other existing results and the method is convergent and consistent.

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CHAPTER I

Introduction

Differential equation is a mathematical tool, which has its application in many branches of knowledge of mankind. Numerous physical, mathematical, economical, chemical, biological, biochemical and many other relations appear mathematically in the form of differential equations that are linear or nonlinear, autonomous or non-autonomous. Generally, in many physical phenomena, such as spring-mass systems, resistor-capacitor-inductor circuits, bending of beams, chemical reactions, the motion of pendulums, the motion of the rotating mass around another body, etc., the differential equations are occurred. Also, in ecology and economics the differential equations are vastly used. Basically, many differential equations involving physical phenomena are nonlinear. Differential equations, which are linear, are comparatively easy to solve and nonlinear are laborious and in some cases it is impossible to solve them analytically. In such situations mathematicians, physicists and engineers convert the nonlinear equations into linear equations by imposing some conditions. In case of small oscillation, linearization is a well-known technique to solve the problems. But, such a linearization is not always possible and when it is not possible, then the original nonlinear equation itself must be used. The study of nonlinear equations is generally confined to a variety of rather special cases, and one must resort to various methods of approximation.

At first Van der Pol [1] paid attention to the new (self-excitations) oscillations and indicated that their existence is inherent in the nonlinearity of the differential systems characterizing the procedure. This nonlinearity appears, thus, as the very essence of these phenomena and by linearizing the differential equation in the sense of the method of small oscillation, one simply eliminates the possibility of investigating such problems. Thus, it is necessary to deal with the nonlinear problems directly instead of evading them by dropping the nonlinear terms.

To solve nonlinear differential equations, there exist some methods such as Perturbation technique, Harmonic Balance, Method of Multiple Scales, Homotopy Perturbation, Iteration method etc. Among the methods, the method of Perturbations, i.e., asymptotic expansions in terms of a small parameter are foremost.

Perturbation methods have received much attention as these methods for accuracy and quickly computing numerical solutions of dynamic, stochastic, economic equilibrium models for both single-agent or rational expectations models and multi-agent or game theory models. A perturbation method is based on the following aspects: the equations to be solved are sufficiently "smooth" or sufficiently differentiable a number of times in the required regions of variables and parameters.

The Krylov-Bogoliubov-Mitropolaskii (KBM) [2- 3] method was developed for obtaining the periodic solutions of second order nonlinear differential equations. Nowadays, the KBM method is used to obtain oscillatory as well as damped, critically damped, over damped, near critically damped, more critically damped oscillatory and non-oscillatory solutions of second, third, fourth etc., order nonlinear differential systems by imposing some restrictions to obtain the uniform solution. The method of KBM [2] is an asymptotic method in the sense that $\varepsilon \to 0$. An asymptotic series itself may not be convergent, but for a fixed number of terms, the approximate solution tends to the exact solution as $\varepsilon \to 0$. It may be noted that the term asymptotic is frequently used in the theory of oscillations in the sense that $\varepsilon \to 0$. It is an important approach to the study of such nonlinear oscillations in the small parameter expansion. Two widely spread methods in this theory are mainly used in literature; one is averaging asymptotic KBM method and other is the Method of Multiple Scales [4]. The KBM method is particularly convenient and is the extensively used technique to obtain the approximate solutions among the methods used to study the nonlinear differential systems with small nonlinearity. The KBM method starts with the solution of linear equation (sometimes called the generating solution of the linear equation), assuming that in the nonlinear case, the amplitude and the phase variables in the solution of the linear differential equations are time dependent functions instead of constants. On the other hand the Method of Multiple Scales is needed for problems in which the solutions depend simultaneously on widely different scales. A typical example is the modulation of an oscillatory solution over time-scales that are much greater than the period of the oscillations.

Harmonic Balance (HB) [11-31] method is a procedure of determining analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series representation. An important advantage of the method is that it can be applied to nonlinear oscillatory problems for which the nonlinear terms are not "small" i.e., no perturbation parameter need to exist. A disadvantage of the method is that it is a priory difficult to predict for a given nonlinear differential equation whether a first order harmonic balance calculation will provide a sufficiently accurate approximation to periodic solution.

The Iteration method [49] was introduced R E Mickens in 1987. The method introduces a reliable and efficient process for wide variety of scientific and engineering application for the case of nonlinear systems. There are two important advantages of Iteration method, one is "Only linear, in homogeneous differential equations are required to be solved at each level of the calculation" and another is "The coefficients of the higher harmonic, for a given value of the iteration index decrease rapidly with increasing harmonic number. This implies that higher order solutions may not be required".

It is noted that the majority of scientists have not been led to their discoveries by a process of deduction from general postulates, or general principles, but rather by a through examination of properly chosen particular cases. The generalizations have come later, because it is far easier to generalize an established result than to discover a new line of argument, Generalization is the temptation of a lot of researchers working now with nonlinear dynamical systems.

The important development of the theory of nonlinear dynamical systems, during these centuries, has essentially its origins in the studies if the "natural effects" encountered in these systems, and the rejection of non-essential generalizations, i.e. the study of concrete nonlinear systems have been possible due to the foundation of results from the theory or nonlinear dynamical system. The main purpose of this thesis is to improve the accuracy of the approximate solution of the 'Inverse Cubic Truly Nonlinear Oscillator' by iterative method so that it will help us to investigate the nature (amplitude, frequency etc.) of the nonlinear dynamical systems.

The chapter outline of this thesis is as follows**: In Chapter II**, some basic conceptions are given. **In Chapter III**, the review of literature is presented. **In Chapter IV,** the Iteration method has been described for obtaining approximate analytic solutions of the Inverse Cubic Truly Nonlinear Oscillator. **In Chapter V,** the convergence and consistency analysis of the adopted method has been shown. Finally, some concluding remarks are included **in Chapter VI**.

CHAPTER II

Basic Conceptions

This chapter introduces the basic, but fundamental concepts relating to the thesis:

Nonlinear Ordinary Differential Equation

A nonlinear ordinary differential equation is an ordinary differential equation that is not linear.

The following ordinary differential equations are all nonlinear:

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y^2=0$$

$$\frac{d^2y}{dx^2}+5\left(\frac{dy}{dx}\right)^3+6y=0$$

$$\frac{d^2y}{dx^2}+5y\frac{dy}{dx}+6y=0$$

but the following ordinary differential equations are all linear:

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0$$

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

Phase Plane

If a plane is such that, each point of this plane describe the position and velocity of a dynamical particle, then this plane is called phase plane.

The differential equation describing many nonlinear oscillators can be written in the form:

$$\frac{d^2x}{dt^2}+f\left(x,\frac{dx}{dt}\right)=0 \tag{2.1}$$

A convenient way to treat equation (2.1) is to rewrite it as a system of two first order ordinary differential equations

$$\frac{dx}{dt}=y, \qquad \frac{dy}{dt}=-f(x,y) \tag{2.2}$$

Equations (2.2) may be generalized in the form

$$\frac{dx}{dt} = F(x, y), \qquad \frac{dy}{dt} = G(x, y) \tag{2.3}$$

A point which satisfies $F(x, y) = 0$ and $G(x, y) = 0$ is called an equilibrium point. The solution to (2.3) may be pictured as a curve in the x-y phase plane [6] passing through the point of initial conditions (x_0, y_0). Each time a motion passes through a given point (x, y), its direction is always the same. This means a given motion may not intersect itself. A periodic motion corresponds to a closed curve in the x-y plane. In the special case that the first equation of (2.3) is $\frac{dx}{dt} = y$, as in the case of equations (2.2), the motion in the upper half-plane $y > 0$ must proceed to the right, that is, x must increase in time for $y > 0$, and vice versa for $y < 0$.

Limit Cycle

A closed trajectory in the phase plane such that other non-closed trajectories spiral toward it, either from the inside or the outside, as $t \to \infty$, is called a limit cycle [7]. If all trajectories that start near a closed trajectory (both inside and outside) spiral toward the closed trajectory as $t \to \infty$, then the limit cycle is asymptotically stable. If the trajectories on both sides of the closed trajectory spiral away as $t \to \infty$, then the closed trajectory is unstable.

Trajectory

If a curve is such that each point of the curve represents the position and velocity of a dynamical particle, the curve is called the path or Trajectory of the particle.

Consider a second order nonlinear differential equation of the form

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right) \tag{2.4}$$

If we put $y = \frac{dx}{dt}$, then the equation (2.4) is replaced by the equivalent system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = f(x, y) \tag{2.5}$$

More generally, we shall consider systems of the form

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y) \tag{2.6}$$

where P and Q have continuous first order partial derivative for all (x, y).

For any number $t = t_0$ and any pair (x_0, y_0) of real number, there exists a unique solution of the equation (2.6), we obtain

$$\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned} \tag{2.7}$$

where $x_0 = f(t_0), \quad y_0 = g(t_0)$

If both f and g are not constant functions, then equation (2.7) defines a curve in the phase plane, which is called a path or orbit or trajectory of the system.

The Autonomous System

Consider the systems of the form

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y)$$

where P and Q have continuous first partial derivatives for all (x, y). Such a system, in which the independent variable t is not explicitly appears in the function P and Q on the right, is called an autonomous system.

The following example is an autonomous system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x$$

The Non-autonomous system

Consider the systems of the form

$$\frac{dx}{dt} = P(x, y, t)$$
$$\frac{dy}{dt} = Q(x, y, t)$$

where P and Q have continuous first partial derivatives for all (x, y). Such a system, in which the independent variable t is explicitly appears in the function P and Q on the right, is called a non-autonomous system.

The following example is a non-autonomous system

$$\frac{dx}{dt} = \sin t$$
$$\frac{dy}{dt} = \cos t$$

Critical Point

Consider the autonomous system of the form

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y)$$

a point (x_0, y_0) at which both $P(x_0, y_0) = 0$ and $Q(x_0, y_0) = 0$ is called a critical point.

Isolated Critical Point

A critical point (x_0, y_0) of the system

$$\frac{dx}{dt} = P(x, y) \qquad \frac{dy}{dt} = Q(x, y) \tag{2.8}$$

is called isolated if there exists a circle $(x - x_0)^2 + (y - y_0)^2 = r^2$ about the point (x_0, y_0) such that (x_0, y_0) is the only critical point of the system (2.8) within this circle.

Classifications of Critical Point:

(a) Centre

The isolated critical point $(0, 0)$ of the system

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y)$$

is called a Centre if there exists a neighborhood of $(0, 0)$ which contains count ably infinite numbers of closed path P_n, $(n = 1, 2, \cdots)$ each of which contains $(0, 0)$ as interior point and which are such that the diameters of the paths approaches to 0 as $n \to \infty$.

(b) Saddle Point

The isolated critical point $(0, 0)$ of the system

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y)$$

is called a saddle point if there exists a neighborhood of $(0, 0)$ in which the following two conditions hold:

(i) There exists two paths which approaches and enter into $(0, 0)$ from a pair of opposite directions as $t \to +\infty$ and there exists two paths which approach and enter into $(0, 0)$ from a different pair of opposite directions as $t \to -\infty$.

(ii) In each of the four domains, between any two of the four directions in (i), there are infinitely many paths which are arbitrarily closed to $(0, 0)$ but which do not approach to $(0, 0)$ either as $t \to +\infty$ or as $t \to -\infty$.

(c) Spiral Point

The isolated critical point $(0, 0)$ of the system

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y)$$

is called a spiral point if there exists a neighborhood of $(0, 0)$ such that every path P in this neighborhood has the following properties:

(i) P is defined for all $t > t_0$ or $t < t_0$, for some number t_0.

(ii) P approaches to $(0, 0)$ as $t \to +\infty$ or as $t \to -\infty$.

(iii) P approaches to $(0, 0)$ in a spiral like manner, winding around $(0, 0)$ an infinite number of times $t \to +\infty$ or as $t \to -\infty$.

(d) Node

The isolated critical point $(0, 0)$ of the system

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y)$$

is called a node point if there exists a neighborhood of $(0, 0)$ such that every path P in this neighborhood has the following properties:

(i) P is defined for all $t > t_0$ or $t < t_0$, for some number t_0.

(ii) P approaches to $(0, 0)$ as $t \to +\infty$ or as $t \to -\infty$.

(iii) P enters into $(0, 0)$ as $t \to +\infty$ or as $t \to -\infty$.

(e) Stable

Consider the system

$$\begin{aligned} \frac{dx}{dt} &= P(x, y) \\ \frac{dy}{dt} &= Q(x, y) \end{aligned} \tag{2.9}$$

Suppose $(0, 0)$ is an isolated critical point of the above system. Let C be a path of the system (1.4) and $x = f(t),\ y = g(t)$ be a solution of (2.9), which define C parametrically. Let $(x, y) = (f(t),\ g(t))$ be a point on C. Define

$$D(t) = \sqrt{\left[f(t)\right]^2 + \left[g(t)\right]^2}$$

where $D(t)$ is the distance between the critical point $(0, 0)$ and $R(f(t), g(t))$, then the critical point $(0, 0)$ is called stable if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$D(t_0) < \delta$, for some t_0

and $D(t_0) < \varepsilon$, for all $t_0 \le t < \infty$.

(f) Asymptotically Stable

Consider the system

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y) \qquad (2.10)$$

Suppose $(0, 0)$ is an isolated critical point of the above system. Let C be a path of the system (2.7) and $x = f(t),\ y = g(t)$ be a solution of the system (2.10), which define C parametrically. Let $(x, y) = (f(t),\ g(t))$ be a point on C. Define

$$D(t) = \sqrt{\left[f(t)\right]^2 + \left[g(t)\right]^2}$$

where $D(t)$ is the distance between the critical point $(0, 0)$ and $R(f(t), g(t))$, then the critical point $(0, 0)$ is called asymptotically stable if it is stable and

$$\lim_{t \to +\infty} f(t) = 0, \quad \lim_{t \to +\infty} g(t) = 0$$

Characteristic Equation

Consider the linear system

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy \qquad (2.11)$$

where a, b, c, d are real constants.

Clearly the origin $(0, 0)$ is critical point of the above system. We assume that

$$\begin{vmatrix} a & d \\ c & b \end{vmatrix} \neq 0$$

and hence $(0, 0)$ is the only critical point of (2.11). By Euler method, the solution of (2.11) is found of the form

$$\begin{cases} x = Ae^{\lambda t} \\ y = Be^{\lambda t} \end{cases} \tag{2.12}$$

where A and B are arbitrary constants. If (2.12) is a solution of (2.11), then we have

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \tag{2.13}$$

Equation (2.13) is called the characteristic equation of (2.11) and its roots are called characteristic roots or eigen values of equation (2.11).

Important Notes:

Nature of the roots	**Nature of the critical point**	**Nature of the stability of critical point (0, 0)**
Real, unequal and of same sign	Node	Asymptotically stable if the roots are negative; unstable if the roots are positive
Real, unequal and of opposite sign	Saddle point	Unstable
Real and equal	Node	Asymptotically stable if the roots are negative; unstable if the roots are positive
Complex conjugate but not purely imaginary	Spiral point	Asymptotically stable if the real part of the roots are is negative; unstable if the real part is positive
Purely imaginary	Centre	Stable but not asymptotically stable

Free Oscillating System

If there are no external forces applied on a system, then the system is called free oscillating system. For a free oscillating system, the applied force is proportional to the restoring force. If $f_1(x)$ is the restoring force and F is the applied force on the system, then

$$F \propto -f_1(x)$$

$$F \propto -k\, f_1(x)$$

where k is the constant of proportionality.

$$m\frac{d^2x}{dt^2} + k\, f_1(x) = 0 \quad [\because F = ma]$$

$$\frac{d^2x}{dt^2}+\frac{k}{m}\,f_1(x)=0\,,$$

$$\frac{d^2x}{dt^2}+f(x)=0 \tag{2.14}$$

Equation (2.14) is the governing equation for a free oscillating system.

Force Oscillating System

If some external forces are applied on system, then the system is called force oscillating system.

Natural Frequency

Without external force every system oscillates together with a frequency, which is called natural frequency.

Duffing Oscillator

The differential equation

$$\frac{d^2x}{dt^2}+x+\varepsilon\alpha x^3=0\,,\quad \varepsilon>0 \tag{2.15}$$

is called the Duffing oscillator [6]. It is a model of a structural system which includes nonlinear restoring forces (for example springs). It is sometimes used as an approximation for the pendulum:

$$\frac{d^2\theta}{dt^2}+\frac{g}{L}\sin\theta=0 \tag{2.16}$$

Expanding $\sin\theta=\theta-\frac{\theta^3}{6}+O(\theta^5)$, and then setting $\theta=\sqrt{\varepsilon}\,x$,

$$\frac{d^2x}{dt^2}+\frac{g}{L}\left(x-\varepsilon\frac{x^3}{6}\right)=O(\varepsilon^2) \tag{2.17}$$

Now we stretch time with $z=\sqrt{\frac{g}{L}t}$,

$$\frac{d^2x}{dz^2}+x-\varepsilon\frac{x^3}{6}=O(\varepsilon^2) \tag{2.18}$$

which is (2.15) with $\alpha=-1/6$

In order to understand the dynamics of Duffing's equation (2.15), we begin by writing it as a first order system:

$$\frac{dx}{dt}=y,\qquad \frac{dy}{dt}=-x-\varepsilon\alpha x^3 \tag{2.19}$$

For a given initial condition $(x(0), y(0))$, equation (2.19) specifies a trajectory in the x-y phase plane, i.e. the motion of a point in time. The integral curve along which the point moves satisfies the differential equation

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-x - \varepsilon\alpha x^3}{y} \tag{2.20}$$

Equation (2.20) may be easily integrated to give

$$\frac{y^2}{2} + \frac{x^2}{2} + \varepsilon\alpha\frac{x^2}{4} = \text{constant} \tag{2.21}$$

Equation (2.21) corresponds to the physical principle of conservation of energy. In the case that α is positive, (2.21) represents a continuum of closed curves surrounding the origin, each of which represents a motion of equation (2.15) which is periodic in time. In the case that α is negative, all motions which start sufficiently close to the origin are periodic. However, in this case equation (2.19) has two additional equilibrium points besides the origin, namely, $x = \pm 1/\sqrt{-\varepsilon\alpha}$, $y = 0$. The integral curves which go through these points separate motions which are periodic from motions which grow unbounded, and are called separatrices. If we were to numerically integrate equation (2.15), we would see that the period of the periodic motions depended on which closed curve in the phase plane we were on. This effect is typical of nonlinear vibrations and is referred to as the dependence of period on amplitude.

Van der Pol Oscillator

The differential equation

$$\frac{d^2x}{dt^2} + x - \varepsilon\left(1 - x^2\right)\frac{dx}{dt} = 0, \qquad \varepsilon > 0 \tag{2.22}$$

is called the van der Pol oscillator [6]. It is a model of a non-conservative system in which energy is added to and subtracted from the system in an autonomous fashion, resulting in a periodic motion called a limit cycle. Here we can see that the sign of the damping term, $-\varepsilon\left(1 - x^2\right)\frac{dx}{dt}$ changes, depending upon whether is $|x|$ larger or smaller than unity.

Van der Pol's equation has been used as a model for stick-slip oscillations, aero-elastic flutter, and numerous biological oscillators, to name but a few of its applications.

Numerical integration of equation (2.22) shows that every initial condition (except $x = \frac{dx}{dt} = 0$) approaches a unique periodic motion. The nature of this limit cycle is dependent on the value of ε. For small values of ε the motion is nearly sinusoidal, whereas for large values of ε it is a relaxation oscillation, meaning that it tends to resemble a series of step functions, jumping between positive and negative values twice per cycle. If we write (2.22) as a first order system,

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = \varepsilon\left(1 - x^2\right)y \tag{2.23}$$

we find that there is no exact closed form solution. Numerical integration shows that the limit-cycle is a closed curve enclosing the origin in the x - y phase plane. From the fact that equation (2.23) are invariant under the transformation $x \to -x$, $y \to -y$ we may conclude that the curve representing the limit cycle is point symmetric about the origin.

Truly Nonlinear Functions

If $f(x)$ has no linear approximation in any neighborhood of $x = 0$, then $f(x)$ is a Truly Nonlinear function.

The following are several explicit examples of Truly Nonlinear functions

$$f_1(x) = x^3,\ f_2(x) = x^{\frac{1}{3}},\ f_3(x) = x + x^{\frac{1}{3}}$$

Truly Nonlinear Oscillators

If $f(x)$ is a Truly Nonlinear function, then the second-order differential equations "$\ddot{x} + f(x) = 0$" is a Truly Nonlinear Oscillator.

The following are particular examples of Truly Nonlinear Oscillators

$$\ddot{x} + x^3 = 0$$

$$\ddot{x} + x^{\frac{1}{3}} = 0$$

$$\ddot{x} + x + x^{\frac{1}{3}} = 0$$

$$\ddot{x} + \frac{1}{x} = 0$$

Nonlinear oscillations occurring in one degree of freedom systems have been studied intensely for almost two centuries [5, 8–9]. The general form that those equations take is

$$\ddot{x} + x = \varepsilon f(x, \dot{x}), \quad 0 < \varepsilon << 1 \tag{2.24}$$

where ε is a small parameter. Harmonic Balance (HB) [52], Averaging Methods [52], Parameter Expansions [52] and Iteration [52] methods are based on expansions in terms of ε which are taken to be asymptotic series. Each particular perturbation method is distinguished by how this feature is accomplished. If from a priori considerations it can be determined that periodic solutions exist, then a major task, for each method, is to eliminate, called the secular terms. Secular terms are expressions in the solutions that are oscillatory, with increasing, time dependent amplitudes [8-10], i.e., for an odd-parity system

$$\text{secular term: } t^n \cos[(2k+1)\Omega t], \tag{2.25}$$

where (n, k) are integers, with $n \geq 1$ and $k \geq 1$. For all of the standard methods, procedures have evolved to resolve this issue.

Inspection of equation (2.24) shows that each of the classical methods has at its foundation the explicit assumption that when $\varepsilon = 0$ the resulting "core" equation is the linear harmonic oscillator differential equation, namely,

$$\ddot{x}_0 + x_0 = 0 \tag{2.26}$$

where the zero indicates $\varepsilon = 0$. This fact presents an immediate difficulty for Truly Nonlinear Oscillators, where equation (2.24) is replaced by, for example,

$$\ddot{x} + x^p = \varepsilon f(x, \dot{x}), \qquad p \neq 1 \tag{2.27}$$

We observe that when $\varepsilon = 0$, this equation reduces to the nonlinear equation

$$\ddot{x}_0 + x_0^p = 0 \tag{2.28}$$

and this type of equation would, at the very least, greatly complicate any solution construction based on expansions in the parameter ε .

CHAPTER III

Literature Review

The characteristics of nonlinear differential equations are peculiar. But mathematical formulations of physical and engineering problems often results in differential equations that are nonlinear. A nonlinear system of equations is a set of simultaneous equations in which the unknowns appear as variables of a polynomial of higher degree than one or in the argument of a function which is not a polynomial of degree one. On the other hand, in a nonlinear system of equations, the equations to be solved cannot be written as a linear combination of the unknown variables or functions that appear in it or others. If nonlinear known functions appear in the equations, it does not matter. Specially, a differential equation is regarded as linear if it gets linear in terms of the unknown function as well as its derivatives, even if nonlinear in terms of the other variables appearing in it.

As nonlinear equations are difficult to solve, nonlinear systems are commonly approximated by linear equations. This works well up to some accuracy and some range for the input values, but some interesting phenomena such as chaos and singularities are hidden by linearization. It follows that some aspects of the behavior of a nonlinear system appear commonly to be chaotic, unpredictable or counterintuitive. Although such chaotic behavior may resemble random behavior, it is absolutely not random. In this position there are several analytical approaches to find approximate solutions to nonlinear problems, such as: Perturbation [32-38], Harmonic Balance (HB) [11-31], Homotophy Perturbation [39], Homotophy [40-45], Energy Balance [46], Cubication [47- 48], Iteration [49-62] methods, etc.

The perturbation method is the most widely utilized method in which the nonlinear term is small. The solution of a differential equation is expanded in a power series of a small parameter in the perturbation method. The method of Lindstedt-Poincare (LP) [2-3, 32], Krylov-Bogoliubov-Mtropolskii (KBM) [2-3], Multiple Scale method [4], Homotopy perturbation [39] and Homotopy [40-45] are most important among all perturbation methods.

The method of Lindstedt-Poincare [32, 52] is an introductory method to solved the following second order nonlinear differential equations

$$\ddot{x} + \omega_0{}^2 x + \varepsilon\, f(\ddot{x}, x) = 0\,, \tag{3.1}$$

where ω_0 is the unperturbed frequency and ε is a small parameter.

The fundamental idea in Lindstedt's technique is based on the observation that the nonlinearities alter the frequency of the system from the linear one ω_0 to $\omega(\varepsilon)$. To account for this change in frequency, he introduces a new variable $\tau = \omega t$ and expand ω and x in power of ε as

$$x = x_0(\tau) + \varepsilon\, x_1(\tau) + \varepsilon^2 x_2(\tau) + \dots\ \dots\ , \tag{3.2}$$

$$\omega = \omega_0 + \varepsilon\,\omega_1 + \varepsilon^2\omega_2 + \dots \quad \dots,$$

where ω_i, $i = 0,1,2,\dots\ \dots$, are unknown constants to be determined.

Substituting equation (3.2) into equation (3.1) and equating the coefficients of the various powers of ε, the following equations are obtained

$$\ddot{x}_0 + x_0 = 0$$

$$\ddot{x}_1 + x_1 = -2\omega_1\ddot{x} - f(x_0, \dot{x}_0)$$

$$\begin{aligned} \ddot{x}_2 + x_2 = -2\omega_1\,\ddot{x}_1 - f(x_0,\, \dot{x}_0) - (\omega_1{}^2 + 2\omega_2)\ddot{x}_0 - f_x(x_0,\, \dot{x}_0)x_1 \\ + f_{\dot{x}}(x_0,\, \dot{x}_0)(\omega_1\,\dot{x}_0 + \dot{x}_1) \end{aligned} \tag{3.3}$$

$$\dots \qquad \dots \qquad \dots$$

$$\ddot{x}_n + x_n = g_n(x_0, x_1, \dots, x_{n-1}; \dot{x}_0, \dot{x}_1, \dots, \dot{x}_{n-1})\,,$$

where over dot represents the differentiation with respect to τ.

Clearly equation (3.3) is a linear system and it is solved by the elementary techniques.

This method is used only for finding the periodic solution, but the method cannot discuss transient case.

Further, Krylov and Bogoliubov [2] introduced a technique to discuss transients of the same equation. This method starts with the solution of the linear equation, assuming that, in the nonlinear case, the amplitude and phase in the solution of the linear equation are time dependent function rather than constants [32].

The solution of corresponding unperturbed equation (i.e., for $\varepsilon = 0$) of equation (3.1) can be written as

$$x = a\cos(\omega_0 t + \theta) \tag{3.4}$$

where a and θ are two arbitrary constants to be determined from the initial conditions $x(0) = x_0$ and $\dot{x}(0) = y_0$. Here a and θ are called amplitude and phase.

Now to determine an approximate solution of equation (3.1) for ε small but different from zero, Krylov and Bogoliubov assumed that the solution is still given by equation (3.4) with varying a and θ subject to the conditions

$$\frac{dx}{dt} = -a\omega_0 \sin\varphi, \qquad \varphi = \omega_0 t + \theta \tag{3.5}$$

Differentiating equation (3.4) with respect to time t and using equation (3.5), we obtain

$$\frac{da}{dt}\cos\varphi - \frac{d\theta}{dt} a\sin\varphi = 0 \tag{3.6}$$

Again differentiating equation (3.5) with respect to time t, we obtain

$$\frac{d^2x}{dt^2} = -a\,\omega_0^2 \cos\varphi - \omega_0 \frac{da}{dt}\sin\varphi - a\omega_0 \frac{d\theta}{dt}\cos\varphi \tag{3.7}$$

Substituting equation (3.7) into equation (3.1) and using equation (3.4) and equation (3.5), we obtain

$$\frac{da}{dt}\omega_0 \sin\varphi + \frac{d\theta}{dt} a\omega_0 \cos\varphi = -\varepsilon\ f\left(a\cos\varphi,\ -a\omega_0 \sin\varphi\right) \tag{3.8}$$

Solving equation (3.6) and equation (3.8) $\frac{da}{dt}$ and $\frac{d\theta}{dt}$ yields

$$\begin{cases} \dfrac{da}{dt} = -\dfrac{\varepsilon}{\omega_0}\sin\varphi\ f\left(a\cos\varphi,\ -a\omega_0 \sin\varphi\right) \\ \dfrac{d\theta}{dt} = -\dfrac{\varepsilon}{a\omega_0}\cos\varphi\ f\left(a\cos\varphi,\ -a\omega_0 \sin\varphi\right) \end{cases} \tag{3.9}$$

Equation (3.4) together with equation (3.9) represents the first approximate solution of equation (3.1). Further, the technique was modified and justified by Bogoliubov and Mitropolskii [3] in 1961. They assumed a solution of the nonlinear differential equation (3.1) of the form

$$x(t,\varepsilon) = a\cos\psi + \varepsilon\, x_1(a,\psi) + \cdots + \varepsilon^n\, x_n(a,\psi) + O\left(\varepsilon^{n+1}\right) \tag{3.10}$$

where x_k, $k = 1, 2, \cdots, n$ is a periodic function of ψ with period 2π , a and ψ very with time t according to

$$\begin{cases} \dfrac{da}{dt} = \varepsilon\, A_1(a) + \cdots + \varepsilon^n\, A_n(a) + O\left(\varepsilon^{n+1}\right) \\ \dfrac{d\psi}{dt} = \omega_0 + \varepsilon\, B_1(a) + \cdots + \varepsilon^n\, B_n(a) + O\left(\varepsilon^{n+1}\right) \end{cases} \tag{3.11}$$

where the function x_k , A_k and B_k are chosen such that equation (3.10) and equation (3.11) satisfy the differential equation (3.1). Later this solution was used by Mitropolskii [63] to investigate similar system (i.e., equation (3.1)) in which the coefficient very slowly with time. Popov [64] extended this method to nonlinear strongly damped oscillatory systems. By Popov's [64] technique, Murty, et al. [65] extended the method to over damped nonlinear system. Murty [66] further presented a unified KBM method to obtain under and over-damped solution of a second- order nonlinear differential equation. Shamsul and Sattar [67] extended Murty's [66] unified KBM method to solve a third-order nonlinear differential equation.

Harmonic Balance method is the most useful technique for finding the periodic solutions of nonlinear system. Which is patented by Mickens [8] and farther work has been done by Lim et al. [16], Wu et al. [18], Hu [21], Gottlieb [20] , Beléndez et al. [28] and so on for solving the strong nonlinear problems. If a periodic solution does not exist of an oscillator, it may be sought in the form of Fourier series, whose coefficients are determined by requiring the series to satisfy the equation of motion. However, in order to avoid solving an infinite system of algebraic equations, it is better to approximate the solution by a suitable finite sum of trigonometric function. This is the main task of harmonic balance method. Thus approximate solutions of an oscillator are obtained by harmonic balance method using a suitable truncated Fourier series.

The method is capable to determining analytic approximate solution to the nonlinear oscillator valid even for the case where the nonlinear terms are not small i.e., no particular parameter need exist.

The formulation of the method of harmonic balance focuses primarily by Mickens [11]. However, it should be indicated that various generalizations of the method of harmonic balance has been made by an intrinsic method of harmonic analysis by Huseyin & Lin [68]. Lately, combining the method of averaging and harmonic balance, Lim & Lai [54] presented analytic technique to obtain first approximate perturbation solution; their solutions gives desired results for some non-conservative systems when the damping force is very small. Another technique is developed by Yamgoue and Kofane [69] to determine approximate solutions of nonlinear problems with strong damping effect, more than two harmonic terms are involved in their solution.

Mickens [52] has given the general procedure for calculating solutions by means of the method of direct Harmonic Balance as follows:

He considered the equation for all Truly Nonlinear (TNL) oscillators as:

$$F(x,\dot{x},\ddot{x})=0, \tag{3.12}$$

where $F(x,\dot{x},\ddot{x})$ is of odd-parity, i.e.

$$F(-x,-\dot{x},-\ddot{x})=-F(x,\dot{x},\ddot{x}). \tag{3.13}$$

A major consequence of this property is that the corresponding Fourier expansions of the periodic solutions only contain odd harmonics (Mickens [70]), i. e.,

$$x(t)=\sum_{k=1}^{\infty}\{A_k\cos[(2k-1)\Omega t]+B_k\sin[(2k-1)\Omega\, t]\}. \tag{3.14}$$

The N -th order harmonic balance approximation to $x(t)$ is the expression

$$x_N(t)=\sum_{k=1}^{N}\{\bar{A}_k^N\cos[(2k-1)\bar{\Omega}_N\, t]+\bar{B}_k^N\sin[(2k-1)\bar{\Omega}_N\, t]\}, \tag{3.15}$$

where $\bar{A}_k^N$, $\bar{B}_k^N$, $\bar{\Omega}_N$ are approximations to A_k, B_k, Ω for $k=1, 2, 3, \ldots\ldots, N$.

For the case of a conservative oscillator, equation (3.12) generally takes the form

$$\ddot{x}+f(x,\lambda)=0, \tag{3.16}$$

where λ denotes the various parameters appearing in $f(x,\lambda)$ and $f(-x,\lambda)=-f(x,\lambda)$.The following initial conditions are selected

$$x(0)=A, \quad \dot{x}(0)=0 \tag{3.17}$$

And this has the consequence that only the cosine terms are needed in the Fourier expansions, and therefore we have

$$x_N(t)=\sum_{k=1}^{N}\bar{A}_k^N\cos[(2k-1)\bar{\Omega}_N\, t] \tag{3.18}$$

Observe that $x_N(t)$ has $(N+1)$ unknowns, the N coefficients, $(\bar{A}_1^N, \bar{A}_2^N, \ldots\ldots \bar{A}_N^N)$ and Ω_N, the angular frequency. These quantities may be calculated by carrying out the following steps:

Step-1: Substitute equation (3.18) into equation (3.16), and expand the resulting form into an expression that has the following structure

$$\sum_{k=1}^{N}H_k\cos[(2k-1)\Omega_N\, t]+HOH\cong 0, \text{ HOH= Higher Order Harmonic} \tag{3.19}$$

where they H_k are functions of the coefficients, the angular frequency, and the parameters, i.e., $H_k=H_k(\bar{A}_1^N, \bar{A}_2^N, \ldots\ldots \bar{A}_N^N, \Omega_N, \lambda)$.

Herein equation (3.19), we only retain as many harmonics in our expansion as initially occur in the assumed approximation to the periodic solution.

Step-2: Set the functions H_k to zero, i.e.,

$$H_k=0, \quad k=1, 2, \ldots\ldots, N. \tag{3.20}$$

The action is justified since the cosine functions are linearly independent, as a result any linear sum of them that is equal to zero must have the property that the coefficient are all zero.

Step-3: Solve the N equations, in equation (3.20), for $(\bar{A}_2^N, \bar{A}_3^N, ...\bar{A}_N^N)$ and Ω_N, in terms of $\bar{A}_1^N$.

Using the initial conditions, equation (3.17), we have for $\bar{A}_1^N$ the relation

$$x_N(0) = A = \bar{A}_1^N + \sum_{k=2}^{N} \bar{A}_k^N (\bar{A}_1^N,\ \lambda). \tag{3.21}$$

An important point is that equation (3.20) will have many distinct solutions and the "one" selected for a particular oscillator equation is that one for which we have known a priori restrictions on the behavior of the approximations to the coefficients. However, as the worked examples in the next section demonstrate, in general, no essential difficulties arise.

For the case of non-conservative oscillators, where $\dot{x}$ appears to an "odd power" the calculation of approximations to periodic solutions follows a procedure modified for the case of conservative oscillators presented above. Many of these equations take the form

$$\ddot{x} + f(x,\ \lambda_1) = g(x,\ \dot{x},\ \lambda_2)\dot{x}, \tag{3.22}$$

where

$$f(-x,\ \lambda_1) = -f(x,\ \lambda_1),\ \ g(-x,\ -\dot{x},\ \lambda_2) = -g(x,\ \dot{x},\ \lambda_2), \tag{3.23}$$

and $(\lambda_1,\ \lambda_2)$ denote the parameters appearing in f and g. For this type of differential equation, a limit-cycle may exist and the initial conditions cannot, in general, be a priori specified.

Harmonic balancing, for systems where limit-cycles [20] may exist, uses the following procedures:

Step-1: The N-th order approximation to the periodic solution to be

$$x_N(t) = \bar{A}_1^N \cos(\bar{\Omega}_N\, t) + \sum_{k=2}^{N} \{\bar{A}_k^N \cos[(2k-1)\bar{\Omega}_N\, t] + \bar{B}_k^N \sin[(2k-1)\bar{\Omega}_N\, t]\}, \tag{3.24}$$

where the $2N$ unknowns $\bar{A}_1^N, \bar{A}_2^N, ..., \bar{A}_N^N$; $\bar{\Omega}_N, \bar{B}_2^N, ..., \bar{B}_N^N$ and $\bar{\Omega}_N$ are to be determined.

Step-2: Substitute equation (3.24) into equation (3.22) and write the result as

$$\sum_{k=1}^{N} \{H_k \cos[(2k-1)\Omega_N\, t] + L_k \sin[(2k-1)\Omega_N\, t]\} + HOH \cong 0, \tag{3.25}$$

where the $\{H_k\}$ and $\{L_k\}$, $k=1$ to N, are functions of the $2N$ unknowns which are mentioned above.

Step-3: Next equate the $2N$ functions $\{H_k\}$ and $\{L_k\}$ to zero and solve them for the $(2N-1)$ amplitudes and the angular frequency. If a "valid" solution exists, then it corresponds to a limit-cycle. In general, the amplitudes and angular frequency will be expressed in terms of the parameters λ_1 and λ_2.

Mickens [52] has presented the following example**:**

Let us consider the nonlinear oscillator given by

$$\ddot{x}+x^3=0,\ \ x(0)=A,\ \ \dot{x}(0)=0 \tag{3.26}$$

This approximation takes the form

$$x_1(t)=A\cos(\Omega_1 t) \tag{3.27}$$

Observe that this expression automatically satisfies the initial conditions. Substituting equation (3.27) into equation (3.26) gives $(\theta=\Omega_1 t)$

$$\left(-A\Omega_1^{\,2}\cos\theta\right)+\left(A\cos\theta\right)^3\cong 0\,,$$

$$-\left(A\Omega_1^{\,2}\right)\cos\theta+A^3\left[\left(\frac{3}{4}\right)\cos\theta+\left(\frac{1}{4}\right)\cos 3\theta\right]\cong 0$$

$$A\left[-\Omega_1^{\,2}+\left(\frac{3}{4}\right)A^2\right]\cos\theta+HOH\cong 0$$

Setting the coefficient of $\cos\theta$ to zero gives the first approximation to the angular frequency

$$\Omega_1(A)=\left(\frac{3}{4}\right)^{1/2}A \tag{3.28}$$

and $$x_1(t)=A\cos\left[\left(\frac{3}{4}\right)^{1/2}At\right] \tag{3.29}$$

The solution for the second approximation takes the form $(\theta=\Omega_2 t)$

$$x_2(t)=A_1\cos\theta+A_2\cos 3\theta \tag{3.30}$$

with $$\ddot{x}_2(t)=-\Omega_2^2\left(A_1\cos\theta+9A_2\cos 3\theta\right) \tag{3.31}$$

Substituting equation (3.30) and equation (3.31) into equation (3.26), we obtain

$$H_1(A_1,A_2,\Omega_2)\cos\theta+H_2(A_1,A_2,\Omega_2)\cos 3\theta+HOH\cong 0\,,$$

where $$H_1=A_1\left[\Omega_2^2-\left(\frac{3}{4}\right)A_1^2-\left(\frac{3}{4}\right)A_1A_2-\left(\frac{3}{2}\right)A_2^2\right] \tag{3.32}$$

and $$H_2 = -9A_2\Omega_2^2 + \left(\frac{1}{4}\right)A_1^3 + \left(\frac{3}{2}\right)A_1^2 A_2 + \left(\frac{3}{4}\right)A_2^3 \tag{3.33}$$

Setting H_1 to zero, and defining z as

$$z \equiv \frac{A_2}{A_1} \tag{3.34}$$

We obtain,

$$\Omega_2 = \left(\frac{3}{4}\right)^{1/2} A_1\left(1 + z + 2z^2\right)^{1/2} = \Omega_1\left(1 + z + 2z^2\right)^{1/2} \tag{3.35}$$

where Ω_1 is that of equation (3.28). Inspection of equation (3.35) shows that the second approximation for the angular frequency is a modification of the first approximation result.

If this value for Ω_2 is substituted into equation (3.33) and this expression is set to zero, and if the definition of z is used, then the following cubic equation must be satisfied by z

$$51z^3 + 27z^2 + 21z - 1 = 0$$

There are three roots, but the one of interest should be real and have a small magnitude, i. e.,

$$|z| << 1$$

The root is $z_1 = 0.044818....,$

And implies that the amplitude, A_2, of the higher harmonic, i. e., the $\cos 3\theta$, is less than 5% of the amplitude of the fundamental mode, $\cos\theta$.

Therefore, the second harmonic balance approximation for equation (3.26) is

$$x_2(t) = A_1[\cos\theta + z_1 \cos 3\theta].$$

For the initial condition, $x_2(0) = A$, we obtain

$$A = A_1(1 + z_1) \text{ or } A_1 = \frac{A}{1 + z_1} = (0.9571)A$$

Using the value of A_1 and z_1 into equation (3.35), we obtain

$$\Omega_2(A) = \left(\frac{3}{4}\right)^{1/2} A\left[\frac{\left(1 + z_1 + 2z_1^{\,2}\right)^{1/2}}{1 + z_1}\right] = (0.8489)A$$

Hence, the second order harmonic balance approximation for the periodic solution of equation (3.26) is

$$x_2(t) = \left(\frac{A}{1 + z_1}\right)[\cos(\Omega_2 t) + z_1 \cos(3\Omega_2 t)]$$

where z_1 and Ω_2 are given above equation.

Recently some authors used iterative technique [49-62] for calculating approximations to the periodic solutions and corresponding frequencies of TNL oscillator differential equations for small and as well as large amplitude of oscillation. The method was originated by Mickens in 1987. In the paper, he provided a general basis for iteration methods as they are currently used in the calculation of approximations to the periodic solutions of various nonlinear oscillatory differential equations successfully.

The general methodology of iteration procedure by Mickens [52] is presented in Chapter IV.

Mickens [52] has presented the following example by iteration procedure**:**
Let us consider the oscillator

$$\ddot{x} + x^{-1} = 0 \tag{3.36}$$

This oscillator can be written as

$$x\ddot{x} + 1 = 0$$

$$\ddot{x} = -(\ddot{x})^2 x \tag{3.37}$$

Adding $\Omega^2 x$ on both sides of equation (3.37), we obtain

$$\ddot{x} + \Omega^2 x = \Omega^2 x - (\ddot{x})^2 x \tag{3.38}$$

The iteration scheme for equation (3.38) as follows

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \Omega_k^2 x_k - (\ddot{x}_k)^2 x_k \tag{3.39}$$

The initial condition is

$$x_0(t) = A\cos\theta \tag{3.40}$$

where $\theta = \Omega_0 t$

For $k = 0$ and substituting equation (3.40) into equation (3.37), we obtain

$$\ddot{x}_1 + \Omega_0^2 x_1 = \left(\Omega_0^2 A\cos\theta\right) - \left(-\Omega_0^2 A\cos\theta\right)^2 \left(A\cos\theta\right)$$

$$= \Omega_0^2 \left[1 - \frac{3A^2\Omega_0^2}{4}\right] A\cos\theta - \left(\frac{A^3\Omega_0^4}{4}\right)\cos 3\theta \tag{3.41}$$

The elimination of secular terms from equation (3.41), we obtain

$$1 - \frac{3A^2\Omega_0^2}{4} = 0$$

$$\Omega_0^2(A) = \left(\frac{4}{3}\right)\frac{1}{A^2} \tag{3.42}$$

Therefore, $x_1(t)$ satisfies the equation

$$\ddot{x}_1 + \Omega_0^2 x_1 = -\left(\frac{A^3\Omega_0^4}{4}\right)\cos 3\theta \tag{3.43}$$

The complementary solution, $x_1^c(t)$ is

$$x_1^c(t) = C\cos\theta .$$

The particular solution, $x_1^{(p)}(t)$ is

$$x_1^{(p)}(t) = \left(\frac{A^3\Omega_0^4}{32}\right)\cos 3\theta = \left(\frac{A}{24}\right)\cos 3\theta$$

Therefore, the full solution is

$$x_1(t) = C\cos\theta + \left(\frac{A}{24}\right)\cos 3\theta \tag{3.44}$$

Using $x_1(0) = A$, then $C = \frac{23}{24}A$

and $$x_1(t) = A\left[\left(\frac{23}{24}\right)\cos\theta + \left(\frac{1}{24}\right)\cos 3\theta\right] \tag{3.45}$$

If the calculation is stopped at this point, then

$$x_1(t) = A\left[\left(\frac{23}{24}\right)\cos(\Omega_0 t) + \left(\frac{1}{24}\right)\cos(3\Omega_0 t)\right]$$

$$\Omega_0(A) = \frac{2}{\sqrt{3}A} = \frac{1.1547}{A}$$

Note that [23]

$$\Omega_{exact}(A) = \frac{\sqrt{2\pi}}{2A} = \frac{1.2533141}{A}$$

and $$\left|\frac{\Omega_{exact}(A) - \Omega_0(A)}{\Omega_{exact}(A)}\right| \times 100 = 7.9\,\%\text{ error}$$

Proceeding to the second level of iteration, $x_2(t)$ must satisfy the equation

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2 x_1 - (\ddot{x}_1)^2 x_1 \tag{3.46}$$

where $x_1(t) = A\left[\left(\frac{23}{24}\right)\cos(\Omega_1 t) + \left(\frac{1}{24}\right)\cos(3\Omega_1 t)\right]$

Let $\theta = \Omega_1 t$ and substitute this $x_1(t)$ into the right hand side of equation (3.46), we obtain

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2\left[\alpha - \left(\frac{3}{4}\right)A^2\Omega_1^2 g(\alpha,\beta)\right]A\cos\theta + HOH\,,$$

where $g(\alpha,\beta) = \alpha^3 + \left(\frac{19}{4}\right)\alpha^2\beta + 66\,\alpha\beta^2 + 27\beta^3\,,$

and $\alpha = \frac{23}{24}, \quad \beta = \frac{1}{24}$

The absence of secular terms gives

$$\Omega_1^2 = \left[\left(\frac{4}{3}\right)\frac{1}{A^2}\right]\left[\frac{\alpha}{g(\alpha,\beta)}\right]$$

$$\Omega_1(A) = \frac{1.0175}{A}$$

with $\left|\frac{\Omega_{exact}(A) - \Omega_1(A)}{\Omega_{exact}(A)}\right| \times 100 = 18.1\%$ error

The existence of such a large percentage error suggests that we should try an alternative iteration scheme and determine if a better result can be found.

Further a generalization of this work was then given by Lim and Wu [53]. Their procedure is as follows:

They assumed the equation in the form

$$\ddot{x} + f(x) = 0\,,\ x(0) = A\,,\ \dot{x}(0) = 0\,, \tag{3.47}$$

where A is given positive constant and $f(x)$ satisfies the condition

$$f(-x) = -f(x)\,. \tag{3.48}$$

Adding $\omega^2 x$ on both sides of equation (3.47), we obtain

$$\ddot{x} + \omega^2 \mathrm{x} = \omega^2 \mathrm{x} - f(x) \equiv g(x)\,, \tag{3.49}$$

where ω is priory unknown frequency of the periodic solution $x(t)$ being sought.

They proposed the iteration scheme of equation (3.49)

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = g(x_{k-1}) + g(x_{k-1})(x_k - x_{k-1}); \text{k = 0, 1, 2,...,} \tag{3.50}$$

where $g_x = \dfrac{\partial g}{\partial x}$ and the inputs of starting functions are

$$x_{-1}(t) = x_0(t) = A\cos(\omega t)\,. \tag{3.51}$$

With the initial conditions

$$x_k(0) = A, \quad \dot{x}_k(0) = 0, \quad k = 1,\ 2,\ 3, \ldots\ \ldots \tag{3.52}$$

Then substituting equation (3.51) into equation (3.50) and expanding the right hand side of equation (3.50) into the Fourier series yields

$$\begin{aligned} g[x_{k-1}(t)] + g_x[x_{k-1}(t)][x_t(t) - x_{k-1}(t)] &= a_1(A,\omega)\cos\omega t \\ &+ \sum_{n=2}^{N} a_{2n-1}(A,\omega)\cos[(2n-1)\omega t], \end{aligned} \tag{3.53}$$

where the coefficients $a_{2n-1}(A,\omega)$ are known functions of A and ω, and the integer N depends upon the function $g(x)$ of the right hand side of equation (3.49) , On view of equation (3.53), the solution of equation is taken to be

$$x_{k+1}(t) = B\cos\omega t - \sum_{n=2}^{N} \frac{a_{2n-1}(A,\omega)}{[(2n-1)^2 - 1]\omega^2}\cos[(2n-1)\omega t]\,, \tag{3.54}$$

where B is, tentatively, an arbitrary constant. In equation (3.54), the particular solution is chosen such that it contains no secular terms [32], which requires that the coefficient $a_1(A,w)$ of right-side term $\cos\omega t$ in equation (3.53) satisfy

$$a_1(A,w) = 0\ . \tag{3.55}$$

Equation (3.55) allows the determination of the frequency as a function A.

Next, the unknown constant B will be computed by imposing the initial conditions in equation (3.52). Finally, putting these steps together gives the solution $x_{k+1}(t)$.

In 2005, this process was extended by Mickens. He consider the equation as

$$\ddot{x} + f(\ddot{x},\dot{x},x) = 0\,,\ x(0) = A\,,\ \dot{x}(0) = 0\,, \tag{3.56}$$

where over dots denote differentiation with respect to time, t.

We choose the natural frequency Ω of this system. Then adding $\Omega^2 x$ on both sides of equation (3.56), we obtain

$$\ddot{x} + \Omega^2 \mathrm{x} = \Omega^2 \mathrm{x} - f(\ddot{x}, \dot{x}, x) \equiv G(x, \dot{x}, \ddot{x}) . \tag{3.57}$$

Now, formulate the iteration scheme as

$$\begin{aligned} \ddot{x}_{k+1} + \Omega_k^2 x_{k+1} &= G(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1});+G_x(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1})(x_k - x_{k-1}) \\ &+ G_{\dot{x}}(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1})(\dot{x}_k - \dot{x}_{k-1}) + G_{\ddot{x}}(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1})(\ddot{x}_k - \ddot{x}_{k-1}) \end{aligned} \tag{3.58}$$

where $G_x = \dfrac{\partial G}{\partial x}$, $G_{\dot{x}} = \dfrac{\partial G}{\partial \dot{x}}$, $G_{\ddot{x}} = \dfrac{\partial G}{\partial \ddot{x}}$. (3.59)

And x_{k+1} satisfies the conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{3.60}$$

The starting function are taken to be [50]

$$x_{-1}(t) = x_0(t) = A\cos(\Omega_0 t) \tag{3.61}$$

The right hand side of equation (3.58) is essentially the first term in a Taylor series expansion of the function $G(x_k, \dot{x}_k, \ddot{x}_k)$ at the point $(x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1})$ [71]. To illustrate this point, note that

$$x_k = x_{k-1} + (x_k - x_{k-1}) , \tag{3.62}$$

and for some function $G(x)$, we have

$$G(x_k) = G[x_{k-1} + (x_k - x_{k-1})] = G(x_{k-1}) + G_x(x_k - x_{k-1}) + \ldots . \tag{3.63}$$

An alternative, but very insightful, modification of above scheme was proposed by Hu [56]. He used the following equation in place of equation (3.62)

$$x_k = x_0 + (x_k - x_o) \tag{3.64}$$

Then, equation (3.63) is changed to

$$G(x_k) = G[x_0 + (x_k - x_0)] = G(x_0) + G_x(x_k - x_0) + \ldots, \tag{3.65}$$

and the corresponding modification to equation (3.58) is

$$\begin{aligned} \ddot{x}_{k+1} + \Omega_k^2 \, x_{k+1} &= G(x_0,\, \dot{x}_0,\, \ddot{x}_0);+G_x(x_0,\, \dot{x}_0,\, \ddot{x}_0)(x_k - x_0) \\ &+ G_{\dot{x}}(x_0,\, \dot{x}_0,\, \ddot{x}_0)(\dot{x}_k - \dot{x}_0) + G_{\ddot{x}}(x_0,\, \dot{x}_0,\, \ddot{x}_0)(\ddot{x}_k - \ddot{x}_0) \end{aligned} \tag{3.66}$$

This scheme is computationally easier to work with, for $k \geq 2$, than the one given in equation (3.58). The essential idea is that if $x_0(t)$ is a good approximation, then the expansion should take place at $x = x_0$. Also, as pointed out by Hu [55], the $x_0(t)$ in $(x_k - x_o)$ is not the same for all k. In particular, $x_0(t)$ in $(x_1 - x_o)$ is the function $A\cos(\Omega_1 t)$, while the $x_0(t)$ in $(x_2 - x_o)$ is the function $A\cos(\Omega_2 t)$.

Further, Mickens [52] used the iterative technique to calculate a higher-order approximation to the periodic solutions of a conservative oscillator for which the elastic force term is proportional to $x^{\frac{1}{3}}$. Hu [72] applied the modified iteration technique of Mickens [52] to find approximate of nonlinear oscillators with fractional powers and quadratic nonlinear oscillator respectively. Recently, Haque [60, 62] has applied Mickens iteration and modified iteration method to determine approximate periodic solutions of a class of nonlinear jerk equations.

CHAPTER IV

Approximate Analytic Solutions of the Inverse Cubic Truly Nonlinear Oscillator by Iterative Method

4.1 Introduction

In this Chapter, we have developed a modified iteration technique for the determination of approximate solution as well as frequency of the Inverse Cubic Truly Nonlinear Oscillator. A particular example governing such a problem is considered and the solution of the problem is obtained using the presented method.

4.2 The method

Assume that the nonlinear oscillator

$$F(\ddot{x}, x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{4.1}$$

and further assume that it can be rewritten to the form

$$\ddot{x} + f(\ddot{x}, x) = 0, \tag{4.2}$$

where over dots denote differentiation with respect to time , t .

We choose the natural frequency Ω of this system. Then adding $\Omega^2 x$ to both sides of equation (4.2), we obtain

$$\ddot{x} + \Omega^2 \mathrm{x} = \Omega^2 \mathrm{x} - f(\ddot{x}, x) \equiv G(x, \ddot{x}) . \tag{4.3}$$

Now, we formulate the iteration scheme as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k, \ddot{x}_k); \quad k = 0, 1, 2, 3, \ldots\ldots\ldots\ldots, \tag{4.4}$$

together with initial condition

$$x_0(t) = A\cos(\Omega_0 t). \tag{4.5}$$

Hence x_{k+1} satisfies the initial conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{4.6}$$

At each stage of the iteration, Ω_k is determined by the requirement that secular terms [32] should not occur in the full solution of $x_{k+1}(t)$.

The above procedure gives the sequence of solutions: $x_0(t),\ x_1(t),\ x_2(t), \cdots$.

The method can be proceed to any order of approximation; but due to growing algebraic complexity the solution is confined to a lower order usually the second [49].

At this point, the following observations should be noted:

(a) The solution for $x_{k+1}(t)$ depends on having the solutions for k less than $(k+1)$.

(b) The linear differential equation for $x_{k+1}(t)$ allows the determination of Ω_k by the requirement that secular terms be absent. Therefore, the angular frequency, "Ω" appearing on the right-hand side of equation (4.4) in the function $x_k(t)$, is Ω_k.

4.3 An Example

Let us consider the Oscillator

$$\ddot{x}+x^{-1/3}=0,$$

$$\ddot{x}=-x^{-1/3} \quad . \tag{4.7}$$

4.4 Solution Procedure

Adding $\Omega^2 x$ on both sides of equation (4.7), we get

$$\ddot{x}+ \Omega^2\mathrm{x}=\Omega^2\mathrm{x}\text{-} x^{-1/3}. \tag{4.8}$$

According to equation (4.4), the iteration scheme of equation (4.8) is

$$\ddot{x}_{k+1}+\Omega_k^2 x_{k+1}=\Omega_k^2 x_k - x_k^{-1/3} \tag{4.9}$$

The initial condition is

$$x_0(t)=Acos\theta, \tag{4.10}$$

where $\theta=\Omega_0\, t$

The first approximation $x_1(t)$ and the frequency Ω_0 will be obtained by putting $k=0$ in equation (4.9) and using equation (4.10), we obtain

$$\ddot{x}_1+\Omega_0^2 x_1=\Omega_0^2 A\cos\theta-(A\cos\theta)^{-1/3} \tag{4.11}$$

Now expanding $(\cos\theta)^{-1/3}$ in a truncated Fourier series then Eq. (4.11) reduces to

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A\cos\theta - \frac{1}{A^{1/3}}\left(1.42635\ \cos\theta - 0.713174\ \cos 3\theta\right.$$

$$+ 0.50941\ \cos 5\theta - 0.407528\ \cos 7\theta + 0.344831\ \cos 9\theta$$

$$\left. -0.301728\ \cos 11\theta\right),$$

$$\ddot{x}_1 + \Omega_0^2 x_1 = (\Omega_0^2 A - \frac{1.42635}{A^{1/3}})\cos\theta + \frac{0.713174}{A^{1/3}}\cos 3\theta$$

$$-\frac{0.50941}{A^{1/3}}\cos 5\theta + \frac{0.407528}{A^{1/3}}\cos 7\theta \qquad (4.12)$$

$$-\frac{0.344831}{A^{1/3}}\cos 9\theta + \frac{0.301728}{A^{1/3}}\cos 11\theta .$$

Secular terms can be eliminated if the coefficient of $\cos\theta$ is set to be zero

i.e. $\Omega_0^2 A - \dfrac{1.42635}{A^{1/3}} = 0$,

$$\Rightarrow \Omega_0^2 = \frac{1.42635}{A^{4/3}},$$

$$\therefore\ \Omega_0 = \frac{1.1942989575}{A^{2/3}}.$$

This is the first approximate frequency of the oscillator (4.7).

Note that $\Omega_{exact}\left(A\right) = \dfrac{1.154700538}{A^{2/3}}$. (4.13)

And $\left|\dfrac{\Omega_{exact} - \Omega_0}{\Omega_{exact}}\right| \times 100 = \left|\dfrac{1.154700538 - 1.1942989575}{1.154700538}\right| \times 100 = 3.43\%$ error.

After reducing the secular term from the equation (4.12), we obtain

$$\ddot{x}_1 + \Omega_0^2 x_1 = \frac{1}{A^{1/3}}\left(0.713174\ \cos 3\theta - 0.50941\ \cos 5\theta + 0.407528\ \cos 7\theta\right.$$

$$\left. -0.344831\ \cos 9\theta + 0.301728\ \cos 11\theta\right). \qquad (4.14)$$

The complementary solution, $x_1^c\left(t\right)$ is

$$x_1^c\left(t\right) = C\cos\theta .$$

The particular solution, $x_1^{(p)}(t)$ is

$$x_1^p(t)=\frac{1}{A^{1/3}}\left(\frac{0.713174}{D^2+\Omega_0^2}\cos 3\theta-\frac{0.50941}{D^2+\Omega_0^2}\cos 5\theta+\frac{0.407528}{D^2+\Omega_0^2}\cos 7\theta\right.$$

$$\left.-\frac{0.344831}{D^2+\Omega_0^2}\cos 9\theta+\frac{0.301728}{D^2+\Omega_0^2}\cos 11\theta\right),$$

$$=\frac{1}{A^{1/3}}\left(\frac{0.713174}{-9\Omega_0^2+\Omega_0^2}\cos 3\theta-\frac{0.50941}{-25\Omega_0^2+\Omega_0^2}\cos 5\theta+\frac{0.407528}{-49\Omega_0^2+\Omega_0^2}\cos 7\theta\right.$$

$$\left.-\frac{0.344831}{-81\Omega_0^2+\Omega_0^2}\cos 9\theta+\frac{0.301728}{-121\Omega_0^2+\Omega_0^2}\cos 11\theta\right),$$

$$=\frac{1}{A^{1/3}}\left(-\frac{0.713174}{8\Omega_0^2}\cos 3\theta+\frac{0.50941}{24\Omega_0^2}\cos 5\theta-\frac{0.407528}{48\Omega_0^2}\cos 7\theta\right.$$

$$\left.+\frac{0.344831}{80\Omega_0^2}\cos 9\theta-\frac{0.301728}{120\Omega_0^2}\cos 11\theta\right),$$

$$=\frac{A^{4/3}}{A^{1/3}}\left(-\frac{0.713174}{11.4108}\cos 3\theta+\frac{0.50941}{34.2324}\cos 5\theta-\frac{0.407528}{68.4648}\cos 7\theta\right.$$

$$\left.+\frac{0.344831}{114.108}\cos 9\theta-\frac{0.301728}{171.162}\cos 11\theta\right),$$

$$=A\left(-0.0625\cos 3\theta+0.014881\cos 5\theta-0.005952\cos 7\theta\right.$$
$$\left.+0.003021\cos 9\theta-0.001762\cos 11\theta\right).$$

Therefore, the complete solution is

$$x_1(t)=C\cos\theta-A\left(-0.0625\cos 3\theta+0.014881\cos 5\theta-0.005952\cos 7\theta\right.$$
$$\left.+0.003021\cos 9\theta-0.001762\cos 11\theta\right). \tag{4.15}$$

Using $x_1(0)=A$ into Eq. (4.15), we have

$$x_1(0)=C+A\left(-0.0625+0.014881-0.005952+0.003021-0.001762\right),$$

$$\Rightarrow A=C-0.052312A,$$

$$C=1.052312A.$$

Therefore,

$$x_1(t) = A\big(1.052312\cos\theta - 0.0625\cos 3\theta + 0.014881\cos 5\theta - 0.005952\cos 7\theta + 0.003021\cos 9\theta - 0.001762\cos 11\theta\big)\,. \tag{4.16}$$

This is the first approximate solution of the oscillator.

Proceeding to the second level of iteration putting $k=1$ and $\theta = \Omega_1 t$ then $x_2(t)$ satisfies the equation

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2 x_1 - (x_1)^{-1/3}\,, \tag{4.17}$$

Substituting equation (4.16) into equation (4.17), we obtain

$$\begin{aligned}\ddot{x}_2 + \Omega_1^2 x_2 &= \Omega_1^2 A\big(1.052312\cos\theta - 0.0625\cos 3\theta + 0.014881\cos 5\theta \\ &\quad - 0.005952\cos 7\theta + 0.003021\cos 9\theta - 0.001762\cos 11\theta\big) \\ &\quad - \frac{1}{A^{1/3}}\big(1.052312\cos\theta - 0.0625\cos 3\theta + 0.014881\cos 5\theta \\ &\quad - 0.005952\cos 7\theta + 0.003021\cos 9\theta - 0.001762\cos 11\theta\big)^{-1/3}\,.\end{aligned} \tag{4.18}$$

Now expanding second term on right hand side of equation (4.17) in a truncated Fourier series, we obtain

$$\begin{aligned}\ddot{x}_2 + \Omega_1^2 x_2 &= \Omega_1^2 A\big(1.052312\cos\theta - 0.0625\cos 3\theta + 0.014881\cos 5\theta \\ &\quad - 0.005992\cos 7\theta + 0.003021\cos 9\theta - 0.001762\cos 11\theta\big) \\ &\quad - \frac{1}{A^{1/3}}\big(1.38767\cos\theta - 0.644734\cos 3\theta + 0.454911\cos 5\theta \\ &\quad - 0.362239\cos 7\theta + 0.306038\cos 9\theta - 0.267895\cos 11\theta\big)\,,\end{aligned}$$

$$\begin{aligned}\ddot{x}_2 + \Omega_1^2 x_2 &= \left(1.052312\,\Omega_1^2 A - \frac{1.38767}{A^{1/3}}\right)\cos\theta + \Omega_1^2 A\big(-0.0625\cos 3\theta + 0.014881\cos 5\theta \\ &\quad - 0.005992\cos 7\theta + 0.003021\cos 9\theta - 0.001762\cos 11\theta\big) \\ &\quad - \frac{1}{A^{1/3}}\big(-0.644734\cos 3\theta + 0.454911\cos 5\theta - 0.362239\cos 7\theta \\ &\quad + 0.306038\cos 9\theta - 0.267895\cos 11\theta\big).\end{aligned} \tag{4.19}$$

Secular terms can be eliminated if the coefficient of $\cos\theta$ is set to be zero

$$1.052312\,\Omega_1^2 A - \frac{1.38767}{A^{1/3}} = 0,$$

$$\Rightarrow \Omega_1^2 = \frac{1.38767}{1.05231A^{4/3}},$$

$$= \frac{1.31869}{A^{4/3}}.$$

$$\Omega_1 = \frac{1.14834}{A^{2/3}}. \tag{4.20}$$

This is the second approximate frequency of the oscillator (4.7).

And $\left|\dfrac{\Omega_{exact} - \Omega_1}{\Omega_{exact}}\right| \times 100 = \left|\dfrac{1.154700538 - 1.148340892}{1.154700538}\right| \times 100 = 0.551\%$ error.

After reducing the secular term from the equation (4.19), we obtain

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2 A\big(-0.0625\cos 3\theta + 0.014881\cos 5\theta - 0.005992\cos 7\theta$$
$$+ 0.003021\cos 9\theta - 0.001762\cos 11\theta\big) - \frac{1}{A^{1/3}}\big(-0.644734\cos 3\theta$$
$$+ 0.454911\cos 5\theta - 0.362239\cos 7\theta + 0.306038\cos 9\theta - 0.267895\cos 11\theta\big),$$

$$= \frac{1.31869A}{A^{4/3}}\big(-0.0625\cos 3\theta + 0.014881\cos 5\theta - 0.005992\cos 7\theta + 0.003021\cos 9\theta$$
$$-0.001762\cos 11\theta\big) + \frac{0.644734}{A^{1/3}}\cos 3\theta - \frac{0.454911}{A^{1/3}}\cos 5\theta + \frac{0.362239}{A^{1/3}}\cos 7\theta$$
$$- \frac{0.306038}{A^{1/3}}\cos 9\theta + \frac{0.267895}{A^{1/3}}\cos 11\theta,$$

$$= \frac{1}{A^{1/3}}\left(0.644734 - \frac{1.31869}{16}\right)\cos 3\theta + \frac{1}{A^{1/3}}\left(\frac{6.59345}{336} - 0.454911\right)\cos 5\theta$$
$$+ \frac{1}{A^{1/3}}\left(0.362239 - \frac{1.31869}{168}\right)\cos 7\theta + \frac{1}{A^{1/3}}\left(\frac{14.50559}{3640} - 0.306038\right)\cos 9\theta$$
$$+ \frac{1}{A^{1/3}}\left(0.267895 - \frac{14.50559}{6240}\right)\cos 11\theta,$$

$$= \frac{1}{A^{1/3}}\left(0.562315\cos 3\theta - 0.435288\cos 5\theta + 0.3543897\cos 7\theta \right.$$
$$\left. -0.302053\cos 9\theta + 0.2655704\cos 11\theta\right) . \quad (4.21)$$

The complementary solution, $x_2^c(t)$ is

$$x_2^c(t) = C_1 \cos\theta$$

The particular solution, $x_2^{(p)}(t)$ is

$$x_2^{(p)}(t) = \frac{1}{A^{1/3}}\left(\frac{0.562315}{D^2+\Omega_1^2}\cos 3\theta - \frac{0.435288}{D^2+\Omega_1^2}\cos 5\theta + \frac{0.3543897}{D^2+\Omega_1^2}\cos 7\theta \right.$$
$$\left. - \frac{0.302053}{D^2+\Omega_1^2}\cos 9\theta + \frac{0.2655704}{D^2+\Omega_1^2}\cos 11\theta\right),$$

$$= \frac{1}{A^{1/3}}\left(\frac{0.562315}{-9\Omega_1^2+\Omega_1^2}\cos 3\theta - \frac{0.435288}{-25\Omega_1^2+\Omega_1^2}\cos 5\theta + \frac{0.3543897}{-49\Omega_1^2+\Omega_1^2}\cos 7\theta \right.$$
$$\left. - \frac{0.302053}{-81\Omega_1^2+\Omega_1^2}\cos 9\theta + \frac{0.2655704}{-121\Omega_1^2+\Omega_1^2}\cos 11\theta\right),$$

$$= \frac{1}{A^{1/3}}\left(-\frac{0.562315}{8\Omega_1^2}\cos 3\theta + \frac{0.435288}{24\Omega_1^2}\cos 5\theta - \frac{0.3543897}{48\Omega_1^2}\cos 7\theta \right.$$
$$\left. + \frac{0.302053}{80\Omega_1^2}\cos 9\theta - \frac{0.2655704}{120\Omega_1^2}\cos 11\theta\right),$$

$$= \frac{1}{A^{1/3}}\left(-\frac{0.562315}{8}\frac{A^{4/3}}{1.31869}\cos 3\theta + \frac{0.435288}{24}\frac{A^{4/3}}{1.31869}\cos 5\theta \right.$$
$$- \frac{0.3543897}{48}\frac{A^{4/3}}{1.31869}\cos 7\theta + \frac{0.302053}{80}\frac{A^{4/3}}{1.31869}\cos 9\theta$$
$$\left. - \frac{0.2655704}{120}\frac{A^{4/3}}{1.31869}\cos 11\theta\right),$$

$$= \frac{A^{4/3}}{A^{1/3}}\left(-0.0533024\cos 3\theta + 0.01320365\cos 5\theta - 0.00559835\cos 7\theta \right.$$
$$\left. + 0.0028631919\cos 9\theta - 0.0016782463\cos 11\theta\right),$$

$$= A\left(-0.0533024\cos 3\theta + 0.01320365\cos 5\theta - 0.00559835\cos 7\theta \right.$$
$$\left. + 0.0028631919\cos 9\theta - 0.0016782463\cos 11\theta\right).$$

Therefore, the complete solution is

$$x_2(t) = C_1\cos\theta + A\big(-0.0533024\cos 3\theta + 0.01320365\cos 5\theta - 0.00559835\cos 7\theta + 0.0028631919\cos 9\theta - 0.0016782463\cos 11\theta\big)\,. \tag{4.22}$$

Using $x_2(0) = A$ into equation (4.20), we obtain

$$x_2(0) = C_1 + A\big(-0.0533024 + 0.01320365 - 0.00559835 + 0.0028631919 - 0.0016782463\big)\,,$$

$$A = C_1 - 0.044512159A,$$

$$C_1 = 1.04451216A\,.$$

Therefore,

$$x_2(t) = A\big(1.0445122\cos\theta - 0.0533024\cos 3\theta + 0.01320365\cos 5\theta - 0.00559835\cos 7\theta + 0.0028631919\cos 9\theta - 0.0016782463\cos 11\theta\big)\,. \tag{4.23}$$

This is the second approximate solution of the oscillator.

Proceeding to the third level of iteration $k = 2$ and $\theta = \Omega_2\, t$ then $x_3(t)$ satisfies the equation

$$\ddot{x}_3 + \Omega_2^2 x_3 = \Omega_2^2 x_2 - (x_2)^{-1/3}\,, \tag{4.24}$$

Substituting equation (4.23) into equation (4.24), we obtain

$$\begin{aligned}\ddot{x}_3 + \Omega_2^2 x_3 = {} & \Omega_2^2 A\big(1.0445122\cos\theta - 0.0533024\cos 3\theta + 0.01320365\cos 5\theta \\ & - 0.00559835\cos 7\theta + 0.0028631919\cos 9\theta - 0.0016782463\cos 11\theta\big) \\ & - \frac{1}{A^{1/3}}(1.0445122\cos\theta - 0.0533024\cos 3\theta + 0.01320365\cos 5\theta \\ & - 0.00559835\cos 7\theta + 0.0028631919\cos 9\theta - 0.0016782463\cos 11\theta)^{-1/3}.\end{aligned} \tag{4.25}$$

Now expanding the term on right hand side of equation (4.25) in a truncated Fourier series, we obtain

$$\ddot{x}_3 + \Omega_2^2 x_3 = \Omega_2^2 A\big(1.0445122\cos\theta - 0.0533024\cos 3\theta + 0.01320365\cos 5\theta$$

$$- 0.00559835\cos 7\theta + 0.0028631919\cos 9\theta - 0.0016782463\cos 11\theta\big)$$

$$- \frac{1}{A^{1/3}}(1.39292\cos\theta - 0.65307\cos 3\theta + 0.460341\cos 5\theta$$

$$- 0.366474\cos 7\theta + 0.309581\cos 9\theta - 0.270977\cos 11\theta),$$

$$\ddot{x}_3 + \Omega_2^2 x_3 = \left(1.0445122\Omega_2^2 A - \frac{1.39292}{A^{1/3}}\right)\cos\theta + \Omega_2^2 A\big(-0.0533024\cos 3\theta$$

$$+ 0.01320365\cos 5\theta - 0.00559835\cos 7\theta + 0.0028631919\cos 9\theta$$

$$- 0.0016782463\cos 11\theta\big) - \frac{1}{A^{1/3}}(-0.65307\cos 3\theta + 0.460341\cos 5\theta$$

$$- 0.366474\cos 7\theta + 0.309581\cos 9\theta - 0.270977\cos 11\theta)\ .$$

Now secular terms can be eliminated if the coefficient of $\cos\theta$ is set to be zero

$$1.0445122\Omega_2^2 A - \frac{1.39292}{A^{1/3}} = 0\,.$$

$$\Omega_2^2 = \frac{1.39292}{1.0445122 A^{4/3}}\,,$$

$$= \frac{1.3335613}{A^{4/3}}\,.$$

$$\Omega_2 = \frac{1.154799}{A^{2/3}}\,.$$

This is the third approximate frequency of the oscillator (4.7).

And $\left|\frac{\Omega_{exact} - \Omega_2}{\Omega_{exact}}\right| \times 100 = \left|\frac{1.154700538 - 1.154799}{1.154700538}\right| \times 100 = 0.00852\,\%$ error.

4.5 Results and discussions

An iterative method is presented to obtain approximate solution of inverse cubic nonlinear oscillator. In order to test the accuracy of the modified approach of iteration method, we compare our results with the other existing results from different methods. To show the accuracy, we have calculated the percentage errors by the definitions

$$\left|\frac{\Omega_e(A)-\Omega_i(A)}{\Omega_e(A)}\right|\times 100, \quad \text{where } i=0,1,2\cdots\cdots$$

We have used a modified iteration method to obtain approximate solutions of the above oscillator. It has been shown that, in most of the cases our solutions give significant results than other existing results.

Herein we have calculated the first, second and third approximate frequencies which are denoted by Ω_0, Ω_1, and Ω_2 respectively . All the results are given in the following table, to compare the approximate frequencies. We have also given the existing results determined by Mickens iteration method [52] and Mickens HB method [52].

4.6 Table

Comparison of the approximate frequencies with exact frequency Ω_e (Mickens [52]) of $\ddot{x} + x^{-1/3} = 0.$

Exact Frequency Ω_e	$\frac{1.154700}{A^{2/3}}$		
Amplitude A	First Approximate Frequency Ω_0 Er(%)	Second Approximate Frequency Ω_1 Er(%)	Third Approximate Frequency Ω_2 Er(%)
Adopted Method	$\frac{1.19429}{A^{2/3}}$ 3.43	$\frac{1.14834}{A^{2/3}}$ 0.55	$\frac{1.154799}{A^{2/3}}$ 0.0085
Mickens HB method [52]	$\frac{1.31329}{A^{2/3}}$ 13.7	$\frac{1.18824}{A^{2/3}}$ 2.9	–
Mickens iteration method [52]	$\frac{1.08148}{A^{2/3}}$ 6.3	$\frac{1.07634}{A^{2/3}}$ 6.78	$\frac{0.988591}{A^{2/3}}$ 14.38

From the table, it is seen that the third-order approximate frequency obtained by adopted method is almost same with exact frequency. It is found that, in each of the cases our solution gives significantly better result than other existing results. The compensation of this method consists of its simplicity, computational efficiency and convergence. It is also observed that the Mickens' iteration technique is convergent for this oscillator.

CHAPTER V

Convergence and Consistency Analysis

The basic idea of iteration methods is to construct a sequence of solutions x_k (as well as frequencies Ω_k) that have the property of convergence

$$x_e = \lim_{k \to \infty} x_k \quad \text{or,} \quad \Omega_e = \lim_{k \to \infty} \Omega_k$$

Here x_e is the exact solution of the given nonlinear oscillator.

In our technique, it has been shown that the solution gives the less error in each iterative step compared to the previous iterative step and finally $|\Omega_2 - \Omega_e| = |1.154799 - 1.154700| < \varepsilon$, where ε is a small positive number and A is chosen to be unity. From this, it is clear that the adopted method is convergent.

An iterative method of the form represented by equation (4.4) with initial guesses given in equation (4.5) and equation (4.6) is said to be consistent if

$$\lim_{k \to \infty} |x_k - x_e| = 0 \quad \text{or,} \quad \lim_{k \to \infty} |\Omega_k - \Omega_e| = 0.$$

In this thesis, we observe that

$$\lim_{k \to \infty} |\Omega_k - \Omega_e| = 0 \ \text{ as } |\Omega_2 - \Omega_e| \approx 0.$$

Thus the consistency of the method is achieved.

CHAPTER VI

Conclusions

In this thesis, we used a simple but effective modification of the iteration method to handle strongly nonlinear oscillators. With the method, the analytical approximate solutions and the corresponding frequency, valid for small as well as large amplitudes of oscillation, can be obtained. An example is given to illustrate the effectiveness and convenience of this method. The approximate frequencies obtained by the method shows a good agreement with the exact frequency. The percentage of error between exact frequency and third approximate frequency of the adopted method is almost equal. The results anticipated were compared with the other methods. The obtained results show that the modification of the iteration method is more accurate than other methods and this method is convergent and consistent. The performance of this method is reliable, simple and gives many new solutions. Moreover the present work can be used as paradigms for many others application in searching for periodic solution of nonlinear oscillations and so can be found widely applicable in engineering and science.

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