An Analytical Technique for Solving Second Order Strongly Damped Nonlinear Oscillator with a Fractional Power Restoring Force

by

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Dedication

I dedicate this thesis work to my beloved Parents

Md. Mokleshur Rahman** & **Nasima Rahman

and

My Elder Brother ***Md. Nazmul Hasan*** & Younger Sister ***Rabeya***

Whose affection, encouragement and pray makes me able to get such success and honor.

Along with all of my respectable

Teachers

Declaration

This is to certify that the thesis work entitled "**An Analytical Technique for Solving Second Order Strongly Damped Nonlinear Oscillator with a Fractional Power Restoring Force**" has been carried out by **Md. Saiful Islam**, Roll No: **1551504**, in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna 9203, Bangladesh. The above thesis work or any part of the thesis work has not been submitted anywhere for the award of any degree or diploma.

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Abstract

In this thesis, an analytical technique has been developed for solving strongly nonlinear damped systems with $x^{1/3}$ restoring force by combining He's homotopy perturbation method (HPM) and the extended form of the Krylov-Bogoliubov-Mitropolskii (KBM) method. The presented method has been justified by an example. We have also established the relationship between amplitude and approximate angular frequency. In this study, the presented technique gives desired results avoiding any numerical complexity. Graphical representation of any physical system is important. So, approximate solutions are compared with those numerical solutions obtained by fourth order Runge-Kutta method in graphically. The results in figures show that the approximations are of extreme accuracy with small and significant damping. The presented method is simple and suitable for solving the above mentioned nonlinear damped systems.

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Nomenclature

ε	Small positive parameter
ω	Angular frequency
a	Amplitude
φ, ψ	Phase
T	Period
t	Time
$\dot{x}$	First derivative of x with respect to time t
$\ddot{x}$	Second derivative of x with respect to time t
Ω	Domain
Γ	The boundary of the domain
$f(r)$	Analytical function
k	Constant
$f^{(k-1)}$	The periodic functions of ψ with period 2π
$(u_k)_\psi$	Partial derivative of functions u_k with respect to ψ
λ	Unknown function which can be evaluated by eliminating the secular terms from particular solution.

CHAPTER I

Introduction

Most of the real-life systems are modeled by nonlinear differential equations. Obtaining exact solutions for these nonlinear problems are very difficult and time consuming for scientists and researchers. Thus, we try to find new approaches to overcome some of these difficulties.

The subject of differential equations constitutes a large and very important branch of modern mathematics. Numerous physical, mechanical, chemical, biological, mechanics in which we want to describe the motion of the body (automobile, electron or satellite) under the action of a given force and many other relations appear mathematically in the form of differential equations that are linear or nonlinear, autonomous or non-autonomous. Also in ecology and economics the differential equations are vastly used. Basically, many differential equations involving physical phenomena are nonlinear such as spring-mass systems, resistor-capacitor-inductor circuits, bending of beams, chemical reactions, the motion of a pendulum, the motion of the rotating mass around another body, population model, etc.

In mathematics and physics, linear generally means "simple" and nonlinear means "complicated". The methods for solving linear equations are very well developed because linear equations are simple enough to be solvable. Usually nonlinear equations can not be solved exactly and are the subject of much on-going research. In such situations mathematicians, physicists and engineers convert the nonlinear equations into linear equations, i.e., they linearize them by imposing some special conditions. Small oscillations are well-known example of the linearization for the physical problems. But, such a linearization is not always possible and when it is not, then the original nonlinear equation itself must be considered. The study of nonlinear equations is generally confined to a variety of rather special cases and one must resort to various methods of approximations.

At first van der Pol [1] paid attention to the new (self-excitations) oscillations and indicated that their existence is inherent in the nonlinearity of the differential equations characterizing the process. Thus, this nonlinearity appears as the very essence of these phenomena and by linearizing the differential equations in the sense of small oscillations, one simply eliminates the possibility of investigating such problems. Thus, it is necessary to deal with the nonlinear differential equations directly instead of evading them by dropping the nonlinear terms. To solve nonlinear differential equations, there exist some methods such as perturbation method [2-25], homotopy perturbation method [26-36], harmonic balance method [37], variational iterative method [39], etc. Among the methods, the method of perturbations, i.e., asymptotic expansions in terms of a small parameter are first and foremost.

A perturbation method known as "the asymptotic averaging method" in the theory of nonlinear oscillations was first introduced by Krylov and Bogoliubov (KB) [2] in 1947. Primarily, the method was developed only for obtaining the periodic solutions of second order weakly nonlinear differential systems. Later, the method of KB has been improved and justified by Bogoliubov and Mitropolskii [3] in 1961. In literature, this method is known as the Krylov-Bogoliubov-Mitropolskii (KBM) [2, 3] method.

A perturbation method is based on the following aspects: the equations to be solved are sufficiently "smooth" or sufficiently differentiable for a number of times in the required regions of variables and parameters. The KBM [2, 3] method was developed for obtaining only the periodic solutions of second order weakly nonlinear differential equations without damping. Now a days, this method is used for obtaining the solutions of second, third and fourth order weakly nonlinear differential systems for oscillatory, damped oscillatory, over damped, critically damped and more critically damped cases by imposing some special restrictions with quadratic and cubic nonlinearities.

Several authors [5-25] have investigated and developed many significant results concerning the solutions of the weakly nonlinear differential systems. Extensive uses have been made and some important works are done by several authors [5-25] based on the **KBM** method.

The method of KB [2] is an asymptotic method in the sense that $\varepsilon \to 0$. An asymptotic series itself may not be convergent but for a fixed number of terms, the approximate solution approaches toward the exact solution. Two widely spread methods in this theory are mainly used in the literature; one is averaging asymptotic KBM method and the other is multiple-time scale method. The KBM method is particularly convenient and extensively used technique for determining the approximate solutions among the methods used for solving the weakly nonlinear differential systems. The KBM method starts with the solution of linear equation (sometimes called the generating solution of the linear equation) assuming that in the nonlinear case, the amplitude and phase variables in the solution of the linear differential equation are time dependent functions instead of constants. This method introduces an additional condition on the first derivative of the assumed solution for determining the solution of second order nonlinear differential systems. The KBM method demands that the asymptotic solutions are free from secular terms. These assumptions are mainly valid for second and third order equations. But sometimes the correction terms for the fourth order differential equations contain secular terms, although the solutions are generated by the classical KBM asymptotic method. For this reason, the traditional solutions fail to explain the proper situation of the systems. To remove the presence of secular terms for obtaining the desired results, one needs to impose some special conditions.

Ji-Huan He [26-29] has developed a homotopy perturbation method for solving second order strongly nonlinear differential systems without considering any damping effects. Uddin *et al.*[30, 31] have presented approximate analytical techniques for solving second order strongly nonlinear oscillatory differential systems with quadratic and cubic nonlinearities in presence of small damping by combing the He's [26-29] homotopy perturbation and the extended form of the KBM [2-4] methods.

The KBM [2, 3] method is failed to tackle the strongly and weakly nonlinear differential systems with high order nonlinearity. Also He's [26-29] homotopy perturbation technique is failed to handle both the strongly and weakly nonlinear differential systems in presence of damping. In this thesis, He's homotopy perturbation method (HPM) has been extended for obtaining the approximate solutions of second order strongly nonlinear differential systems with a $x^{1/3}$ restoring force in presence of small and significant damping based on the extended form of the KBM method. The results may be used in mechanics, physics, chemistry, plasma physics, circuit and control theory, population dynamics, economics, etc.

In **Chapter II**, the review of literature is presented. An approximate analytical technique has been developed for solving second order strongly nonlinear damped oscillator with a $x^{1/3}$ restoring force in **Chapter III.** Results are discussed in **Chapter IV.** Finally, in **Chapter V**, the concluding remarks are given.

CHAPTER II

Literature Review

In generally, nonlinear differential equations are difficult to solve and their exact solutions are often difficult to determine. But mathematical formulations of many physical problems often results in differential equations that are linear or nonlinear. In many situations, linear differential equation is substituted for a nonlinear differential equation, which approximates the original equation closely enough to give expected results. In many cases such a linearization is not always possible and when it is not, the original nonlinear differential equation must be considered directly. During the last several decades in the $20th$ century, some famous Russian scientists like Krylov and Bogoliubov [2], Bogoliubov and Mitropolskii [3], Mitropolskii [4], have investigated the nonlinear dynamics. For solving nonlinear differential equations, there exists some methods. Among the methods, the method of perturbations, i.e., an asymptotic expansion in terms of small parameter is well known. Firstly, Krylov and Bogoliubov (KB) [2] considered the equation of the form

$$\ddot{x} + \omega^2 x = \varepsilon\, f(x, \dot{x}, t, \varepsilon), \tag{2.1}$$

where $\ddot{x}$ denotes the second order derivative with respect to *t*, ε is a small positive parameter and *f* is a power series in ε, whose coefficients are polynomials in $x, \dot{x}, \sin t$ and $\cos t$ and the procedure was proposed by Krylov and Bogoliubov [2]. In general, f does not contain either ε or t explicitly. In literature, the method presented in [2, 3] is known as Krylov-Bogoliubov-Mitropolskii (KBM) method. Poincare [5] discussed only periodic solutions to describe the behavior of the oscillators by the perturbation method.

The KBM [2, 3] method started with the solution of the linear equation, assuming that in the nonlinear systems, the amplitude and phase variables in the solutions of the linear equations are time dependent functions rather than constants. This procedure introduces an additional condition on the first derivative of the assumed solution for determining the desired results. Duffing [6] has investigated many significant results for obtaining the solutions of the following nonlinear damped system

$$\ddot{x} + 2k\,\dot{x} + \omega^2 x = -\varepsilon\, x^3. \tag{2.2}$$

Sometimes, different types of nonlinear phenomena occur when the amplitude of the dynamic system is less than or greater than unity. The damping is negative when the amplitude is less than unity and the damping is positive when the amplitude is greater than unity. The governing equation having these phenomena is

$$\ddot{x} - \varepsilon (1 - x^2)\, \dot{x} + x = 0. \tag{2.3}$$

In literature, this equation is known as van der Pol [1] equation and which is used in electrical circuit theory. Kruskal [7] has extended the KB [2] method to solve the fully nonlinear differential equation of the following form

$$\ddot{x} = F(x, \dot{x}, \varepsilon). \tag{2.4a}$$

Cap [8] has studied nonlinear system of the form

$$\ddot{x} + \omega^2 x = \varepsilon F(x, \dot{x}). \tag{2.4b}$$

Generally, F does not contain ε or t, thus the equation (2.1) becomes

$$\ddot{x} + \omega^2 x = \varepsilon\, f(x, \dot{x}). \tag{2.5}$$

In the treatment of nonlinear oscillations by the perturbation method, only periodic solutions are discussed, transients are not considered by different investigators, where as KB [2] have discussed transient response.

When $\varepsilon = 0,$ the equation (2.5) reduces to linear equation and its solution is obtained by

$$x = a \cos(\omega t + \varphi), \tag{2.6}$$

where a and φ are arbitrary constants and the values of a and φ are determined by using the given initial conditions.

When $\varepsilon \neq 0,$ but is sufficiently small, then KB [2] have assumed that the solution of equation (2.5) is still given by equation (2.6) together with the derivative of the form

$$\dot{x} = -a\, \omega \sin(\omega t + \varphi), \tag{2.7}$$

where a and φ are functions of t, rather than being constants.

In this case, the solution of equation (2.5) is considered as the following form

$$x = a(t)\cos(\omega t + \varphi(t)), \tag{2.8}$$

and the derivative of the solution is

$$\dot{x} = -a(t)\,\omega\sin(\omega t + \varphi(t)). \tag{2.9}$$

Differentiating the assumed solution equation (2.8) with respect to time *t*, we obtain

$$\dot{x} = \dot{a}\cos\psi - a\,\omega\sin\psi - a\,\dot{\varphi}\sin\psi, \quad \psi = \omega t + \varphi(t). \tag{2.10}$$

Using the equations (2.7) and (2.10), we get

$$\dot{a}\cos\psi = a\,\dot{\varphi}\sin\psi. \tag{2.11}$$

Again, differentiating equation (2.9) with respect to *t*, we have

$$\ddot{x} = -\dot{a}\,\omega\sin\psi - a\,\omega^2\cos\psi - a\,\omega\dot{\varphi}\cos\psi. \tag{2.12}$$

Putting the value of $\ddot{x}$ from equation (2.12) into the equation (2.5) and using equations (2.8) and (2.9), we obtain

$$\dot{a}\,\omega\sin\psi + a\,\omega\dot{\varphi}\cos\psi = -\varepsilon\, f(a\cos\psi, -a\,\omega\sin\psi). \tag{2.13}$$

Solving equations (2.11) and (2.13), we have

$$\dot{a} = -\frac{\varepsilon}{\omega}\sin\psi\, f(a\cos\psi, -a\omega\sin\psi), \tag{2.14}$$

$$\dot{\varphi} = -\frac{\varepsilon}{a\,\omega}\cos\psi\, f(a\cos\psi, -a\,\omega\sin\psi). \tag{2.15}$$

It is observed that a basic differential equation (2.5) of the second order in the unknown x, reduces to two first order differential equations (2.14) and (2.15) in the unknowns a and φ.

Moreover, $\dot{a}$ and $\dot{\varphi}$ are proportional to ε; a and φ are slowly varying functions of the time period $T = 2\pi/\omega$. It is noted that these first order equations are now written in terms of the amplitude a and phase φ as dependent variables. Therefore, the right sides of equations (2.14) and (2.15) show that both a and φ are periodic functions of period

T. In this case, the right-hand terms of these equations contain a small parameter ε and also contain both a and φ, which are slowly varying functions of the time t with period $T = 2\pi/\omega$. We can transform the equations (2.14) and (2.15) into more convenient form. Now, expanding $\sin\psi\, f(a\cos\psi, -a\omega\sin\psi)$ and $\cos\psi\, f(a\cos\psi, -a\omega\sin\psi)$ in Fourier series with phase ψ, the first approximate solution of equation (2.5) is obtained by averaging equations (2.14) and (2.15) with period $T = 2\pi/\omega$ in the following form

$$\langle \dot{a} \rangle = -\frac{\varepsilon}{2\pi\,\omega}\int_0^{2\pi} \sin\psi\; f(a\cos\psi, -a\,\omega\,\sin\psi)\,d\psi, \qquad \text{(2.16 a, b)}$$
$$\langle \dot{\varphi} \rangle = -\frac{\varepsilon}{2\pi\,\omega\, a}\int_0^{2\pi} \cos\psi\; f(a\cos\psi, -a\,\omega\,\sin\psi)\,d\psi,$$

where a and φ are independent of time t under the integrals. KB [2] have called their method asymptotic in the sense that $\varepsilon \to 0$. Later, this technique has been extended by Bogoliubov and Mitropolskii [3], and has been extended to non-stationary vibrations by Mitropolskii [4]. They have assumed the solution of equation (2.5) in the following form

$$x = a\cos\psi + \varepsilon u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + ... + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}), \qquad \text{(2.17)}$$

where u_k, $(k = 1, 2, ..., n)$ are periodic functions of ψ with a period 2π, and the terms a and ψ are functions of time t and the following set of first order ordinary differential equations are satisfied by a and ψ

$$\dot{a} = \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \cdots\cdots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}), \qquad \text{(2.18 a, b)}$$
$$\dot{\psi} = \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \cdots\cdots + \varepsilon^n B_n(a) + O(\varepsilon^{n+1}).$$

The functions $u_{k,}$ A_k and B_k, $(k = 1, 2, ..., n)$ are to be chosen in such a way that the equation (2.17), after replacing a and ψ by the functions defined in equation (2.18), is a solution of equation (2.5). Since there are no restrictions in choosing functions A_k and B_k, so it generates the arbitrariness in the definitions of the functions u_k (Bogoliubov and Mitropolskii [3]). To remove this arbitrariness, the following additional conditions are imposed

$$\int_0^{2\pi} u_k(a,\psi)\cos\psi\, d\psi = 0, \qquad \text{(2.19 a, b)}$$
$$\int_0^{2\pi} u_k(a,\psi)\sin\psi\, d\psi = 0,$$

Secular terms are removed by using these conditions in all successive approximations. Differentiating equation (2.17) two times with respect to t, substituting the values of $\ddot{x}$, $\dot{x}$ and x into equation (2.5), and using the relations equation (2.18) and equating the coefficients of ε^k, $(k = 1, 2, ..., n)$, we obtain the following relations

$$\omega^2((u_k)_{\psi\psi} + u_k) = f^{(k-1)}(a,\psi) + 2\omega(a B_k \cos\psi + A_k \sin\psi), \tag{2.20}$$

and

$$\begin{aligned} f^{(0)}(a,\psi) &= f(a\cos\psi, -a\,\omega\sin\psi), \\ f^{(1)}(a,\psi) &= u_1 f_x(a\cos\psi, -a\,\omega\sin\psi) + (A_1\cos\psi - a\,B_1\sin\psi + \omega(u_1)_\psi) \\ &\quad \times f_{\dot{x}}(\cos\psi, -a\omega\sin\psi) + (aB_1^2 - A_1\frac{dA_1}{da})\cos\psi \\ &\quad + (2A_1B_1 - a\,A_1\frac{dB_1}{da})\sin\psi - 2\omega(A_1(u_1)_{a\psi} + B_1(u_1)_{\psi\psi}). \end{aligned} \tag{2.21 a,b}$$

where $(u_k)_\psi$ denotes partial derivative with respect to ψ and u_k, $f^{(k-1)}$ are periodic functions of ψ with period 2π which also depends on the amplitude a. Therefore, $f^{(k-1)}$ and u_k can be expanded in a Fourier series in the following form

$$\begin{aligned} f^{(k-1)}(a,\psi) &= g_0^{(k-1)}(a) + \sum_{n=1}^{\alpha}(g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi), \\ u_k(a,\psi) &= v_0^{(k-1)}(a) + \sum_{n=1}^{\alpha}(v_n^{(k-1)}(a)\cos n\psi + \omega_n^{(k-1)}(a)\sin n\psi), \end{aligned} \tag{2.22 a, b}$$

where

$$g_0^{(k-1)}(a) = \frac{1}{2\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\,\omega\sin\psi)\,d\psi. \tag{2.23}$$

Here, $v_1^{(k-1)} = \omega_1^{(k-1)} = 0$ for all values of k, since both integrals of equation (2.19) are vanished. Substituting these values into the equation (2.20), we obtain

$$\begin{aligned} &\omega^2 v_0^{(k-1)}(a) + \sum_{n=2}^{\alpha}\omega^2(1-n^2)[v_n^{(k-1)}(a)\cos n\psi + \omega_n^{(k-1)}(a)\sin n\psi] \\ &= g_0^{(k-1)}(a) + (g_1^{(k-1)}(a) + 2\omega\,a\,B_k)\cos n\psi + (h_1^{(k-1)}(a) + 2\omega\,A_k)\sin\psi \\ &\quad + \sum_{n=2}^{\alpha}[g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi]. \end{aligned} \tag{2.24}$$

Now, equating the coefficients of the harmonics of the same order, yield

$$g_1^{(k-1)}(a) + 2\omega a B_k = 0, \quad h_1^{(k-1)}(a) + 2\omega A_k = 0, \quad v_0^{(k-1)}(a) = \frac{g_0^{(k-1)}(a)}{\omega^2},$$
$$v_n^{(k-1)}(a) = \frac{g_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad \omega_n^{(k-1)}(a) = \frac{h_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad n \geq 1. \tag{2.25}$$

These are the sufficient conditions for obtaining the desired order of approximations. For the first order approximation, we have

$$A_1 = -\frac{h_1^{(0)}(a)}{2\omega} = -\frac{1}{2\pi\omega}\int_0^{2\pi} f(a\cos t\psi, -a\omega\sin\psi)\sin\psi\, d\psi,$$
$$B_1 = -\frac{g_1^{(0)}(a)}{2a\omega} = -\frac{1}{2\pi a\omega}\int_0^{2\pi} f(a\cos t\psi, -a\omega\sin\psi)\cos\psi\, d\psi, \tag{2.26 a, b}$$

Thus, the variational equations (2.18) becomes

$$\dot{a} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\, d\psi,$$
$$\dot{\psi} = \omega - \frac{\varepsilon}{2\pi a\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi\, d\psi, \tag{2.27 a, b}$$

It is seen that, the equation (2.27) are similar to the equation (2.16). Thus, the first approximate solution obtained by Bogoliubov and Mitropolskii [3] is identical to the original solution obtained by KB [2]. The correction term u_1 is obtained from equation (2.22) by using equation (2.25) as

$$u_1 = \frac{g_0^{(0)}(a)}{\omega^2} + \sum_{n=2}^{\alpha} \frac{g_n^{(0)}(a)\cos n\psi + h_n^{(0)}(a)\sin n\psi}{\omega^2(1-n^2)} \tag{2.28}$$

The solution equation (2.17) together with u_1 is known as the first order improved solution in which a and ψ are obtained from equation (2.27). If the values of the functions A_1 and B_1 are substituted from equation (2.26) into the second relation of equation (2.21b), the function $f^{(1)}$ and in a similar way, the functions A_2, B_2 and u_2 can be found. Therefore, the determination of the second order approximation is completed. The KB [2] method is very similar to that of van der Pol [1] and related to it. van der Pol has applied the method of variation of constants to the basic solution

$x = a\cos\omega t + b\sin\omega t$ of $\ddot{x} + \omega^2 x = 0$, on the other hand, KB [2] have applied the same method to the basic solution $x = a\cos(\omega t + \varphi)$ of the same equation. Thus, in the KB [2] method the varied constants are a and φ, while in the van der Pol's method the constants are a and b. The KB [2] method seems more interesting from the point of view of applications, since it deals directly with the amplitude and phase of the quasi-harmonic oscillation. The solution of the equation (2.4 a) is based on recurrent relations and is given as the power series of the small parameter. Cap [8] has solved the equation (2.4 b) by using elliptical functions in the sense of KB [2]. The KB [2] method has been extended by Popov [9] to nonlinear damped systems represented by the following equation

$$\ddot{x} + 2k\,\dot{x} + \omega^2 x = \varepsilon\, f(\dot{x}, x), \tag{2.29}$$

where $2k\,\dot{x}$ is the linear damping force and $0 < k < \omega$. It is noteworthy that, because of the importance of the Popov's method in the physical systems involving damping force, Mendelson [10] and Bojadziev [11] have retrieved Popov's [9] results. In case of nonlinear damped systems, the first equation of equation (2.18) has been replaced by

$$\dot{a} = -k\,a + \varepsilon\, A_1(a) + \varepsilon^2 A_2(a) + \dots\dots\dots\dots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}). \tag{2.18a}$$

Murty and Deekshatulu [12] have developed a simple analytical method to find the time response of second order nonlinear over damped systems with small nonlinearity represented by the equation (2.29), based on the KB [2] method of variation of parameters. In accordance to the KBM [2, 3] method, Murty *et al.* [13] have found a hyperbolic type asymptotic solution of an over damped system represented by the nonlinear differential equation (2.29), i.e., in the case $k > \omega$. They have used hyperbolic functions $\cosh\varphi$ and $\sinh\varphi$ instead of their circular counterpart, which are used by KBM [2, 3], Popov [9] and Mendelson [10]. Murty [14] has presented a unified KBM method for solving the nonlinear systems represented by the equation (2.29), which cover the undamped, damped and over-damped cases. Bojadziev and Edwards [15] have investigated solutions of oscillatory and non-oscillatory systems represented by equation (2.29) when k and ω are slowly varying functions of time *t*. Initial conditions may be used arbitrarily for the case of oscillatory or damped oscillatory process. But, in case of non-oscillatory systems $\cosh\varphi$ or $\sinh\varphi$ should be used depending on the given set of initial conditions (Murty *et al.* [13], Murty [14], Bojadziev and Edwards [15]). Sattar [16] has developed an asymptotic method to solve a second order critically damped nonlinear system represented by equation (2.29). He has found the asymptotic solution of the equation (2.29) in the following form

$$x = a(1+\psi) + \varepsilon u_1(a,\psi) + ... + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}), \tag{2.30}$$

where a is defined by the equation (2.18a) and ψ is defined by

$$\dot{\psi} = 1 + \varepsilon C_1(a) + \varepsilon^2 C_2(a) + ... + \varepsilon^n C_n(a) + O(\varepsilon^{n+1}). \tag{2.18 b}$$

Osiniskii [17] has extended the KBM method to the following third order nonlinear differential equation

$$\dddot{x} + c_1 \ddot{x} + c_2 \dot{x} + c_3 x = \varepsilon f(\ddot{x}, \dot{x}, x), \tag{2.31}$$

where ε is a small positive parameter and f is a given nonlinear function. He has assumed the asymptotic solution of equation (2.31) in the form

$$x = a + b\cos\psi + \varepsilon u_1(a,b,\psi) + \cdots\cdots + \varepsilon^n u_n(a,b,\psi) + O(\varepsilon^{n+1}), \tag{2.32}$$

where each u_k $(k = 1, 2, ..., n)$ is a periodic function of ψ with period 2π and a, b and ψ are functions of time *t*, and they are given by

$$\begin{aligned} \dot{a} &= -\lambda a + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \cdots\cdots\cdots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}), \\ \dot{b} &= -\mu b + \varepsilon B_1(b) + \varepsilon^2 B_2(b) + \cdots\cdots\cdots + \varepsilon^n B_n(b) + O(\varepsilon^{n+1}), \\ \dot{\psi} &= \omega + \varepsilon C_1(b) + \varepsilon^2 C_2(b) + \cdots\cdots\cdots + \varepsilon^n C_n(b) + O(\varepsilon^{n+1}), \end{aligned} \tag{2.33 a, b, c}$$

where $-\lambda, -\mu \pm i\omega$ are the eigen values of the equation (2.31) when $\varepsilon = 0$.

Lardner and Bojadziev [18] have investigated the solutions of nonlinear damped oscillations governed by a third order partial differential equation. They have introduced the concept of "couple amplitude" where the unknown functions A_k, B_k and C_k depend on both the amplitudes a and b.

Alam *et al.* [19] have investigated a general Struble's technique for solving *nth* order weakly nonlinear differential systems with damping. Alam *et al.* [20] have developed an analytical tecnique to find approximate solutions of nonlinear damped oscillatory systems. Alam *et al.* [21] have investigated a new analytical technique to find the periodic solutions of nonlinear systems. Nayfeh [22] has interpreted the introduction to perturbation techniques. Murdock [23] has interpreted the theory and methods of perturbation techniques. Alam [24] has presented some special conditions of over-damped nonlinear systems. Sachs *et al.* [25] have developed a simple ODE model for tumor growth and anti-angiogenic or radiation treatment.

The HPM was first proposed by the Chinese mathematician Ji Huan He [26]. The essential idea of this method is to introduce a homotopy parameter, say p, which varies from 0 to 1. At $p=0$, the system of equations usually has been reduced to a simplified form which normally admits a rather simple solution. As p gradually increases continuously toward 1, the system goes through a sequence of deformations, and the solution at each stage is closed to that at the previous stage of the deformation. Eventually at $p=1$ the system takes the original form of the equation and the final stage of the deformation give the desired solution.

He [26] has investigated a novel homotopy perturbation technique for obtaining a periodic solution of a general nonlinear oscillator for conservative systems. He [26] has considered the nonlinear differential equation in the form

$$A(u)-f(r)=0,\ \ r\in\Omega, \tag{2.34}$$

with the boundary conditions

$$B(u,\frac{\partial u}{\partial t})=0,\quad r\in\Gamma, \tag{2.35}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is known as analytical function, Γ is the boundary of the domain Ω.Then He [26] has written equation (2.34) in the following form

$$L(u)+N(u)-f(r)=0, \tag{2.36}$$

where L is linear part, while N is nonlinear part. He [26] has constructed a homotopy $v(r,p):\Omega\times[0,1]\to\Re$ which satisfies

$$\mathrm{H}(v,p)=(1-p)[L(v)-L(u_0)]+p[A(u)-f(r)]=0,\ \ p\in[0,1],\ \ r\in\Omega \tag{2.37 a}$$

or, $\mathrm{H}(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0,$ (2.37 b)

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation of equation (2.34), which satisfies the boundary conditions. Obviously, from equation (2.37), it becomes

$$\mathrm{H}(v, 0) = L(v) - L(u_0) = 0, \tag{2.38}$$

$$\mathrm{H}(v, 1) = A(v) - f(r) = 0. \tag{2.39}$$

The changing process of p from zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. He [26] has assumed the solution of equation (2.37) as a power series of p in the following form

$$v = v_0 + p\,v_1 + p^2\,v_2 + \cdots. \tag{2.40}$$

The approximate solution of equation (2.34) is given by setting $p = 1$ in the form

$$u = v_0 + v_1 + v_2 + \cdots. \tag{2.41}$$

The series (2.41) is convergent for most of the cases, and also the rate of convergence depends on how one choose $A(u)$.

He [27] has developed a coupling method of a homotopy technique and a perturbation technique for nonlinear problems. He [28] has presented a new nonlinear analytical technique. Also, He [29] has presented a new interpretation of homotopy perturbation method. Uddin *et al.* [30] have presented an analytical technique for solving second order strongly nonlinear differential systems with damping by combing the He's [26-29] homotopy perturbation and the extended form of the KBM [2-4] methods. Uddin *et al.* [31] have also developed an analytical approximate technique for solving a certain type of fourth order strongly nonlinear oscillatory differential systems with small damping and cubic nonlinearity based on He's homotopy perturbation [26-29] and the extended form of the KBM [2-4] methods. Ludeke and Wagner [32] have obtained the solutions of generalized Duffing equation with cubic nonlinearity in presence of large damping effects. Lim *et al.* [33] have obtained accurate high-order analytical approximate solutions to large amplitude oscillating systems with a general non-rational restoring force. Bojadziev [34] has presented the damped nonlinear oscillations modeled by a 3-

dimensional differential system. Belendez *et al.* [35] have applied He's homotopy perturbation method to obtain higher order approximation of an $x^{1/3}$ force nonlinear oscillator. Arya and Bojadziev [36] have presented a time depended oscillating systems with damping, slowly varying parameters and delay. Chatterjee [37] has also presented the harmonic balance method based on averaging method for solving strongly nonlinear cubic oscillator with damping effects. Mickens [38] has analyzed the nonlinear oscillators having non-polynomial elastic terms. In another paper, Mickens [39] has developed an iteration method for solving second order conservative nonlinear differential systems with an $x^{1/3}$ force oscillators. Yamgoue and Kofane [40] have investigated an approximate analytical technique for determining the solutions of damped nonlinear systems. Recently, Dey *et al.* [41] have developed an analytical technique for solving second order strongly nonlinear damped system with slowly varying coefficients.

CHAPTER III

An Analytical Technique for Solving Strongly Nonlinear Damped Systems with Fractional Power Restoring Force

3.1 Introduction

It is very difficult to solve nonlinear problems either numerically or theoretically. Nonlinear oscillation problems are very important in the physical science, mechanical structures and other kind of mathematical sciences. This is so due to the fact that nonlinear phenomena play vital role in applied mathematics, physics, plasma physics, economics and engineering. Mathematical methods are aimed for obtaining approximate solutions of nonlinear differential equations arising in various fields of science and engineering and have appeared in the research literature [2-5, 7-41]. However, most of them require a tedious analysis and laborious work to handle these problems [7-25].

The most common methods for constructing the approximate analytical solutions to the nonlinear differential systems are the perturbation methods. Some well known perturbation methods are the Krylov-Bogoliubov-Mitropolskii (KBM) [2, 3] method, the Lindstedt-Poincare (LP) method [22, 23], the method of multiple time scales [22] and the harmonic balance [22, 23] method which are valid even for rather large amplitudes of oscillations. Almost all perturbation methods are based on an assumption that small parameter and linear term must exist in the equations. In general, the perturbation approximations are valid only for weak nonlinear differential systems in presence of linear term in the equation. The perturbation techniques [1-24] which are, in principle, applicable if there exists small parameters in the equations. The parameters are expanded into power series of the parameters. The coefficients of the series are obtained as solutions of a set of linear equations. But in both science and engineering, there exist many nonlinear differential systems without small parameters.

Most of the authors [27-29, 33, 35, 38, 39] have developed analytical techniques for solving strongly nonlinear differential systems without considering any damping effects for which the elastic restoring forces are non-polynomial functions of the displacement. But numerous physical and oscillating systems encounter in presence of small damping in nature. To fill this gap, we have developed an analytical technique for solving second order strongly nonlinear damped systems with a $x^{1/3}$ restoring force based on the HPM [27-29, 30, 31, 35] and the extended form of the KBM [2-4] method. The solutions are obtained containing only two lower order harmonic terms, which measure satisfactory results with small and significant damping forces. The advantage of the presented method is that the first approximate solutions show a good agreement with the corresponding numerical solutions in presence of significant small damping in the whole solution domain.

3.2 The method

Let us consider a second order strongly nonlinear damped system with a fractional power restoring force [33, 35, 38, 39] in the following form

$$\ddot{x} + 2k\,\dot{x} = -f(x), \tag{3.1}$$

with the initial conditions

$$x(0) = a_0,\ \dot{x}(0) = 0, \tag{3.2}$$

where "dots" denote differentiation with respect to time t, $2k$ is the linear damping coefficients, $f(x)$ is a given $x^{1/3}$ restoring force and a_o is usually a given constant and known as the initial amplitude.

To changing the independent variable, we are going to use the following substitution [30]

$$x(t) = y(t)e^{-k\,t}. \tag{3.3}$$

Differentiating equation (3.3) twice with respect to time t and putting $\ddot{x}$, $\dot{x}$ and x into the original equation (3.1) and then simplifying them, we obtain

$$\ddot{y} - k^2 y = -e^{k\,t} f(y e^{-k\,t}). \tag{3.4}$$

Now equation (3.4) can be re-written as

$$\ddot{y} + \omega^2 y = \lambda\, y - e^{k\,t} f(y e^{-k\,t}), \tag{3.5}$$

where

$$\omega^2 = \lambda - k^2. \tag{3.6}$$

Herein ω is known as the angular frequency of the nonlinear oscillator and the unknown constant λ can be obtained by eliminating the secular terms. The angular frequency ω is a time dependent function for a nonlinear damped systems and it varies slowly with time t.

In this method, an artificial perturbation equation is constructed by embedding an artificial parameter $p \in [0,1]$. According to the homotopy perturbation method [27-29, 30, 31, 35], equation (3.5) can be written as

$$\ddot{y} + \omega^2 y = p[\lambda y - e^{kt} f(y e^{-kt})], \tag{3.7}$$

where p is the homotopy parameter.

When $p = 0$, equation (3.7) becomes a linear differential equation for which an exact solution can be calculated as $x_0 = a e^{-kt} \cos\varphi$. When $p = 1$, equation (3.7) turns out to be the original one. To handle this situation, we are going to use the extended form of the KBM [2, 3] method which was developed by Mitropolskii [4]. According to this technique, we choose the first approximate solution of equation (3.7) in the following form

$$y = a\cos\varphi + p\,u_1(a,\varphi), \tag{3.8}$$

where a and φ represent the amplitude and phase variable of the system and u_1 is the correction term. In presence of damping, a and φ are changing functions with time t.

The functions for time varying amplitude a and phase φ are obtained by the following set of first order ordinary differential equations

$$\begin{aligned} \dot{a} &= p\,A_1(a,\tau) + p^2 A_2(a,\tau) + \cdots, \\ \dot{\varphi} &= \omega(\tau) + p\,B_1(a,\tau) + p^2 B_2(a,\tau) + \cdots, \end{aligned} \tag{3.9}$$

where $A_1, A_2, \cdots$ and $B_1, B_2, \cdots$ are unknown functions, and $\tau = pt$ is the slow time.

Differentiating equation (3.8) twice with respect to time t, utilizing the relations equation (3.9) and neglecting $O(p^2)$, we get

$$\ddot{y}+\omega^2 y = p\omega' a\sin\varphi - 2p\,\omega(A_1\sin\varphi + a\,B_1\cos\varphi) + p\omega^2\left(\frac{\partial^2 u_1}{\partial\varphi^2}+u_1\right), \tag{3.10}$$

where ω' denotes the differentiation with slow time τ.

By using equations (3.10) and (3.8) into equation (3.7) and then equating the coefficients of p, we obtain

$$\begin{aligned}&\omega' a\sin\varphi - 2\,\omega(A_1\sin\varphi + a\,B_1\cos\varphi) + \omega^2\left(\frac{\partial^2 u_1}{\partial\varphi^2}+u_1\right)\\ &= \lambda\,a\cos\varphi - e^{kt} f(a\,e^{-kt}\cos\varphi).\end{aligned} \tag{3.11}$$

It was early imposed by Krylov and Bogoliubov [2] that u_1 does not contain secular terms. In general, $f(x_0)$ can be expanded as a Fourier series [33, 35] in φ, where $x_0 = a\,e^{-kt}\cos\varphi$ and then we impose a restriction that the no secular terms required in the particular solution. Then the variational parameters $A_1, A_2, \cdots$ and $B_1, B_2, \cdots$ are obtained by equating the fundamental terms on both sides and the rest of the terms are included to the partial differential equation involving the correction term u_1 and solving them for these unknown.

In general, KBM solutions are useful when nonlinearity is very small. Sometimes it gives desired results even for strong nonlinear differential systems for a damped solution [10, 20]. Usually, the integration of equation (3.9) is performed by well-known techniques of calculus [22] with the values of the variational parameters $A_1, A_2, \cdots$ and $B_1, B_2, \cdots$; but sometimes they are also solved by a numerical procedure [9-21, 26-30]. Finally putting equation (3.8) in equation (3.3) we get

$$x(t) = e^{-kt}\left(a\cos\varphi + p\,u_1(a,\varphi)\right). \tag{3.12}$$

Thus, the first approximate analytical solution of equation (3.1) is completed.

3.3. Example

As an example of the above procedure, we are going to consider a second order strongly nonlinear damped system for which the elastic force term is proportional to $x^{1/3}$ [33, 35, 38, 39] in the following form

$$\ddot{x} + 2k\,\dot{x} = -x^{1/3}, \tag{3.13}$$

where $f(x) = x^{1/3}$. Now using the transformation equation (3.3) into equation (3.13) and then simplifying them, we get

$$\ddot{y} - k^2 y = -y^{1/3} e^{2k\,t/3}. \tag{3.14}$$

According to the homotopy perturbation method [27-29, 30, 31, 35], equation (3.14) can be rewritten as

$$\ddot{y} + \omega^2 y = p[\lambda\, y - y^{1/3} e^{2k\,t/3}], \tag{3.15}$$

where ω is given by equation (3.6).

Now according to the extended form of the KBM [2-4] method, the solution of equation (3.15) is given by equation (3.8) and the amplitude a and the phase φ are obtained by equation (3.9). Substituting equation (3.8) into the right side of equation (3.15) and neglecting $O(p^2)$, we get

$$\ddot{y} + \omega^2 y = p[\lambda\, a \cos\varphi - a^{1/3} e^{2k\,t/3} \cos^{1/3}\varphi]. \tag{3.16}$$

Expanding $\cos^{1/3}\varphi$ as the Fourier series [33, 35] in the following form

$$\cos^{1/3}\varphi = \sum_{n=0}^{\infty} b_{2n+1} \cos[(2n+1)\varphi], \tag{3.17}$$

where

$$b_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} \cos^{1/3} \cos[(2\,n+1)\varphi]\,d\varphi; \quad n = 0,1,2,3\cdots. \tag{3.18}$$

Inserting equation (3.17) into equation (3.16), we obtain

$$\ddot{y} + \omega^2 y = p[\lambda\, a \cos\varphi - a^{1/3} e^{2k\,t/3}(b_1 \cos\varphi + b_3 \cos 3\varphi + \cdots)]. \tag{3.19}$$

To avoid the secular terms in particular solution of equation (3.19) requires that the coefficients of the $\cos\varphi$ term must be zero. Then we have

$$\lambda\, a - b_1\, a^{1/3} e^{2k\,t/3} = 0. \tag{3.20}$$

As we are seeking non-trivial solution $(a \neq 0)$ then equation (3.20) leads to

$$\lambda = b_1\, a^{-2/3} e^{2k\,t/3}. \tag{3.21}$$

Putting the value of λ from equation (3.21) into equation (3.6), we obtain

$$\omega^2 = b_1\, a^{-2/3} e^{2k\,t/3} - k^2. \tag{3.22}$$

This is a time dependent angular frequency equation of the given nonlinear damped system with a $x^{1/3}$ restoring force. As $t \to 0$, equation (3.22) yields

$$\omega_0 = \omega(0) = \sqrt{\,b_1\, a_0^{-2/3} - k^2}\,, \tag{3.23}$$

where ω_0 is known as the constant angular frequency.

The Fourier coefficients can be evaluated by using symbolic software such as *MTHEMATICA*, then we obtain

$$\begin{aligned} b_1 &= \frac{4}{\pi}\int_0^{\pi/2} \cos^{1/3}\varphi\, \cos\varphi\; d\varphi = \frac{3\Gamma(7/6)}{\sqrt{\pi}\,\Gamma(2/3)}, \\ b_3 &= \frac{4}{\pi}\int_0^{\pi/2} \cos^{1/3}\varphi\, \cos 3\varphi\; d\varphi = -\frac{3\Gamma(7/6)}{5\sqrt{\pi}\,\Gamma(2/3)}. \end{aligned} \tag{3.24}$$

Relation between b_1 and b_3 are obtained as

$$\frac{b_3}{b_1} = -\frac{1}{5}. \tag{3.25}$$

Now equation (3.19) can be rewritten in the following form with the help of equation (3.11)

$$\begin{aligned} &\omega'\, a\sin\varphi - 2\,\omega\,(A_1 \sin\varphi + a\, B_1 \cos\varphi) + \omega^2\left(\frac{\partial^2 u_1}{\partial\varphi^2} + u_1\right) \\ &= -a^{1/3} b_3 e^{2k\,t/3} \cos 3\varphi. \end{aligned} \tag{3.26}$$

Equating the coefficients of $\sin\varphi$ and $\cos\varphi$ on both sides and then solving them, we get

$$A_1 = -\frac{\omega' a}{2\omega}, \qquad B_1 = 0, \tag{3.27}$$

and

$$\omega^2 \left(\frac{\partial^2 u_1}{\partial \varphi^2} + u_1 \right) = -\, a^{1/3} b_3 e^{2k\,t/3} \cos 3\varphi. \tag{3.28}$$

Solving equation (3.28) we obtain

$$u_1 = \frac{a^{1/3} b_3 e^{2k\,t/3} \cos 3\varphi}{8\omega^2}. \tag{3.29}$$

By integrating equation (3.9) with the help of equation (3.27), we get

$$\begin{aligned} a &= a_0 \sqrt{\frac{\omega_0}{\omega}}, \\ \varphi &= \varphi_0 + \int_0^t \omega(\tau)\, dt, \end{aligned} \tag{3.30 a, b }$$

where $a(0) = a_0$ and $\varphi(0) = \varphi_0$ are constants of integration which represent the initial amplitude and initial phase of the nonlinear systems.

Now putting equation (3.30 a) into equation (3.22), we obtain the following frequency equation

$$\omega^2 = b_1\, a_0^{-2/3} \left(\frac{\omega}{\omega_0} \right)^{1/3} e^{2k\,t/3} - k^2. \tag{3.31}$$

Neglecting k^2 as the damping is small, then solving equation (3.31) for ω with the help of *MTHEMATICA* and taking $\omega(0) = \omega_0 = b_1 / a_0^{1/3}$, we obtain

$$\omega = \sqrt{b_1}\, a_0^{-1/3} e^{2k\,t/5}. \tag{3.32}$$

Using the equations (3.25) and (3.32) in equation (3.29), we get

$$u_1 = \frac{a^{1/3} b_3 e^{2k\,t/3} \cos 3\varphi}{8\omega^2} = -\frac{a \cos 3\varphi}{40}. \tag{3.33}$$

Finally putting equation (3.33) in equation (3.12), the first approximate solution of equation (3.13) becomes

$$x(t) = a\, e^{-kt} (\cos\varphi - \frac{p}{40}\cos 3\varphi). \tag{3.34}$$

Using equation (3.32) and solving equation (3.30 b), the amplitude and phase equations are reduced to

$$\begin{aligned} a &= a_0 e^{-kt/3}, \\ \varphi &= \varphi_0 - \frac{3b_1\, a_0^{-1/3}}{2k}\left(1 - e^{-2kt/3}\right). \end{aligned} \qquad (3.35\ \text{a, b})$$

Thus, the first approximate solutions of equation (3.13) is given by the equation (3.34) with the help of equation (3.35).

CHAPTER IV

Results and Discussion

The approximate solutions of equation (3.13) are compared with the numerical (considered to be exact) solutions for testing the accuracy of the presented technique. The approximate solutions have been obtained for several small and significant damping and it is seen that approximate results are converging rapidly to numerical solutions which are shown in graphically. Furthermore, the presented method is simple and the advantage of this method is that the first approximate solutions show good agreement (see also **Figs. 4.1, 4.2**) with the corresponding numerical solutions for strong nonlinear damped systems with imbedding parameter $p = 1.0$ but the method presented in [24] is laborious and tedious work. The figures indicate that the approximate solutions almost coincide with the numerical solutions for several small and significant damping (**Figs. 4.1, 4.2**) but it deviates from the numerical solutions for large damping (**Fig. 4.3**). The initial approximation can be freely chosen, which is identified via various methods [10-21, 26-37].

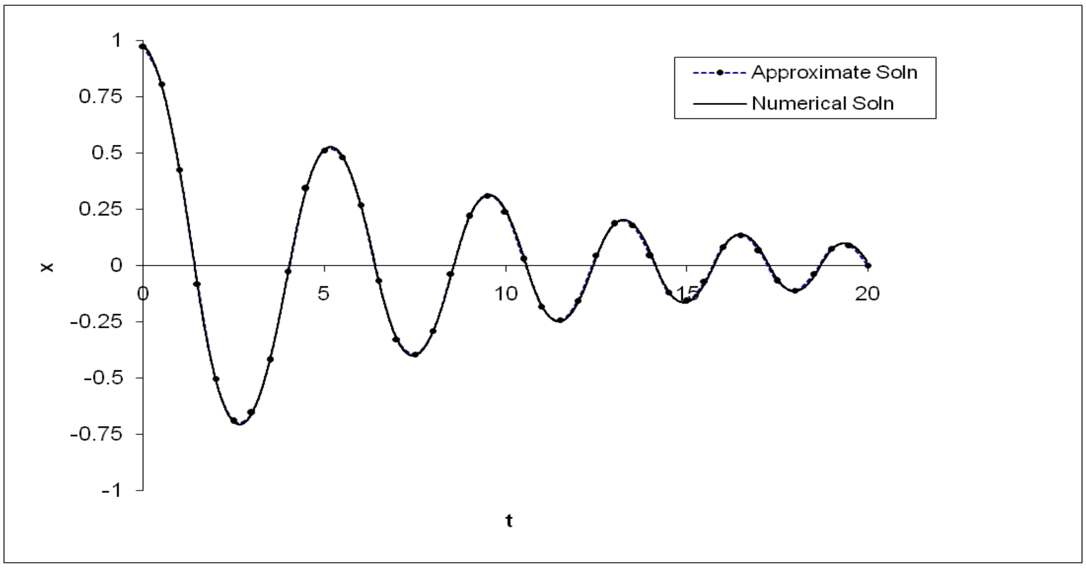

Fig. 4.1: First approximate solution of equation (3.13) is denoted by $-\bullet-$ (dashed lines) by the presented method with the initial conditions $[x(0) = 0.97500,\ \dot{x}(0) = -0.10175]$ or $a_0 = 1.0,\ \varphi_0 = 0$ when $p = 1.0,\ k = 0.1,\ b_1 = 1.15960$ and $f = x^{1/3}$. Corresponding numerical solution is denoted by ——— (solid line).

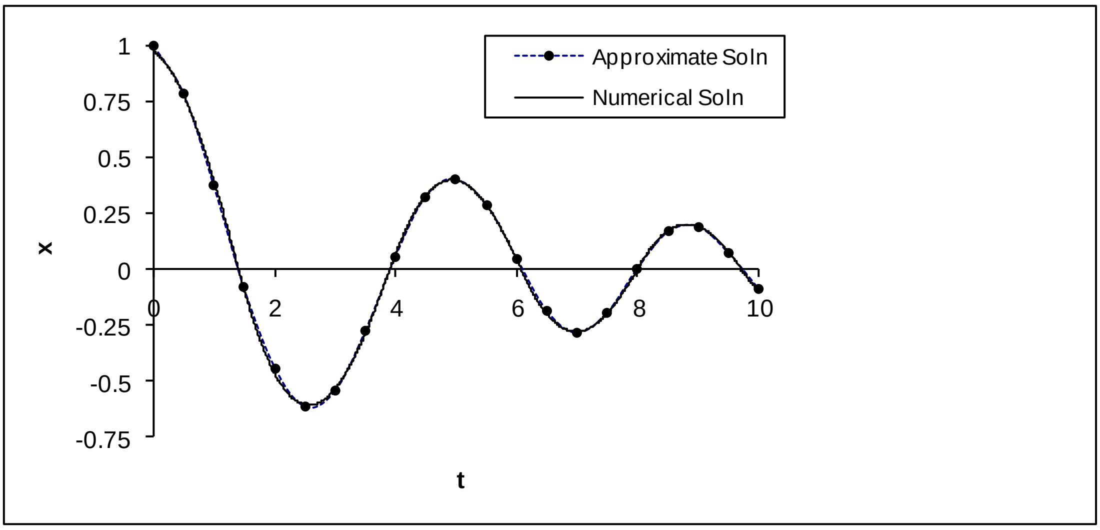

Fig. 4.2: First approximate solution of equation (3.13) is denoted by –•– (dashed lines) by the presented method with the initial conditions $[x(0) = 0.97500,\ \dot{x}(0) = -0.15265]$ or $a_0 = 1.0,\ \varphi_0 = 0$ when $p = 1.0,\ k = 0.2,\ b_1 = 1.15960$ and $f = x^{1/3}$. Corresponding numerical solution is denoted by ——— (solid line).

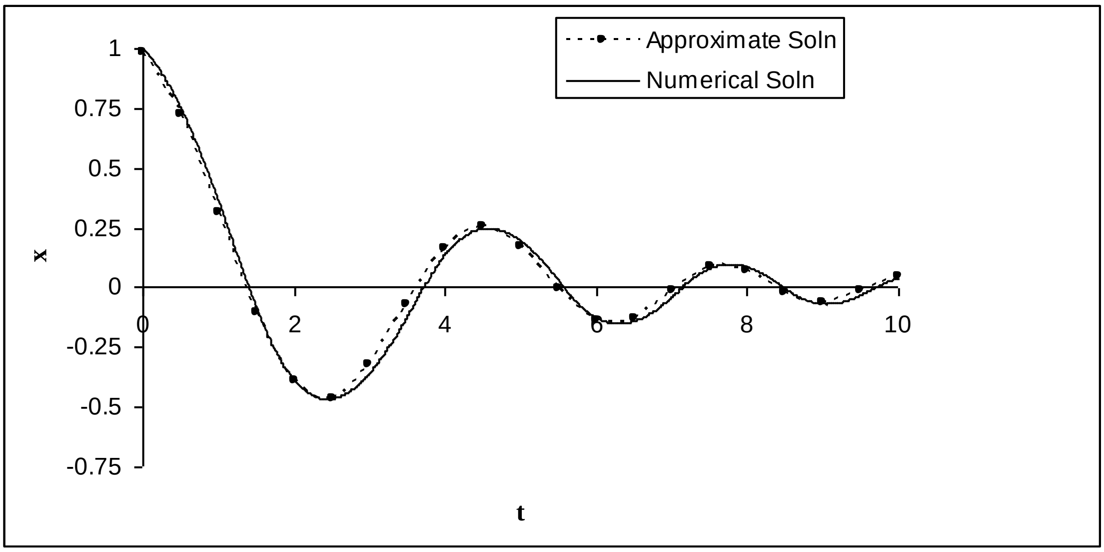

Fig. 4.3: First approximate solution of equation (3.13) is denoted by –•– (dashed lines) by the presented method with the initial conditions $[x(0) = 0.99750,\ \dot{x}(0) = -0.25471]$ or $a_0 = 1.0,\ \varphi_0 = 0$ when $p = 1.0,\ k = 0.25,\ b_1 = 1.15960$ and $f = x^{1/3}$. Corresponding numerical solution is denoted by ——— (solid line).

CHAPTER V

Conclusions

The determination of solution, amplitude and phase of nonlinear differential systems is very crucial in mechanics. In this thesis, we have applied an analytical approch to find an appoximate solutions for strongly nonlinear differential systems with a $x^{1/3}$ elastic force in presence of significant small damping. The presented method does not require a small parameter and linear term in the equation like the classical one and the solution procedure is very simple and easy to understand but the method presented in [24] is complicated. The presented method has been successfully implemented to illustrate the effectiveness and convenience for solving second order strongly nonlinear damped systems for which the elastic force is proportional to $x^{1/3}$.

From the figures (**Figs. 4.1, 4.2**), it is clear that the first approximate solutions show good agreement with those solutions obtained by the fourth order Runge-Kutta method with the several small and significant damping in the whole solution domain for imbedding parameter $p = 1.0$. It is also noticed that He's HPM is incapable for solving nonlinear differential systems in presence of any damping and KBM method is fail to handle strongly nonlinear differential systems. Both limitations have been overcome by the presented method.

This method is effective for solving second order strongly nonlinear damped physical problems with a $x^{1/3}$ restoring force and converging rapidly to the exact solutions. Whereas Lim *et al.* [33], Belendez *et al.* [35] and Mickens [38, 39] have developed approximate techniques for handling nonlinear differential systems with a $x^{1/3}$ restoring force without damping. So our presented method can serve as a useful mathematical tool for dealing damped nonlinear oscillators with a $x^{1/3}$ restoring force.

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