Similarity Solution of Unsteady Natural Convection Boundary Layer Flow above a Semi-infinite Porous Horizontal Plate with Suction and Blowing

by

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Declaration

This is to certify that the thesis work entitled "Similarity Solution of Unsteady Natural Convection Boundary Layer Flow above a Semi-infinite Porous Horizontal Plate with Suction and Blowing" has been carried out by Rita Mojumder in the Department of Mathematics, Khulna university of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

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This is to certify that the thesis work submitted by Rita Mojumder entitled "Similarity Solution of Unsteady Natural Convection Boundary Layer Flow above a Semi-infinite Porous Horizontal Plate with Suction and Blowing" has been approved by the board of examiners for the partial fulfillment of the requirements for the degree of Master of Philosophy (M. Phil.) in the Department of Mathematics, Khulna university of Engineering & Technology, Khulna, Bangladesh in December 2011.

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ABSTRACT

The present study deals with the similarity solutions of laminar boundary layer equations for the unsteady free convection flow over a heated horizontal semi-infinite porous plate. The Boussinesq approximation is employed firstly in order to simplify the governing boundary layer equations. Secondly, similarity requirements for an incompressible fluid are sought on the basis of detailed analysis in order to reduce the governing coupled partial differential equations into a set of ordinary differential equations. The influence of suction and blowing on the flow and temperature fields and other flow factors like skin friction and heat transfer coefficients are extensively investigated under different similarity cases. Sixth order R-K method is used to solved the simplified equations and the obtained numerical results are displayed graphically for some selected values of the controlling parameters provided by the similarity transformation. It is found that a small value of suction or blowing play a vital role on the patterns of flow and temperature fields as well as on the coefficients of skin friction and heat transfer and pressure distribution.

Publications

The following papers have been extracted from this thesis:

1. M. M. Touhid Hossain, Rita Mojumder and Mohammad Arif Hossain, Solution of natural convection boundary layer flow above a semi-infinite porous horizontal plate under similarity transformations with suction and blowing.
   Daffodil International University Journal of Science and Technology (2011), 6(1), 43–51.

2. M. M. Touhid Hossain, Mohammad Arif Hossain and Rita Mojumder, A study of similarity solutions for the unsteady natural convection flow above a semi-infinite heated horizontal porous plate with transpiration.
   Proceedings of 13th Asian Congress of Fluid Mechanics (*ACFM*), 17–21 December, 2010 Vol.-II, 962–965.

3. Mohammad Arif Hossain, M. M. Touhid Hossain, Rita Mojumder, Further study of similarity solutions of unsteady natural convection flow above a heated horizontal semi-infinite porous plate with suction and blowing.
   1st International Conference on Mechanical, Industrial and Energy Engineering (*ICMIEE*) *2010*, 23–24 December, 2010, IE10-095-1–6.

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CHAPTER I

Introduction and Literature Review

Fluid dynamics is a subject of widespread interest to researchers and it becomes an obvious challenge for the scientists, engineers as well as users to understand more about fluid motion. An important contribution to the fluid dynamics is the concept of boundary layer introduced first by L. Prandtl [32]. The concept of the boundary layer is the consequence of the fact that flows at high Reynolds numbers can be divided into unequally spaced regions. A very thin layer (called boundary layer) in the vicinity (of the object) in which the viscous effects dominate, must be taken into account, and for the bulk of the flow region, the viscosity can be neglected and the flow corresponds to the inviscid outer flow. Although the boundary layer is very thin, it plays a vital role in the fluid dynamics. Boundary layer theory has become an essential study now-a-days in analysing the complex behaviors of real fluids. The concept of boundary layer can be used to simplify the Navier-Stokes' equations to such an extent that the viscous effects of flow parameters are evaluated, and these are useable in many practical problems (viz. the drag on ships and missiles, the efficiency of compressors and turbines in jet engines, the effectiveness of air intakes for ram and turbojets and so on).

Further, the boundary layer effects caused by free convection are frequently observed in our environmental happenings and engineering devices. We know that if externally induced flow is provided and flows arising naturally solely due to the effect of the differences in density, caused by temperature or concentration differences in the body force field (such as gravitational field). These types of flows are called 'free convection' or 'natural convection' flows. The density difference causes buoyancy effects and these effects act as 'driving forces' due to which the flow is generated. Hence free convection is the process of heat transfer which occurs due to movement of the fluid particles by density differences associated with temperature differences in a fluid. In such cases, the free stream velocity falls away, in deed, no reference velocity does a priori exist. If the density in the vicinity of the object is kept constant, natural convection flow can not be formed. Thus, this is an effect of variable properties, where there is a mutual coupling between momentum and heat transport. The direct origin of the formation of natural convection flows is a heat transfer via conduction through the fixed surfaces surrounding the fluid. If the surface temperature is greater than that of ambient fluid, heat is transferred from the plate to the fluid leads to an increase in temperature of the fluid close to the surfaces and to a change in the density, because it is temperature dependent. If the density decreases with increasing temperature, buoyancy forces arise close to the surface and warmer fluid moves upwards. Such buoyant forces are proportional to the coefficient of thermal expansion β_T, defined as $\beta_T = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{p=\text{constant}}$, where ρ, T and p are density, temperature and pressure respectively. It is observed that $\beta_T = \frac{1}{T}$ for a perfect gas and we see that stream is well approximated by the perfect-gas result $\beta_T T = 1$ at low pressure and high temperature. Also $\beta_T < \frac{1}{T}$ for a liquid and may even be negative, and $\beta_T > \frac{1}{T}$ for imperfect gas, particularly at high pressure. β_T is also useful in estimating the dependence of enthalpy 'h' on pressure, from the thermodynamic relation $dh = c_p dT + (1 - \beta_T T)\frac{dp}{\rho}$, where T is the absolute temperature. For the perfect gas, the second term vanishes, so that $h = h(T)$ only.

The natural convection studies begun in the year 1881 with Lorentz and continued at a relatively constant rate until recently. This mode of heat transfer occurs very commonly, the cooling of transmission lines, electric transformers and rectifiers, the heating of rooms by use of radiators, the heat transfer from hot pipes and ovens surrounded by cooled air, cooling the reactor core (in nuclear power plant) and carry out the heat generated by nuclear fission, etc. Bulks of information are now available in literature about the boundary layer form of natural convection flows over bodies of different shapes. The theoretical, experimental and numerical analysis for the natural convection boundary layer flow about isothermal, vertical flat plates have been carried out widely by many authors (viz. [11, 26, 28, 36, 39, 41]).

Schmidt [35] was apparently the first researcher who investigated experimentally the behavior of the flow near the leading edge above a flat horizontal surface. The theoretical analysis of the laminar, two-dimensional, steady natural convection boundary layer flow on a semi-infinite horizontal flat plate was first considered by Stewartson [40] (later corrected by Gill, Zeh and Del-Casal [14]). In that analysis he used the Boussinesq approximation to show how the boundary layer analysis could be incorporated with the natural convection on rectangular plates, which are of high planform aspect ratio.

Rotem and Claassen [33] investigated the boundary layer equation over a semi-infinite horizontal surface of uniform temperature and results were presented for some specific values of Prandtl number with its limits form zero to infinity. The effect of buoyancy forces that exist in boundary layer flow, over a horizontal surface, where the surface temperature differs from that of ambient fluid, was studied by Sparrow and Minkowycz [38]. The free convection above a heated and almost horizontal plate has been treated by Jones [21].

The boundary layer type of the natural convection flow, which occurs on the upper surface of heated horizontal, surfaces has been investigated theoretically and experimentally by amongst other, Rotem and Claassen [34], Pera and Gebhart [29, 30] and Goldstein, Sparrow and Jones [15]. It is seen from their experiments and also from the flow visualization of Husar and Sparrow [19] that a boundary layer starts from each edge of a plate edge, each boundary layer having its leading at a straight-side plate edge. The boundary layer development occurs normal to the corresponding edge so that collisions between opposing boundary layer flows occur on the plate surface. After collision, the fluid checked in the boundary layer forms a rising buoyant plume.

Furthermore, the solution of a system of coupled partial differential equations with boundary conditions is often difficult and even impossible with the usual classical method. Thus, it is imperative to reduce the number of variables from the system which reached in a stage of great extent. Similarity solution is one of the important means for the reduction of a number of independent variables with simplifying assumptions and finally the system of partial differential equations reduces to a set of ordinary differential equations successfully. A vast literature of similarity solution has appeared in the area of fluid mechanics, heat transfer, and mass transfer, etc.

In 1978, Johnson and Cheng [20] examined the necessary and sufficient conditions under which similarity solutions exist for free convection boundary layers adjacent to flat plates in porous media. The solutions obtained in their work were more general than those appearing in the previous studies. With a parameter associated with the body shapes a similarity solution on the natural convection flow has also been studied by Pop and Takhar [31]. Ferdows *et al.* [12] have been made a similarity analysis for the forced as well as free convection boundary layer flow of an electrically conducting viscous incompressible fluid past a semi-infinite con-conducting vertical porous plate by introducing a time dependent suction.

Most of the above analyses were based on the Buossinesq approximation and have been concerned with the seeking of similarity solutions in which the plate temperature varies with the distance from plate leading edge. In this approximation thus density, viscosity, thermal conductivity and specific heat variations are ignored except for the necessary inclusion of the density-variation in the body force term.

An analysis is performed by Chen *et al.* [8] to study the flow and heat transfer characteristic of laminar natural convection in boundary layer flows from horizontal, inclined and vertical plates with power law variation of the wall temperature.

In most of the above analyses the boundary layer of the natural convection flows were considered over heated or uniformly heated horizontal vertical, horizontal or near horizontal, semi-infinite, rectangular porous plates. The surface is impermeable to the fluid, so that there is no transpiration i.e., suction or blowing velocity normal to the surface. This led to the kinematic boundary condition $v_w = 0$.

The problem of boundary layer control has become very important factor; in actual application it is often necessary to prevent separation. The separation of the boundary layer is generally undesirable, since separated flow causes a great increase in the drag experienced by the body. So it is often necessary to prevent separation in order to reduce pressure drag and attain high lift.

Suction (or blowing) is one of the useful means in preventing boundary layer separation. The effect of suction consists in the removal of decelerated particles from the boundary layer before they are given a chance to cause separation. The surface is considered to be permeable to the fluid, so that the surface will allow a non-zero normal velocity and fluid is either sucked or blown through it. In doing this however, no-slip condition $u_w = 0$ at the surface (non-moving) shall continue to remain valid.

In driving the boundary layer equation, it is anticipated that the *v*-component of the velocity is a small quantity of the order of magnitude $O\left(\mathrm{Re}^{-\frac{1}{2}}\right)$ and it is assumed that the suction (or blowing) velocity v_w normal to the surface has its magnitude of order (characteristic Reynolds number)$^{-1/2}$. The consequence of this is that outer flow is independent of v_w and the boundary condition at the surface is given by $y=0$; $u=0$, $v=v_w(x)$.

Suction or blowing causes double effects with respect to the heat transfer. On the one hand, the temperature profile is influenced by the changed velocity field in the boundary-layer, leading to a change in the heat conduction at the surface. On the other hand, convective heat transfer occurs at the surface along with the heat conduction for $v_w \neq 0$. A summary of flow separation and its control are found in Chang [5, 6].

The study of natural convection on a horizontal plate with suction and blowing is of huge interest in many engineering applications, for instance, transpiration cooling, boundary layer control and other diffusion operations. The effects of blowing and suction on forced or free convection flow over vertical as well as horizontal plates were analyzed in a symmetric way by Gortler [16], Sparrow and Cess [37], Koh and Hartnett [22], Gersten and Gross [13], Merkin [24, 25], Vedhanaygam, Altenkirch and Eichhorn [42], Hasio-Tsung and Wen-Shing [18], Merkin [27] and Acharya, Shingh and Dash [1] etc.

Using the usual asymptotic approach, the similar solutions of the steady natural convection boundary layer for a non-similar flow situation on a horizontal plate with large suction approximation has been developed by Afzal and Hussain [3]. A detailed study on similarity solutions for free convection boundary layer flow over a permeable wall in a fluid saturated porous medium was carried out by Chaudhary *et al.* [7]. They have shown that the system depends on the power law exponent and the dimensionless surface mass transfer rate. They also examined the range of exponent under which the solution exists. With constant plate temperature and a particular distribution of blowing rate Clarke and Riley [10] obtained a special case of similarity solution, allowing variable fluid density. But there is still a shortage of accurate data for a wide range of both suction and blowing rate. Lin and Yu [23] presented a non-similar solution for the laminar free convection flow over a semi-infinite heated upward-facing horizontal porous plate with suitable transpiration rate as a power-law variation. Emphasis was given for an isothermal plate under the condition of uniform blowing or suction. Lately, using a parameter concerned pseudo-similarity technique of time and position coordinates, Cheng and Huang [9] studied the unsteady laminar boundary layer flow and heat transfer in the presence and absence of heat source or sink on a continuous moving and stretching isothermal surface with suction and blowing. In their analysis they paid attention on the temporal developments of the hydrodynamic and thermal characteristics after the sudden simultaneous changes in momentum and heat transfer. Recently, an analysis is performed by Aydin and Kayato [4] for the laminar boundary layer flow over a porous horizontal flat plate, particularly, to study the effect of uniform suction/injection on the heat transfer. Using the constant surface temperature as thermal boundary condition they also investigated the effect of Prandtl number on heat transfer.

Recently, Hossain and Mojumder [17] presented the similarity solution for the steady, laminar, free convection boundary layer flow generated above a heated horizontal rectangular surface. They investigated the effect of suction and blowing on fluid flow and heat transfer as well as skin friction coefficients. They also found that suction increased skin-friction and heat transfer coefficients whereas injection caused a decrease in both.

In our present study, we confined our discussion about the unsteady, laminar, free convection boundary layer flow above a semi-infinite heated, horizontal porous plate and investigated the effects of suction and blowing on the flow and temperature fields and other important flow parameters like pressure distribution, skin friction and heat transfer coefficients.

In order to solve the laminar natural convection boundary layer equations, the general Navier-Stokes' and energy equations are transformed into convenient simplified forms using the usual method of dimensional analysis. At the outset attempts are made to incorporate the idea of similarity analysis. Because, the objectives of seeking similarity solutions are manifold, the governing differential equations relevant to the problem have been solved by using the similarity technique. The Boussinesq approximation is employed to deal with the possible requirements of unsteady solution. Similarity requirements for an incompressible fluid are sought on the basis of detailed analysis in order to reduce the governing coupled partial differential equations into a set of ordinary differential equations.

Here we adopt the method of classical 'separation of variables' which is of the simplest and most straightforward method of determining similarity solutions. This method was first initiated by Abbott and Kline (1960). In this method, once a form of similarity variable is chosen, the given PDE is changed under the selected co-ordinate system. The dependent variables are to be expressed in terms of the product of separable functions of the new independent variables where each function is dependent on the single variable. Substitution of the product form of the dependent variables into the original PDE generally leads to an equation in which no functions of single variable can be isolated on the two sides of the equation unless certain parameters are to be specified. Usually, these parameters can be specified quite readily and “separation of the variables” is achieved. On this way the separation proceeds until the one side becomes an ODE. Four different similarity cases arise here, viz. Case A, Case B, Case C and Case D, on the basis of our assumptions.

Thus, this thesis is composed of 5 chapters. An introduction of basic principles of boundary layer theory, natural convection flows, suction and blowing phenomena with historical review of earlier researches and background of our problem are presented in Chapter I.

Basic equations governing the problem, dimensional analysis with simplifying assumptions and similarity transformations with possible similarity cases are given in Chapter II.

In Chapter III, a detailed discussion of one of the four similarity cases, namely, Case A has given. Under the considered condition, the numerical solutions with graphs and tables have also been given here for some selected values of the established parameters. The effects of these parameters on several variables will also be exhibited in the analysis.

Chapter IV is concerned with the study of another similarity case (Case B). The numerical solutions with graphs and tables for this case are also displayed here. We also have predicted the role of small suction or blowing velocity on these parameters concerned.

In Chapter V, the conclusions gained from this work and brief descriptions for further works related to our present research are discussed.

CHAPTER II

Basic Equations

The unsteady laminar two-dimensional boundary layer flows above a heated horizontal porous surface, maintained at a temperature different to that of the ambient fluid conditions are governed by the continuity, momentum and energy equations as follows:

Continuity equation-

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0 \tag{2.1}$$

x-component of momentum equation-

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{\rho}(\rho-\rho_0)g_x-\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x}+\frac{1}{\rho}\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2.2}$$

y-component of momentum equation-

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{1}{\rho}(\rho-\rho_0)g_y-\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial y}+\frac{1}{\rho}\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right) \tag{2.3}$$

Energy equation-

$$\begin{aligned}\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)&=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\left(u\frac{\partial \tilde{p}}{\partial x}+v\frac{\partial \tilde{p}}{\partial y}\right)\\&\quad+\mu\left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right\}+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2\right]\end{aligned} \tag{2.4}$$

In order to derive the boundary layer equations, it is anticipated that the *v*-component of the velocity is small enough and assumed that the suction or blowing velocity normal to the surface has its order of magnitude $O(\mathrm{Re}^{-\frac{1}{2}})$. Consequently, the flow outer the boundary layer is independent of v_w, so that, the boundary conditions at the surface is given by

$$y=0;\quad u=0,\quad v=v_w(x) \tag{2.5}$$

The schematic view and coordinate geometry of the problem are shown in Fig. 1 below:

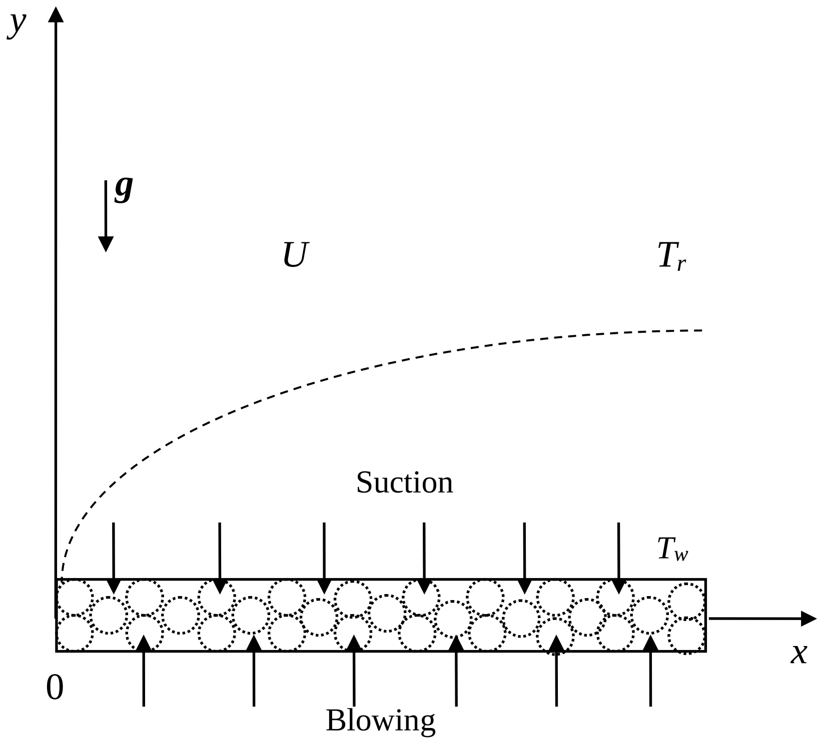

Fig. 2.1: Schematic representation and coordinate system of the problem.

Dimensional Analysis

The following non-dimensional variables are introduced in order to reduce the less significant terms in the above boundary layer equations (2.1) – (2.4):

$$t' = \frac{Ut}{L}, x' = \frac{x}{L}, y' = \mathrm{Re}^{\frac{1}{2}}\frac{y}{L}, u' = \frac{u}{U}, v' = \mathrm{Re}^{\frac{1}{2}}\frac{v}{U}, p' = \frac{\tilde{p}}{\rho_r U^2}, \rho' = \frac{\rho}{\rho_r},$$
$$\mu' = \frac{\mu}{\mu_r}, k' = \frac{k}{k_r}, c_p' = \frac{c_p}{c_{p_r}}, T' = \frac{T}{T_r}, g_x{}' = \frac{g_x}{g}, g_y{}' = \frac{g_y}{g} \tag{2.6}$$

where dashed variables are non-dimensional and U, L represent convenient characteristic velocity and length scales, $\mathrm{Re} = \frac{UL}{\upsilon_r}$ is a characteristic Reynolds number based on U and L. Suffix r refers to a convenient constant reference condition at a fixed point outside the boundary layer. The Cartesian co- ordinates x, y are chosen to lie along and normal to the plate, g_x and g_y are the components of the gravity vector in the x and y-directions. The perturbation pressure $\tilde{p}$ is related to the absolute pressure p by the equation $p = \tilde{p} + p_0$, where $\frac{\partial p_0}{\partial x} = \rho_0 g_x$, $\frac{\partial p_0}{\partial y} = \rho_0 g_y$. Here suffix 0 is considered to denote conditions in a fluid at rest and $\tilde{p}$ is termed as the motion pressure. The gradients of the hydrostatic pressure p_0 are balanced by the body force terms.

Now from equations (2.6) we get

$$\frac{\partial}{\partial t}=\frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}=\frac{\partial}{\partial t}\left(\frac{Ut}{L}\right)\frac{\partial}{\partial t'}=\frac{U}{L}\frac{\partial}{\partial t'}$$

$$\frac{\partial}{\partial x}=\frac{\partial x'}{\partial x}\frac{\partial}{\partial x'}=\frac{\partial}{\partial x}\left(\frac{x}{L}\right)\frac{\partial}{\partial x'}=\frac{1}{L}\frac{\partial}{\partial x'}$$

$$\frac{\partial}{\partial y}=\frac{\partial y'}{\partial y}\frac{\partial}{\partial y'}=\frac{\partial}{\partial y}\left(\mathrm{Re}^{\frac{1}{2}}\frac{y}{L}\right)\frac{\partial}{\partial y'}=\frac{\mathrm{Re}^{\frac{1}{2}}}{L}\frac{\partial}{\partial y'}$$

Then the above equations (2.1) – (2.4) become:

Continuity equation-

$$\frac{U}{L}\frac{\partial}{\partial t'}(\rho_r\rho')+\frac{1}{L}\frac{\partial}{\partial x'}(\rho_r\rho' Uu')+\frac{\mathrm{Re}^{\frac{1}{2}}}{L}\frac{\partial}{\partial y'}\left(\frac{\rho_r\rho' Uv'}{\mathrm{Re}^{\frac{1}{2}}}\right)=0$$

i.e., $$\frac{\partial\rho'}{\partial t'}+\frac{\partial}{\partial x'}(\rho' u')+\frac{\partial}{\partial y'}(\rho' v')=0 \qquad (2.7)$$

x-component of momentum equation-

$$\frac{U}{L}\frac{\partial}{\partial t'}(Uu')+\frac{Uu'}{L}\frac{\partial}{\partial x'}(Uu')+\frac{Uv'}{\mathrm{Re}^{\frac{1}{2}}}\frac{\mathrm{Re}^{\frac{1}{2}}}{L}\frac{\partial}{\partial y'}(Uu')=\frac{1}{\rho_r\rho'}(\rho_r\rho'-\rho_r\rho_0')gg_x'-\frac{1}{\rho_r\rho' L}\frac{\partial}{\partial x'}(\rho_r U^2 p')$$
$$+\frac{1}{\rho_r\rho' L}\frac{\partial}{\partial x'}\left\{\frac{\mu_r\mu'}{L}\frac{\partial}{\partial x'}(Uu')\right\}+\frac{\mathrm{Re}^{\frac{1}{2}}}{\rho_r\rho' L}\frac{\partial}{\partial y'}\left\{\frac{\mu_r\mu'\mathrm{Re}^{\frac{1}{2}}}{L}\frac{\partial}{\partial y'}(Uu')\right\}$$

Or,

$$\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}=\frac{gL}{U^2\rho'}(\rho'-\rho_0')g_x'-\frac{1}{\rho'}\frac{\partial p'}{\partial x'}+\frac{\mu_r}{LU\rho_r\rho'}\frac{\partial}{\partial x'}\left(\mu'\frac{\partial u'}{\partial x'}\right)+\frac{\mu_r\,\mathrm{Re}}{LU\rho_r\rho'}\frac{\partial}{\partial y'}\left(\mu'\frac{\partial u'}{\partial y'}\right)$$

Or,

$$\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}=\frac{gL}{U^2}\left(1-\frac{\rho_0'}{\rho'}\right)g_x'-\frac{1}{\rho'}\frac{\partial p'}{\partial x'}+\frac{\upsilon_r}{LU}\frac{1}{\rho'}\frac{\partial}{\partial x'}\left(\mu'\frac{\partial u'}{\partial x'}\right)+\frac{\upsilon_r}{LU}\frac{\mathrm{Re}}{\rho'}\frac{\partial}{\partial y'}\left(\mu'\frac{\partial u'}{\partial y'}\right)$$

Or, $$\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}=\frac{1}{\mathrm{Fr}}\left(1-\frac{\rho_0'}{\rho'}\right)g_x'-\frac{1}{\rho'}\frac{\partial p'}{\partial x'}+\frac{1}{\mathrm{Re}}\frac{1}{\rho'}\frac{\partial}{\partial x'}\left(\mu'\frac{\partial u'}{\partial x'}\right)+\frac{1}{\rho'}\frac{\partial}{\partial y'}\left(\mu'\frac{\partial u'}{\partial y'}\right)$$

i.e., $$\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}=\frac{1}{\mathrm{Fr}}\left(1-\frac{\rho_0'}{\rho'}\right)g_x'-\frac{1}{\rho'}\frac{\partial p'}{\partial x'}+\frac{1}{\rho'}\frac{\partial}{\partial y'}\left(\mu'\frac{\partial u'}{\partial y'}\right)+O\left(\mathrm{Re}^{-1}\right) \qquad (2.8)$$

y-component of momentum equation-

$$\frac{U}{L}\frac{\partial}{\partial t'}\left(\frac{Uv'}{\mathrm{Re}^{\frac{1}{2}}}\right)+\frac{Uu'}{L}\frac{\partial}{\partial x'}\left(\frac{Uv'}{\mathrm{Re}^{\frac{1}{2}}}\right)+\frac{Uv'}{\mathrm{Re}^{\frac{1}{2}}}\frac{\mathrm{Re}^{\frac{1}{2}}}{L}\frac{\partial}{\partial y'}\left(\frac{Uv'}{\mathrm{Re}^{\frac{1}{2}}}\right)=\frac{1}{\rho_r\rho'}\left(\rho_r\rho'-\rho_r\rho_0'\right)gg_y'$$

$$-\frac{1}{\rho_r\rho'}\frac{\mathrm{Re}^{\frac{1}{2}}}{L}\frac{\partial}{\partial y'}\left(\rho_r U^2 p'\right)+\frac{1}{\rho_r\rho' L}\frac{\partial}{\partial x'}\left\{\frac{\mu_r\mu'}{L}\frac{\partial}{\partial x'}\left(\frac{Uv'}{\mathrm{Re}^{\frac{1}{2}}}\right)\right\}+\frac{\mathrm{Re}^{\frac{1}{2}}}{\rho_r\rho' L}\frac{\partial}{\partial y'}\left\{\frac{\mu_r\mu'\mathrm{Re}^{\frac{1}{2}}}{L}\frac{\partial}{\partial y'}\left(\frac{Uv'}{\mathrm{Re}^{\frac{1}{2}}}\right)\right\}$$

Or,

$$\frac{1}{\mathrm{Re}^{\frac{1}{2}}}\frac{\partial v'}{\partial t'}+\frac{1}{\mathrm{Re}^{\frac{1}{2}}}u'\frac{\partial v'}{\partial x'}+\frac{1}{\mathrm{Re}^{\frac{1}{2}}}v'\frac{\partial v'}{\partial y'}=\frac{gL}{U^2}\left(1-\frac{\rho_0'}{\rho'}\right)g_y'-\frac{\mathrm{Re}^{\frac{1}{2}}}{\rho'}\frac{\partial p'}{\partial y'}$$
$$+\frac{1}{\mathrm{Re}^{\frac{1}{2}}}\frac{\mu_r}{LU\rho_r\rho'}\frac{\partial}{\partial x'}\left(\mu'\frac{\partial v'}{\partial x'}\right)+\frac{\mu_r\mathrm{Re}^{\frac{1}{2}}}{LU\rho'\rho_r}\frac{\partial}{\partial y'}\left(\mu'\frac{\partial v'}{\partial y'}\right)$$

Dividing both sides by $\mathrm{Re}^{\frac{1}{2}}$, we obtain

$$\mathrm{Re}^{-1}\left(\frac{\partial v'}{\partial t'}+u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}\right)=\frac{\mathrm{Re}^{-\frac{1}{2}}}{\mathrm{Fr}}\left(1-\frac{\rho_0'}{\rho'}\right)g_y'-\frac{1}{\rho'}\frac{\partial p'}{\partial y'}$$
$$+\frac{\upsilon_r}{LU}\frac{\mathrm{Re}^{-1}}{\rho'}\frac{\partial}{\partial x'}\left(\mu'\frac{\partial v'}{\partial x'}\right)+\frac{\upsilon_r}{LU}\frac{1}{\rho'}\frac{\partial}{\partial y'}\left(\mu'\frac{\partial v'}{\partial y'}\right)$$

i.e., $$\frac{1}{\rho'}\frac{\partial p'}{\partial y'}=\frac{\mathrm{Re}^{-\frac{1}{2}}}{\mathrm{Fr}}\left(1-\frac{\rho_0'}{\rho'}\right)g_y'+O\left(\mathrm{Re}^{-1}\right) \tag{2.9}$$

Energy equation-

$$\rho_r\rho' c_p' c_{p_r}\left\{\frac{U}{L}\frac{\partial}{\partial t'}\left(T_rT'\right)+\frac{Uu'}{L}\frac{\partial}{\partial x'}\left(T_rT'\right)+\frac{Uv'}{\mathrm{Re}^{\frac{1}{2}}}\frac{\mathrm{Re}^{\frac{1}{2}}}{L}\frac{\partial}{\partial y'}\left(T_rT'\right)\right\}=\frac{1}{L}\frac{\partial}{\partial x'}\left\{\frac{k_rk'}{L}\frac{\partial}{\partial x'}\left(T_rT'\right)\right\}$$
$$+\frac{\mathrm{Re}^{\frac{1}{2}}}{L}\frac{\partial}{\partial y'}\left\{\frac{k_rk'\mathrm{Re}^{\frac{1}{2}}}{L}\frac{\partial}{\partial y'}\left(T_rT'\right)\right\}+\left\{\frac{Uu'}{L}\frac{\partial}{\partial x'}\left(\rho_rU^2p'\right)+\frac{Uv'}{\mathrm{Re}^{\frac{1}{2}}}\frac{\mathrm{Re}^{\frac{1}{2}}}{L}\frac{\partial}{\partial y'}\left(\rho_rU^2p'\right)\right\}$$
$$+\mu_r\mu'\left[2\left\{\frac{1}{L}\frac{\partial}{\partial x'}\left(Uu'\right)\right\}^2+2\left\{\frac{\mathrm{Re}^{\frac{1}{2}}}{L}\frac{\partial}{\partial y'}\left(\frac{Uv'}{\mathrm{Re}^{\frac{1}{2}}}\right)\right\}^2+\left\{\frac{\mathrm{Re}^{\frac{1}{2}}}{L}\frac{\partial}{\partial y'}\left(Uu'\right)+\frac{1}{L}\frac{\partial}{\partial x'}\left(\frac{Uv'}{\mathrm{Re}^{\frac{1}{2}}}\right)\right\}^2\right.$$

$$-\frac{2}{3}\left\{\frac{1}{L}\frac{\partial}{\partial x}(Uu')+\frac{\mathrm{Re}^{\frac{1}{2}}}{L}\frac{\partial}{\partial y'}\left(\frac{Uv'}{\mathrm{Re}^{\frac{1}{2}}}\right)\right\}^2\Bigg]$$

Or,

$$\rho' c_p'\left(\frac{\partial T'}{\partial t'}+u'\frac{\partial T'}{\partial x'}+v'\frac{\partial T'}{\partial y'}\right)=\frac{\upsilon_r}{UL}\frac{k_r}{\mu_r c_{p_r}}\frac{\partial}{\partial x'}\left(k'\frac{\partial T'}{\partial x'}\right)+\mathrm{Re}\frac{\upsilon_r}{UL}\frac{k_r}{\mu_r c_{p_r}}\frac{\partial}{\partial y'}\left(k'\frac{\partial T'}{\partial y'}\right)$$

$$+\frac{U^2}{c_{p_r}T_r}\left(u'\frac{\partial p'}{\partial x'}+v'\frac{\partial p'}{\partial y'}\right)+\frac{\upsilon_r}{UL}\frac{U^2}{c_{p_r}T_r}\mu'\left[2\left\{\left(\frac{\partial u'}{\partial x'}\right)^2+\left(\frac{\partial v'}{\partial y'}\right)^2\right\}\right.$$

$$\left.\left(\mathrm{Re}^{\frac{1}{2}}\frac{\partial u'}{\partial y'}+\frac{1}{\mathrm{Re}^{\frac{1}{2}}}\frac{\partial v'}{\partial x'}\right)^2-\frac{2}{3}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y'}\right)^2\right]$$

Or,

$$\rho' c_p'\left(\frac{\partial T'}{\partial t'}+u'\frac{\partial T'}{\partial x'}+v'\frac{\partial T'}{\partial y'}\right)=\mathrm{Re}^{-1}\,\mathrm{Pr}^{-1}\frac{\partial}{\partial x'}\left(k'\frac{\partial T'}{\partial x'}\right)+\mathrm{Pr}^{-1}\frac{\partial}{\partial y'}\left(k'\frac{\partial T'}{\partial y'}\right)+\mathrm{E_c}\left(u'\frac{\partial p'}{\partial x'}+v'\frac{\partial p'}{\partial y'}\right)$$

$$+\mathrm{E_c}\,\mathrm{Re}^{-1}\mu'\left[2\left\{\left(\frac{\partial u'}{\partial x'}\right)^2+\left(\frac{\partial v'}{\partial y'}\right)^2\right\}+\left(\mathrm{Re}^{\frac{1}{2}}\frac{\partial u'}{\partial y'}+\frac{1}{\mathrm{Re}^{\frac{1}{2}}}\frac{\partial v'}{\partial x'}\right)^2-\frac{2}{3}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y'}\right)^2\right]$$

i.e., $$\rho' c_p'\left(\frac{\partial T'}{\partial t'}+u'\frac{\partial T'}{\partial x'}+v'\frac{\partial T'}{\partial y'}\right)=\mathrm{Pr}^{-1}\frac{\partial}{\partial y'}\left(k'\frac{\partial T'}{\partial y'}\right)+O(\mathrm{E_c})+O\left(\mathrm{Re}^{-1}\right) \qquad (2.10)$$

where $\rho_r=\frac{\mu_r}{\upsilon_r}$ and $\mathrm{Fr}\left(=\frac{U^2}{gL}\right)$, $\mathrm{Pr}\left(=\frac{\mu_r c_{p_r}}{k_r}\right)$ and $\mathrm{E_c}\left(=\frac{U^2}{c_{p_r}T_r}\right)$ are the characteristic Froude number, Prandtl number and Eckert number of the flow respectively .

Thus the flow above the horizontal flat plate is governed by the following non – dimensional boundary – layer equations:

$$\frac{\partial \rho'}{\partial t'}+\frac{\partial}{\partial x'}(\rho' u')+\frac{\partial}{\partial y'}(\rho' v')=0 \qquad (2.7)$$

$$\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}=\frac{1}{\mathrm{Fr}}\left(1-\frac{\rho_0'}{\rho'}\right)g_x'-\frac{1}{\rho'}\frac{\partial p'}{\partial x'}+\frac{1}{\rho'}\frac{\partial}{\partial y'}\left(\mu'\frac{\partial u'}{\partial y'}\right)+O\left(\mathrm{Re}^{-1}\right) \qquad (2.8)$$

$$\frac{1}{\rho'}\frac{\partial p'}{\partial y'}=\frac{\mathrm{Re}^{-\frac{1}{2}}}{\mathrm{Fr}}\left(1-\frac{\rho_0'}{\rho'}\right)g_y'+O\left(\mathrm{Re}^{-1}\right) \qquad (2.9)$$

$$\rho' c_p'\left(\frac{\partial T'}{\partial t'}+u'\frac{\partial T'}{\partial x'}+v'\frac{\partial T'}{\partial y'}\right)=\mathrm{Pr}^{-1}\frac{\partial}{\partial y'}\left(k'\frac{\partial T'}{\partial y'}\right)+O(\mathrm{E_c})+O\left(\mathrm{Re}^{-1}\right) \qquad (2.10)$$

The order of magnitude of $\mathrm{E_c} = \dfrac{U^2}{c_p \Delta T}$ for these flows will be determined presently by $\Pr = \dfrac{\mu_r c_{p_r}}{k_r}$. Taking the traditional approach, if we consider the limit $\mathrm{Re}^{-\frac{1}{2}} \to 0$ with Fr finite, according to first order boundary layer theory, the *v*-momentum equation asserts that $p' = p'(x', t')$. However, if we consider $\varepsilon = \mathrm{Re}^{-\frac{1}{2}}$ and imposes the condition that $\underset{\varepsilon \to 0}{\mathrm{Lt}} \dfrac{\varepsilon g'_y}{\mathrm{Fr}}$ remains finite, then the gravity dependent term must be retained in the *v*-momentum equation, resulting in $p' = p'(x', y', t')$. In the present investigation we are concerned with those boundary layer flows for which $\dfrac{\mathrm{Re}^{-\frac{1}{2}}}{F_r}\left(1 - \dfrac{\rho'_0}{\rho'}\right) g'_y \cong O(1)$. Since the variation in the buoyancy force normal to the surface is the only means of producing boundary layer motion, on a horizontal surface, the component of the buoyancy force parallel to the surface is zero, i.e., $g_x{}' = 0$. In natural convection flow the order of magnitude of velocity created by the density differences across the boundary layer is determined as $U \cong O\left\{\dfrac{\rho'_w - \rho'_0}{\rho'_w} g'_y (\upsilon_r L)^{\frac{1}{2}}\right\}^{\frac{2}{5}}$. In all such situations $\rho' = \rho'(x', y', t')$ inside the first order boundary layer provides the mechanism for flow generation. If Re is large then Re^{-1} is treated as very small in magnitude. The pressure gradient normal to the surface caused by the density difference $(= \rho_w - \rho_0)$ generates the perturbation pressure field $\tilde{P}(x', y', t')$ inside the boundary layer, x' - variations of this field being sufficient to cause the motion in the boundary layer. This motion occurs irrespective of any exterior forcing flow (caused, say, by a body- force component along the surface) and natural convection flow. Since the differential of p' occurs in the momentum and energy equations, it is convenient to write the general equation of state for a fluid $p = p(\rho, T) = \tilde{p}(x, y, t) + p_0$ as

$$d\tilde{p} = \left(\frac{\partial p}{\partial \rho}\right)_T d\rho + \left(\frac{\partial p}{\partial T}\right)_\rho dt - dp_0 = \frac{1}{k}\frac{\partial \rho}{\rho} + \frac{\beta_T}{k} dT - dp_0,$$

which in the above non-dimensional form becomes $k\rho_r U^2 dp' = \dfrac{d\rho'}{\rho'} + \beta_T T_r dT' - kdp_0$,

where $k = \dfrac{1}{\rho}\left(\dfrac{\partial \rho}{\partial p}\right)_T$, $\beta_T = \dfrac{1}{\rho}\left(\dfrac{\partial \rho}{\partial T}\right)_p$.

For those cases in which p_0 is determined by the condition that a given function of state is constant, it can be shown that (Ackroyd [2]). $kdp_0 \approx O\left(\dfrac{l_0 g\beta_T}{c_p}\right), dp_0 \approx O\left(\dfrac{l_0 g\beta_T}{c_p}\right)$. Typically, $\dfrac{c_p}{g\beta_T}$ is a length scale of order 10^4 meters for air and 10^6 meters for water under normal pressure and temperature, whereas l_0 represents the vertical scale of the flow field, which can considerably be taken to be the maximum boundary layer thickness. Consequently, with the additional provision $k_r\rho_r U^2 << 1$, it follows that $\rho = \rho(T)$, $\rho_0 = \rho_r$, so that variations in ρ_0 etc. with altitude, due to hydrostatic relations can be ignored .

In view of the above discussions and omitting the dashes, the governing boundary layer equations (i.e., the continuity, momentum and energy equations) (2.7) – (2.10) in dimensional form for a variable properties fluid over a semi-infinite horizontal flat surface are as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{2.11}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2.12}$$

$$\frac{\partial \tilde{p}}{\partial y'} = (\rho - \rho_0)\, g_y \tag{2.13}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \mathrm{Pr}^{-1}\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \tag{2.14}$$

In the energy equation (2.14) the pressure and viscous work contributions have been ignored because of the relatively small Eckert number usually encountered in the free convection flow. The Eckert number Ec, which governs the significance of these terms, is

$$\mathrm{E_c} = \frac{U^2}{c_p\Delta T} = \frac{U^2}{c_p(T_w - T_r)} \approx O\left\{\frac{\beta_T g_y L}{c_{p_r}}\frac{\dfrac{(\rho_w - \rho_r)}{\rho_w}}{\beta_T(T_w - T_r)}\left(\frac{UL}{\upsilon_r}\right)^{\frac{1}{2}}\right\}.$$

Now $\dfrac{\dfrac{(\rho_w - \rho_r)}{\rho_w}}{\beta_T (T_w - T_r)}$ is of order unity whereas $\dfrac{\beta_T g_y L}{c_{p_r}}$, as we have seen above, is extremely small compared with unity. However the occurrence of $\left(\dfrac{UL}{\upsilon_r}\right)^{\frac{1}{2}} = \mathrm{Re}^{\frac{1}{2}}$ in the above expression for the Eckert number indicates that terms involving the Eckert number should not appear in the first order boundary layer theory.

Since the present study is concerned solely with the possible self-similar flow situations for a Boussinesq fluid, without loss of generality we have been introduced the effect of buoyancy by means of the Boussinesq approximation. Thus, fluid property variations other than the essential density variation are ignored completely in this approximation. The density difference $(= \rho_r - \rho)$ is indispensable to the free convection motion and must be retained where they appear in the body force term (i.e., term multiplied by g, the acceleration due to gravity), but elsewhere the density variation is considered to be small enough and is to be neglected. In view of the above discussions, the governing boundary layer equations of laminar two-dimensional unsteady flow over a semi-infinite heated horizontal porous surface in dimensional form are simplified to the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.15}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho_r}\frac{\partial \tilde{p}}{\partial x} = \upsilon_r \frac{\partial^2 u}{\partial y^2} \tag{2.16}$$

$$\frac{\partial \tilde{p}}{\partial y} = \pm(\rho - \rho_r)g \tag{2.17}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\upsilon_r}{\Pr}\frac{\partial^2 T}{\partial y^2} \tag{2.18}$$

where $g_y = \pm g$ is constants and $\rho \propto \dfrac{1}{T}$. There are, however, boundary conditions, which are imposed in order to determine the solution of the boundary layer equations (2.15) – (2.18).

$$\begin{aligned} &(i)\ u = 0,\ v = v_w \text{ at } y = 0,\ (ii)\ T = T_w \text{ at } y = 0 \Rightarrow \theta = 1 \text{ at } y = 0, \\ &(iii)\ u = 0 \text{ when } y \to \infty \text{ and } (iv)\ T = T_r \Rightarrow \theta = 0 \text{ when } y \to \infty \end{aligned} \tag{2.19}$$

(suffix '*w*' represents the condition at the surface of the plate and suffix '*r*' is the constant reference condition in the fluid at rest exterior the boundary layer).

Similarity Transformations

In order to reduce the above system of equations into convenient forms, we adopt the method of seeking similarity solutions. Hence the following substitutions are introduced–

$$\begin{aligned}&\tau = t,\ \xi = x,\ \phi = \frac{y}{\gamma(x,t)},\ u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x},\\ &T - T_r = \Delta T(\tau,\xi)U(\tau,\xi)\theta(\tau,\xi,\phi),\ \Delta T = T_w(\tau,\xi) - T_r\\ &\psi(\tau,\xi,\phi) = \gamma(\tau,\xi)U(\tau,\xi)F(\tau,\xi,\phi),\ \tilde{p} = p(\tau,\xi)G(\tau,\xi,\phi)\\ &(\rho - \rho_r) = -\rho_r\beta_T\Delta T(\tau,\xi)\theta(\tau,\xi,\phi)\end{aligned} \tag{2.20}$$

Guided by the idea of the similarity procedure, we also use the traditional substitution

$$\int_0^{\phi} \frac{u}{U(x,t)} d\phi_1 = F(\tau,\xi,\phi) \tag{2.21}$$

Now

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \tau}{\partial y}.\frac{\partial}{\partial \tau}\{\psi(\tau,\xi,\phi)\} + \frac{\partial \xi}{\partial y}.\frac{\partial}{\partial \xi}\{\psi(\tau,\xi,\phi)\} + \frac{\partial \phi}{\partial y}.\frac{\partial}{\partial \phi}\{\psi(\tau,\xi,\phi)\}$$

i.e., $u = \frac{1}{\gamma(\tau,\xi)}\frac{\partial}{\partial \phi}\psi(\tau,\xi,\phi)$, since $\phi = \frac{y}{\gamma(\tau,\xi)} \Rightarrow \frac{\partial \phi}{\partial y} = \frac{1}{\gamma}$.

Therefore,

$$\frac{u}{U(\tau,\xi)} = \frac{1}{\gamma(\tau,\xi)U(\tau,\xi)}\frac{\partial}{\partial \phi}\psi(\tau,\xi,\varphi).$$

Hence

$$\int_0^{\phi} \frac{u}{U(\tau,\xi)} d\phi_1 = \frac{1}{\gamma(\tau,\xi)U(\tau,\xi)}\left[\psi(\tau,\xi,\phi)\right]_0^{\phi} = \frac{1}{\gamma(\tau,\xi)U(\tau,\xi)}\left[\psi(\tau,\xi,\phi) - \psi(\tau,\xi,0)\right]$$

,

and by (2.21) implies

$$F(\tau,\xi,\phi) = \frac{1}{\gamma(\tau,\xi)U(\tau,\xi)}\left[\psi(\tau,\xi,\phi) - \psi(\tau,\xi,0)\right] \tag{2.22}$$

Then we obtain

$$\psi(\tau,\xi,\phi) = \gamma(\tau,\xi)U(\tau,\xi)F(\tau,\xi,\phi) + \psi(\tau,\xi,0) \tag{2.23}$$

Also by (2.21)

$$\frac{u}{U(\tau,\xi)} = \frac{\partial}{\partial \phi}F(\tau,\xi,\phi)$$

i.e., $u = U(\tau,\xi)F_{\phi}(\tau,\xi,\phi)$ (2.24)

Again

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial \tau}.\frac{\partial \tau}{\partial t}+\frac{\partial}{\partial \xi}.\frac{\partial \xi}{\partial t}+\frac{\partial}{\partial \phi}.\frac{\partial \phi}{\partial t}=\frac{\partial}{\partial \tau}-\frac{\gamma_\tau}{\gamma}\phi\frac{\partial}{\partial \phi} \tag{2.25}$$

$$\left[\because \phi=\frac{y}{\gamma(t,x)}\Rightarrow\frac{\partial \phi}{\partial t}=-\frac{y\gamma_t}{\gamma^2}=-\frac{\gamma_\tau}{\gamma}\frac{y}{\gamma}=-\frac{\gamma_\tau}{\gamma}\phi\right]$$

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \tau}.\frac{\partial \tau}{\partial x}+\frac{\partial}{\partial \xi}.\frac{\partial \xi}{\partial x}+\frac{\partial}{\partial \phi}.\frac{\partial \phi}{\partial x}=\frac{\partial}{\partial \xi}-\frac{\gamma_\xi}{\gamma}\phi\frac{\partial}{\partial \phi} \tag{2.26}$$

$$\left[\because \phi=\frac{y}{\gamma(t,x)}\Rightarrow\frac{\partial \phi}{\partial x}=-\frac{y\gamma_x}{\gamma^2}=-\frac{\gamma_\xi}{\gamma}\frac{y}{\gamma}=-\frac{\gamma_\xi}{\gamma}\phi\right]$$

and

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial \tau}.\frac{\partial \tau}{\partial y}+\frac{\partial}{\partial \xi}.\frac{\partial \xi}{\partial y}+\frac{\partial}{\partial \phi}.\frac{\partial \phi}{\partial y}=\frac{1}{\gamma}\frac{\partial}{\partial \phi} \tag{2.27}$$

$$\left[\because \phi=\frac{y}{\gamma(t,x)}\Rightarrow\frac{\partial \phi}{\partial y}=\frac{1}{\gamma}\right]$$

$$\begin{aligned}-v=\frac{\partial \psi}{\partial x}&=\left(\frac{\partial}{\partial \xi}-\frac{\gamma_\xi}{\gamma}\phi\frac{\partial}{\partial \phi}\right)\left[\gamma UF(\tau,\xi,\phi)+\psi(\tau,\xi,0)\right]\\&=\frac{\partial}{\partial \xi}\left[\gamma UF(\tau,\xi,\phi)\right]+\frac{\partial}{\partial \xi}\left[\psi(\tau,\xi,0)\right]-\frac{\gamma_\xi}{\gamma}\phi\frac{\partial}{\partial \phi}\left[\gamma UF(\tau,\xi,\phi)\right]-\frac{\gamma_\xi}{\gamma}\phi\frac{\partial}{\partial \phi}\left[\psi(\tau,\xi,0)\right]\\&=(\gamma UF)_\xi-\gamma_\xi U\phi F_\phi+\frac{\partial}{\partial \xi}\left[\psi(\tau,\xi,0)\right]\end{aligned}$$

Hence $v=\gamma_\xi U\phi F_\phi-(\gamma UF)_\xi-\frac{\partial}{\partial \xi}\left[\psi(\tau,\xi,0)\right]=\gamma_\xi U\phi F_\phi-(\gamma UF)_\xi+v_w$ (2.28)

where $v_w=-\frac{\partial \psi(\tau,\xi,0)}{\partial \xi}$ represents the non-zero wall velocity called the suction or blowing velocity normal to the porous surface, so that fluid can either be sucked or blown through it. Physically, $v_w<0$ and $v_w>0$ represent the suction and blowing velocity through the porous surface, respectively. For uniform suction (or blowing) $v_w=\text{constant}$. However, $v_w=0$ implies that the surface is impermeable to the fluid.

Again

$$\frac{\partial u}{\partial t}=\left(\frac{\partial}{\partial \tau}-\frac{\gamma_\tau}{\gamma}\phi\frac{\partial}{\partial \phi}\right)UF_\phi=\frac{\partial}{\partial \tau}\left(UF_\phi\right)-\frac{\gamma_\tau}{\gamma}\phi\frac{\partial}{\partial \phi}\left(UF_\phi\right)$$

i.e., $$\frac{\partial u}{\partial t}=U_\tau F_\phi+UF_{\phi\tau}-\frac{\gamma_\tau U}{\gamma}\phi F_{\phi\phi} \tag{2.29}$$

$$u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}=UF_\phi\left(\frac{\partial}{\partial\xi}-\frac{\gamma_\xi}{\gamma}\phi\frac{\partial}{\partial\phi}\right)+\left[\gamma_\xi U\phi F_\phi-(\gamma UF)_\xi+v_w\right]\frac{1}{\gamma}\frac{\partial}{\partial\phi}$$

$$=UF_\phi\frac{\partial}{\partial\xi}-\frac{\gamma_\xi U}{\gamma}\phi F_\phi\frac{\partial}{\partial\phi}+\frac{\gamma_\xi U}{\gamma}\phi F_\phi\frac{\partial}{\partial\phi}-\frac{(\gamma UF)_\xi}{\gamma}\frac{\partial}{\partial\phi}+\frac{v_w}{\gamma}\frac{\partial}{\partial\phi}$$

$$=UF_\phi\frac{\partial}{\partial\xi}-\frac{(\gamma UF)_\xi}{\gamma}\frac{\partial}{\partial\phi}+\frac{v_w}{\gamma}\frac{\partial}{\partial\phi}$$

$$\therefore u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)u=\left[UF_\phi\frac{\partial}{\partial\xi}-\frac{(\gamma UF)_\xi}{\gamma}\frac{\partial}{\partial\phi}+\frac{v_w}{\gamma}\frac{\partial}{\partial\phi}\right]UF_\phi$$

$$=UF_\phi\frac{\partial}{\partial\xi}(UF_\phi)-\frac{(\gamma UF)_\xi}{\gamma}\frac{\partial}{\partial\phi}(UF_\phi)+\frac{v_w}{\gamma}\frac{\partial}{\partial\phi}(UF_\phi)$$

$$=UF_\phi\left(U_\xi F_\phi+UF_{\phi\xi}\right)-\frac{(\gamma UF)_\xi}{\gamma}UF_{\phi\phi}+\frac{v_w}{\gamma}UF_{\phi\phi}$$

i.e., $$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=UU_\xi F_\phi^{\,2}+U^2F_\phi F_{\phi\xi}-\frac{(\gamma UF)_\xi}{\gamma}UF_{\phi\phi}+\frac{v_w}{\gamma}UF_{\phi\phi} \tag{2.30}$$

$$\frac{1}{\rho_r}\frac{\partial\tilde{p}}{\partial x}=\frac{1}{\rho_r}\left(\frac{\partial}{\partial\xi}-\frac{\gamma_\xi}{\gamma}\phi\frac{\partial}{\partial\phi}\right)\left[p(\tau,\xi)G(\tau,\xi,\phi)\right]$$

$$=\frac{1}{\rho_r}\frac{\partial}{\partial\xi}\left[p(\tau,\xi)G(\tau,\xi,\phi)\right]-\frac{\gamma_\xi}{\rho_r\gamma}\phi\frac{\partial}{\partial\phi}\left[p(\tau,\xi)G(\tau,\xi,\phi)\right]$$

$$=\frac{1}{\rho_r}\left(p_\xi G+pG_\xi\right)-\frac{\gamma_\xi}{\rho_r\gamma}p(\tau,\xi)\phi G_\phi$$

i.e., $$\frac{1}{\rho_r}\frac{\partial\tilde{p}}{\partial x}=\frac{p_\xi G}{\rho_r}+\frac{pG_\xi}{\rho_r}-\frac{\gamma_\xi p}{\rho_r\gamma}\phi G_\phi \tag{2.31}$$

$$\upsilon_r\frac{\partial^2u}{\partial y^2}=\upsilon_r\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\upsilon_r}{\gamma}\frac{\partial}{\partial\phi}\left[\frac{1}{\gamma}\frac{\partial}{\partial\phi}(UF_\phi)\right]$$

i.e., $$\upsilon_r\frac{\partial^2u}{\partial y^2}=\frac{\upsilon_r}{\gamma^2}UF_{\phi\phi\phi} \tag{2.32}$$

With the help of (2.29) – (2.32) equation (2.16) becomes

$$U_\tau F_\phi+UF_{\phi\tau}-\frac{\gamma_\tau U}{\gamma}\phi F_{\phi\phi}+UU_\xi F_\phi^{\,2}+U^2F_\phi F_{\phi\xi}-\frac{(\gamma UF)_\xi}{\gamma}UF_{\phi\phi}+\frac{v_w}{\gamma}UF_{\phi\phi}$$

$$+\frac{p_\xi G}{\rho_r}+\frac{pG_\xi}{\rho_r}-\frac{\gamma_\xi p}{\rho_r\gamma}\phi G_\phi=\frac{\upsilon_r}{\gamma^2}UF_{\phi\phi\phi}$$

Or, $\frac{\upsilon_r}{\gamma^2}UF_{\phi\phi\phi}+\frac{\gamma_\tau U}{\gamma}\phi F_{\phi\phi}+U^2F_\xi F_{\phi\phi}+UU_\xi FF_{\phi\phi}+\frac{\gamma_\xi U^2}{\gamma}FF_{\phi\phi}-\frac{v_w}{\gamma}UF_{\phi\phi}$

$$-UF_{\phi\tau}-U^2F_\phi F_{\phi\xi}-UU_\xi F_\phi{}^2-U_\tau F_\phi=\frac{p_\xi G}{\rho_r}+\frac{pG_\xi}{\rho_r}-\frac{\gamma_\xi p}{\rho_r\gamma}\phi G_\phi$$

as $\frac{(\gamma UF)_\xi}{\gamma}UF_{\phi\phi}=\frac{1}{\gamma}\left[\gamma(UF)_\xi+\gamma_\xi UF\right]UF_{\phi\phi}=U^2F_\xi F_{\phi\phi}+UU_\xi FF_{\phi\phi}+\frac{\gamma_\xi U^2}{\gamma}FF_{\phi\phi}$

Multiplying both sides by $\frac{\gamma^2}{U}$ we get-

$$\upsilon_r F_{\phi\phi\phi}+\gamma\gamma_\tau\phi F_{\phi\phi}+\gamma\gamma_\xi UFF_{\phi\phi}+\gamma^2U_\xi FF_{\phi\phi}+\gamma^2UF_\xi F_{\phi\phi}-\gamma v_w F_{\phi\phi}-\gamma^2UF_\phi F_{\phi\xi}-\gamma^2F_{\phi\tau}$$

$$-\gamma^2U_\xi F_\phi{}^2-\frac{\gamma^2U_\tau}{U}F_\phi=\frac{\gamma^2}{\rho_r U}p_\xi G+\frac{\gamma^2}{\rho_r U}pG_\xi-\frac{\gamma\gamma_\xi}{\rho_r U}\phi pG_\phi$$

Or, $\upsilon_r F_{\phi\phi\phi}+\gamma\gamma_\tau\phi F_{\phi\phi}+\frac{1}{2}\left(2\gamma\gamma_\xi U+2\gamma^2U_\xi\right)FF_{\phi\phi}+\gamma^2UF_\xi F_{\phi\phi}-\gamma v_w F_{\phi\phi}-\gamma^2UF_\phi F_{\phi\xi}-\gamma^2F_{\phi\tau}$

$$-\gamma^2U_\xi F_\phi{}^2-\frac{\gamma^2U_\tau}{U}F_\phi=\frac{\gamma^2}{\rho_r U}p_\xi G+\frac{\gamma^2}{\rho_r U}pG_\xi-\frac{\gamma\gamma_\xi}{\rho_r U}\phi pG_\phi$$

Or,

$$\upsilon_r F_{\phi\phi\phi}+\gamma\gamma_\tau\phi F_{\phi\phi}+\frac{1}{2}\left\{\left(2\gamma\gamma_\xi U+\gamma^2U_\xi\right)+\gamma^2U_\xi\right\}FF_{\phi\phi}+\gamma^2UF_\xi F_{\phi\phi}-\gamma v_w F_{\phi\phi}-\gamma^2UF_\phi F_{\phi\xi}-\gamma^2F_{\phi\tau}$$

$$-\gamma^2U_\xi F_\phi{}^2-\frac{\gamma^2U_\tau}{U}F_\phi=\frac{\gamma^2}{\rho_r U}p_\xi G+\frac{\gamma^2}{\rho_r U}pG_\xi-\frac{\gamma\gamma_\xi}{\rho_r U}\phi pG_\phi$$

Or,

$$\upsilon_r F_{\phi\phi\phi}+\gamma\gamma_\tau\phi F_{\phi\phi}+\frac{1}{2}\left\{\left(\gamma^2U\right)_\xi+\gamma^2U_\xi\right\}FF_{\phi\phi}+\gamma^2UF_\xi F_{\phi\phi}-\gamma v_w F_{\phi\phi}-\gamma^2UF_\phi F_{\phi\xi}-\gamma^2F_{\phi\tau}$$

$$-\gamma^2U_\xi F_\phi{}^2-\frac{\gamma^2U_\tau}{U}F_\phi=\frac{\gamma^2}{\rho_r U}p_\xi G+\frac{\gamma^2}{\rho_r U}pG_\xi-\frac{\gamma\gamma_\xi}{\rho_r U}\phi pG_\phi \quad (2.33)$$

Further

$$\frac{\partial\tilde{p}}{\partial y}=\frac{1}{\gamma}\frac{\partial}{\partial\phi}\left[p(\tau,\xi)G(\tau,\xi,\phi)\right]=\frac{1}{\gamma}pG_\phi \quad (2.34)$$

and $(\rho-\rho_r)=-\rho_r\beta_T\Delta T(\tau,\xi)\theta(\tau,\xi,\phi)$ (2.35)

In view of (2.34) – (2.35) equation (2.17) becomes

$$pG_\phi = -\rho_r \gamma \beta_T \Delta T(\tau,\xi)\theta(\tau,\xi,\phi)g$$

Or, $G_\phi = -\dfrac{\rho_r \gamma \beta_T g \Delta T}{p(\tau,\xi)}\theta(\tau,\xi,\phi)$ (2.36)

Also we have $T - T_r = \{T_w(\tau,\xi) - T_r\}U(\tau,\xi)\theta(\tau,\xi,\phi)$

$$\therefore \frac{\partial T}{\partial t} = \left(\frac{\partial}{\partial \tau} - \frac{\gamma_\tau}{\gamma}\phi\frac{\partial}{\partial \phi}\right)\left[\{T_w(\tau,\xi) - T_r\}U(\tau,\xi)\theta(\tau,\xi,\phi)\right]$$

$$= \frac{\partial}{\partial \tau}\left[\{T_w(\tau,\xi) - T_r\}U(\tau,\xi)\theta(\tau,\xi,\phi)\right] - \frac{\gamma_\tau}{\gamma}\phi\frac{\partial}{\partial \phi}\left[\{T_w(\tau,\xi) - T_r\}U(\tau,\xi)\theta(\tau,\xi,\phi)\right]$$

$$= U\theta(T_w)_\tau + (T_w - T_r)(U\theta)_\tau - \frac{\gamma_\tau}{\gamma}\phi(T_w - T_r)U\theta_\phi$$

i.e., $\dfrac{\partial T}{\partial t} = U(T_w)_\tau\,\theta + U\Delta T\theta_\tau + U_\tau \Delta T\theta - \dfrac{\gamma_\tau U\Delta T}{\gamma}\phi\theta_\phi$ (2.37)

$$\therefore u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)T$$

$$= \left[UF_\phi\frac{\partial}{\partial \xi} - \frac{(\gamma UF)_\xi}{\gamma}\frac{\partial}{\partial \phi} + \frac{v_w}{\gamma}\frac{\partial}{\partial \phi}\right]\left[\{T_w(\tau,\xi) - T_r\}U(\tau,\xi)\theta(\tau,\xi,\phi)\right]$$

$$= UF_\phi\frac{\partial}{\partial \xi}\left[\{T_w(\tau,\xi) - T_r\}U(\tau,\xi)\theta(\tau,\xi,\phi)\right]$$
$$- \frac{(\gamma UF)_\xi}{\gamma}\frac{\partial}{\partial \phi}\left[\{T_w(\tau,\xi) - T_r\}U(\tau,\xi)\theta(\tau,\xi,\phi)\right]$$
$$+ \frac{v_w}{\gamma}\frac{\partial}{\partial \phi}\left[\{T_w(\tau,\xi) - T_r\}U(\tau,\xi)\theta(\tau,\xi,\phi)\right]$$

$$= U^2(T_w)_\xi F_\phi\theta + U^2\Delta TF_\phi\theta_\xi + UU_\xi\Delta TF_\phi\theta - \frac{(\gamma UF)_\xi U\Delta T}{\gamma}\theta_\phi + \frac{v_w U\Delta T}{\gamma}\theta_\phi$$

i.e., $u\dfrac{\partial T}{\partial x} + v\dfrac{\partial T}{\partial y} = U^2(T_w)_\xi F_\phi\theta + U^2\Delta TF_\phi\theta_\xi + UU_\xi\Delta TF_\phi\theta$

$$-U^2\Delta TF_\xi\theta_\phi - UU_\xi\Delta TF\theta_\phi - \frac{\gamma_\xi U^2\Delta T}{\gamma}F\theta_\phi + \frac{v_w U\Delta T}{\gamma}\theta_\phi \qquad (2.38)$$

and $\dfrac{\partial^2 T}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial T}{\partial y}\right) = \dfrac{1}{\gamma}\dfrac{\partial}{\partial \phi}\left[\dfrac{1}{\gamma}\dfrac{\partial}{\partial \phi}\{T_w(\tau,\xi) - T_r\}U(\tau,\xi)\theta(\tau,\xi,\phi)\right] = \dfrac{U\Delta T}{\gamma^2}\theta_{\phi\phi}$ (2.39)

Substituting (2.37) – (2.39) in equation (2.18) we get

$$U\left(T_w\right)_\tau \theta+U\Delta T\theta_\tau+U_\tau\Delta T\theta-\frac{\gamma_\tau U\Delta T}{\gamma}\phi\theta_\phi+U^2\left(T_w\right)_\xi F_\phi\theta+U^2\Delta TF_\phi\theta_\xi+UU_\xi\Delta TF_\phi\theta$$

$$-U^2\Delta TF_\xi\theta_\phi-UU_\xi\Delta TF\theta_\phi-\frac{\gamma_\xi U^2\Delta T}{\gamma}F\theta_\phi+\frac{v_w U\Delta T}{\gamma}\theta_\phi=\frac{\upsilon_r U\Delta T}{\Pr\gamma^2}\theta_{\phi\phi}$$

Or, $$\frac{\upsilon_r}{\Pr}\frac{U\Delta T}{\gamma^2}\theta_{\phi\phi}+U^2\Delta TF_\xi\theta_\phi+UU_\xi\Delta TF\theta_\phi+\frac{\gamma_\xi U^2\Delta T}{\gamma}F\theta_\phi-\frac{v_w U\Delta T}{\gamma}\theta_\phi+\frac{\gamma_\tau U\Delta T}{\gamma}\phi\theta_\phi$$

$$-U^2\Delta TF_\phi\theta_\xi-U\Delta T\theta_\tau-U\left(T_w\right)_\tau\theta-U_\tau\Delta T\theta-U^2\left(T_w\right)_\xi F_\phi\theta-UU_\xi\Delta TF_\phi\theta=0$$

Multiplying both sides by $\dfrac{\gamma^2}{U\Delta T}$ we get-

$$\frac{\upsilon_r}{\Pr}\theta_{\phi\phi}+\gamma\gamma_\tau\phi\theta_\phi+\gamma\gamma_\xi UF\theta_\phi+\gamma^2U_\xi F\theta_\phi+\gamma^2UF_\xi\theta_\phi-\gamma v_w\theta_\phi-\gamma^2UF_\phi\theta_\xi-\gamma^2\theta_\tau$$

$$-\gamma^2U_\xi F_\phi\theta-\frac{\gamma^2U_\tau}{U}\theta-\frac{\gamma^2}{\Delta T}\left(T_w\right)_\tau\theta-\frac{\gamma^2U}{\Delta T}\left(T_w\right)_\xi F_\phi\theta=0$$

Or, $$\frac{\upsilon_r}{\Pr}\theta_{\phi\phi}+\gamma\gamma_\tau\phi\theta_\phi+\frac{1}{2}\left\{\left(\gamma^2U\right)_\xi+\gamma^2U_\xi\right\}F\theta_\phi+\gamma^2UF_\xi\theta_\phi-\gamma v_w\theta_\phi-\gamma^2UF_\phi\theta_\xi-\gamma^2\theta_\tau$$

$$-\gamma^2U_\xi F_\phi\theta-\frac{\gamma^2U_\tau}{U}\theta-\frac{\gamma^2}{\Delta T}\left(T_w-T_r\right)_\tau\theta-\frac{\gamma^2U}{\Delta T}\left(T_w-T_r\right)_\xi F_\phi\theta=0$$

Or, $$\frac{\upsilon_r}{\Pr}\theta_{\phi\phi}+\gamma\gamma_\tau\phi\theta_\phi+\frac{1}{2}\left\{\left(\gamma^2U\right)_\xi+\gamma^2U_\xi\right\}F\theta_\phi+\gamma^2UF_\xi\theta_\phi-\gamma v_w\theta_\phi-\gamma^2UF_\phi\theta_\xi-\gamma^2\theta_\tau$$

$$-\gamma^2U_\xi F_\phi\theta-\frac{\gamma^2U_\tau}{U}\theta-\frac{\gamma^2\left(\Delta T\right)_\tau}{\Delta T}\theta-\frac{\gamma^2U\left(\Delta T\right)_\xi}{\Delta T}F_\phi\theta=0$$

Or, $$\frac{\upsilon_r}{\Pr}\theta_{\phi\phi}+\gamma\gamma_\tau\phi\theta_\phi+\frac{1}{2}\left\{\left(\gamma^2U\right)_\xi+\gamma^2U_\xi\right\}F\theta_\phi+\gamma^2UF_\xi\theta_\phi-\gamma v_w\theta_\phi-\gamma^2UF_\phi\theta_\xi-\gamma^2\theta_\tau$$

$$-\left[\left\{\frac{\gamma^2U_\tau}{U}+\gamma^2\left(\log\Delta T\right)_\tau\right\}+\left\{\gamma^2U_\xi+\gamma^2U\left(\log\Delta T\right)_\xi\right\}F_\phi\right]\theta=0 \quad (2.40)$$

In view of similarity solutions, the functions $F\left(\tau,\xi,\phi\right)$, $\theta\left(\tau,\xi,\phi\right)$ and $G\left(\tau,\xi,\phi\right)$ are assumed at this stage to be functions of ϕ alone, so that, equations (2.33), (2.36) and (2.40) are in simplified forms as follows:

$$\upsilon_r F_{\phi\phi\phi}+\gamma\gamma_\tau\phi F_{\phi\phi}+\frac{1}{2}\left\{\left(\gamma^2U\right)_\xi+\gamma^2U_\xi\right\}FF_{\phi\phi}-\gamma v_w F_{\phi\phi}-\gamma^2U_\xi F_\phi^{\ 2}-\frac{\gamma^2U_\tau}{U}F_\phi$$

$$=\frac{\gamma^2p_\xi}{\rho_r U}G-\frac{\gamma\gamma_\xi}{\rho_r U}\phi pG_\phi$$

$$G_\phi = -\frac{\rho_r \gamma\, \beta_T g \Delta T}{p(\tau,\xi)}\theta$$

$$\frac{\upsilon_r}{\Pr}\theta_{\phi\phi} + \gamma\gamma_\tau \phi\theta_\phi + \frac{1}{2}\left\{\left(\gamma^2 U\right)_\xi + \gamma^2 U_\xi\right\} F\theta_\phi - \gamma v_w \theta_\phi$$

$$-\left[\left\{\frac{\gamma^2 U_\tau}{U} + \gamma^2\left(\log \Delta T\right)_\tau\right\} + \left\{\gamma^2 U_\xi + \gamma^2 U\left(\log\Delta T\right)_\xi\right\} F_\phi\right]\theta = 0$$

Or,

$$\upsilon_r F_{\phi\phi\phi} + a_0\phi F_{\phi\phi} + \frac{1}{2}\left(a_1 + a_2\right) FF_{\phi\phi} - a_2 F_\phi{}^2 + a_3 F_{\phi\phi} - a_4 F_\phi = a_5 G - a_6 \phi G_\phi \tag{2.41}$$

$$G_\phi = a_7 \theta \tag{2.42}$$

$$\frac{\upsilon_r}{\Pr}\theta_{\phi\phi} + a_0\phi\theta_\phi + \frac{1}{2}\left(a_1 + a_2\right)F\theta_\phi + a_3\theta_\phi + \left[\left(a_8 - a_4\right) + \left(a_9 - a_2\right)F_\phi\right]\theta = 0 \tag{2.43}$$

where

$$\begin{aligned}
&i)\quad a_0 = \gamma\gamma_\tau \\
&ii)\quad a_1 = \left(\gamma^2 U\right)_\xi \\
&iii)\quad a_2 = \gamma^2 U_\xi \\
&iv)\quad a_3 = -\gamma v_w \\
&v)\quad a_4 = \frac{\gamma^2 U_\tau}{U} \\
&vi)\quad a_5 = \frac{\gamma^2 p_\xi}{\rho_r U} \\
&vii)\quad a_6 = \frac{\gamma\gamma_\xi p}{\rho_r U} \\
&viii)\quad a_7 = -\frac{\rho_r \gamma\, \beta_T g\Delta T}{p(\tau,\xi)} \\
&ix)\quad a_8 = -\gamma^2\left(\log\Delta T\right)_\tau \\
&x)\quad a_9 = -\gamma^2 U\left(\log\Delta T\right)_\xi
\end{aligned} \tag{2.44}$$

The transformed boundary conditions are now

$$\begin{aligned}
&F(0) = F_\phi(0) = 0,\ F_\phi(\infty) = 0 \\
&\text{and } \theta(0) = 1,\ \theta(\infty) = 0
\end{aligned} \tag{2.45}$$

The equations (2.45) furnish us with the conditions under which similarity solutions are obtained provided that all a’s must be constants and thus equations (2.41) to (2.43) will finally become non-linear ordinary differential equations in the limiting situations for the remaining variable other than the similarity variable. Consequently, the relations given by equations (2.45) are the treated conditions which provide us the equations for $U(\tau,\xi)$ and $\gamma(\tau,\xi)$, the scale factors for the velocity component and the ordinate *y*. Uniquely, these scale factors together with the suction or blowing parameter will determine the flow characteristics of the boundary layer. We shall now proceed to find $U(\tau,\xi)$, $\gamma(\tau,\xi)$ and consequently the suction velocity v_w for the possible requirements of similarity solution in the case of Boussinesq fluid.

From condition *ii*) of equation (2.44), we have

$$a_1 = (\gamma^2 U)_\xi$$

Integrating with respect to ξ, we obtain

$$\gamma^2 U = a_1\xi + A(\tau) \tag{2.46}$$

where $A(\tau)$ is either a function of τ or constant.

Again we get from condition *i*) of equation (2.44) that

$$a_0 = \gamma\gamma_\tau \Rightarrow a_0 = \frac{1}{2}\left(\gamma^2\right)_\tau \Rightarrow 2a_0 = \left(\gamma^2\right)_\tau$$

Integrating with respect to τ, we obtain

$$\gamma^2 = 2a_0\tau + B(\xi) \tag{2.47}$$

where $B(\xi)$ is either a function of ξ or constant.

Further, differentiating (2.46) with respect to τ and using *i*) and *iv*) of equation (2.44), we obtain

$$2\gamma\gamma_\tau U + \gamma^2 U_\tau = \frac{dA}{d\tau} \Rightarrow U\left(2\gamma\gamma_\tau + \frac{\gamma^2 U_\tau}{U}\right) = \frac{dA}{d\tau} \Rightarrow 2a_0 + a_4 = \frac{1}{U}\frac{dA}{d\tau}$$

Hence

$$\frac{dA}{d\tau} = U\left(2a_0 + a_4\right) \tag{2.48}$$

Also differentiating (2.47) with respect to ξ, we obtain

$$\left(\gamma^2\right)_\xi = \frac{dB}{d\xi}$$

But from conditions *i*) and *ii*) of equation (2.44), we have

$$a_1 = \left(\gamma^2 U\right)_\xi = \gamma^2 U_\xi + \left(\gamma^2\right)_\xi U = a_2 + \left(\gamma^2\right)_\xi$$

i.e., $\left(\gamma^2\right)_\xi = \frac{1}{U}\left(a_1 - a_2\right)$

Hence

$$\frac{dB}{d\xi} = \frac{1}{U}\left(a_1 - a_2\right) \tag{2.49}$$

The above two relations (2.48) and (2.49) yield

$$\frac{dA}{d\tau}\cdot\frac{dB}{d\xi} = \left(2a_0 + a_4\right)\left(a_1 - a_2\right) \tag{2.50}$$

Therefore, the forms of the similarity equations, the scale factors $U(\tau,\xi)$ and $\gamma(\tau,\xi)$ entirely depend on relation (2.50).

Possible Similarity Cases

Equation (2.50) yields possibilities of four similarity cases, namely,

Case A: both $\frac{dA}{d\tau}$ and $\frac{dB}{d\xi}$ are finite constants

Case B: both $\frac{dA}{d\tau}$ and $\frac{dB}{d\xi}$ are zero

Case C: $\frac{dA}{d\tau} \neq 0$, but $\frac{dB}{d\xi} = 0$ and

Case D: $\frac{dA}{d\tau} = 0$, but $\frac{dB}{d\xi} \neq 0$.

But for the sake of brevity the two most significant cases, namely, Case A and Case B are studied in the next following chapters, Chapter 3 and Chapter 4, respectively.

CHAPTER III

Study of Case A

In this chapter we will discuss the similarity case, viz., case A which is obtained in the similarity analysis as given in Chapter 2.

Case A :

When both $\dfrac{dA(\tau)}{d\tau}$ and $\dfrac{dB(\xi)}{d\xi}$ are finite consents, then from equation (2.48) we have

$$U = U_0 \text{ (constant)} \tag{3.1}$$

From equation (2.48) we have

$$A(\tau) = U_0 (2a_0 + a_4)\tau + C_1 .$$

Then equation (2. 46) gives

$$\gamma^2 U_0 = a_1\xi + U_0 (2a_0 + a_4)\tau + C_1$$

i.e., $$\gamma^2 = (2a_0 + a_4)\tau + \frac{a_1}{U_0}\xi + \frac{C_1}{U_0} \tag{3.2}$$

where C_1 is a constant of integration.

Similarly, from equation (2.49) we have

$$B(\xi) = \frac{1}{U_0}(a_1 - a_2)\xi + C_2 .$$

Then equation (2. 47) gives

$$\gamma^2 = 2a_0\tau + \frac{1}{U_0}(a_1 - a_2)\xi + C_2 \tag{3.3}$$

Comparing (3.2) and (3.3), we get

$$a_2 = 0,\ a_4 = 0 \text{ and } C_2 = \frac{C_1}{U_0}.$$

Therefore, we have

$$\gamma^2 = 2a_0\tau + \frac{a_1}{U_0}\xi + \frac{C_1}{U_0} \tag{3.4}$$

Again from *vii*) of equation (2.44), we get

$$a_6 = \frac{\gamma\gamma_\xi p}{\rho_r U} = \frac{a_1 p}{\rho_0 U_0^2}.$$ as $U = U_0$, $\rho_r = \rho_0$ and from (3.4) $\gamma\gamma_\xi = \dfrac{a_1}{2U_0}$

Hence

$$p = \frac{\rho_0 U_0^2 a_6}{a_1} \Rightarrow \frac{\partial p}{\partial \xi} = 0 \, .$$

Then *vi*) of equation (2.44) gives $a_5 = 0$

Further, from *ix*) of equation (2.44), we have

$$a_8 = -\gamma^2 \left(\log \Delta T\right)_\tau \Rightarrow \left(\log \Delta T\right)_\tau = -\frac{a_8}{\gamma^2} \Rightarrow \log \Delta T = -a_8 \int \frac{d\tau}{\gamma^2}$$

Or, $\log \Delta T = -a_8 \int \dfrac{d\tau}{2a_0\tau + \dfrac{a_1}{U_0}\xi + \dfrac{C_1}{U_0}} = -\dfrac{a_8}{2a_0} \log\left(2a_0\tau + \dfrac{a_1}{U_0}\xi + \dfrac{C_1}{U_0}\right) = -\dfrac{a_8}{2a_0}\log\gamma^2$

i.e., $\log \Delta T = -\dfrac{a_8}{a_0}\log\gamma$ (3.5)

Also, from *x*) of equation (2.44), we have

$$a_9 = -\gamma^2 U \left(\log \Delta T\right)_\xi \Rightarrow \left(\log \Delta T\right)_\xi = -\frac{a_9}{\gamma^2 U_0} \Rightarrow \log \Delta T = -\frac{a_9}{U_0} \int \frac{d\xi}{\gamma^2}$$

Or, $\log \Delta T = -\dfrac{a_9}{U_0} \int \dfrac{d\xi}{2a_0\tau + \dfrac{a_1}{U_0}\xi + \dfrac{C_1}{U_0}} = -\dfrac{a_9}{U_0} . \dfrac{U_0}{a_1} \log\left(2a_0\tau + \dfrac{a_1}{U_0}\xi + \dfrac{C_1}{U_0}\right) = -\dfrac{a_9}{a_1}\log\gamma^2$

i.e., $\log \Delta T = -\dfrac{2a_9}{a_1}\log\gamma$ (3.6)

Comparing (3.5) and (3.6), we obtain

$$a_8 = a_0 \, , \; a_9 = \frac{a_1}{2}$$

Thus, conditions *i*) – *x*) of equation (2.44) yield relations between the constants as follows:

a_0 and a_1 are arbitrary and $a_2 = a_4 = a_5 = 0$, $a_8 = a_0$, $a_9 = \dfrac{a_1}{2}$, a_3, a_6 and a_7 are disposable constants.

Substituting the constants through equations (2.41) – (2.43), we obtain

$$\upsilon_r F_{\phi\phi\phi} + a_0 \phi F_{\phi\phi} + \frac{a_1}{2} F F_{\phi\phi} + a_3 F_{\phi\phi} + a_6 \phi G_\phi = 0 \tag{3.7}$$

$$G_\phi = a_7 \theta \tag{3.8}$$

$$\frac{\upsilon_r}{\Pr}\theta_{\phi\phi} + a_0 \phi \theta_\phi + \frac{a_1}{2} F \theta_\phi + a_3 \theta_\phi + \left(a_0 + \frac{a_1}{2} F_\phi\right)\theta = 0 \tag{3.9}$$

Now writing $F=\alpha_1 f$, $\phi=\alpha_2\eta$ and $G=\alpha_3\bar{g}$, we get

$$\frac{d}{d\phi}=\frac{d}{d\eta}.\frac{d\eta}{d\phi}=\frac{d}{d\eta}.\frac{1}{\alpha_2}=\frac{1}{\alpha_2}\frac{d}{d\eta}$$

$$F_\phi=\frac{dF}{d\phi}=\frac{1}{\alpha_2}\frac{dF}{d\eta}=\frac{1}{\alpha_2}\frac{d}{d\eta}(\alpha_1 f)=\frac{\alpha_1}{\alpha_2}\frac{df}{d\eta}$$

i.e., $F_\phi=\dfrac{\alpha_1}{\alpha_2}f_\eta$

$$F_{\phi\phi}=\frac{d^2F}{d\phi^2}=\frac{d}{d\phi}\left(\frac{dF}{d\phi}\right)=\frac{1}{\alpha_2}\frac{d}{d\eta}\left(\frac{\alpha_1}{\alpha_2}f_\eta\right)=\frac{\alpha_1}{\alpha_2^2}\frac{d}{d\eta}(f_\eta)=\frac{\alpha_1}{\alpha_2}f_{\eta\eta}$$

i.e., $F_{\phi\phi}=\dfrac{\alpha_1}{\alpha_2^2}f_{\eta\eta}$

Similarly, $F_{\phi\phi\phi}=\dfrac{\alpha_1}{\alpha_2^3}f_{\eta\eta\eta}$

and $G_\phi=\dfrac{dG}{d\phi}=\dfrac{1}{\alpha_2}\dfrac{d}{d\eta}(\alpha_3\bar{g})=\dfrac{\alpha_3}{\alpha_2}\dfrac{d\bar{g}}{d\eta}$

i.e., $G_\phi=\dfrac{\alpha_3}{\alpha_2}\bar{g}_\eta$

Substituting above in equations (3.7) to (3.9) we have

$$\frac{\upsilon_r\alpha_1}{\alpha_2^3}f_{\eta\eta\eta}+a_0\alpha_2\eta.\frac{\alpha_1}{\alpha_2^2}f_{\eta\eta}+\frac{a_1}{2}\alpha_1 f.\frac{\alpha_1}{\alpha_2^2}f_{\eta\eta}+a_3\frac{\alpha_1}{\alpha_2^2}f_{\eta\eta}+a_6\alpha_2\eta.\frac{\alpha_3}{\alpha_2}\bar{g}_\eta=0$$

Or, $\dfrac{\upsilon_r\alpha_1}{\alpha_2^3}f_{\eta\eta\eta}+\dfrac{a_0\alpha_1}{\alpha_2}\eta f_{\eta\eta}+\dfrac{a_1\alpha_1^2}{2\alpha_2^2}ff_{\eta\eta}+\dfrac{a_3\alpha_1}{\alpha_2^2}f_{\eta\eta}+a_6\alpha_3\eta\bar{g}_\eta=0$

Multiplying both sides by $\dfrac{\alpha_2^3}{\upsilon_r\alpha_1}$, we obtain

$$f_{\eta\eta\eta}+\frac{a_0\alpha_2^2}{\upsilon_r}\eta f_{\eta\eta}+\frac{a_1\alpha_1\alpha_2}{2\upsilon_r}ff_{\eta\eta}+\frac{a_3\alpha_2}{\upsilon_r}f_{\eta\eta}+\frac{a_6\alpha_2^3\alpha_3}{\upsilon_r\alpha_1}\eta\bar{g}_\eta=0 \tag{3.10}$$

$$\frac{\alpha_3}{\alpha_2}\bar{g}_\eta=a_7\theta$$

Or, $\bar{g}_\eta=\dfrac{a_7\alpha_2}{\alpha_3}\theta$ (3.11)

and

$$\frac{\upsilon_r}{\Pr}\frac{1}{\alpha_2^2}\theta_{\eta\eta}+a_0\alpha_2\eta.\frac{1}{\alpha_2}\theta_\eta+\frac{a_1}{2}\alpha_1 f.\frac{1}{\alpha_2}\theta_\eta+a_3.\frac{1}{\alpha_2}\theta_\eta+\left(a_0+\frac{a_1}{2}\frac{\alpha_1}{\alpha_2}f_\eta\right)\theta=0$$

Or, $\dfrac{\upsilon_r}{\Pr\alpha_2^2}\theta_{\eta\eta}+a_0\eta\theta_\eta+\dfrac{a_1\alpha_1}{2\alpha_2}f\theta_\eta+\dfrac{a_3}{\alpha_2}\theta_\eta+\left(a_0+\dfrac{a_1\alpha_1}{2\alpha_2}f_\eta\right)\theta=0$

Multiplying both sides by $\dfrac{\alpha_2^2}{\upsilon_r}$, we get

$$\frac{1}{\Pr}\theta_{\eta\eta}+\frac{a_0\alpha_2^2}{\upsilon_r}\eta\theta_\eta+\frac{a_1\alpha_1\alpha_2}{2\upsilon_r}f\theta_\eta+\frac{a_3\alpha_2}{\upsilon_r}\theta_\eta+\left(\frac{a_0\alpha_2^2}{\upsilon_r}+\frac{a_1\alpha_1\alpha_2}{2\upsilon_r}f_\eta\right)\theta=0$$

Or, $\Pr^{-1}\theta_{\eta\eta}+\dfrac{a_0\alpha_2^2}{\upsilon_r}\eta\theta_\eta+\dfrac{a_1\alpha_1\alpha_2}{2\upsilon_r}f\theta_\eta+\dfrac{a_3\alpha_2}{\upsilon_r}\theta_\eta+\dfrac{a_0\alpha_2^2}{\upsilon_r}\left(1+\dfrac{a_1\alpha_1}{2a_0\alpha_2}f_\eta\right)\theta=0 \qquad (3.12)$

For a complete similarity solution choosing $\alpha_1=\alpha_2$, $\dfrac{a_0\alpha_1^2}{\upsilon_r}=1$, $\dfrac{a_6\alpha_1^2\alpha_3}{\upsilon_r}=1$ and $\dfrac{a_1}{2a_0}=\beta$

and writing $\dfrac{a_3\alpha_1}{\upsilon_r}=-\dfrac{\gamma v_w}{\sqrt{a_0\upsilon_r}}=f_w$, the above equations (3.10) – (3.12) yield

$$f_{\eta\eta\eta}+\eta f_{\eta\eta}+\beta ff_{\eta\eta}+f_w f_{\eta\eta}+\eta\bar{g}_\eta=0 \qquad (3.13)$$

$$\bar{g}_\eta=\left(\frac{U_F}{U_0}\right)^{\frac{5}{2}}\theta \qquad (3.14)$$

$$\Pr^{-1}\theta_{\eta\eta}+\eta\theta_\eta+\beta f\theta_\eta+f_w\theta_\eta+\left(1+\beta f_\eta\right)\theta=0 \qquad (3.15)$$

where

$$\frac{a_1\alpha_1\alpha_2}{2\upsilon_r}=\frac{a_1\alpha_1^2}{2\upsilon_r}=\frac{a_1}{2\upsilon_r}.\frac{\upsilon_r}{a_0}=\frac{a_1}{2a_0}=\beta$$

$$\frac{a_3\alpha_2}{\upsilon_r}=\frac{a_3\alpha_1}{\upsilon_r}=\frac{-\gamma v_w}{\upsilon_r}.\sqrt{\frac{\upsilon_r}{a_0}}=\frac{-\gamma v_w}{\sqrt{a_0\upsilon_r}}=f_w \qquad \Big|\because\frac{a_0\alpha_1^2}{\upsilon_r}=1\Rightarrow\alpha_1=\sqrt{\frac{\upsilon_r}{a_0}}$$

$$\frac{a_7\alpha_2}{\alpha_3}=\frac{a_7\alpha_1}{\alpha_3}=a_7.\frac{a_6}{a_0}\sqrt{\frac{\upsilon_r}{a_0}} \qquad \Big|\because\frac{a_6\alpha_1^2\alpha_3}{\upsilon_r}=1\Rightarrow\frac{a_0\alpha_1^2}{\upsilon_r}\frac{a_6\alpha_3}{a_0}=1\Rightarrow\frac{a_6\alpha_3}{a_0}=1\Rightarrow\frac{1}{\alpha_3}=\frac{a_6}{a_0}$$

$$=\frac{\gamma\gamma_\xi p}{\rho_0U_0}\left(-\frac{\rho_0\gamma\beta_T g\Delta T}{p}\right).\frac{1}{a_0}\sqrt{\frac{\upsilon_r}{a_0}}$$

$$=\gamma.\gamma\gamma_\xi\left(-\beta_T g\Delta T\right).\frac{1}{a_0U_0}\sqrt{\frac{\upsilon_r}{a_0}} \qquad \Big|\because\gamma^2=2a_0\tau+\frac{a_1}{U_0}\xi+\frac{C_1}{U_0}\Rightarrow\ \gamma\gamma_\xi=\frac{a_1}{2U_0}$$

$$=\sqrt{2a_0\tau+\frac{a_1}{U_0}\xi+\frac{C_1}{U_0}}\,\frac{a_1}{2U_0}.\left(-\beta_T g\Delta T\right).\frac{1}{a_0U_0}\sqrt{\frac{\upsilon_r}{a_0}}$$

$$=\frac{1}{\sqrt{U_0}}\sqrt{2a_0U_0\tau+a_1\xi+C_1}\cdot\frac{a_1}{2a_0}\cdot\left(-\beta_T g\Delta T\right)\cdot\frac{1}{U_0^2}\sqrt{\frac{\upsilon_r}{a_0}}$$

$$=\frac{\sqrt{2a_0}}{\sqrt{U_0}}\sqrt{U_0\tau+\frac{a_1}{2a_0}\xi+\frac{C_1}{2a_0}}\cdot\frac{a_1}{2a_0}\cdot\left(-\beta_T g\Delta T\right)\cdot\frac{1}{U_0^2}\sqrt{\frac{\upsilon_r}{a_0}}$$

$$=\sqrt{U_0\left(\tau+\tau_0\right)+\beta\left(\xi+\xi_0\right)}\sqrt{2}\beta\left(-\beta_T g\Delta T\right)\cdot\frac{1}{U_0^2}\sqrt{\frac{\upsilon_r}{U_0}}$$

where we choose $\frac{C_1}{2a_0}=U_0\tau_0+\beta\xi_0$

$$=\sqrt{U_0\left(\tau+\tau_0\right)+\beta\left(\xi+\xi_0\right)}\sqrt{2}\beta\left(-\beta_T g\Delta T\right)\cdot\frac{1}{U_0^2}\sqrt{\frac{\upsilon_r}{U_0}}$$

$$=\frac{\sqrt{2}\beta\left(-\beta_T g\Delta T\right)\sqrt{\upsilon_r\left[U_0\left(\tau+\tau_0\right)+\beta\left(\xi+\xi_0\right)\right]}}{U_0^{\frac{5}{2}}}$$

$$=\left(\frac{U_F}{U_0}\right)^{\frac{5}{2}}$$

where

$$U_F=\left\{\sqrt{2}\beta\left(-\beta_T g\Delta T\right)\sqrt{\upsilon_r\left[U_0\left(\tau+\tau_0\right)+\beta\left(\xi+\xi_0\right)\right]}\right\}^{\frac{2}{5}}$$

$$=\left\{\sqrt{2}\beta\left(-\beta_T g\Delta T\right)\sqrt{\upsilon_r\times\text{Characteristic Length}}\right\}^{\frac{2}{5}}\qquad(3.16)$$

is called free convection velocity associated with the local characteristic length

$$L=U_0\left(\tau+\tau_0\right)+\beta\left(\xi+\xi_0\right)\qquad(3.17)$$

Since we are concerned with a purely free convection flow, without loss of generality we may put $U_F=U_0$ and hence the similarity equations are given by

$$f_{\eta\eta\eta}+\left(\eta+\beta f+f_w\right)f_{\eta\eta}+\eta\bar{g}_\eta=0\qquad(3.18)$$

$$\bar{g}_\eta=\theta\qquad(3.19)$$

$$\Pr^{-1}\theta_{\eta\eta}+\left(\eta+\beta f+f_w\right)\theta_\eta+\left(1+\beta f_\eta\right)\theta=0\qquad(3.20)$$

subject to the transformed boundary conditions:

$$f(0)=f_\eta(0)=f_\eta(\infty)=0;\ \theta(0)=1,\ \theta(\infty)=0;\ \bar{g}_n(0)=1,\ \bar{g}_n(\infty)=0\qquad(3.21)$$

where the boundary conditions for $\bar{g}$ is attained from that prescribed for θ according to equation (3.19).

Again

$$\gamma^2 = 2a_0\tau + \frac{a_1}{U_F}\xi + \frac{C_1}{U_F}$$

$$= \frac{2a_0}{U_F}\left(U_F\tau + \frac{a_1}{2a_0}\xi + \frac{C_1}{2a_0}\right)$$

$$= \frac{2a_0}{U_F}\left(U_F\tau + \beta\xi + \frac{C_1}{2a_0}\right) = \frac{2a_0}{U_F}\left(U_F\tau + \beta\xi + U_F\tau_0 + \beta\xi_0\right), \text{ where } \frac{C_1}{2a_0} = U_F\tau_0 + \beta\xi_0$$

$$= \frac{2a_0}{U_F}\left[U_F\left(\tau+\tau_0\right) + \beta\left(\xi+\xi_0\right)\right]$$

$$= \frac{\sqrt{2a_0}}{\sqrt{U_F}}\sqrt{U_F\left(\tau+\tau_0\right) + \beta\left(\xi+\xi_0\right)}$$

i.e., $\gamma = \dfrac{\sqrt{2a_0 L}}{\sqrt{U_F}}$ (3.22)

where L is the local characteristic length defined by (3.17).

Hence the similarity function $f(\eta)$, the similarity variable η and the pressure function $\bar{g}(\eta)$ are related to the stream function ψ, the physical co-ordinate y and perturbation pressure $\tilde{p}$ by the following equations respectively

$$\psi(\tau,\xi,\phi) = \gamma U_F F + \psi(\tau,\xi,0)$$

$$\psi(\tau,\xi,\eta) = \frac{\sqrt{2a_0 L}}{\sqrt{U_F}} U_F \alpha_1 f(\eta) + \psi(\tau,\xi,0)$$

$$= \frac{\sqrt{2a_0 L}}{\sqrt{U_F}} U_F \sqrt{\frac{\upsilon_r}{a_0}} f(\eta) + \psi(\tau,\xi,0)$$

$$= \sqrt{2\upsilon_r U_F L}\, f(\eta) + \psi(\tau,\xi,0)$$

$$= \sqrt{2}\upsilon_r \sqrt{\frac{U_F L}{\upsilon_r}}\, f(\eta) + \psi(\tau,\xi,0)$$

i.e., $\psi(\tau,\xi,\eta) = \sqrt{2}\upsilon_r\sqrt{\mathrm{Re_F}}\, f(\eta) + \psi(\tau,\xi,0)$ (3.23)

Here $\mathrm{Re_F}$ is the dimensionless Reynolds number based on free convection velocity U_F given by equation (3.16) and the local characteristic length L given by equation (3.17) as

$$\mathrm{Re_F} = \frac{U_F L}{\upsilon_r} \tag{3.24}$$

Further, we have

$$\phi = \alpha_2 \eta$$

$$\therefore \eta = \frac{\phi}{\alpha_2} = \frac{1}{\alpha_2}\cdot\frac{y}{\gamma}$$

$$= \frac{1}{\sqrt{\frac{\upsilon_r}{a_0}}}\cdot\frac{y}{\frac{\sqrt{2a_0 L}}{\sqrt{U_F}}}$$

$$= \frac{\sqrt{U_F}}{\sqrt{2\upsilon_r}}\cdot\frac{y}{\sqrt{L}}$$

$$= \frac{\sqrt{U_F L}}{\sqrt{2\upsilon_r}}\cdot\frac{y}{L}$$

$$= \sqrt{\frac{U_F L}{\upsilon_r}}\,\frac{y}{\sqrt{2}L}$$

i.e., $$\eta = \frac{\sqrt{\mathrm{Re_F}}}{\sqrt{2}L}\,y \tag{3.25}$$

and

$$\tilde{p} = pG(\phi)$$

$\Rightarrow \tilde{p} = p\alpha_3 \bar{g}(\eta)$ | since from condition *vii*) of equation (2.44) $a_6 = \dfrac{\gamma\gamma_\xi}{\rho_r U}p$

$$\Rightarrow p = \frac{a_6 \rho_r U_F}{\gamma\gamma_\xi}$$

$$= \frac{a_6 \rho_r U_F \alpha_3}{\gamma\gamma_\xi}\,\bar{g}(\eta)$$

$$= \frac{\rho_r \upsilon_r U_F}{\alpha_1^2 \cdot \frac{a_1}{2U_F}}\,\bar{g}(\eta) \qquad \left|\because \frac{a_6 \alpha_1^2 \alpha_3}{\upsilon_r} = 1 \Rightarrow a_6 = \frac{\upsilon_r}{\alpha_1^2 \alpha_3} \text{ and } \gamma\gamma_\xi = \frac{a_1}{2U_F}\right.$$

$$= \frac{\rho_r U_F^2}{\frac{a_1}{2a_0}}\,\bar{g}(\eta) \qquad \left|\because \frac{a_0 \alpha_1^2}{\upsilon_r} = 1 \Rightarrow \alpha_1^2 = \frac{\upsilon_r}{a_0}\right.$$

i.e., $\tilde{p} = \dfrac{\rho_r U_F^2}{\beta}\, \bar{g}(\eta)$ (3.26)

The velocity components, the skin friction and the local heat transfer coefficients associated with the equations (3.18) – (3.20) are now

$$u = UF_\phi = U_F \cdot \frac{\alpha_1}{\alpha_2}\, f_\eta$$

i.e., $u = U_F\, f_\eta(\eta)$ (3.27)

$$v = \gamma_\xi U \phi F_\phi - \gamma_\xi UF + v_w$$

Or, $v = \gamma_\xi U_F \alpha_2 \eta . \dfrac{\alpha_1}{\alpha_2}\, f_\eta - \gamma_\xi U_F . \alpha_1 f - \dfrac{\sqrt{a_0 \upsilon_r}}{\gamma}\, f_w$

Or, $v = \gamma_\xi U_F \alpha_1 \eta\, f_\eta - \gamma_\xi U_F \alpha_1 f - \dfrac{\sqrt{a_0 \upsilon_r}}{\gamma}\, f_w$

Or, $v = \alpha_1 \gamma_\xi U_F \left(\eta\, f_\eta - f\right) - \dfrac{\sqrt{a_0 \upsilon_r}}{\gamma}\, f_w$

Or, $-v = \sqrt{\dfrac{\upsilon_r}{a_0}}\, \gamma_\xi U_F \left(f - \eta\, f_\eta \right) + \dfrac{\sqrt{a_0 \upsilon_r}}{\gamma}\, f_w$ $\quad\Bigg|\quad \because \gamma_\xi = \dfrac{1}{\sqrt{2a_0\tau + \dfrac{a_1}{U_F}\xi + \dfrac{C_1}{U_F}}} \cdot \dfrac{a_1}{2U_F}$

$$= \sqrt{\frac{\upsilon_r}{a_0}}\, \frac{U_F}{\sqrt{2a_0\tau + \dfrac{a_1}{U_F}\xi + \dfrac{C_1}{U_F}}} \cdot \frac{a_1}{2U_F} . \left(f - \eta\, f_\eta \right) + \frac{\sqrt{a_0 \upsilon_r}}{\sqrt{2a_0\tau + \dfrac{a_1}{U_F}\xi + \dfrac{C_1}{U_F}}}\, f_w$$

$$= \sqrt{\frac{\upsilon_r U_F}{2}}\, \frac{1}{\sqrt{U_F\tau + \dfrac{a_1}{2a_0}\xi + \dfrac{C_1}{2a_0}}} \cdot \frac{a_1}{2a_0} . \left(f - \eta\, f_\eta \right) + \sqrt{\frac{\upsilon_r U_F}{2}}\, \frac{1}{\sqrt{U_F\tau + \dfrac{a_1}{2a_0}\xi + \dfrac{C_1}{2a_0}}}\, f_w$$

$$= \sqrt{\frac{\upsilon_r U_F}{2}}\, \frac{1}{\sqrt{U_F\left(\tau + \tau_0\right) + \beta\left(\xi + \xi_0\right)}}\, \beta\left(f - \eta\, f_\eta \right) + \sqrt{\frac{\upsilon_r U_F}{2}}\, \frac{1}{\sqrt{U_F\left(\tau + \tau_0\right) + \beta\left(\xi + \xi_0\right)}}\, f_w$$

$$= \sqrt{\frac{\upsilon_r U_F}{2}}\, \frac{1}{\sqrt{U_F\left(\tau + \tau_0\right) + \beta\left(\xi + \xi_0\right)}} \left\{ \beta\left(f - \eta\, f_\eta \right) + f_w \right\}$$

$$= \sqrt{\frac{\upsilon_r U_F}{2L}} \left\{ \beta\left(f - \eta\, f_\eta \right) + f_w \right\}$$

$$= \frac{\upsilon_r}{L\sqrt{2}} \sqrt{\frac{U_F L}{\upsilon_r}} \left\{ \beta\left(f - \eta\, f_\eta \right) + f_w \right\}$$

i.e., $$-v = \frac{\upsilon_r \sqrt{\mathrm{Re_F}}}{L\sqrt{2}} \left[\beta \left\{ f(\eta) - \eta f_\eta(\eta) \right\} + f_w \right] \tag{3.28}$$

The skin friction coefficient is defined by

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0}$$

Now

$$\frac{\partial u}{\partial y} = \frac{1}{\gamma} \frac{d}{d\phi}\left(U_F F_\phi\right) = \frac{U_F}{\gamma} F_{\phi\phi} = \frac{U_F}{\gamma} \frac{\alpha_1}{\alpha_2^2} f_{\eta\eta}(\eta) = \frac{U_F}{\gamma \alpha_1} f_{\eta\eta}(\eta)$$

Hence

$$\tau_w = \frac{\mu U_F}{\gamma \alpha_1} f_{\eta\eta}(0)$$

$$= \frac{\mu U_F}{\dfrac{\sqrt{2a_0 L}}{\sqrt{U_F}} \sqrt{\dfrac{\upsilon_r}{a_0}}} f_{\eta\eta}(0)$$

$$= \frac{\mu U_F}{\dfrac{\sqrt{2\upsilon_r L}}{\sqrt{U_F}}} f_{\eta\eta}(0)$$

$$= \frac{\mu U_F}{\sqrt{2} L \sqrt{\dfrac{\upsilon_r}{U_F L}}} f_{\eta\eta}(0)$$

$$= \frac{\mu U_F}{\sqrt{2} L \sqrt{\dfrac{1}{\mathrm{Re_F}}}} f_{\eta\eta}(0)$$

i.e., $$\tau_w = \frac{\mu U_F \sqrt{\mathrm{Re_F}}}{\sqrt{2} L} f_{\eta\eta}(0) \tag{3.29}$$

The heat transfer coefficient is defined by

$$q_w = -k\Delta T \left[\frac{\partial \theta}{\partial y} \right]_{y=0}$$

Now

$$\frac{\partial \theta}{\partial y} = \frac{1}{\gamma} \frac{\partial \theta}{\partial \phi} = \frac{1}{\gamma} \frac{1}{\alpha_2} \frac{\partial \theta}{\partial \eta} = \frac{1}{\gamma \alpha_1} \theta_\eta(\eta)$$

Hence

$$q_w = -\frac{k\Delta T}{\gamma \alpha_1}\theta_\eta(0)$$

i.e., $$q_w = -\frac{k\Delta T\sqrt{\mathrm{Re_F}}}{\sqrt{2}L}\theta_\eta(0) \tag{3.30}$$

Numerical scheme and procedure

The set of ordinary differential equations (3.18) to (3.20) with the boundary conditions (3.21) are non-linear and coupled. Thus, to solve them analytically is very intricate. Therefore, a numerical procedure based on the standard initial-value solver shooting method, namely, Runge-Kutta shooting method in collaboration with the Runge-Kutta Merson method is adopted to obtain the solution of the problem. An extension of the Nachtsheim-Swigert iteration scheme (guessing the missing value) (Nachtsheim & Swigert (1965)) is implemented. It is clear that the numbers of initial conditions are not sufficient to obtain the particular solution of the differential equations, so we require assuming additional missing/unspecified initial conditions. Thus, in this method, the missing initial conditions at the initial point of the interval are assumed and with all the initial conditions (given and assumed) the equations are integrated numerically in steps as an initial value problem to the terminal point. These are to be so assumed that the solution of the outer prescribed points also matches. The accuracy of the assumed missing initial condition is checked by comparing the calculated value of the dependent variable at the terminal point with its given value there. If match is not found (a difference exists) at the outer end then another set of missing initial conditions are considered and the process is repeated. This trial and error process is taken care through Nachtsheim-Swigert iteration technique and the process is continued until the agreement between the calculated and the given condition at the terminal point is within the specified degree of accuracy. For this type of iterative approach, one naturally inquires whether or not there is a systematic way of finding each succeeding (assumed) value of the missing initial condition.

The boundary conditions (3.21) associated with the system are of the two-point asymptotic class. Two-point boundary conditions have values of the dependent variable specified at two different values of the independent variable, where the outer boundary conditions are specified at infinity. There are three asymptotic boundary conditions as well as three unknown surface conditions $f_{\eta\eta}(0)$, $\theta_{\eta}(0)$ and $\bar{g}(0)$ here. Specification of an asymptotic boundary condition implies that the value of velocity approaches to zero and the value of temperature approaches from unity to zero as the outer specified value of the independent variable is approached. The governing differential equations are then integrated with these assumed surface boundary conditions. If the required outer boundary condition is satisfied, a solution has been achieved. However, this is not generally the case. Hence a method must be devised to logically estimate the new surface boundary conditions for the next trial integration. Asymptotic boundary value problems such as those governing the boundary layer equations are further complicated by the fact that the outer boundary condition is specified at infinity. In the trial integrations, infinity is numerically approximated by some large specified value of the independent variable. There is no a priori general method of estimating this value. Selection of too small a maximum value for the independent variable may not allow the solution to asymptotically converge to the required accuracy. Selecting a large value may result in divergence of the trial integration or in slow convergence of surface boundary conditions required satisfying the asymptotic outer boundary condition. Selecting too large a value of the independent variable is expensive in terms of computer time. Nachtsheim-Swigert developed an iteration method, which overcomes these difficulties. Extension of the Nachtsheim-Swigert iteration shell to above system of differential equations (3.18) – (3.20) is very straightforward.

Based on the integration done with the aforementioned numerical technique, the velocity f_{η}, temperature θ and pressure function $\bar{g}$ are determined as function of the co-ordinate η. In the process of integration the skin friction coefficient $f_{\eta\eta}(0)$ and the heat transfer rate $\theta_{\eta}(0)$ are also evaluated. The numerical results thus obtained in terms of the similarity variables are displayed in graphs and tables for several selected values of the established parameters f_w, β and Pr below:

Numerical Results and Discussion

On the basis of the numerical results, the dimensionless velocity and temperature profiles and the pressure distributions are plotted. The effects of suction parameter f_w, driving parameter β and Prandtl number Pr on velocity f_{η}, temperature θ and pressure $\bar{g}$ against η are illustrated in Fig. 3.1 through Fig.3.9. Also their effects on the coefficient of skin-friction and heat transfer coefficient are tabulated in Table- 3.1 and Table- 3.2, respectively. To observe the effect of f_w, β and Pr are kept as constants. Similarly, to observe the effect of β and Pr, f_w and Pr and f_w and β respectively are kept as constants. In all cases where f_w is kept as constant, its value is chosen as 0.30. Similarly, when Pr is kept constant, its value is chosen as 0.71 and when β is kept fixed, its value is chosen as 0.33.

Fig. 3.1 to Fig. 3.3 represent the effects of f_w, β and Pr on the velocity profiles respectively. From figures it is observed that the velocity decreases with the increase in all the controlling parameters, i.e., the velocity decreases with the increase in f_w (for fixed β and Pr) or increase in β (for fixed f_w and Pr) or increase in Pr (for fixed f_w and β).

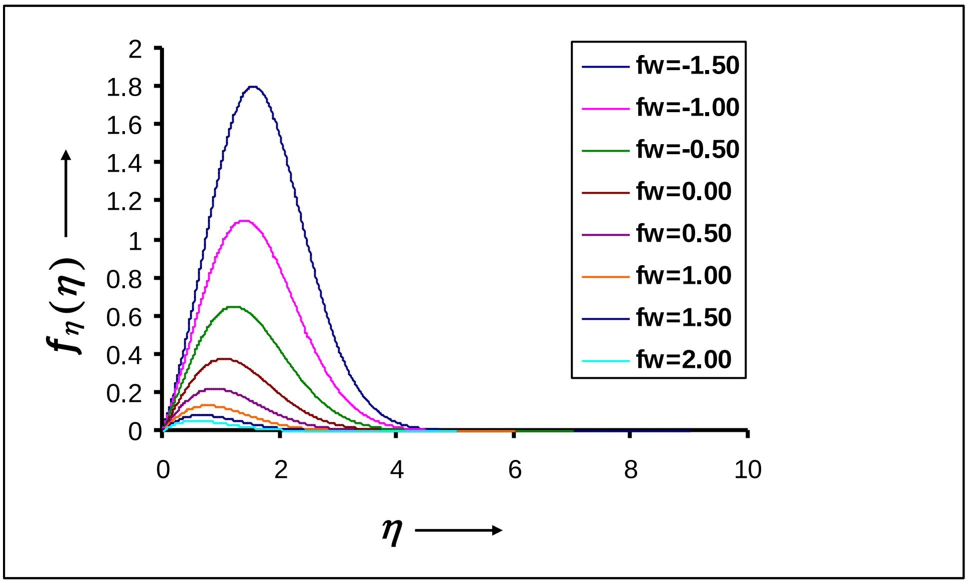

Figure 3.1: Velocity profiles for different values of the suction parameter f_w with fixed values of β and Pr.

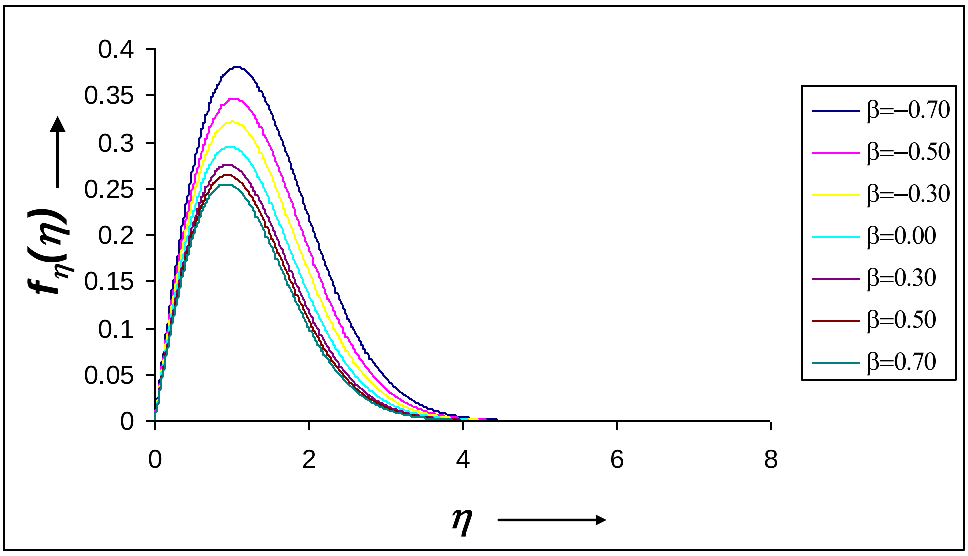

Figure 3.2: Velocity profiles for different values of the parameter β with fixed values of f_w and Pr.

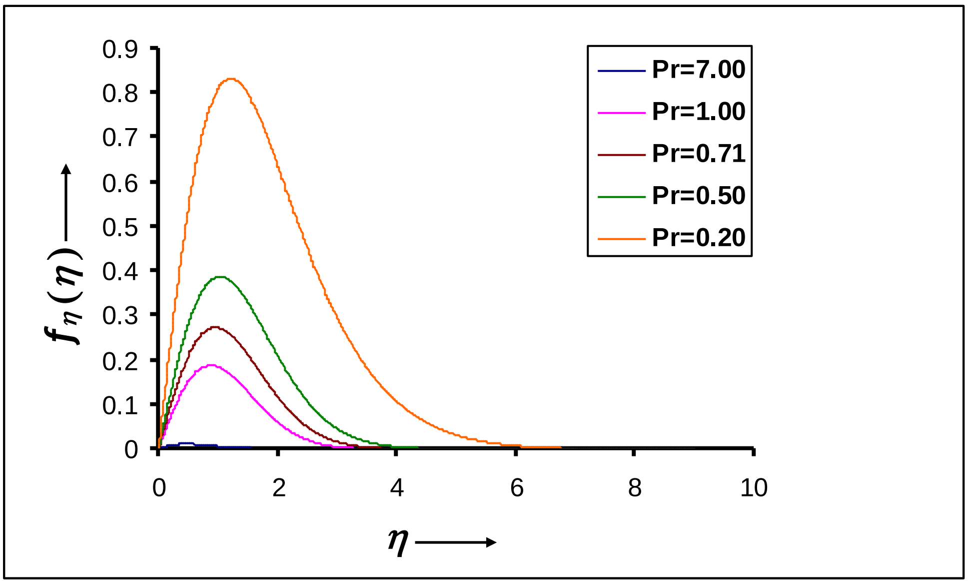

Figure 3.3: Velocity profiles for different values of the Prandtl number Pr with fixed values of f_w and β.

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Fig. 3.4 to Fig. 3.6 represent the effects of f_w, β and Pr on the temperature profiles respectively. The effect of the controlling parameters on temperature is the same as that on the velocity, i.e., with their increase (increasing one of them keeping the other two as fixed), temperature (θ) decreases.

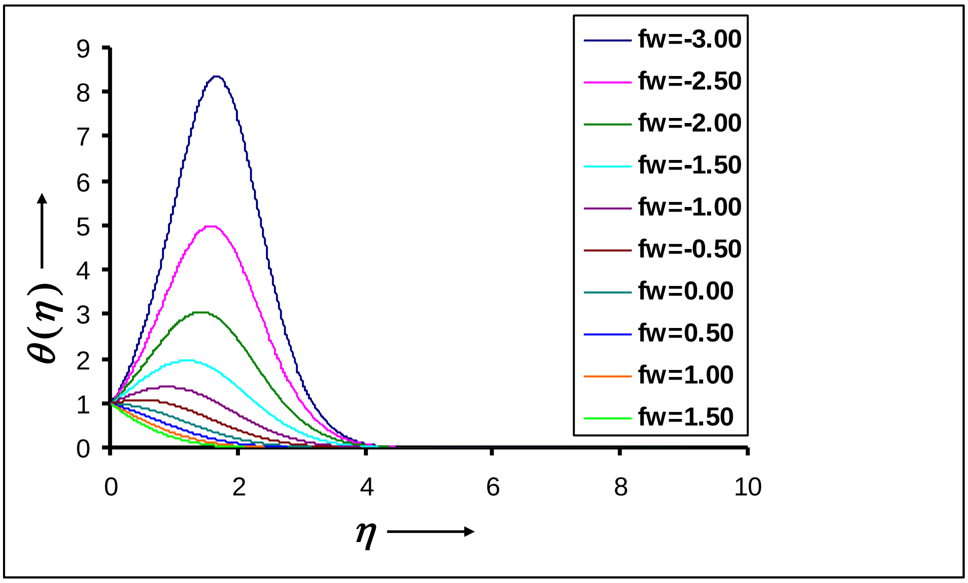

Figure 3.4: Temperature profiles for different values of the suction parameter f_w with fixed values of β and Pr.

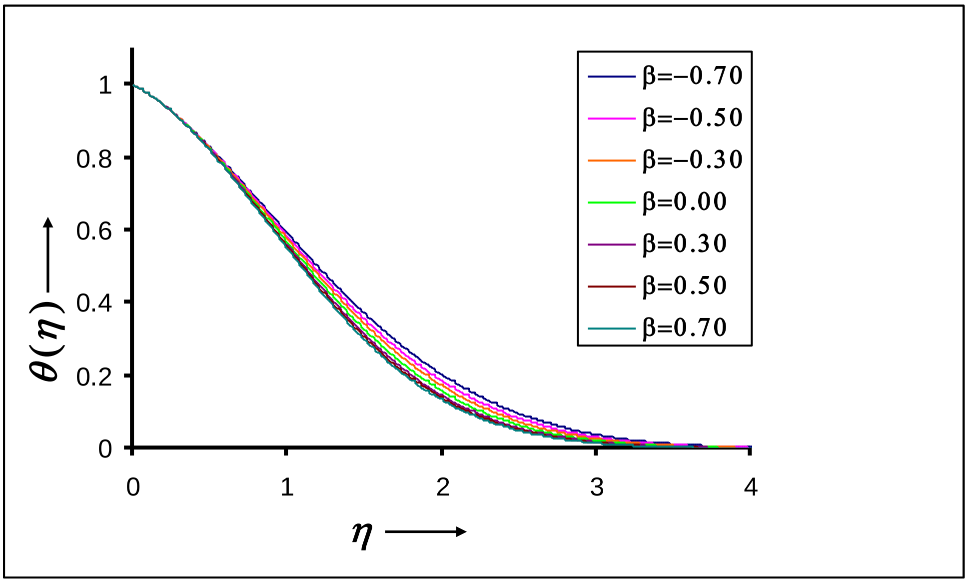

Figure 3.5: Temperature profiles for different values of the parameter β with fixed values of f_w and Pr.

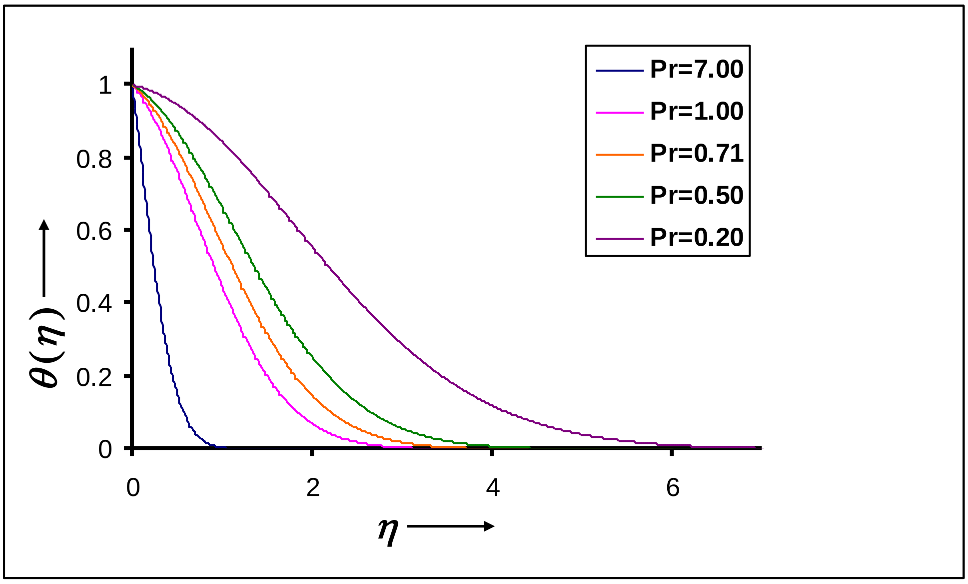

Figure 3.6: Temperature profiles for different values of the Prandtl number Pr with fixed values of f_w and β.

Fig. 3.7 to Fig. 3.9 representing the effect of f_w, β and Pr respectively, but on the pressure variable $\bar{g}$. In these cases their effects are reversed i.e., with the increase in the controlling parameters (increasing one of them keeping the other two as fixed), pressure increases. This increase-increase behavior is observed for all three parameters.

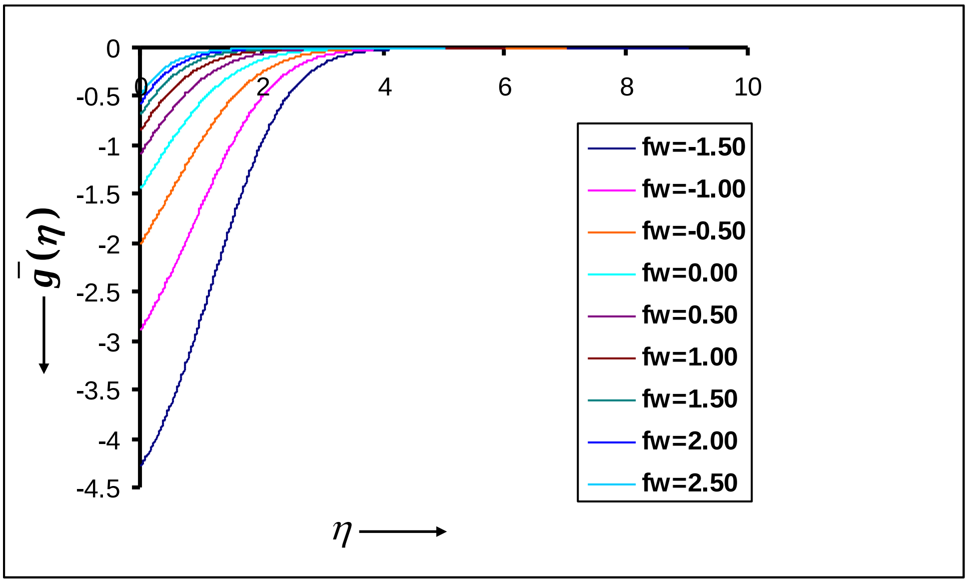

Figure 3.7: Pressure distributions for different values of the suction parameter f_w with fixed values of β and Pr.

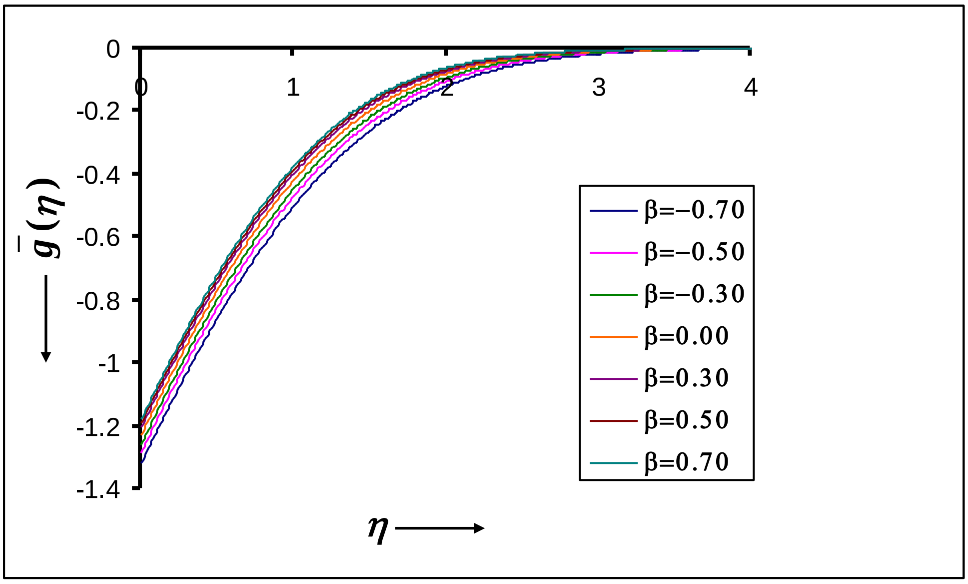

Figure 3.8: Pressure distributions for different values of the parameter β with fixed values of f_w and Pr.

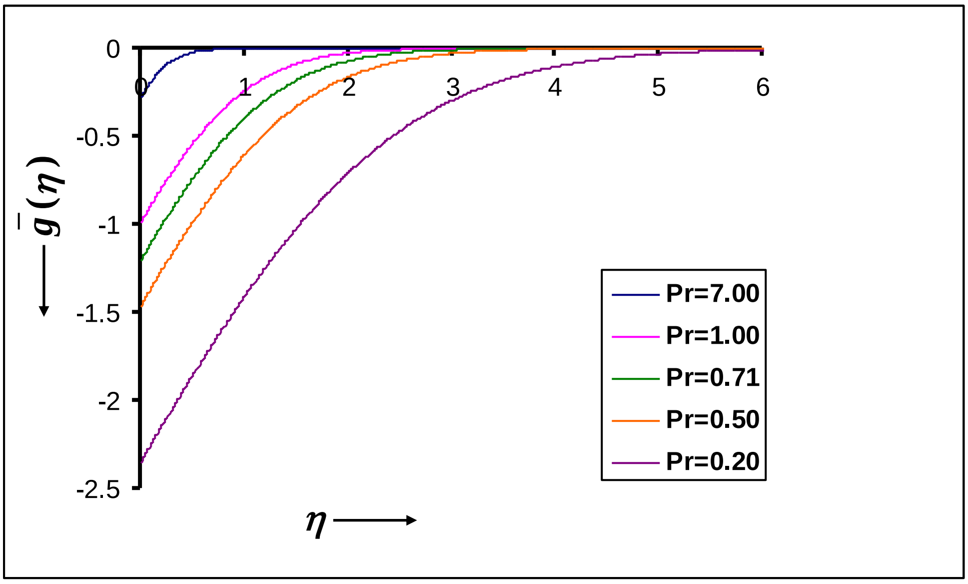

Figure 3.9: Pressure distributions for different values of the Prandtl number Pr with fixed values of f_w and β.

The values proportional to the coefficient of skin-friction are tabulated in Table- 3.1. From this table the effect of f_w, β and Pr on the skin-friction can be observed. It is seen that for fixed β and Pr with the increase in the f_w, the coefficient of skin-friction decreases. The same behaviour is being observed for the other controlling parameters (increasing one of them keeping the other two as fixed).

Table- 3.2 contains the values proportional to the heat transfer coefficient. The effect of the controlling parameters on it is as that to the skin-friction except for β variation i.e., with the increase in the controlling parameters f_w and Pr the coefficient of heat transfer decreases. But in case of β variation it remains unchanged. It indicates the heat transfer coefficient is independent of the controlling parameter β.

Table- 3.1: Variation of the coefficient of skin-friction with f_w, β and Pr

Values Proportional to the coefficient of skin-friction with the variation of					
f_w (for β =0.33 and Pr = 0.71)		β (for f_w = 0.30 and Pr = 0.71)		Pr (for β =0.33 and f_w = 0.30)	
f_w	$f_{\eta\eta}(0)$	β	$f_{\eta\eta}(0)$	Pr	$f_{\eta\eta}(0)$
-1.50	0.937023	-0.70	0.648051	0.20	1.265479
-1.00	0.865262	-0.50	0.605335	0.50	0.675216
-0.50	0.739413	-0.30	0.574441	0.71	0.511229
0.00	0.594089	0.00	0.539857	1.00	0.381323
0.50	0.460483	0.30	0.513543	7.00	0.047620
1.00	0.352563	0.50	0.498975		
1.50	0.270811	0.70	0.486150		
2.00	0.210395				

Table- 3.2: Variation of the heat transfer coefficient with f_w, β and Pr

Values Proportional to the heat transfer coefficient with the variation of					
f_w (for β =0.33 and Pr = 0.71)		β (for f_w = 0.30 and Pr = 0.71)		Pr (for β = 0.33 and f_w = 0.30)	
f_w	$\theta_\eta(0)$	β	$\theta_\eta(0)$	Pr	$\theta_\eta(0)$
-1.50	1.065006	-0.70	-0.21300	0.20	-0.06016
-1.00	0.710004	-0.50	-0.21300	0.50	-0.15000
-0.50	0.355002	-0.30	-0.21300	0.71	-0.21300
0.00	0.000001	0.00	-0.21300	1.00	-0.30000
0.50	-0.355000	0.30	-0.21300	7.00	-2.10011
1.00	-0.710000	0.50			
1.50	-1.065000	0.70			
2.00	-1.420010				

CHAPTER IV

Study of Case B

In this chapter we will discuss the similarity case, viz., case B which is obtained in the similarity analysis as given in Chapter 2.

<u>Case B</u> :

When both $\frac{dA(\tau)}{d\tau}$ and $\frac{dB(\xi)}{d\xi}$ are zero, i.e., $\frac{dA(\tau)}{d\tau}=0$ and $\frac{dB(\xi)}{d\xi}=0$.

Then *A* and *B* are both constants.

Therefore, from equations (2.46) and (2.47), we have

$$\gamma^2 U = a_1\xi + A \tag{4.1}$$

and

$$\gamma^2 = 2a_0\tau + B \tag{4.2}$$

With the help of (4.2) equation (4.1) becomes

$$U = \frac{a_1\xi + A}{\gamma^2} = \frac{a_1\xi + A}{2a_0\tau + B} \tag{4.3}$$

Equations (2.48) and (2.49) now give

$$2a_0 + a_4 = 0 \Rightarrow a_4 = -2a_0$$

and

$$a_1 - a_2 = 0 \Rightarrow a_2 = a_1$$

Again differentiating equation (4.2) with respect to ξ , we have

$$\gamma\gamma_\xi = 0$$

Then from condition *vii*) of equation (2.44), we have

$$a_6 = \frac{\gamma\gamma_\xi}{p_r U}\, p \Rightarrow a_6 = 0$$

Again from condition *ix*) of equation (2.44), we have

$$a_8 = -\gamma^2\left(\log\Delta T\right)_\tau \Rightarrow \left(\log\Delta T\right)_\tau = -\frac{a_8}{\gamma^2} \Rightarrow \log\Delta T = -a_8\int\frac{d\tau}{\gamma^2}$$

Or, $$\log\Delta T = -a_8\int\frac{d\tau}{2a_0\tau + B} \tag{4.4}$$

Now

$$U=\frac{a_1\xi+A}{2a_0\tau+B}\Rightarrow dU=\frac{a_1\xi+A}{(2a_0\tau+B)^2}(-2a_0)\,d\tau$$

Or, $dU=\frac{a_1\xi+A}{2a_0\tau+B}\left(-\frac{2a_0}{2a_0\tau+B}\right)d\tau$

Or, $dU=U\left(-\frac{2a_0}{2a_0\tau+B}\right)d\tau$

Or, $\frac{1}{2a_0}\frac{dU}{U}=-\frac{d\tau}{2a_0\tau+B}$

Then equation (4.4) gives

$$\log\Delta T=\frac{a_8}{2a_0}\int\frac{dU}{U}$$

i.e., $\log\Delta T=\frac{a_8}{2a_0}\log U$ (4.5)

Also, from *x*) of equation (2.44), we have

$$a_9=-\gamma^2U(\log\Delta T)_\xi\Rightarrow(\log\Delta T)_\xi=-\frac{a_9}{\gamma^2U}$$

Or, $\log\Delta T=-\frac{a_9}{\gamma^2}\int\frac{d\xi}{U}$ (4.6)

Now from (4.1), we have

$$\gamma^2U=a_1\xi+A$$

Or, $\gamma^2dU=a_1d\xi$

Or, $\frac{1}{a_1}dU=\frac{1}{\gamma^2}d\xi$

Then equation (4.6) gives

$$\log\Delta T=-\frac{a_9}{a_1}\int\frac{dU}{U}$$

i.e., $\log\Delta T=-\frac{a_9}{a_1}\log U$ (4.7)

Comparing (4.5) and (4.7), we obtain

$a_8=4a_0$ and $a_9=-2a_1$.

Thus, conditions *i*) – *x*) of equation (2.44) yield relations between the constants as follows: a_0 and a_1 are arbitrary and $a_2 = a_1$, $a_4 = -2a_0$, $a_6 = 0$, $a_8 = 2a_0$, $a_9 = -a_1$, a_3, a_5 and a_7 are disposable constants.

Substituting the constants through equations (2.41) – (2.43), we obtain

$$\upsilon_r F_{\phi\phi\phi} + a_0\phi F_{\phi\phi} + a_1 F F_{\phi\phi} + a_3 F_{\phi\phi} - a_1 F_\phi^{\ 2} + 2a_0 F_\phi = a_5 G \tag{4.8}$$

$$G_\phi = a_7\theta \tag{4.9}$$

$$\frac{\upsilon_r}{\Pr}\theta_{\phi\phi} + a_0\phi\theta_\phi + a_1 F\theta_\phi + a_3\theta_\phi + \left(4a_0 - 2a_1 F_\phi\right)\theta = 0 \tag{4.10}$$

Now writing $F = \alpha_1 f$, $\phi = \alpha_2\eta$ and $G = \alpha_3\overline{g}$, we get

$$\frac{d}{d\phi} = \frac{d}{d\eta}.\frac{d\eta}{d\phi} = \frac{d}{d\eta}.\frac{1}{\alpha_2} = \frac{1}{\alpha_2}\frac{d}{d\eta}$$

$$F_\phi = \frac{dF}{d\phi} = \frac{1}{\alpha_2}\frac{dF}{d\eta} = \frac{1}{\alpha_2}\frac{d}{d\eta}\left(\alpha_1 f\right) = \frac{\alpha_1}{\alpha_2}\frac{df}{d\eta}$$

i.e., $F_\phi = \dfrac{\alpha_1}{\alpha_2} f_\eta$

$$F_{\phi\phi} = \frac{d^2F}{d\phi^2} = \frac{d}{d\phi}\left(\frac{dF}{d\phi}\right) = \frac{1}{\alpha_2}\frac{d}{d\eta}\left(\frac{\alpha_1}{\alpha_2} f_\eta\right) = \frac{\alpha_1}{\alpha_2^2}\frac{d}{d\eta}\left(f_\eta\right) = \frac{\alpha_1}{\alpha_2} f_{\eta\eta}$$

i.e., $F_{\phi\phi} = \dfrac{\alpha_1}{\alpha_2^2} f_{\eta\eta}$

Similarly, $F_{\phi\phi\phi} = \dfrac{\alpha_1}{\alpha_2^3} f_{\eta\eta\eta}$

and $G_\phi = \dfrac{dG}{d\phi} = \dfrac{1}{\alpha_2}\dfrac{d}{d\eta}\left(\alpha_3\overline{g}\right) = \dfrac{\alpha_3}{\alpha_2}\dfrac{d\overline{g}}{d\eta}$

i.e., $G_\phi = \dfrac{\alpha_3}{\alpha_2}\overline{g}_\eta$

Substituting above in equations (4.8) to (4.10) we have

$$\frac{\upsilon_r\alpha_1}{\alpha_2^3} f_{\eta\eta\eta} + a_0\alpha_2\eta.\frac{\alpha_1}{\alpha_2^2} f_{\eta\eta} + a_1\alpha_1 f.\frac{\alpha_1}{\alpha_2^2} f_{\eta\eta} + a_3\frac{\alpha_1}{\alpha_2^2} f_{\eta\eta} - a_1\frac{\alpha_1^2}{\alpha_2^2} f_\eta^{\ 2} + 2a_0\frac{\alpha_1}{\alpha_2} f_\eta - a_5\alpha_3\overline{g} = 0$$

Or, $\dfrac{\upsilon_r\alpha_1}{\alpha_2^3} f_{\eta\eta\eta} + \dfrac{a_0\alpha_1}{\alpha_2}\eta f_{\eta\eta} + \dfrac{a_1\alpha_1^2}{\alpha_2^2} ff_{\eta\eta} + \dfrac{a_3\alpha_1}{\alpha_2^2} f_{\eta\eta} - \dfrac{a_1\alpha_1^2}{\alpha_2^2} f_\eta^{\ 2} + \dfrac{2a_0\alpha_1}{\alpha_2} f_\eta - a_5\alpha_3\overline{g} = 0$

Multiplying both sides by $\dfrac{\alpha_2^3}{\upsilon_r\alpha_1}$, we obtain

$$f_{\eta\eta\eta}+\frac{a_0\alpha_2^2}{\upsilon_r}\eta f_{\eta\eta}+\frac{a_1\alpha_1\alpha_2}{\upsilon_r}ff_{\eta\eta}+\frac{a_3\alpha_2}{\upsilon_r}f_{\eta\eta}-\frac{a_1\alpha_1\alpha_2}{\upsilon_r}f_\eta^2+\frac{2a_0\alpha_2^2}{\upsilon_r}f_\eta-\frac{a_5\alpha_2^3\alpha_3}{\upsilon_r\alpha_1}\bar{g}=0 \quad (4.11)$$

$$\frac{\alpha_3}{\alpha_2}\bar{g}_\eta=a_7\theta$$

Or, $\bar{g}_\eta=\dfrac{a_7\alpha_2}{\alpha_3}\theta$ (4.12)

and

$$\frac{\upsilon_r}{\Pr}\frac{1}{\alpha_2^2}\theta_{\eta\eta}+a_0\alpha_2\eta.\frac{1}{\alpha_2}\theta_\eta+a_1\alpha_1 f.\frac{1}{\alpha_2}\theta_\eta+a_3.\frac{1}{\alpha_2}\theta_\eta+\left(6a_0-3a_1.\frac{\alpha_1}{\alpha_2}f_\eta\right)\theta=0$$

Or, $\dfrac{\upsilon_r}{\Pr\alpha_2^2}\theta_{\eta\eta}+a_0\eta\theta_\eta+\dfrac{a_1\alpha_1}{\alpha_2}f\theta_\eta+\dfrac{a_3}{\alpha_2}\theta_\eta+\left(6a_0-\dfrac{3a_1\alpha_1}{\alpha_2}f_\eta\right)\theta=0$

Multiplying both sides by $\dfrac{\alpha_2^2}{\upsilon_r}$, we get

$$\frac{1}{\Pr}\theta_{\eta\eta}+\frac{a_0\alpha_2^2}{\upsilon_r}\eta\theta_\eta+\frac{a_1\alpha_1\alpha_2}{\upsilon_r}f\theta_\eta+\frac{a_3\alpha_2}{\upsilon_r}\theta_\eta+\left(\frac{6a_0\alpha_2^2}{\upsilon_r}-\frac{3a_1\alpha_1\alpha_2}{\upsilon_r}f_\eta\right)\theta=0$$

Or, $\Pr^{-1}\theta_{\eta\eta}+\dfrac{a_0\alpha_2^2}{\upsilon_r}\eta\theta_\eta+\dfrac{a_1\alpha_1\alpha_2}{\upsilon_r}f\theta_\eta+\dfrac{a_3\alpha_2}{\upsilon_r}\theta_\eta+\dfrac{3a_0\alpha_2^2}{\upsilon_r}\left(2-\dfrac{a_1\alpha_1}{a_0\alpha_2}f_\eta\right)\theta=0$ (4.13)

For a complete similarity solution choosing $\alpha_1=\alpha_2$, $\dfrac{a_0\alpha_1^2}{\upsilon_r}=1$, $\dfrac{a_5\alpha_1^2\alpha_3}{\upsilon_r}=1$ and $\dfrac{a_1}{a_0}=\beta$

and writing $\dfrac{a_3\alpha_1}{\upsilon_r}=-\dfrac{\gamma v_w}{\sqrt{a_0\upsilon_r}}=f_w$, the above equations (4.11) – (4.13) yield

$$f_{\eta\eta\eta}+\eta f_{\eta\eta}+\beta ff_{\eta\eta}+f_w f_{\eta\eta}-\beta f_\eta^2+2f_\eta-\bar{g}=0 \quad (4.14)$$

$$\bar{g}_\eta=\left(\frac{U_F}{U_0}\right)^{\frac{5}{2}}\theta \quad (4.15)$$

$$\Pr^{-1}\theta_{\eta\eta}+\eta\theta_\eta+\beta f\theta_\eta+f_w\theta_\eta+3\left(2-\beta f_\eta\right)\theta=0 \quad (4.16)$$

where

$$\frac{a_1\alpha_1\alpha_2}{\upsilon_r}=\frac{a_1\alpha_1^2}{\upsilon_r}=\frac{a_1}{\upsilon_r}.\frac{\upsilon_r}{a_0}=\frac{a_1}{a_0}=\beta$$

$$\frac{a_3\alpha_2}{\upsilon_r}=\frac{a_3\alpha_1}{\upsilon_r}=\frac{-\gamma v_w}{\upsilon_r}.\sqrt{\frac{\upsilon_r}{a_0}}=\frac{-\gamma v_w}{\sqrt{a_0\upsilon_r}}=f_w \qquad \Bigg|\ \because\frac{a_0\alpha_1^2}{\upsilon_r}=1\Rightarrow\alpha_1=\sqrt{\frac{\upsilon_r}{a_0}}$$

$$\frac{a_7\alpha_2}{\alpha_3}=\frac{a_7\alpha_1}{\alpha_3}$$

$$=-\frac{\rho_r\gamma\,\beta_T g\Delta T}{p\left(\tau,\xi\right)}.\frac{\alpha_1}{\alpha_3}$$

Now, from condition *vi*) of equation (2.44), we have

$$a_5=\frac{\gamma^2 p_\xi}{\rho_r U}$$

$$\Rightarrow p_\xi=\frac{a_5\rho_r U}{\gamma^2}$$

$$\Rightarrow p_\xi=\frac{a_5\rho_r}{\gamma^2}.\frac{a_1\xi+A}{\gamma^2}$$

$$\Rightarrow p=\frac{a_5\rho_r}{\gamma^2.\gamma^2}\int\left(a_1\xi+A\right)d\xi$$

$$\Rightarrow p=\frac{a_5\rho_r}{\gamma^2.\gamma^2}\frac{\left(a_1\xi+A\right)^2}{2a_1}$$

$$\Rightarrow p=\frac{a_5\rho_r}{2a_1}\left(\frac{a_1\xi+A}{\gamma^2}\right)^2$$

i.e., $p=\dfrac{a_5\rho_r}{2a_1}U^2$

Then

$$\frac{a_7\alpha_2}{\alpha_3}=-\frac{\rho_r\gamma\,\beta_T g\Delta T}{\frac{a_5\rho_r}{2a_1}U^2}.\frac{\alpha_1}{\alpha_3}$$

$$=-\frac{\gamma\,\beta_T g\Delta T}{U^2}.\frac{2a_1\alpha_1}{a_5\alpha_3}$$

$$=-\frac{\gamma\,\beta_T g\Delta T}{U^2}.\frac{2a_1\alpha_1}{a_0}$$

$$\Big|\because\frac{a_5\alpha_1^2\alpha_3}{\upsilon_r}=1\Rightarrow\frac{a_0\alpha_1^2}{\upsilon_r}\frac{a_5\alpha_3}{a_0}=1\Rightarrow\frac{a_5\alpha_3}{a_0}=1\Rightarrow\frac{1}{a_5\alpha_3}=\frac{1}{a_0}$$

$$=-\frac{\gamma\,\beta_T g\Delta T}{U^2}.2\beta\sqrt{\frac{\upsilon_r}{a_0}}$$

Again

$$\gamma^2 = \frac{a_1\xi + A}{U}$$

$$\Rightarrow \gamma = \sqrt{\frac{a_1\xi + A}{U}}$$

$$= \sqrt{a_1}\sqrt{\frac{\xi + \frac{A}{a_1}}{U}}$$

$$= \sqrt{a_1}\sqrt{\frac{\xi + \xi_0}{U}} \text{ , where we choose } \frac{A}{a_1} = \xi_0$$

Hence

$$\frac{a_7\alpha_2}{\alpha_3} = -\frac{\beta_T g\Delta T}{U^2}.\sqrt{a_1}\sqrt{\frac{\xi + \xi_0}{U}}2\beta\sqrt{\frac{\upsilon_r}{a_0}}$$

$$= -\frac{\beta_T g\Delta T\sqrt{\upsilon_r\left(\xi + \xi_0\right)}}{U^{\frac{5}{2}}}2\beta\sqrt{\frac{a_1}{a_0}}$$

$$= -\frac{\beta_T g\Delta T\sqrt{\upsilon_r\left(\xi + \xi_0\right)}}{U^{\frac{5}{2}}}2\beta\sqrt{\beta}$$

$$= -\frac{2\beta^{\frac{3}{2}}\beta_T g\Delta T\sqrt{\upsilon_r\left(\xi + \xi_0\right)}}{U^{\frac{5}{2}}}$$

$$= \left(\frac{U_F}{U}\right)^{\frac{5}{2}}$$

where

$$U_F = \left\{2\beta^{\frac{3}{2}}\left(-\beta_T g\Delta T\right)\sqrt{\upsilon_r\left(\xi + \xi_0\right)}\right\}^{\frac{2}{5}}$$

$$= \left\{\sqrt{2}\beta\left(-\beta_T g\Delta T\right)\sqrt{\upsilon_r \times \text{Characteristic Length}}\right\}^{\frac{2}{5}} \tag{4.17}$$

is called free convection velocity associated with the local characteristic length

$$L = \xi + \xi_0 \tag{4.18}$$

Since we are concerned with a purely free convection flow, without loss of generality we may put $U_F = U$ and hence the similarity equations are given by

$$f_{\eta\eta\eta} + \eta f_{\eta\eta} + \beta ff_{\eta\eta} + f_w f_{\eta\eta} - \beta f_\eta^2 + 2 f_\eta - \bar{g} = 0 \tag{4.19}$$

$$\bar{g}_\eta = \theta \tag{4.20}$$

$$\mathrm{Pr}^{-1}\theta_{\eta\eta} + \eta\theta_\eta + \beta f \theta_\eta + f_w\theta_\eta + 3\left(2 - \beta f_\eta\right)\theta = 0 \tag{4.21}$$

subject to the transformed boundary conditions:

$$f(0) = f_\eta(0) = f_\eta(\infty) = 0;\ \theta(0) = 1,\ \theta(\infty) = 0;\ \bar{g}_n(0) = 1,\ \bar{g}_n(\infty) = 0 \tag{4.22}$$

where the boundary conditions for $\bar{g}$ is obtained form that described for θ according to equation (4.20).

Further, we can write

$$\gamma = \sqrt{\frac{a_1\left(\xi + \xi_0\right)}{U_F}}$$

$$\Rightarrow \gamma = \sqrt{\frac{a_1 L}{U_F}} \tag{4.23}$$

where L is the local characteristic length defined by (4.18).

Hence the similarity function $f\left(\eta\right)$, the similarity variable η and the pressure function $\bar{g}\left(\eta\right)$ are related to the stream function ψ, the physical co-ordinate y and perturbation pressure $\tilde{p}$ by the following equations respectively

$$\psi\left(\tau,\xi,\phi\right) = \gamma U_F F + \psi\left(\tau,\xi,0\right)$$

$$\psi\left(\tau,\xi,\eta\right) = \sqrt{\frac{a_1 L}{U_F}}.U_F \alpha_1 f\left(\eta\right) + \psi\left(\tau,\xi,0\right)$$

$$= \sqrt{\frac{L}{U_F}}\sqrt{a_1}U_F\sqrt{\frac{\upsilon_r}{a_0}} f\left(\eta\right) + \psi\left(\tau,\xi,0\right)$$

$$= \upsilon_r\sqrt{\frac{U_F L}{\upsilon_r}}\sqrt{\frac{a_1}{a_0}} f\left(\eta\right) + \psi\left(\tau,\xi,0\right)$$

i.e., $\psi\left(\tau,\xi,\eta\right) = \upsilon_r\sqrt{\mathrm{Re_F}}\sqrt{\beta} f\left(\eta\right) + \psi\left(\tau,\xi,0\right)$ (4.24)

Here $\mathrm{Re_F}$ is the dimensionless Reynolds number based on free convection velocity U_F given by equation (4.17) and the local characteristic length L given by equation (4.18) as

$$\mathrm{Re_F} = \frac{U_F L}{\upsilon_r} \tag{4.25}$$

Further, we have

$$\phi = \alpha_2 \eta$$

$$\therefore \eta = \frac{\phi}{\alpha_2} = \frac{1}{\alpha_1} . \frac{y}{\gamma}$$

$$= \frac{1}{\sqrt{\frac{\upsilon_r}{a_0}}} . \frac{y}{\frac{\sqrt{a_1 L}}{\sqrt{U_F}}}$$

$$= \frac{1}{\sqrt{\frac{a_1}{a_0}}} \sqrt{\frac{U_F}{\upsilon_r}} . \frac{y}{\sqrt{L}}$$

$$= \frac{1}{\sqrt{\beta}} \sqrt{\frac{U_F L}{\upsilon_r}} \frac{y}{L}$$

i.e., $\eta = \frac{\sqrt{\mathrm{Re}_\mathrm{F}}}{\sqrt{\beta} L} y$ (4.26)

and

$$\tilde{p} = pG(\phi)$$

$\Rightarrow \tilde{p} = p\alpha_3 \bar{g}(\eta)$ | since from condition *vi*) of equation (2.44) $p = \frac{a_5 \rho_r}{2a_1} U^2$

$$= \frac{a_5 \alpha_3 \rho_r U^2}{2a_1} \bar{g}(\eta)$$

$$= \frac{a_0 \rho_r U_F^2}{2a_1} \bar{g}(\eta) \left| \because \frac{a_5 \alpha_1^2 \alpha_3}{\upsilon_r} = 1 \Rightarrow \frac{a_0 \alpha_1^2}{\upsilon_r} \frac{a_5 \alpha_3}{a_0} = 1 \Rightarrow \frac{a_5 \alpha_3}{a_0} = 1 \Rightarrow a_5 \alpha_3 = a_0 \right.$$

$$= \frac{\rho_r U_F^2}{\frac{2a_1}{a_0}} \bar{g}(\eta) \qquad \left| \because \beta = \frac{a_1}{a_0} \right.$$

i.e., $\tilde{p} = \frac{\rho_r U_F^2}{2\beta} \bar{g}(\eta)$ (4.27)

The velocity components, the skin friction and the local heat transfer coefficients associated with the equations (4.19) – (4.21) are now

$$u = UF_\phi = U_F . \frac{\alpha_1}{\alpha_2} f_\eta$$

i.e., $u = U_F f_\eta(\eta)$ (4.28)

$$v = \gamma_\xi U \phi F_\phi - \gamma_\xi UF + v_w$$

Or, $v = \gamma_\xi U_F \alpha_2 \eta . \dfrac{\alpha_1}{\alpha_2} f_\eta - \gamma_\xi U_F . \alpha_1 f - \dfrac{\sqrt{a_0 \upsilon_r}}{\gamma} f_w$

Or, $v = \gamma_\xi U_F \alpha_1 \eta f_\eta - \gamma_\xi U_F \alpha_1 f - \dfrac{\sqrt{a_0 \upsilon_r}}{\gamma} f_w$

Or, $v = \alpha_1 \gamma_\xi U_F \left(\eta f_\eta - f\right) - \dfrac{\sqrt{a_0 \upsilon_r}}{\gamma} f_w$

$$-v = \sqrt{\frac{\upsilon_r}{a_0}} \gamma_\xi U_F \left(f - \eta f_\eta\right) + \frac{\sqrt{a_0 \upsilon_r}}{\gamma} f_w$$

$$\left| \because \gamma = \sqrt{\frac{a_1 \xi + A}{U}} = \frac{\sqrt{a_1 \xi + A}}{\sqrt{U_F}} \Rightarrow \gamma_\xi = \frac{1}{\sqrt{a_1 \xi + A}} . \frac{a_1}{2\sqrt{U_F}} \right.$$

$$= \sqrt{\frac{\upsilon_r}{a_0}} \frac{U_F}{\sqrt{a_1 \xi + A}} . \frac{a_1}{2\sqrt{U_F}} . \left(f - \eta f_\eta\right) + \frac{\sqrt{a_0 \upsilon_r}}{\sqrt{\dfrac{a_1 \xi + A}{U_F}}} f_w$$

$$= \sqrt{\frac{\upsilon_r}{a_0}} \frac{\sqrt{U_F}}{\sqrt{a_1}\sqrt{\xi + \dfrac{A}{a_1}}} . \frac{a_1}{2} \left(f - \eta f_\eta\right) + \frac{\sqrt{U_F}\sqrt{a_0 \upsilon_r}}{\sqrt{a_1}\sqrt{\xi + \dfrac{A}{a_1}}} f_w$$

$$= \sqrt{\frac{a_1}{a_0}} \sqrt{\frac{U_F\left(\xi + \xi_0\right)}{\upsilon_r}} \frac{\upsilon_r}{2\left(\xi + \xi_0\right)} \left(f - \eta f_\eta\right) + \frac{\upsilon_r}{\xi + \xi_0} \frac{1}{\sqrt{\dfrac{a_1}{a_0}}} \sqrt{\frac{U_F\left(\xi + \xi_0\right)}{\upsilon_r}} f_w$$

$$= \frac{\upsilon_r \sqrt{\beta}\sqrt{\mathrm{Re_F}}}{2\left(\xi + \xi_0\right)} \left(f - \eta f_\eta\right) + \frac{\upsilon_r \sqrt{\mathrm{Re_F}}}{\sqrt{\beta}\left(\xi + \xi_0\right)} f_w$$

$$= \frac{\upsilon_r \sqrt{\beta}\sqrt{\mathrm{Re_F}}}{2\left(\xi + \xi_0\right)} \left(f - \eta f_\eta\right) + \frac{\upsilon_r \sqrt{\mathrm{Re_F}}}{\sqrt{\beta}\left(\xi + \xi_0\right)} f_w$$

i.e., $-v = \dfrac{\upsilon_r \sqrt{\mathrm{Re_F}}}{2\sqrt{\beta}L} \left\{\beta\left(f - \eta f_\eta\right) + f_w\right\}$ (4.29)

The skin friction coefficient is defined by

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

Now

$$\frac{\partial u}{\partial y}=\frac{1}{\gamma}\frac{d}{d\phi}\left(U_F F_\phi\right)=\frac{U_F}{\gamma}F_{\phi\phi}=\frac{U_F}{\gamma}\frac{\alpha_1}{\alpha_2^2}f_{\eta\eta}\left(\eta\right)=\frac{U_F}{\gamma\alpha_1}f_{\eta\eta}\left(\eta\right)$$

Hence

$$\tau_w=\frac{\mu U_F}{\gamma\alpha_1}f_{\eta\eta}\left(0\right)$$

$$=\frac{\mu U_F}{\frac{\sqrt{a_1 L}}{\sqrt{U_F}}\sqrt{\frac{\upsilon_r}{a_0}}}f_{\eta\eta}\left(0\right)$$

$$=\frac{\mu U_F}{\frac{\sqrt{\upsilon_r L}}{\sqrt{U_F}}\sqrt{\frac{a_1}{a_0}}}f_{\eta\eta}\left(0\right)$$

$$=\frac{\mu U_F}{\sqrt{\beta}L\sqrt{\frac{\upsilon_r}{U_F L}}}f_{\eta\eta}\left(0\right)$$

$$=\frac{\mu U_F}{\sqrt{\beta}L\sqrt{\frac{1}{\mathrm{Re_F}}}}f_{\eta\eta}\left(0\right)$$

i.e., $\tau_w=\dfrac{\mu U_F\sqrt{\mathrm{Re_F}}}{\sqrt{\beta}L}f_{\eta\eta}\left(0\right)$ (4.30)

The heat transfer coefficient is defined by

$$q_w=-k\Delta T\left[\frac{\partial\theta}{\partial y}\right]_{y=0}$$

Now

$$\frac{\partial\theta}{\partial y}=\frac{1}{\gamma}\frac{\partial\theta}{\partial\phi}=\frac{1}{\gamma}\frac{1}{\alpha_2}\frac{\partial\theta}{\partial\eta}=\frac{1}{\gamma\alpha_1}\theta_\eta\left(\eta\right)$$

Hence

$$q_w=-\frac{k\Delta T}{\gamma\alpha_1}\theta_\eta\left(0\right)$$

i.e., $q_w=-\dfrac{k\Delta T\sqrt{\mathrm{Re_F}}}{\sqrt{\beta}L}\theta_\eta\left(0\right)$ (4.31)

Numerical scheme and procedure

The set of ordinary differential equations (4.19) to (4.21) with the boundary conditions (4.22) are solved based on the same numerical procedure as stated in Case A, that is, using the Runge-Kutta shooting method in collaboration with the Runge-Kutta Merson method. Like Case A, here the velocity f_η, temperature θ and pressure function $\bar{g}$ are determined as function of the co-ordinate η. The skin friction coefficient $f_{\eta\eta}(0)$ and the heat transfer rate $\theta_\eta(0)$ are also evaluated for this case and the numerical results thus obtained in terms of the similarity variables are displayed in graphs and tables for several selected values of the established parameters f_w, β and Pr below:

Numerical Results and Discussion

On the basis of the numerical results, the dimensionless velocity and temperature profiles, and the pressure distributions are plotted. The effects of suction parameter f_w, driving parameter β and Prandtl number Pr on velocity f_η, temperature θ and pressure $\bar{g}$ against η are illustrated in Fig. 4.1 through Fig.4.9. Also their effects on the coefficient of skin-friction and heat transfer coefficient are tabulated in Table- 4.1 and Table- 4.2, respectively. To observe the effect of f_w, the other two parameters i.e., β and Pr are kept as constants. Similarly, to observe f_w and Pr and f_w and β are kept as constants the effect of β and Pr, respectively. In all cases where f_w is kept as constant, its value is chosen as 0.50. Similarly, when Pr is kept constant, its value is chosen as 0.71 and when β is kept fixed, its value is chosen as 0.30.

Figs. 4.1 to 4.3 represent the effect of f_w, β and Pr on the velocity profiles, respectively. From Fig.4.1 it is observed that the velocity remains negative for f_w reduces from 0.50 to – 0.10. The magnitude of the velocity increases with the increase in f_w. The positive values of the velocities are found when f_w is -0.3 or less but they become again negative before being vanished. The positive value of the velocity increase with the decrease in f_w.

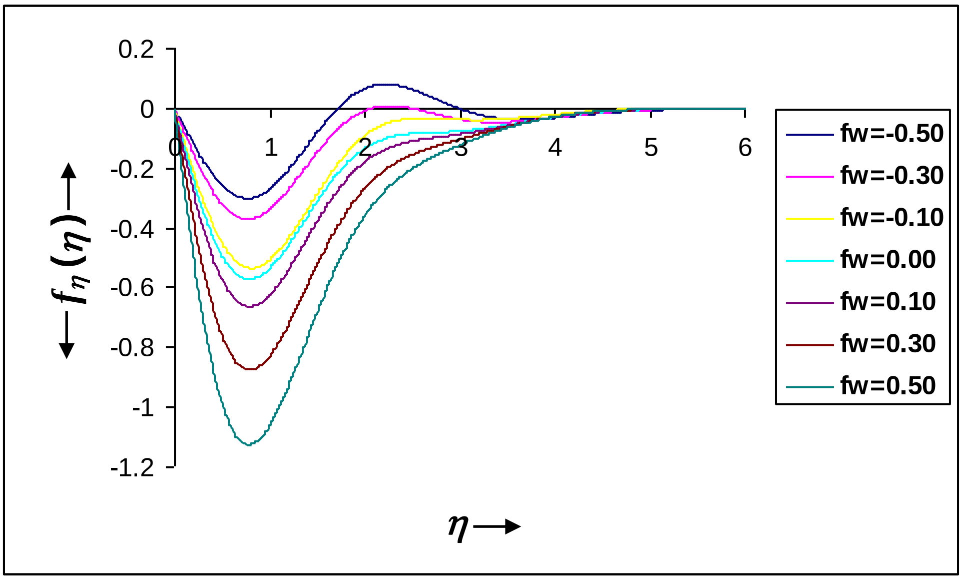

Figure 4.1: Velocity profiles for different values of the suction parameter f_w with fixed values of β and Pr.

In Fig. 4.2 we observe that the behaviour of the velocity differs when β is positive with those when β is zero or negative. If β equals zero or negative, the velocities go to positive from negative before finally approaching to zero. Also when β increasing in magnitude, the velocity is decreasing in magnitude for those values of β. But when β is positive, with its increase the velocity decreases in magnitude. For this domain of β the velocity remains negative all through.

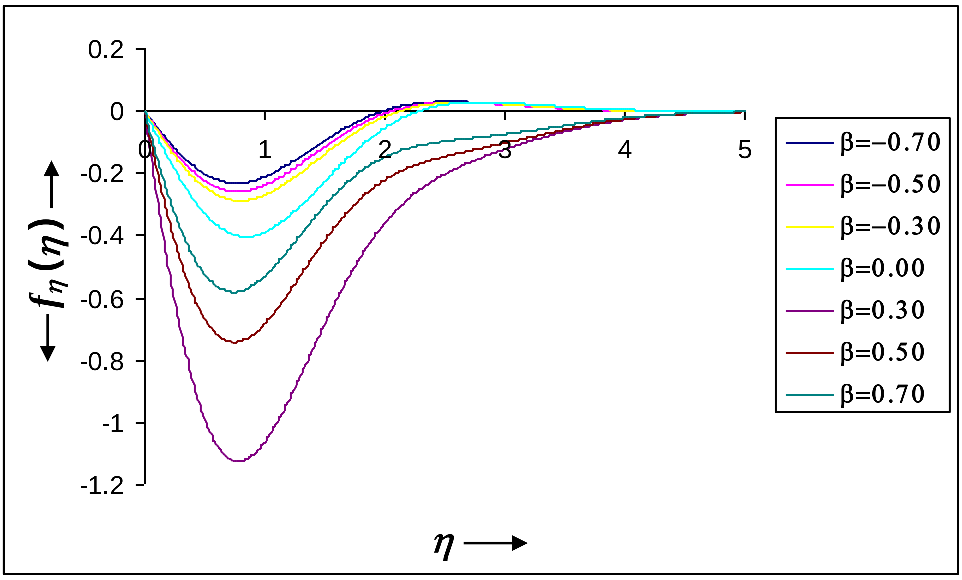

Figure 4.2: Velocity profiles for different values of the parameter β with fixed values of f_w and Pr.

From Fig. 4.3 we see that the velocity decreases in magnitude with the increase in Pr. The velocity is observed negative for all values of Pr. It may be note here that for lighter gases than air a fluctuation in velocity is observed far away from the wall before finally approaching to zero.

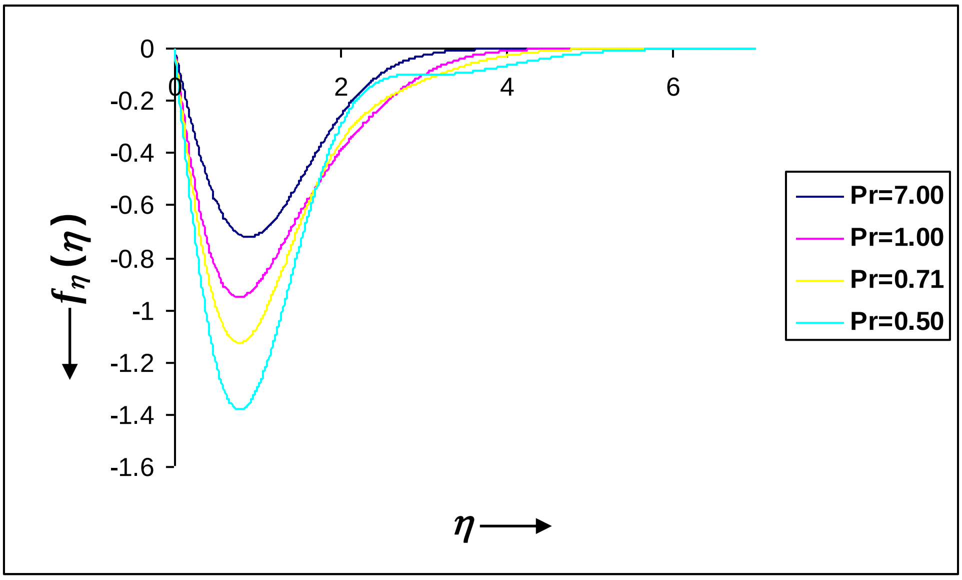

Figure 4.3: Velocity profiles for different values of the Prandtl number Pr with fixed values of f_w and β.

Figs. 4.4 to 4.6 represent the effect of f_w, β and Pr on the temperature profiles respectively. From Fig. 4.4 we see that in the region where $\eta < 1.33$, the temperature decreases with the increasing f_w. After that, it reverses the direction till $\eta < 2.9$, that is, the temperature increases with the decrease in f_w. Then again it reverses the order before reaching to zero.

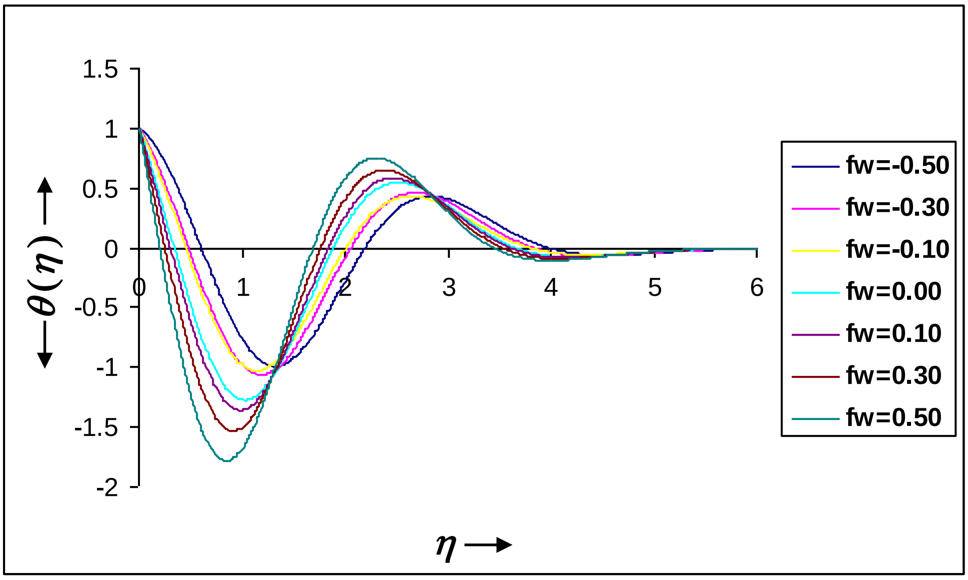

Figure 4.4: Temperature profiles for different values of the suction parameter f_w with fixed values of β and Pr.

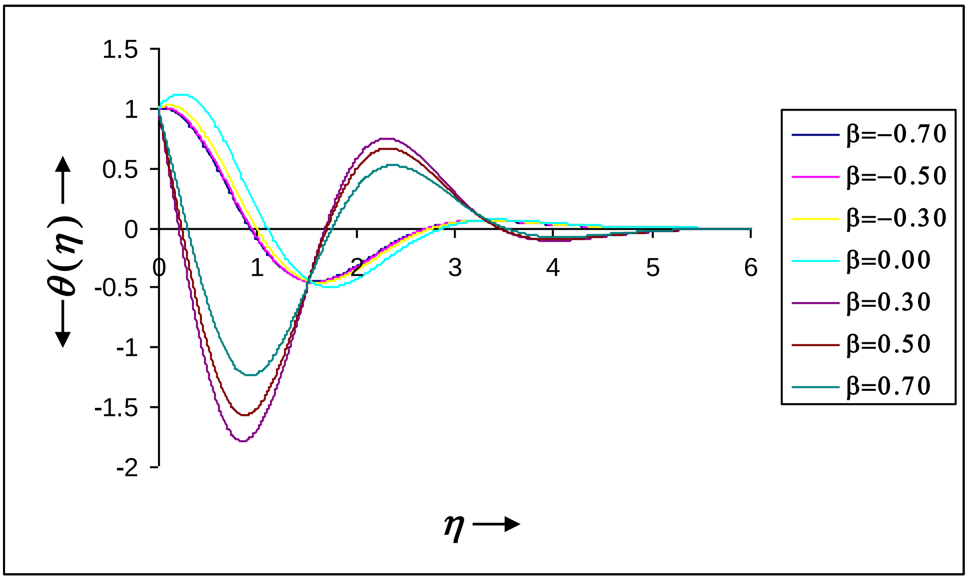

Figure 4.5: Temperature profiles for different values of the parameter β with fixed values of f_w and Pr.

The effect of β on the temperature profiles are shown in Fig. 4.5. Here again the behaviour is different when β is 0 or less and when $\beta > 0$. When β is zero or negative, close to the wall, the temperature goes up or remains same and then falls slowly. When η is about 3.0 the temperature again become positive and finally reaches to zero. Where as when β is positive, temperature falls sharply and reverse its direction of change when η is about 1.0 and reaches to the positive region after η being 1.7. Before being zero the temperature again becomes negative. In the first region, that is while temperature was decreasing it decreases much for smaller value of β. Consequently, when it is becoming positive, the magnitude is higher for that smaller value.

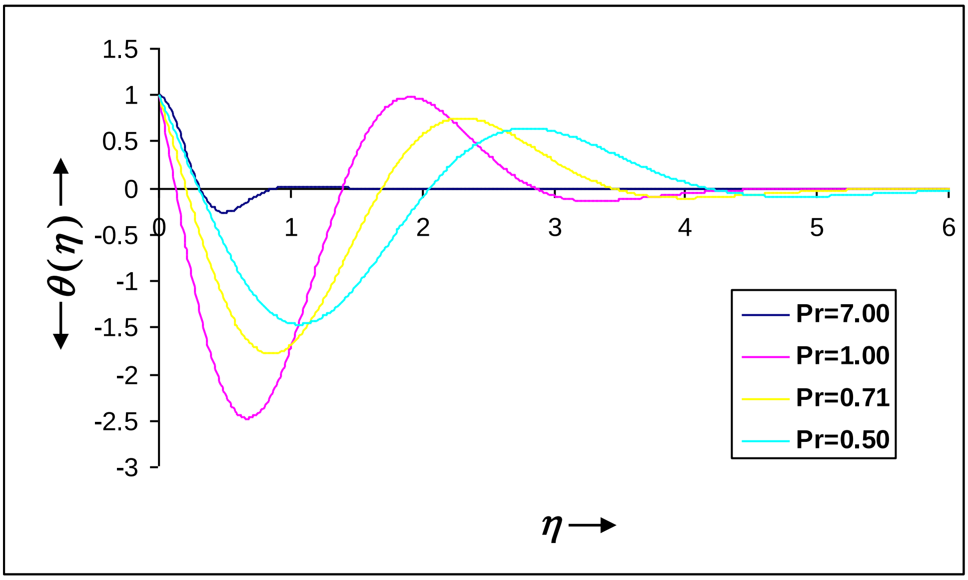

Figure 4.6: Temperature profiles for different values of the Prandtl number Pr with fixed values of f_w and β.

From Fig. 4.6 it is observed that the temperature decreases more with the increase in Pr near the wall. For relative higher values of η the temperature goes to positive except Pr is equal to 7.0. Before reaching zero finally, the temperature goes to negative again in the region where, $\eta \angle 1.3$ the temperature decreases with the increase in Pr except when Pr is 7.0. After that temperature increases with the increases in P_r ,it again decrease when P_r increase. This peculiar behavior for Pr = 7.0 is observed may due to the constituents of the fluid.

Figs. 4.7 to 4.9 representing the effects of f_w, β and Pr respectively on the pressure variable $\bar{g}$. The initial value of the dependent variable $\bar{g}$ is not known rather it was subject to guessing to satisfy the boundary conditions at the other end. As a result the graphs for $\eta = 0$ (on the wall) starts from different points and for small value of η (close to the wall) the patterns are different from the other values of η as shown in Figs. 4.7 to 4.9.

Fig. 4.7 represents the variation of pressure in terms f_w. For smaller values of f_w, the pressure near the wall is small and increase with the increase in f_w and starts to decrease after a short distance from the wall. This distance decreases with the increase in f_w. For higher value of f_w the pressure goes higher near the wall and then falls sharply and attains the minimum negative value before it goes to positive and finally approaches to zero. The behaviour for the smaller value of f_w is same that is initially it goes up and then returns to negative value and before being zero it again becomes positive. From the figure is also seen that the curves are intersecting at $\eta = 0.66$ and $\eta = 2.27$.

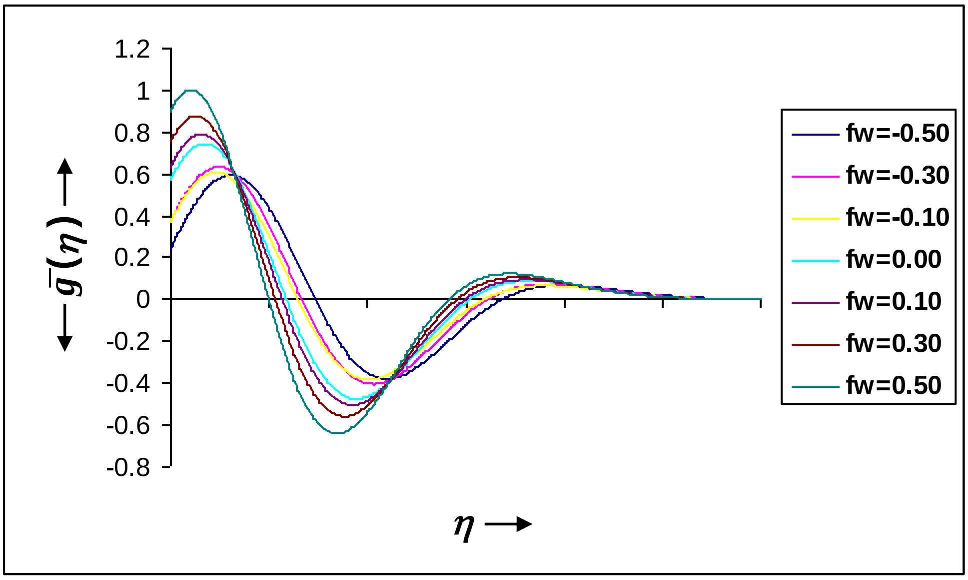

Figure 4.7: Pressure distributions for different values of the suction parameter f_w with fixed values of β and Pr.

The effect of β on the pressure variable $\bar{g}$ is shown in Fig. 4.8. Like velocity and temperature we also observe different behaviour when β is 0 or less and when $\beta > 0$. When β is zero or negative, close to the wall, the pressure is found negative with leading to the positive value far away. Here with the increase in magnitude of β the pressure decreases in absolute value. But this effect is observed till $\eta \approx 0.94$. After that the reveres effect is observed and again goes to negative value before being asymptotically approaches to zero. An alternate situation is found when $\beta > 0$, that is, pressure is positive near the wall, then goes to negative value and finally again goes to positive one before being zero. Also the pressure decreases primarily with increase in β but finally it increases with the decrease in β.

From Fig. 4.9 it is observed that the rate of decrease of pressure is more with the decrease in Pr near the wall except for Pr is equal to 7.0. For relatively higher values of η the pressure goes to negative with decreasing-increasing behaviour and further become positive before reaching to zero with the exception for Pr is equal to 7.0.

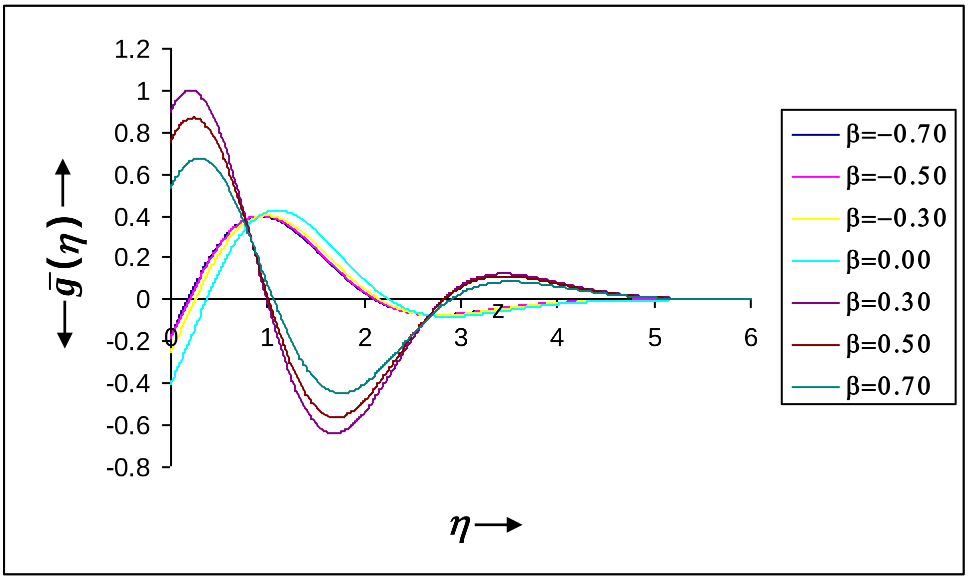

Figure 4.8: Pressure distributions for different values of the parameter β with fixed values of f_w and Pr.

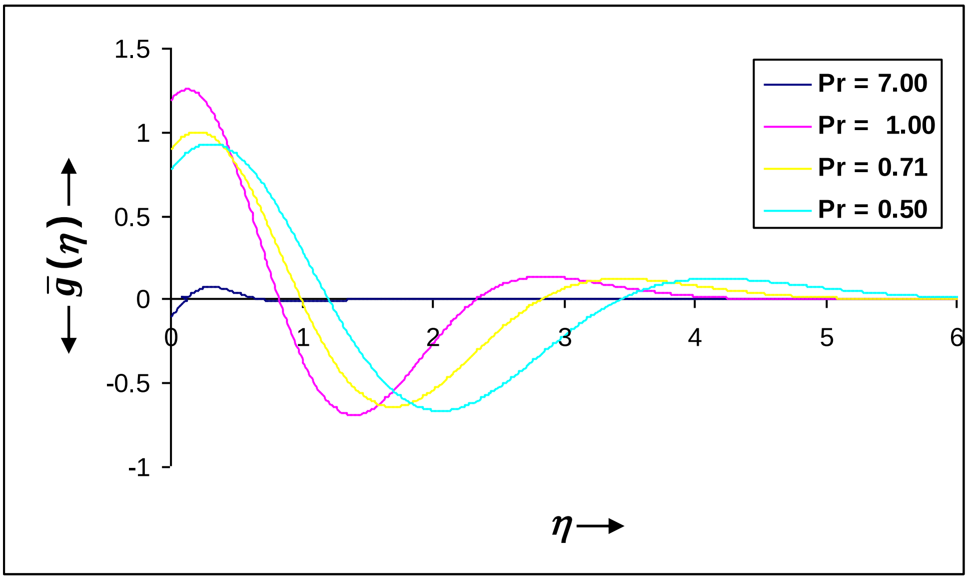

Figure 4.9: Pressure distributions for different values of the Prandtl number Pr with fixed values of f_w and β.

The values proportional to the coefficients of skin friction and heat transfer are tabulated in Table 4.1 and 4.2 respectively. From the Table 4.1, it is seen that for fixed β and Pr, the values proportional to the coefficient of skin friction decreases for increase of suction parameter f_w . Further, it is observed that for fixed f_w and Pr the values proportional to the coefficient of skin friction decreases with the increase of β where β is zero or negative but it increases with the increases in β when β is greater than 0.3.

Again, for fixed β and f_w, the values proportional to the skin friction decreases with decreasing Pr.

Table- 4.1. Variation of the coefficient of skin-friction with f_w, β and Pr:

Values Proportional to the coefficient of skin-friction with the variation of					
f_w (for β =0.30 and Pr = 0.71)		β (for f_w = 0.50 and Pr = 0.71)		Pr (for f_w = 0.50 and β =0.30)	
f_w	$f_{\eta\eta}(0)$	β	$f_{\eta\eta}(0)$	Pr	$f_{\eta\eta}(0)$
-0.50	-0.612297	-0.70	-0.512443	0.50	-3.332221
-0.30	-0.803193	-0.50	-0.564585	0.71	-2.799246
-0.10	-1.126024	-0.30	-0.601379	1.00	-2.492708
0.00	-1.299355	0.00	-0.782206	7.00	-1.502655
0.10	-1.528126	0.30	-2.799246		
0.30	-2.082459	0.50	-1.937616		
0.50	-2.799246	0.70	-1.518315		

Table 4.2 shows the values proportional to the heat transfer coefficient, varying one parameter keeping the other two fixed. It is observed that for fixed β and Pr, the heat transfer coefficient decreases with the increase in f_w. For fixed f_w and Pr the values proportional to the heat transfer coefficient decreases with the decrease of β where β is zero or negative but it behaves like skin friction coefficient for β variation, when β is greater than 0.3..

For fixed β and f_w, the values proportional to the heat transfer coefficient increases with decreasing Pr except for Pr = 7.0.

Table- 4.2. Variation of the heat transfer coefficient with f_w, β and Pr:

Values Proportional to the heat transfer coefficient with the variation of					
f_w (for β =0.30 and Pr = 0.71)		β (for f_w = 0.50 and Pr = 0.71)		Pr (for f_w = 0.50) and β = 0.30	
f_w	$\theta_\eta(0)$	β	$\theta_\eta(0)$	Pr	$\theta_\eta(0)$
-0.50	-0.567070	-0.70	0.182250	0.50	-3.145284
-0.30	-1.270933	-0.50	0.254844	0.71	-4.754354
-0.10	-1.513344	-0.30	0.493686	1.00	-8.064734
0.00	-2.431226	0.00	1.086548	7.00	-0.057409
0.10	-2.841108	0.30	-4.754354		
0.30	-3.680210	0.50	-4.160390		
0.50	-4.754354	0.70	-3.181425		

CHAPTER V

Conclusions

With the technique of the similarity solutions, we have solved the governing boundary layer equations of the laminar two-dimensional unsteady natural convection flow over a semi-infinite heated horizontal porous surface, taking into account the effect of suction and blowing. Four different similarity cases arise with the choice of $\frac{dA}{d\tau}$ and $\frac{dB}{d\xi}$ either zero or constant. Similarity solutions for two of the cases have been studied in this thesis. It is being observed that for Case A, the similarity variables related to the velocity and temperature have inverse relations with the controlling parameters. Whereas the variables related to the pressure has direct relation with the controlling parameters. It is also observed that the coefficient of skin-friction and heat transfer have also inverse relationship with the controlling parameters. But for Case B, no systematic relationship of controlling parameters on the flow variables is observed. Therefore, study of Case A is more suitable than Case B. In future, we shall study further two cases.

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