AN EXPERIMENTAL STUDY ON LINEAR FRACTIONAL PROGRAMMING PROBLEMS

By

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Declaration

This is to certify that the thesis work entitled **" An experimental study on linear fractional programming problems"** has been carried out by **Kazi Tanzila Ferdaus** in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

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Dedication

To

My parents

Md. Abbdur Razzaque & Most. Arifa Khatun

Acknowledgement

I am highly grateful to the Almighty for granting me time and effort to undertake this thesis and complete it.

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Abstract

Much effort has been concentrated on transportation problems (TP) with equality constraints. In real life, however, most problems have mixed constraints accommodating many applications that go beyond transportation related problems to include job scheduling, production inventory, production distribution, allocation problems, and investment analysis. A literature search revealed no systematic method for finding an optimal solution or addressing more-for-less situations in transportation problems with mixed constraint. Here we consider modified VAM method to solve TP with mixed constraints to find out more-for-less situations in transportation problems. Several numerical examples have been considered to justify the effectiveness of the method.

On the other hand Linear Fraction Programming (LFP) (i.e. ratio objective that have numerator and denominator) have attracted the interest of many researches due to its application in many important fields such as production planning, financial and corporate planning, health care and hospital planning. Also various optimization problems in engineering and economics involve maximization (or minimization) of the ratio of physical or economical function, for instances cost/time, cost/volume, cost/benefit, profit/cost or other quantities measuring the efficiency of the system. This study presents a new approach for solving a fractional linear programming problem in which the objective function is a linear fractional function, while the constraint functions are in the form of linear inequalities. The LFP problem is converted it into a regular linear programming (LP) problem by an efficient way. The proposed approach is able to reduce some significant limitations of the existing methods. To test the effectiveness and efficiency of the algorithm some hard instances are considered. The proposed approach is able to solve the problem efficiently whereas in some cases the existence approaches are failed to solve the problems.

Publications

The following articles have been extracted from this thesis work:

1. **Kazi Tanzila Ferdaus,** A. R. M. Jalal Uddin Jamali and Md. Rezwan Hossain, "**A proposed modified algorithm on linear fractional programming problems**", 19th International Conference on Mathematics, Brac University, Dhaka, Bangladesh. December 18-20, 2015.
2. **Kazi Tanzila Ferdaus,** A. R. M. Jalal Uddin Jamali and Md. Rezwan Hossain, "**A New Approach to Solve Linear Fractional Programming Problems**", Proceedings of 1st International Conference on Mathematics and Its Applications (ICMA - 2015) Khulna University (KU), Khulna, Bangladesh. December 23, 2015, pp. 83- 86.

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CHAPTER I

Introduction

1.1 Background

Linear fraction maximum problems (i.e. ratio objective that have numerator and denominator) have attracted considerable research and interest, since they are useful in production planning, financial and corporate planning, health care and hospital planning. Also various optimization problems in engineering and economics involve maximization (or minimization) of the ratio of physical or economical function, for instances cost/time, cost/volume, cost/benefit, profit/cost or other quantities measuring the efficiency of the system. Naturally there is a need for generalizing the simplex technique for linear programming to the ratio of linear functions or to the case of the ratio of quadratic functions in such a situation. All these problems are fragments of a general class of optimization problems, termed in the literature as Fractional Programming Problems (FPP) and are the subject of the field of operation Research.

1.2 History of Operations Research

As a formal discipline, operations research (OR) were initiated in England during World War II when a term of British scientists set out to make decisions regarding the best utilization of war material.

Immediately after the war, the success of military teams attracted the attention of industrial managers who were seeking solutions to their problems. Industrial operations reaches in U.K. and U.S.A. developed along different lines. In U.K., the critical economic situation required drastic increase in production efficiency and creation of new markets. Nationalization of a few key industries further increased the potential field for OR. Consequently OR soon spread from military to government, industrial, social and economic planning.

In U.S.A. the situation was different. Impressed by its dramatic success in U.K. defense operations research in U.S.A. was increased. Most of the war-experienced OR workers remained in military services. Industrial executives did not call for much help because they were returning to the peace-time situation and many of them believed that it was merely a new application of an old technique. The progress of industrial operations research in U.S.A. was due to advent of second industrial revolution, which resulted in automation- the replacement of man by machine as a source of control. This new revolution began around 1940s electric computer becomes commercially available. These electronic brains possessed tremendous computational speed and information storage. But for these digital computers, operations research with its complex computational problems could not have achieved its promising place in all kinds of operational environments.

Today, the impact of operations research can be felt in many areas. This is shown by the ever increasing number of educational institutions offering this subject at degree level. The fast increasing number of management consulting firms speaks of the popularity of the subject. Of late, OR activities have spread to diverse fields such as hospitals, libraries, city planning, transportation systems, crime investigation, etc. Some of the Indian organization using OR techniques are: Indian Airlines, Railways, Defense Organizations, Fertilizer Corporation of India, Delhi Cloth Mills, Tata Iron & Steel Co., etc. So we can use OR in our organization for better solution.

1.3 Mathematical Model

Many application of science makes use of models. The term ‘model’ is usually used for structure has been built purposely to exhibit features and characteristics of some other object. Generally only some of these features and characteristics will be retrained in the model depending upon the use to which it is to be put. More often in Operation Research we will be concerned with abstract models. These models will usually be mathematical in that algebraic symbolism will be used to mirror the internal relationships in that object (often an organization) being modeled. Our attention will mainly be confined to such mathematical models although the term ‘model’ is sometimes used more widely to include purely descriptive models.

The essential feature of a mathematical model in Operation Research is that it involves a set of mathematical relationship (such as equations, inequalities, logical dependencies, etc) which corresponds to some down-to-earth relationships in a real world (such as technological relationships, physical laws, marketing constraints, etc).

There are a number of motives for building such models:

a) The actual exercise of building a model often reveals relationships, which were not apparent to many people. As a result a greater understanding is achieved of the object being modeled.
b) Having built a model it is usually possibly to analysis it mathematically to help suggest courses of action, which might not otherwise be apparent.
c) Experimentation is possible with a model whereas it is often not possible or desirable to experiment with the object being modeled. It would clearly be politically difficult as well as undesirable to experiment with unconventional economic measures in a country if there was a high probability of disastrous failure.

It is important to realize that a model is really defined by the relationships which it incorporates. These relationships are to large extent, independent of data in the model. A model may be used on many different occasions with differing data, e.g. cost, technological coefficients, resource availability's, etc. We would usually still think of it as the same model even though some coefficients had changed. This distinction is not, of course, total radical changes in the data would usually be thought of as a changing the relationships and therefore the model.

1.4 Mathematical Programming

Mathematical programming is one of the most widely used techniques in Operations Research. In many cases its application has been so successful that its use has passed out of Operations Research departments to become an accepted routine planning tool. It is therefore rather surprising that comparatively little attention has been paid in the literature to the problems of formulating and building mathematical programming models even deciding when such model is applicable.

It should be pointed out immediately that mathematical programming is very different from Computer Programming. Mathematical programming is ‘Programming’ in the sense of ‘planning’. As such it need have nothing to do with Computers. The confusion over the use of world ‘programming’ is widespread and unfortunate. Inevitably mathematical programming becomes involved with computing since practical problems almost always involves large quantities of data and arithmetic which can only reasonably be tackle by the calculating power of a computer. The correct relationship between Computers and Mathematical Programming should, however, be understood.

The common feature which mathematical programming models have is they all involve Optimization. We want to maximize something. The quantity by which we want to maximize or minimize is known as an objective function. Unfortunately the realization that Mathematical Programming is concerned with optimizing an objective often leads people to summarily dismiss Mathematical programming as being inapplicable in practical situation where there is no clear objective or there are a multiplicity of objectives. In this thesis we confine our attention to a special sort of a Mathematical Programming Model, called a linear programming model and its related problems.

1.5 Linear Fractional Programming

The mathematical model of a linear programming problem (in its canonical form) is as follows:

$$\text{Maximize} \quad Z = \mathbf{c}^T\mathbf{x} \tag{1.1}$$

$$\text{Subject to} \quad \mathbf{Ax}(\leq, =, \geq)\mathbf{b} \; : \mathbf{x} \in S \quad \text{and } S \subseteq \boldsymbol{R}^n \tag{1.2}$$

where, $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{c} \in R^n$, $\mathbf{b} \in R^n$, $\mathbf{c}^{\mathrm{T}}$ denotes transpose of $\mathbf{c}$, $\mathbf{x}$ be the solution vector and S is the domain of feasible solution space.

On the other hand a linear fractional programming (LFP) problem is one whose objective function is the ratio of a numerator and a denominator with certain linear constraints. Linear fractional programming (LFP) deals with that class of mathematical programming problems in which the relations among the variables are linear: the constraint relations must be in linear form and the objective function to be optimized must be a ratio of two linear functions. This field of LFP was first developed by Hungarian mathematician

Matros [Matros (1960), Matros(1996)]. The linear fractional programming problem arises when a ratio of linear function has to be maximized over a compact set X and can be written as follows [Bajalinov (2003)]

$$\text{Maximize} \quad F(\mathbf{x}) = \frac{\mathbf{c}^T\mathbf{x}+\gamma}{\mathbf{d}^T\mathbf{x}+\beta} \tag{1.3}$$

$$\text{Subject to} \quad \mathbf{x} \in X = \{\mathbf{x}, \mathbf{A}\mathbf{x} \leq \mathbf{b}\} \tag{1.4}$$

where **c**, **d** and $\mathbf{x} \in R^n$, **A** is an $(m + n) \times n$ matrix, $\mathbf{b} \in R^{m+n}$, γ and β are scalars. We point out that the nonnegative conditions are included in the set of constraints and that $\mathbf{d}^T\mathbf{x} + \beta > 0$ has to be satisfied over the compact set X.

1.6 Goal of the Thesis

In this thesis, our first goal is to implement modified VAM in TP with mixed constraints and justify the effectiveness of the method. Our main target is to develop a new technique for solving any type of LFP problem by converting it into a single linear programming (LP) problem because some cases of denominator and numerator and the negative value of β, other method is failed to solve the linear fractional problems. The final goal of the study is to develop a FORTRAN based computer program for implementation of the new technique and analyzes the solution of the problem.

1.7 Structure of the Thesis

After **Chapter I** in which the introduction of the research works is presented, the literature review is discussed in **Chapter II**. **Chapter III** presents the review of the properties of Operation Research basically LP and TP. The algorithm of VAM and preliminaries of LFP is also discussed in this chapter. A modified VAM method is presented in **Chapter IV.** Some Numerical examples are elaborately presented in the chapter to test the validity the method. A new Technique, for solving LPP especially LFP, is introduced in **Chapter V**. Moreover a FORTRAN based code is developed to implement the new technique for solving LFP. Moreover this chapter displays some numerical illustrations to verify the robustness as well as effectiveness of the proposed technique and the computer program. Finally concluding remarks and brief discussion about the research works are given in **Chapter VI**.

CHAPTER II

Literature Review

2.1 Introduction

Operations research is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control. It's encompasses a wide range of problem-solving techniques and methods applied in the pursuit of improved decision-making and efficiency, such as simulation, mathematical optimization, queuing theory and other stochastic-process models, Markov decision processes, econometric methods, data envelopment analysis, neural networks, expert systems, decision analysis, and the analytic hierarchy process. Nearly all of these techniques involve the construction of mathematical models that attempt to describe the system. Because of the computational and statistical nature of most of these fields, OR also has strong ties to computer science and analytics. Operational researchers faced with a new problem must determine which of these techniques are most appropriate given the nature of the system, the goals for improvement, and constraints on time and computing power.

2.2 Literature Review of Transportation Problems

The first linear programming formulation of a problem that is equivalent to the general linear programming problem was given by Kantorovich (1939), who also proposed a method for solving it. He developed it during World War II as a way to plan expenditures and returns so as to reduce costs to the army and increase losses incurred by the enemy. About the same time as Kantorovich, the Dutch-American economist Koopmans (1949) formulated classical economic problems as linear programs. Kantorovich and Koopmans later shared the Nobel Prize in economics. The Simplex method is the most popular method to solve the general linear programming problem. Dentzing (1947) formulated the general LPP and devised the simplex method for solving these LPP. Khachiyan (1979) devised the ellipsoid method. More recent, Karmarkar (1984a-1984b) developed a new method to solved LPP.

Hitchcock (1941) formulated transportation problems as linear programs and gave a solution very similar to the later Simplex method; Hitchcock had died in 1957 and the Nobel Prize is not awarded posthumously. The transportation problems one of the earliest and most useful applications of linear programming techniques has been the formulation and solution of the Transportation Problem (TP). The mathematical formulation of this problem gives us an LPP which in turn can be solved by the simplex method or by the revised simplex method or dual simplex method but the special structure of the coefficient matrix is such that more efficient methods can be used to solve these problems easily. The basic transportation problem was originally stated by Hitchcock (1941) and later discussed in detail by Kopman (1949).

The More-for-less (MFL) paradox in a TP occurs when it is possible to ship more 'total goods' for less (or equal) 'total cost' while shipping the same amount or more from each origin and to each destination, keeping all shipping costs non-negative. The occurrence of MFL in distribution problems is observed in nature. The mixed constraints TP have extensively been studied by many researchers in the past years, for example [Arora and Khurana (2002), Lev and Intrator (1977), Lev (1972)]. Again Gupta et al. (1992) and Arsham (1992) obtained the more-for-less solution for the TPs with mixed constraints by relaxing the constraints and introducing new slack variables. While yielding the best more-for-less solution, their method is very hard to understand since it introduces more variables and requires solving sets of complex equations. Later Adlakha et al. (2010), Adlakha et al. (2007), Adlakha et al. (2006) and Adlakha & Kowalski (2001) developed heuristic algorithm for solving TP with mixed constraints, which is based on the theory of shadow price. In the succeeding year, Pandian and Natarajan (2010a -2010d) also studied the approaches for solving TP with mixed constraints.

2.3 Literature Review of Linear Fractional Programming

Various optimization problems in engineering and economics involve maximization (or minimization) of the ratio of physical or economical function, for instances cost/time, cost/volume, cost/benefit, profit/cost or other quantities measuring the efficiency of the system. Naturally there is a need for generalizing the simplex technique for linear programming to the ratio of linear functions or to the case of the ratio of quadratic functions in such a situation. All these problems are fragments of a general class of optimization problems, termed in the literature as Fractional programming problems.

Linear fraction problems (i.e. ratio objective that have numerator and denominator) have attracted considerable research and interest, since they are useful in production planning, financial and corporate planning, health care and hospital planning. The field of LFP was developed by Hungarian mathematician Matros (1960). After that several methods are proposed to solve this problem. Charnes and Kooper (1962) have proposed their method depends on transforming this (LFP) to an equivalent linear program. On the other hand, the simplex type algorithm introduced by Swarup (1964) and Swarup et al. (2003). Another method is called updated objective function method derived from Bitran and Novas' (1973) is used to solve this linear fractional program by solving a sequence of linear programs only re-computing the local gradient of the objective function. Singh (1981) in his paper made a useful study about the optimality condition in fractional programming. Tantawy (2007) developed a technique with the dual solution. Hasan and Acharjee (2011) also develop a method for solving LFP by converting it into a single LP. But for the negative value of β their method fails. Tantawy (2008) developed another technique for solving LFP which can be used for sensitivity analysis. Effati and Pakdaman (2012) proposed a method for solving the interval-valued linear fractional programming problem. Pramanik et al. (2011) develop a method for solving multi-objective linear plus linear fractional programming problem based on Taylor Series approximation.

Nowadays linear fractional criterion is frequently encountered in business and economics such as Min [debt-to-equity ratio], Max [return on investment], Min [Risk asset to capital], Max [Actual capital to required capital] etc. So, the importance of linear fractional programming (LFP) problems is evident.

2.4. Limitations of Existing Methods

For many of us, modern-day linear programming (LP) started with the work of Dantzig (1947). However, it must be said that many other scientists have also made seminal contributions to the subject, and some would argue that the origins of LP predate

Dantzig's contribution. After Dantzing other scientist try to develop LP in different way, some of this algorithm are very easy to understand and some are very difficult. Also for requirement we have to add new parameter. In TP, the More-for-less (MFL) paradox occurs when it is possible to ship more 'total goods' for less (or equal) 'total cost' while shipping the same amount or more from each origin and to each destination, keeping all shipping costs non-negative. There are several methods available in the literature dealing with TP & LFP problems. But each method has some limitations as well as limited fields of application. Our intent is to develop new technique or modified technique for solving any type of TP & LFP problem and try to develop a computer technique for those methods. Finally by comparing the proposed methods with existing methods by considering some instance problems we can say that our technique is very effective to solve LFP.

CHAPTER 3

Preliminaries

3.1 Introduction

Transportation Problem (TP) is a special branch of Linear programming which is an important part of Operations Research. Operations Research (OR) is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control. It's encompasses a wide range of problem-solving techniques and methods applied in the pursuit of improved decision-making and efficiency, such as simulation, mathematical optimization, queuing theory and other stochastic-process models, Markov decision processes, econometric methods, data envelopment analysis, neural networks, expert systems, decision analysis, and the analytic hierarchy process. Nearly all of these techniques involve the construction of mathematical models that attempt to describe the system. Because of the computational and statistical nature of most of these fields, OR also has strong bond to computer science and analytics. Operational researchers faced with a new problem must determine which of these techniques are most appropriate given the nature of the system, the goals for improvement, and constraints on time and computing power.

Therefore, below we will present some preliminaries as well as necessary definitions and properties of Linear programming mainly relevant to TP.

3.2 Mathematical Model

Mathematical Programming (MP) problem or Mathematical Program deals with the optimization (maximization or minimization) of a function of several variables subject to a set of constraints (inequalities or equalities) imposed on the values of variables. A general mathematical programming problem can be stated as follows:

$$\text{Maximize } F(\mathbf{x}) \tag{3.1}$$

Subject to $\quad g_i(\mathbf{x}) \leq 0 : \mathbf{x} \in S \subseteq R^m, \quad i = 1,2,\ldots,m, \qquad (3.2)$

where, $\mathbf{x} = (x_1, x_2, x_3, \cdots, x_m)^T$ is the vector of unknown decision variables; $F(\mathbf{x})$ and $g_i(\mathbf{x}),\ i = 1,2,\ldots,m,$, are real valued functions of m real variables $\mathbf{x} = x_1, x_2, x_3, \cdots, x_m$. Here the function F is called objective function and condition (3.2) is referred to it constraints.

Linear programming problem: If both the objective function F($\mathbf{x}$) and the constraint set (3.2) are linear, then MP is called a **Linear Programming (LP)** problem or a linear program. Among the mathematical programs the linear programming problem is a well-known optimization technique. Linear programming is a mathematical technique applied for identifying optimal maximum or minimum values of a problem subject to certain linear constraints. The mathematical model of a linear programming problem (in its canonical form) is as follows:

Maximize $\quad Z = \mathbf{c}^T\mathbf{x} \qquad (3.3)$

Subject to $\quad \mathbf{Ax}\{\leq, =, \geq\}\mathbf{b} \ : \mathbf{x} \in S$ and $S \subseteq R^n \qquad (3.4)$

where, $\mathbf{A}$ is an $m\times n$ matrix, $\mathbf{c} \in \mathbf{R}^n$, $\mathbf{b} \in \mathbf{R}^n$ and $\mathbf{c}^T$ denotes transpose of $\mathbf{c}$.
The above equations can be rewritten in algebraic form as follows:

Maximize (or Minimize) $\quad Z = \sum_{j=1}^{n} c_j x_j \qquad (3.5)$

Subject to $\quad \sum_{j=1}^{n} a_{ij} x_j \ \{\leq, =, \geq\} b \ ; i = 1,2, \ .. \ , m \qquad (3.6)$

It is noted that one and only one of the signs $\{\leq, =, \geq\}$ is hold for each i in the constraint set (3.6).

We have stated here the MP as maximization one. This has been done without any loss of generality, since a minimization problem can always be converted into a maximization problem using the identity

$$min\, F(\mathbf{x}) \ = \ max\, (-F(\mathbf{x}))$$

i.e. the minimization of $F(\mathbf{x})$ is equivalent to the maximization of - $F(\mathbf{x})$.
The set S is normally taken as a connected subset of $\mathbf{R}^n$. Here the set S is taken as the entire space $\mathbf{R}^n$. The set $\mathbf{X} = \{\mathbf{x} \in S \subseteq R^n : g_i(\mathbf{x}) \leq 0; i = 1,2, \ldots, m\}$ is known to as the feasible **region, feasible set or constraint set** of the program MP and any point $\mathbf{x} \in \mathbf{X}$ is a feasible solution or feasible point of the program MP which satisfies all the constraints of

MP. If the constraint set $\mathbf{X}$ is empty $(\mathbf{X} = \varphi)$ then there is no feasible solution; in this case the program MP is **inconsistent**. A feasible point $\mathbf{x}^0 \in X$ is known as a global optimal solution to the program MP if

$$F(\mathbf{x}) \leq F(\mathbf{x}^{\mathbf{0}}) \ \forall \ \mathbf{x} \in \mathbf{X} \tag{3.7}$$

A point $\mathbf{x}^0$ is said to be a strictly global maximum point of $F(\mathbf{x})$ over $\mathbf{X}$ if the strict inequality $(<)$ in (3.5) holds for all $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{x} \neq \mathbf{x}^0$. A point $\mathbf{x}^0 \in \mathbf{X}$ is a local or relative maximum point of $F(\mathbf{x})$ over $\mathbf{X}$ if there exists some $\in > 0$ such that

$$F(\mathbf{x}) \leq F(\mathbf{x}^{\mathbf{0}}) \ \forall \ \mathbf{x} \in \mathbf{X} \cap N_{\in}(\mathbf{x}^{\mathbf{0}}) \tag{3.8}$$

Where $N_{\in}(\mathbf{x}^{\mathbf{0}})$ is the neighborhood of $\mathbf{x}^{\mathbf{0}}$ having radius $\in > 0$. Similarly, global minimum and local minimum can define by changing the sense of inequality.

The MP can be broadly classified into two categories: (a) **unconstrained optimization** problem and (b) **constrained optimization** problem. If the constraint set $\mathbf{X}$ is the whole space $\mathbf{R}^n$, program MP is then known as an unconstrained optimization problem, in this case, we are interested in finding a point of $\mathbf{R}^n$ at which the objective function has an optimum value. On the contrary, if $\mathbf{X}$ is a proper subset of $\mathbf{R}^n$, then MP is known as constrained optimization problem. In this case, we are interested in finding a point on $\mathbf{X}$ at which the objective function has an optimum value.

Non-linear programming problem: If at least one of the objective function and the constraint set are not linear then MP is called a **Non-Linear Programming (NLP)** problem or a non-linear program. Several algorithms have been developed to solve certain NLP.

Standard Linear Programming:

$$\text{Maximize} \quad \boldsymbol{Z} = \mathbf{c}^T\mathbf{x} \tag{3.9}$$

$$\text{Subject to} \quad \mathbf{Ax} = \mathbf{b} \tag{3.10}$$

$$x_i \geq 0 \text{ and } b_i \geq 0 : i = 1,2,\cdots,n \tag{3.11}$$

A LP problem of the above form is known as a LP in **standard form**. The characteristics of this form are:

i) All the constraints are expressed in the form of equations, except for the non-negative restrictions condition (3.10).

ii) The right hand side of each constraint equation is non-negative.

In linear programming, the matrix $\mathbf{A} = (a_{ij})$ is the coefficient matrix of order $m \times n$, the quality constraints, $\mathbf{b} = (b_{1,} b_{2,} b_{3,} \cdots, b_{n,})^T$ is the vector of right hand side constants, the component of **c** are the profit factors, $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in R^n$ is the vector of variables, called the decision variables and condition (3.10) are the non-negativity constraints. The column vectors of the matrix **A** are referred to as activity vectors. Now we recall the following definition for standard linear program.

Feasible Solution:

$\mathbf{x} = (x_1, x_2, \cdots, x_n) \in R^n$ is a feasible solution of the standard linear programming if it satisfies conditions (3.9) and (3.10).

Basic Solution:

A basic solution to the eq. (3.9) is a solution obtained by setting *(n-m)* variables equal to zero and solving for the remaining *m* variables, provided the determinant of the coefficients of these *m* variables are non-zero. The *m* variables are called basic variables.

Basic Feasible Solution:

A basic feasible solution is a basic solution, which also satisfies (3.10) that is, all basic variables are non-negative.

Degenerate Solution:

A basic feasible solution to (3.9) is called degenerate if one or more the basic variables are zero.

Non- degenerate Basic Feasible Solution:

A non-degenerate basic feasible solution is a basic feasible solution with exactly m positive x_i, that is, all basic variables arc positive.

Optimal Solution:

A basic feasible solution is said to be optimal solution if it maximize the objective function (3.9) while satisfying condition (3.10) and (3.11) provided the maximum value exists.

Reduction to Standard Form:

Every general linear program can be reduced to an equivalent standard linear program as explained below.

(i) Conversion of right hand side constraint to non-negative. If a right hand side constant of a constraint is negative, it can be made non-negativity by multiplying both sides of the constraints by –1(if necessary).

(ii) Conversion of inequality constraint to equality.

Slack Variable:

For an inequality constraint of the form

$$\sum_{ij}^{n} a_{ij}x_j \leq b_i : (\mathrm{i} = 1,2,\ldots,m;\ b_{\mathrm{i}} \geq 0\ ) \quad (3.12)$$

by adding a non-negative variable x_{n+1} on the left of the inequality (3.12) we have eq.(3.13).

$$\sum_{ij}^{n} a_{ij}x_j + x_{n+1} = b_i : (\mathrm{i} = 1,2,\ldots,m;\ b_{\mathrm{i}} \geq 0\ ) \quad (3.13)$$

Here the non-negative variable x_{n+1} is called slack variable.

Surplus Variable:

Again for an inequality constraint of the form

$$\sum_{ij}^{n} a_{ij}x_j \geq b_i : (\mathrm{i} = 1,2,\ldots,m;\ b_{\mathrm{i}} \geq 0\ ) \quad (3.14)$$

By subtracting a non-negative variables $\mathrm{x}_{\mathrm{n+1}}$ x_{n+1} on the left of the inequality (3.14) we have eq. (3.15).

$$\sum_{ij}^{n} a_{ij}x_j - x_{n+1} = b_i : (\mathrm{i} = 1,2,\ldots,m;\ b_{\mathrm{i}} \geq 0\ ) \quad (3.15)$$

Here the non-negative variable x_{n+1} is called the surplus variable .

Making All Variables Non-Negative:

All variables in the equivalent linear program can be made non-negative as follows:

(a) If $x_j \leq 0$, then put $\acute{x}_j = -x_j$ clearly $\acute{x}_j \geq 0$.

(b) If x_i is unrestricted in sign (i.e. a free variables), then put $x_i = x_i' - x_i''$ where $x_i' - x_i'' \geq 0$

3.3 Some Important Theorems related to LP

Now we will state below some important theorems related to the LP problems (3.1. – 3.5) without any proof [Taha (1997)].

Theorem 3.1: The constraint set F is convex.

Theorem 3.2: The set optimal solution to the linear program (LP) is convex.

Theorem 3.3: (Fundamental theorem) Let the constraint sets F be non-empty, closed and bounded, then, an optimal solution to the LP exists and it is attained at a vertex of F.

Theorem 3.4: If standard linear program with the constraints $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq 0$, where **A** is an $m \times n$ matrix of rank m, has a feasible solution, then it also has a basic feasible solution.

Theorem 3.5: Let **F** be a convex polyhedron consisting of all vectors $\mathbf{x} \in \mathbf{R}^n$ satisfying the system $\mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq 0$ where **A** is an $m \times n$ matrix of rank m. Then, **x** is an extreme point of F if and only if **x** is a basic feasible solution to the system.

The above theorem ensures that every basic feasible solution to a (LP) is an extreme point of the convex set of feasible solutions to the problem and that every extreme point is a basic feasible solution corresponding to one and only one extreme point of the convex set of feasible solution and vice versa.

3.4 Formulation of General Transportation Problem

One of the most important and successful application of quantities analysis to solving business problems has been in the physical distribution of products, commonly referred to as Transportation Problems (TPs). Basically, the purpose is to minimize the cost of shipping goods from one location to another so that the needs of each arrival area are met and every shipping location operates within its capacity. The TP finds application in industry, planning, communication network, scheduling, transportation and allotment etc. A homogeneous product, available at a finite number of origins, is to be transported to a finite number of destinations. The total amount available at each of these origins is known and also the total quantity required at each of the destination is known. The unit transportation cost from each origin to each destination is given. The question here is to determine the amount of the products to be transported from these origins to be destinations so, as to minimize the total transportation cost. A schematic view of Transportation problem is shown Figure 3.1. It aims to find the best way to fulfill the demand of *n* demand points using the capacities of *m* supply points. If total supply equals total demand then the problem is said to be a balanced transportation problem otherwise the problem be imbalanced [Taha (1997)

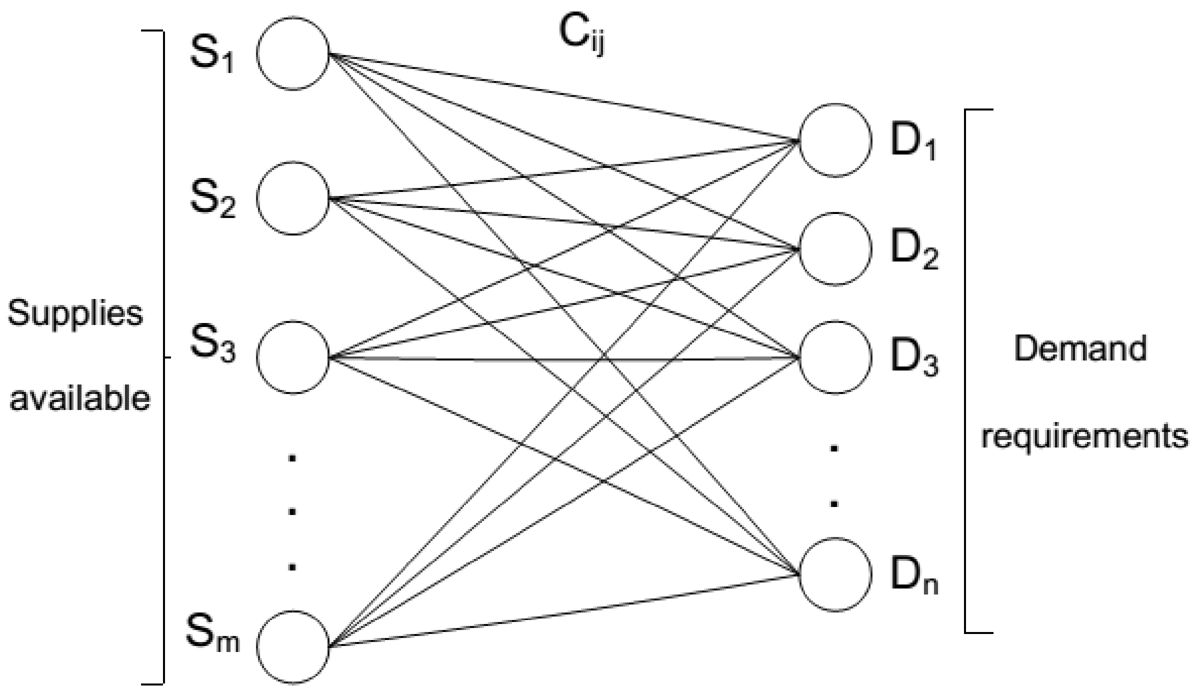

Figure 3.1 Network model of the transportation problem

For formulation of mathematical model of TP, let m be the number or origin and n be the number of destinations and the cost of transporting one unit of the commodity from origin i to to destination j is c_{ij}. Let a_i be the quantity of the commodity available at origin i and b_j be the quantity required at destination j. Therefore obviously thus $a_i \geq 0$ for i and $b_j \geq 0$ for each j. Also let x_{ij} be the quantity transported from origin i to destination j. Then the general formulation of the transportation problem is as follows:

Minimize $\mathbf{Z} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \, x_{ij}$ (Total transportation cost) (3.16)

Subject to $\sum_{j=1}^{n} x_{ij} = a_i \, , i = 1,2, \cdots, m$ (supplies at origin) (3.17)

$\sum_{i=1}^{n} x_{ij} = b_i \, , i = 1,2, \cdots, n$ (demands at destination) (3.18)

$x_{ij} \geq 0 \; \forall \; i, j$ (quantities) (3.19)

Equation (3.16) indicates that the above LP has mn variable, Eqs. (3.17) have n constraint and Eqs. (3.18) have m constraints. That is the above LP has $m+n$ constraint excluding the non-negativity constraints (3.19). With a special structure the above TP can be represented as matrix notation:

$$\text{Minimize} \quad \mathbf{Z} = \mathbf{Cx} \tag{3.20}$$

$$\text{Subject to} \quad \mathbf{Ax} = \mathbf{b} \tag{3.21}$$

$$\mathbf{x} \geq 0 \tag{3.22}$$

Where **C** be mn order row matrix, **x** be mn order column vector. Moreover **b** be $m+n$ order column matrix and **A** be a matrix of order $(m+n) \times mn$.

To see the special structure of the coefficient matrix **C** and **A** here we take a special case for $m = 2$ and $n = 3$. Then vector $\mathbf{x} = (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})^T$ (i.e mn =6 variables) The vector $\mathbf{b} = (a_1, a_2, b_1, b_2, b_3)^T$ (i.e. 2+3 components) and $\mathbf{C} = (c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23})$ (i.e. 6 components). The exact form of **A** is

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

which is of order 5×6. Note that, each column of the matrix **A** contains 1 exactly in two places as $m = 2$.

3.5 Feasible Canonical Form

Consider the constraints (3.21) i.e. $\mathbf{Ax} = \mathbf{b}$, are consistent and rank of $\mathbf{A} = m(\leq n)$. Let **B** be any non-singular $m \times m$ sub-matrix made up of the columns of **A** and **R** is the remainder portion of **A**. Further, suppose that $\mathbf{X}_B$ is the vector of variables associated with columns of **B**. Then eq. (3.20) can be written as

$$[\mathbf{B}, \mathbf{R}] \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_{NB} \end{bmatrix} = \mathbf{b}$$

$$\text{or} \quad \mathbf{Bx}_B + \mathbf{Rx}_{NB} = \mathbf{b}$$

$$\text{or} \quad \mathbf{x}_\mathrm{B} = \boldsymbol{B}^{-1}\mathbf{b} - \boldsymbol{B}^{-1}\mathrm{R}\mathbf{x}_{NB} \tag{3.23}$$

or $\mathbf{x}_B - \boldsymbol{B}^{-1}\mathrm{R}\mathbf{x}_{NB} = \boldsymbol{B}^{-1}\mathbf{b}$ (3.25)

where the *(n-m)* variables $\mathbf{x}_{NB}$ can be assigned arbitrary values. The form (3.24) of constraint is called the canonical form in the variables $\mathbf{x}_B$. The particular solution of (3.23*)* given by

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}, \qquad \mathbf{x}_{NB} = \mathbf{0} \tag{3.26}$$

Which is called the basic solution to the system $\mathbf{Ax} = \mathbf{b}$ with respect to the basic matrix $\mathbf{B}$. The variables $\mathbf{x}_{NB}$ are known as the **non-basic variables** and the variables $\mathbf{x}_B$ are said to be the **basic variables**. The basic solution given by (3.26) is feasible if $\mathbf{x}_B \geq 0$.

Transportation problems can be solved by using general simplex based integer programming methods, however it involves time-consuming computations. There are specialized algorithms for transportation problem that are much more efficient than the simplex algorithm. The basic steps to solve transportation problem are:

Step 1. *Find the initial feasible solution,*

Step 2. *Find optimal solution using the initial feasible solution.*

3.6 Relative Profit Factors

Suppose that there exists a feasible solution to the constraint (3.21). The coefficients of the variables in the objective function **Z** (correspond to basic initial feasible solution) are called relative profit factors. In order to find relative profit factors corresponding to basis matrix **B**, we partition the profit vector **c** as $\mathbf{c}^T = \left(\mathbf{c}_B{}^T, \mathbf{c}_{NB}{}^T\right)$, where $\mathbf{c}_B$ and $\mathbf{c}_{NB}$ are the profit vectors corresponding to the variables $\mathbf{x}_B$ and $\mathbf{x}_{NB}$ respctively.

Then the objective function becomes

$$\mathbf{Z} = \mathbf{c}^T\,\mathrm{B}$$

i.e. $$\mathbf{Z} = \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_{NB}^T\mathbf{x}_{NB} \tag{3.27}$$

or $$\mathbf{Z} = \mathbf{c}_B^T\boldsymbol{B}^{-1}\mathbf{b} - \mathbf{c}_B^T\boldsymbol{B}^{-1}\mathrm{R}\mathbf{x}_{NB} + \mathbf{c}_{NB}^T\mathbf{x}_{NB} \quad \text{(from eq. 3.23)}$$

$$= \overline{\boldsymbol{Z}} - (\mathbf{c}_B^T\boldsymbol{B}^{-1}\mathrm{R} - \mathbf{c}_{NB}^T)\mathbf{x}_{NB}$$

$$= \overline{\boldsymbol{Z}} - \bar{\mathbf{c}}_B\boldsymbol{B}^T\mathbf{x}_B - \bar{c}_{NB}{}^T\mathbf{x}_{NB}$$

$$= \overline{\boldsymbol{Z}} - \bar{\mathbf{c}}^T\mathbf{x}$$

where, $\bar{\mathbf{c}} = (\overline{\mathbf{c}_B}, \overline{\mathbf{c}_{NB}})^T = (\bar{\mathbf{c}}_{\mathrm{B}}, \bar{\mathbf{c}}_{\mathrm{NB}})^{\mathrm{T}}$

$\bar{\mathbf{c}}_{\mathrm{B}} = 0.$

$\bar{\mathbf{c}}_{NB}{}^{T} = \mathbf{c}_B{}^{T}\,\mathbf{B}^{-1}\,\mathbf{R} - \mathbf{c}_{NB}{}^{T}$

$\bar{\mathbf{Z}} = \mathbf{c}_B^T \boldsymbol{B}^{-1}\mathbf{b}\ \bar{\mathbf{z}}$

Here $\bar{\mathbf{c}}$ is the vector of relative profit factors corresponding to, the basis matrix **B** and $\bar{z}$ is the value of the objective function at the basic solution given by (3.27). It is bbserved that the components of $\bar{\mathbf{c}}$ corresponding to the basic variables are zero, which ought to be as is evident from the definition of $\bar{\mathbf{c}}$.

3.7 Formulation of TP for finding Initial Basic Feasible Solution (IBFS)

A general Transportation Problem (TP), which is off course a special type of LPP, can be formulated as follows:

Table 3.1 Pictorial view of a Transportation Problem (TP)

Origin \ Destinations	D_1	D_2		D_j	D_n	Availability
O_1	c_{11}	c_{12}		c_{ij}	c_{in}	a_1
O_2	c_{21}	c_{22}		c_{2j}	c_{2n}	a_2
O_i	c_{i1}	c_{i2}		c_{ij}	c_{in}	a_i
O_m	c_{m1}	c_{m2}		c_{mj}	c_{mn}	a_m
Requirement	b_1	b_2		b_j	b_n	

In the table O_i denotes *i*th origin (production unit) where i=1, 2, . . . , m) and D_j indicates *j*th destination (storehouse/show room) where j=1, 2, . . ., n . In the last column a_i indicates that origin O_i has ability to product a_i units. On the other hand in the last row *bi* indicates that the destination D_j has ability to store b_i units. This table has mn cells. In each cell C_{ij}

means cost to carry unit item from origin O_i to destination D_j. In general it is considered that the TP is balanced and there exist a finite minimal solution in each such balance TP. This table has mn cells.

3.8 Definition of Loop

In a transportation table, an ordered set of four or more cells is said to form a loop if

(1) Any two adjacent cells in the ordered set lie either lie in the same column.
(2) Any three or more adjacent cells in the ordered set do not lie in the same row or the same column.

The first cell of the set is considered to follow the last in the set. It may be noted here that the vectors associated with the cells of a loop are linearly dependent.

3.9 Some Important Theorems Related to TP

Theorem 3.6: A necessary and sufficient condition for the existence of a feasible solution to the transportation problem is balanced. i.e.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

Theorem 3.7: A finite minimum feasible solution always exists.

Now we will discuss some methods to find one initial basic feasible solution for a balanced transportation problem:

(1) The North-West Corner rule
(2) The Matrix-Minima method
(3) The Row-Minima method
(4) The Column-Minima method
(5) The Vogel's approximation method.

Note that if a problem is not balanced we can first balance it by taking fictitious origin or destination as required and then use the methods.

3.10 Vogel's Approximation Method:

Vogel's Approximation Method (VAM) method is a well-known method for solving linear Transportation problems where all constraints are of equal (=) sign i.e. all constraints are equation rather than mixed constraints. The main steps of the method are given briefly below:

Note that this method also takes costs into account in allocation. Five steps are involved in applying this heuristic:

Step 1: Determine the difference between the lowest two cells in all rows and columns, including dummies.

Step 2: Identify the row or column with the largest difference. Ties may be broken arbitrarily.

Step 3: Allocate as much as possible to the lowest-cost cell in the row or column with the highest difference. If two or more differences are equal, allocate as much as possible to the lowest-cost cell in these rows or columns.

Step 4: Stop the process if all row and column requirements are met. If not, go to the next step.

Step 5: Recalculate the differences between the two lowest cells remaining in all rows and columns. Any row and column with zero supply or demand should not be used in calculating further differences. Then go to Step 2.

The Vogel's approximation method (VAM) usually produces an optimal or near- optimal starting solution. One study found that VAM yields an optimum solution in 80 percent of the sample problems tested.

Here we introduced a problem of transportation problem. We will try to solve the problem by VAM method for finding an optimal solution.

3.10.1 Example

Table-3.1

	A	*B*	*C*	*D*	*Supply*
1	6	4	1	5	14
2	8	9	2	7	18
3	4	3	6	2	7
Demand	**6**	10	15	8	

Solution:

Table-3.2

	A	*B*	*C*	*D*	*Supply*
1	6	4	1	5	14 [3]
2	8	9	2(15)	7	18/3; [5] ←
3	4	3	6	2	7 [1]
Demand	6 [2]	10 [1]	15 [1]	8 [3]	**39/39**

Table-3.3

	A	*B*	*C*	*D*	*Supply*
1	6	4	1	5	14 [1]
2	8	9	2(15)	7	3 [1]
3	4	3	6	2(7)	7 [1]
Demand	6 [2]	10 [1]		8/1 [3] ↑	

Table-3.4

	A	*B*	*C*	*D*	*Supply*
1	6(4)	4(10)	1	5	14/4; [1]
2	8(2)	9	2(15)	7(1)	3 [1]
3	4	3	6	2(7)	
Demand	6/2 [2]	10 [5] ↑		8/1 [2]	

Therefore,

The total transportation cost is $= 6 \times 4 + 4 \times 10 + 8 \times 2 + 2 \times 15 + 7 \times 1 + 2 \times 7$

$$= 131$$

3.11 Linear Fractional Program (LFP)

Recently various optimization problems, involving the optimization of the ratio of functions, *e.g.* time/cost, volume/cost, profit/cost, loss/cost or other quantities measuring the efficiency of the system have been the subject of wide interest in non-linear programming problem. Such problems are known as LFP.

If the objective function of a mathematical programming problem is the ratio of two linear functions and the constraints are linear, it is called a linear fractional programming problem (LFPP). Likewise LP, a standard LFPP can be expressed as follows:

$$\textit{Maximize} \;\; F(x) = \frac{\mathbf{c}^T\mathbf{x} + \alpha}{\mathbf{d}^T\mathbf{x} + \beta} \tag{3.28}$$

$$\textit{Subject to} \quad \mathbf{x} \in X = \left\{ \mathbf{x} \in \mathbf{R}^n : \mathbf{Ax} = \mathbf{b},\; \mathbf{x} \geq 0 \right\} \tag{3.29}$$

Where $\mathbf{x}, \mathbf{c}, \mathbf{d} \in \mathbf{R}^n$; $\mathbf{b} \in \mathbf{R}^m$; $\alpha, \beta \in \mathbf{R}$; $\mathbf{A}$ is an $m \times n$ matrix and superscript T denotes transpose.

For simplicity of notation, throughout this chapter and hence forth, we can omit the transpose sign T over vectors. In an inner product of two vectors, one can assume that the left hand side vectors are a row vector and right side vector be a column vector.

Fractional objectives occur in areas of marine transportation. Instead of maximizing profit from a given unit voyage a more relevant measure is profit divided by the duration of the unit voyage. In water resources, we may wish to minimize water temperature elevation in a river due to the cooling of power generation plants in the basin. The objective would then be to minimize the B.T.U.'s (British Thermal Units) to be dissipated divided by the volume of the flow. In health care we may have cost-to-bed, nurse-to-doctor and doctor-to-patient ratio. In University planning, we may have student-teacher ratio, tenured-to-untenured faculty ratio and so forth.

Linear fractional criteria are frequently encountered in finance as illustrated by the following situation.

Corporate Planning:

Min [debt –to-equity ratio]

Max [return on investment]

Max [output per employee]

Min [actual cost –to-standard cost]

Bank Balance Sheet Management:

Min [risk assets –to-capital]

Max [actual capital –to-required capital]

Min [foreign loans –to-total loans]

Min [residential mortgages –to-total mortgages]

Linear fractional objective also occur in other areas of science, engineering and social sciences.

CHAPTER IV

Transportation Problem with Mixed Constraints

4.1 Introduction:

One of the most important and successful applications of quantities analysis to solving business problems has been in the physical distribution of products, commonly referred to as transportation problems (TP). Basically, the purpose is to minimize the cost of shipping goods from one location to another so that the needs of each arrival area are met and every shipping location operates within its capacity. The TP finds application in industry, planning, communication network, scheduling, transportation and allotment etc. In real life, however, most of the problems have mixed constraints but we used TPs for optimal solutions with equality constraints. The TPs with mixed constraints are not addressed in the literature because of the rigorous required to solve these problems optimally. A literature search reveals no systematic method available to find an optimal solution for TPs with mixed constraints.

4.2 Formulation of Transportation Problem with Mixed Constraints

Table 4.1: The schematic view of the transportation problem with mixed

	1	2	3	……	n	$Supply$
1	c_{11}	c_{12}	c_{13}	……	c_{1n}	$\leq/=/\geq\ a_1$
2	c_{21}	c_{22}	c_{23}	……	c_{2n}	$\leq/=/\geq\ a_2$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
m	c_{m1}	c_{m2}	c_{m3}	……	c_{mn}	$\leq/=/\geq\ a_m$
$Demand$	$\leq/=/\geq b_1$	$\leq/=/\geq\ b_2$	$\leq/=/\geq\ b_3$	……	$\leq/=/\geq\ b_n$	

Let m be the number of sources and n be the number of destinations. Suppose that the cost of transporting one unit of the commodity from source i to the destination j is c_{ij}. Let a_i be the quantity of the commodity available at source i and b_j be the quantity required at destination j. Thus $a_j \geq 0$ and $b_j \geq 0$ for all i and j. Then the general formulation of the transportation problem with mixed constraints, as described by Pandian and Natarajan (2010), is given in the Table 4.1.

If x_{ij} is the quantity transported from source i to destination j then the transportation problem is written with the help of Adlakha et al. (2010) and Pandian and Natarajan (2010) as

$$Minimize\ Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$Subject\ to\ \sum_{j=1}^{n} x_{ij} \leq / = / \geq a_i, i = 1, 2, \ldots\ldots\ldots\ldots m$$

$$\sum_{i=1}^{m} x_{ij} \leq / = / \geq b_j,\ j = 1, 2, \ldots\ldots\ldots\ldots n$$

$$x_{ij} \geq 0$$

The above formulation represents a Linear Programming Problem (LPP) with $m \times n$ variables and $m + n$ constraints. If the LPP is small, we can solve the problem by using any simplex method, but in practical life LPP can be very large, which is difficult to solve by analytically. This type of problem can be solved very easily by using computer programming.

Remark 1: If all constraints are of equal (=) sign, then the problem becomes the transportation problem with equality constraints.

4.3 Modified Vogel Approximation Method

Here we introduce a modified algorithm based on VAM method which is first developed by Mondal et al. (2012) for finding an optimal solution to a transportation problem with mixed constraints. Because we cannot solved mixed constraints problem by using VAM method. Below we introduce the Modified Vogel Approximation Method (MVAM) method for solving TPs with mixed constraints in More-For-Less (MEL) paradoxical situation. The main steps of the proposed algorithm are given bellow:

Step 1: *For each row and column of the transportation table, find the difference between the two lowest unit shipping costs.* These numbers represent the difference between the distribution cost on the best route in the row or column and the second best route in the row or column.

Step 2: *Identify the row or column with the greatest opportunity cost or difference.*

Step 3: *Assign possible units to the lowest cost square in the row or column selected which satisfy the inequalities.*

We can follow the following chart to assign the supply and demand unit.

Chart	
$\boldsymbol{Supply(a_i)}$, $\boldsymbol{Demand(b_i)}$	Transport Unit
$(\boldsymbol{a_i}, \boldsymbol{b_i})$: $(=,=), (=,\leq), (\leq,=)$	$min\left(a_i, b_j\right)$
$(\boldsymbol{a_i}, \boldsymbol{b_i})$: $(\geq,\geq)$	$max\left(a_i, b_j\right)$
$(\boldsymbol{a_i}, \boldsymbol{b_i})$: $(=,\geq), (\leq,\geq)$	a_i
$(\boldsymbol{a_i}, \boldsymbol{b_i})$: $(\geq,=), (\geq,\leq)$	b_j
$(\boldsymbol{a_i}, \boldsymbol{b_i})$: $(\leq,\leq)$	0

Step 4: *Eliminate any row or column that has just been completely satisfied by the assignment just made.*

Step 5: *Recomputed the cost difference for the transportation table.*

Step 6: *Return to **step 2** and repeat the steps until an initial feasible solution has been obtained.*

Here we introduced some instances of transportation problems with mixed constraints. We will try to solve the problem by the modified VAM method to examine the efficiency of the method.

4.4 Numerical Examples

Example 4.1:

Table-4.2

	A	B	C	$Supply$
1	9	12	11	$= 75$
2	13	10	8	≥ 90
3	15	16	9	≤ 80
$Demand$	$= 95$	≤ 70	≥ 90	

We have solved the above problem by using MVAM method which is given below in Tables 4.3-4.5:

Table 4.3

	A	B	C	$Supply$
1	9	12	11	$= 75[2]$
2	13	10	8	$\geq 90[2]$
3	**15**	**16**	**9 (80)**	$\geq$ **80[6]** $\leftarrow$
$Demand$	$= 95$ [4]	≤ 70 [2]	$\leq$ **90/10** [**1**]	

In Table 4.3, first we calculate the difference between the two lowest unit shipping costs in each row and column and find that row 3 has the largest difference as shown in the table by $\leftarrow$ mark. We find that 9 is the lowest cost square in row 3. Now we try to assign possible in C_{33} . Since (demand, supply): $(\geq, \leq)$, so according to the chart the assignment unit is 80. Now we recomputed the cost difference in the same way and proceed in the next step until we get the feasible solution.

Table 4.4

	A	B	C	$Supply$
1	**9 (75)**	**12**	**11**	= **75[2]**
2	13	10	8	$\geq 90[2]$
3	**15**	**16**	**9 (80)**	$\leq$ **80**
$Demand$	$= 95/20$ [**4**] $\uparrow$	≤ 70 [2]	$\leq 90/10$ [3]	

In Table 4.4, we find that 9 is the lowest cost square in column 1. Now we assign possible units in C_{11} . We see that the demand and supply are both $=$ sign, so using the chart we get our assignment unit as 75.

Combining all of the work in table 4.5 and have

Table 4.5

	A	B	C	$Supply$
1	$\mathbf{9}(\mathbf{75})$	12	11	$= \mathbf{75}[2][2]$
2	$\mathbf{13}(\mathbf{20})$	$\mathbf{10}(\mathbf{60})$	$\mathbf{8}(\mathbf{10})$	$\geq 80[2][2]$
3	15	16	$\mathbf{9}(\mathbf{80})$	$\leq 80[\mathbf{6}]$
$Demand$	$= 95/20$ [4] [**4**]	≤ 70 [2] [2]	$\leq 90/10$ [1] [3]	

Therefore, the solution of the given problem is $x_{11} = 75, x_{21} = 20, x_{22} = 60, x_{23} = 10$, $x_{33} = 80$ and all other $x_{ij} = 0$. Total supply = 245 and the total transportation cost as= 2335.

Example 4.2: The Best way Group owns factories in five towns that distribute to five warehouses. Factory availabilities, projected demands and unit shipping costs are summarized in the table below:

Table-4.6

	A	B	C	D	E	$Factory\ Availability$
1	4	3	1	2	6	≤ 75
2	5	2	3	4	5	≥ 60
3	3	5	6	3	2	≥ 45
4	2	4	4	5	3	$= 50$
5	3	4	6	5	2	≥ 30
$Store\ Demand$	≥ 40	≤ 70	≥ 65	$= 30$	≥ 45	

Now we solve the above problem by MVAM method by hand calculation. Hand calculation solution is shown in Tables 2.7-2.10.

Table-4.7						
	A	***B***	***C***	***D***	***E***	***Factory Availability***
1	**4**	**3**	1(75)	**2**	**6**	$\leq$ **75[1]**
2	5	2	**3**	4	5	$\geq$ 60[1]
3	3	5	**6**	3	2	$\geq$ 45[1]
4	2	4	**4**	5	3	= 50[1]
5	3	4	**6**	5	2	$\geq$ 30[1]
Store Demand	$\geq$ 40 [1]	$\leq$ 70 [1]	$\geq$ **65** [**2**] ↑	= 30 [1]	$\geq$ 45 [0]	

Table-4.8						
	A	***B***	***C***	***D***	***E***	***Factory Availability***
1	**4**	**3**	1(75)	**2**	**6**	$\leq$ **75**
2	**5**	2(70)	**3**	**4**	**5**	$\geq$ **60[1]**
3	3	**5**	**6**	3	2	$\geq$ 45[1]
4	2	**4**	**4**	5	3	= 50[1]
5	3	**4**	**6**	5	2	$\geq$ 30[1]
Store Demand	$\geq$ 40 [1]	$\leq$ **70** [**2**]	$\geq$ **65**	= 30 [1]	$\geq$ 45 [0]	

Table-4.9						
	A	***B***	***C***	***D***	***E***	***Factory Availability***
1	**4**	**3**	1(75)	**2**	**6**	$\leq$ **75**
2	**5**	2(70)	**3**	**4**	**5**	$\geq$ **60**
3	3	**5**	**6**	3(30)	2	$\geq$ 45/15[1]
4	2	**4**	**4**	**5**	3	= 50[1]
5	3	**4**	**6**	**5**	2	$\geq$ 30[1]
Store Demand	$\geq$ 40 [1]	$\leq$ **70**	$\geq$ **65**	= **30** [**2**]	$\geq$ 45 [0]	

Table-4.10						
	A	B	C	D	E	$Factory\ Availability$
1	**4**	**3**	$1(75)$	**2**	**6**	$\leq \mathbf{75}$
2	**5**	$2(70)$	**3**	**4**	**5**	$\geq \mathbf{60}$
3	3	**5**	**6**	3(30)	$2(15)$	$\geq 45/15[1]$
4	$2(50)$	**4**	**4**	**5**	3	$= 50[1]$
5	3	**4**	**6**	**5**	$2(45)$	$\geq 30[1]$
$Store\ Demand$	≥ 40 [1]	$\leq \mathbf{70}$	$\geq \mathbf{65}$	$= \mathbf{30}$	≥ 45 [0]	

The solution of the given transportation problem with mixed constraints is:

$x_{13} = 75, x_{22} = 70, x_{34} = 30, x_{35} = 15, x_{41} = 50, x_{55} = 45$, and all others $x_{ij} = 0$

The minimum transportation cost as 525 unit. It is observed that the modified VAM [Mondol 2012] approach efficiently find the IBFS in the case of mixed constraint TP problems.

CHAPTER V

A New Technique for Solving Linear Fractional Programming Problem

5.1 Introduction

Though there are much researches have been carried out in the field of linear programming, but there are few research works are available in the field of linear fractional problem. Moreover existing methods have many draw backs. In this chapter, we will discuss a new method for solving linear fractional programming (LFP) problems to overcome some drawbacks. We will also develop computer program for this method.

5.2 Mathematical form of LP and LFP

The mathematical form of an LP is as follows:

Maximize (Minimize) $F(\mathbf{x}) = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots\ldots\ldots\ldots + c_nx_n$

$$F(\mathbf{x}) = c_ix_i \quad i = 1,2,3,\ldots\ldots\ldots\ldots, n \tag{5.1}$$

Subject to
$$A_{ji}x_i = b_j \tag{5.2}$$

$$x_i \geq 0 \tag{5.3}$$

$$b_j \geq 0 \tag{5.4}$$

The mathematical form of an LFP is as follows:

Maximize $$F(\mathbf{x}) = \frac{c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots\ldots\ldots\ldots + c_nx_n + \alpha}{d_1x_1 + d_2x_2 + d_3x_3 + \ldots\ldots\ldots\ldots\ldots + d_nx_n + \beta} = \frac{c_ix_i + \alpha}{d_ix_i + \beta}$$

Subject to
$$A_{ji}x_i = b_j$$

$$x_i \geq 0$$

$$b_j \geq 0$$

Where, $i = 1, 2, 3\ldots\ldots n,\ j = 1, 2, 3\ldots\ldots m$, $\mathbf{A}$ is a $\mathrm{m}\times\mathrm{n}$ matrix, $\mathrm{b}_i \in \mathrm{R}^{\mathrm{m}}$; $\mathrm{x}_i, \mathrm{c}_i, d_i \in \mathrm{R}^n$; $\alpha, \beta \in R$

5.3 Conversion from LFP to LP

We can convert the LFP into a LP in the following way (where $\beta \neq 0$, $c \neq 0$ and $d \neq 0$)

Case I : $\beta > 0$

Conversion of objective function

$$Maximize \;\; F(\mathbf{x}) = \frac{c_i x_i + \alpha}{d_i x_i + \beta}$$

$$= \left(\frac{c_i\beta - d_i\alpha}{\beta}\right)\left(\frac{x_i}{d_i x_i + \beta}\right) + \frac{\alpha}{\beta}$$

$$Let \;\; c'_i = \frac{c_i\beta - d_i\alpha}{\beta}; y_i = \frac{x_i}{d_i x_i + \beta}; \alpha' = \frac{\alpha}{\beta}$$

Now we get

$$\therefore G(\mathbf{y}) = c'_i\, y_i + \alpha'$$

Conversion of the constraint

Subject to $A_{ji} x_i \leq b_j$

$$\Rightarrow A_{ji} \times \frac{y_i\beta}{1 - y_i d_i} \leq b_j \;\; \text{;where } x_i = \frac{y_i\beta}{1 - y_i d_i}$$

$$\Rightarrow \left(A_{ji}\beta + b_j d_i\right) y_i \leq b_j$$

$$Let \;\; A'_{ji} = A_{ji}\beta + b_j d_i$$

$$\therefore A'_{ji} y_i \leq b_j$$

So the new LP is:

Maximize $G(\mathbf{y}) = c'_i\, y_i + \alpha'$

Subject to $A'_{ji} y_i \leq b_j$, $y_i \geq 0$

Calculation of unknown variable:

We will get the value of y_i after solving the converted LP.

Calculate $x_i = \frac{y_i\beta}{1 - y_i d_i}$

Putting the values of x_i in the objective function $F(\mathbf{x}) = \frac{c_i x_i + \alpha}{d_i x_i + \beta}$.

Then, we will get the optimal solution.

Case II: $\beta < 0$ **,** $\alpha \geq 0$

Conversion of objective function

The objective function, $F(\mathbf{x}) = \frac{c_i x_i + \alpha}{d_i x_i - \beta}$

$$\Rightarrow \frac{F(\mathbf{x}) + 1}{F(\mathbf{x}) - 1} = \frac{\left(c_i + d_i\right) x_i + \alpha - \beta}{\left(c_i - d_i\right) x_i + \alpha + \beta}$$

Now we get converted variable which are

$$c_i = c_i + d_i \;,\; \mathrm{d}_i = c_i - d_i$$

$$\alpha = \alpha - \beta \text{ and } \beta = \alpha + \beta$$

$$\text{and } F'(\mathbf{x}) = \frac{F(\mathbf{x}) + 1}{F(\mathbf{x}) - 1}$$

Now we get

$$F'(\mathbf{x}) = \frac{c_i x_i + \alpha}{d_i x_i - \beta}$$

And Subject to $A_{ji} x_i \leq b_j$

Which is looks like the case-I. Now we can convert it into LP by the case-I procedure. Then we get the LP problem as follow:

Maximize $G(\mathbf{y}) = c'_i\, y_i + \alpha'$

Subject to $A'_{ji} y_i \leq b_j$, $y_i \geq 0$

Where,

$$c'_i = \frac{c_i\beta - d_i\alpha}{\beta}; y_i = \frac{x_i}{d_i x_i + \beta}; \alpha' = \frac{\alpha}{\beta}$$

$$\text{and } \; A'_{ji} = A_{ji}\beta + b_j d_i$$

Calculation of unknown variable:
We will get the value of y_i after solving the converted LP.

Calculate $x_i = \dfrac{y_i \beta}{1 - y_i d_i}$

Putting the values of x_i in the objective function $F(\mathbf{x}) = \dfrac{c_i x_i + \alpha}{d_i x_i - \beta}$.

Then, we will get the optimal solution.

<table>
<tr><td colspan="2">Case III: $\beta < 0$, $\alpha < 0$</td></tr>
<tr><td>Conversion of objective function
The objective function,
$$F(\mathbf{x}) = \frac{c_i x_i - \alpha}{d_i x_i - \beta} = \frac{-c_i x_i + \alpha}{-d_i x_i + \beta}$$
And Subject to $A_{ji} x_i \le b_j$</td><td>Same as above procedure, we have
Then we get the LP problem as follow:
Maximize $G(\mathbf{y}) = c'_i\, y_i + \alpha'$
Subject to $\mathrm{A}'_{ji} y_i \le b_j$, $y_i \ge 0$
Where,
$c'_i = \dfrac{c_i \beta - d_i \alpha}{\beta}; y_i = \dfrac{x_i}{d_i x_i + \beta}; \alpha' = \dfrac{\alpha}{\beta}$
and $\mathrm{A}'_{ji} = A_{ji}\beta - b_j d_i$</td></tr>
<tr><td colspan="2">Calculation of unknown variable:
We will get the value of y_i after solving the converted LP.
Calculate $x_i = \dfrac{y_i \beta}{1 + y_i d_i}$
Putting the values of x_i in the objective function $F(\mathbf{x}) = \dfrac{c_i x_i - \alpha}{d_i x_i - \beta}$.
Then, we will get the optimal solution.</td></tr>
</table>

5.4 Proposed Algorithm

Here we have presented an algorithm to implement the program to solve FLP.

Step 1: READ numerator (c_i, α) & Denominator (d_i, β) and all constraints (A_{ji}, b_j)

Step 2: IF β > 0, Then

Calculate i. $c'_i = \dfrac{c_i \beta - d_i \alpha}{\beta}; \alpha' = \dfrac{\alpha}{\beta}$; $\mathrm{A}'_{ji} = A_{ji}\beta + b_j d_i$

ii. $G(\mathbf{y}) = c'_i\, y_i + \alpha'$

iii. $\mathrm{A}'_{ji} y_i \le b_j$, If $y_i \ge 0$ **Go to Step 5**

Step 3: ELSE IF β < 0 & α > 0 then

Calculate i. $c_i = c_i + d_i$; $\mathrm{d}_i = c_i - d_i; \alpha = \alpha + \beta\ ; \beta = \alpha - \beta$.

ii. $c'_i = \dfrac{c_i\beta - d_i\alpha}{\beta}; \alpha' = \dfrac{\alpha}{\beta}$ and $\mathrm{A}'_{ji} = A_{ji}\beta + b_j d_i$

iii. $G'(\mathbf{y}) = c'_i\, y_i + \alpha'$

iv. $\mathrm{A}'_{ji} y_i \le b_j$, If $y_i \ge 0$ **Go to Step 5**

Step 4: ELSE IF β < 0 & α < 0 then

Calculate i. $c_i = -c_i$; $\mathrm{d}_i = -d_i; \alpha = -\alpha\ ; \beta = -\beta$

ii. $c'_i = \dfrac{c_i\beta - d_i\alpha}{\beta}; \alpha' = \dfrac{\alpha}{\beta}$ and $\mathrm{A}'_{ji} = A_{ji}\beta - b_j d_i$

iii. $G(\mathbf{y}) = c'_i\, y_i + \alpha'$

iv. $\mathrm{A}_{ji} y_i \le b_j$, If $y_i \ge 0$ **Go to Step 5**

Step 5: Solving this LP by using any Simplex Method.

Step 6: IF β > 0, Then

Calculate i. $x_i = \dfrac{y_i\beta}{1 - y_i d_i}$

ii. $F(\mathbf{x}) = \dfrac{c_i x_i + \alpha}{d_i x_i + \beta}$

ELSE IF β < 0 & α > 0 then

Calculate i. $x_i = \dfrac{y_i\beta'}{1 - y_i d'_i}$

ii. $F'(\mathbf{x}) = \dfrac{c_i x_i + \alpha}{d_i x_i - \beta}$

iii) $F(\mathbf{x}) = \dfrac{F'(x) + 1}{F'(x) - 1}$

ELSE IF β < 0 & α < 0 then

Calculate i. $x_i = \dfrac{y_i\beta}{1 + y_i d_i}$

ii. $F(\mathbf{x}) = \dfrac{c_i x_i - \alpha}{d_i x_i - \beta}$

Then we get the optimal solution.

Step-7: END

5.5 Numerical illustrations

Here we have illustrated some numerical examples to justify the effectiveness ond efficience of the proposed method.

Example 5.1:

Let us consider a company those manufactures two products P_1 and P_2. By assumption, the costs arising and the capital demands required are with proportional to the individual activities, furthermore, regardless of the production program to be determined, there are fixed charges amounting to Tk. 200 and a fixed capital demand amounting to Tk. 400. Furthermore the data for the production are fixed as follows:

Capacity available		Demand (per unit of product)	
		P_1	P_2
Raw material (units of quantity):	200	-1	1
Machines (hours):	800	1	3
Owned capital (Tk.):	1400	4	2
Profit per unit Tk. :		3	2

Formulation

Let us consider that x_1 units of product P_1, and x_2 units of product P_2 are to be produced.

Profit from product P_1 is Tk. $3x_1$

Profit from product P_2 is Tk. $2x_2$.

Fixed profit Tk. 200.

Therefore, total profit is Tk. $(3x_1 + 2x_2 + 200)$.

Capital needed for P_1 is Tk. $4x_1$.

Capital needed for P_2 is Tk. $2x_2$.

Fixed capital demand Tk. 400

Therefore total capital needed is Tk. $(4x_1 + 2x_2 + 400)$.

Availability of raw materials is 200 units.

Hence constraint for raw material is: $-x_1 + x_2 < 200$.

Similarly the constraint for machine is: $x_1 + 3x_2 < 800$.

The constraint for capital investment is: $4x_1 + 2x_2 + 400 < 1400$.

$$\text{or } 2x_1 + x_2 < 500.$$

(Due to the fixed capital demand, there is Tk. 500 left for the variable capital demand). Moreover the company either produces some units of P_1 and P_2 or not. So $x_1 > 0, x_2 > 0$

Therefore, profitability $= \frac{profit}{capital} = \frac{3x_1 + 2x_2 + 200}{4x_1 + 2x_2 + 400}$

Hence the linear fractional program is

$$\text{Maximize } F(x) = \frac{3x_1 + 2x_2 + 200}{4x_1 + 2x_2 + 400}$$

Subject to

$$-x_1 + x_2 \le 200$$
$$x_1 + 3x_2 \le 800$$
$$4x_1 + 2x_2 \le 1000$$
$$x_{1,} x_2 \ge 0$$

Solution:

Here we have, $c_1 = 3,\ c_2 = 2\ and\ \alpha = 200$

$$d_1 = 4, d_2 = 2\ and\ \beta = 400$$

$$A_{11} = -1, A_{12} = 1, b_1 = 200$$
$$A_{21} = 1, A_{22} = 3, b_2 = 800$$
$$A_{31} = 2, A_{32} = 1, b_3 = 500$$

Now

$$c_1' = \frac{c_1\beta - d_1\alpha}{\beta} = \frac{3 \times 400 - 4 \times 200}{400} = 1$$

$$c_2' = \frac{c_2\beta - d_2\alpha}{\beta} = \frac{2 \times 400 - 2 \times 200}{400} = 1$$

$$\alpha' = \frac{\alpha}{\beta} = \frac{200}{400} = \frac{1}{2}$$

So, we have the new objective function

$$\therefore Maximize\ G(y) = c_1'y_1 + c_2'y_2 + \alpha'$$

$$= y_1 + y_2 + \frac{1}{2}$$

For the constraint,

$$A'_{11} = A_{11}\beta + b_1 d_1 \qquad A'_{12} = A_{12}\beta + b_1 d_2 \qquad A'_{21} = A_{21}\beta + b_2 d_1$$

$$= -1\times 400 + 200\times 4 \qquad = 1\times 400 + 200\times 2 \qquad = 1\times 400 + 800\times 400$$

$$= 400 \qquad = 800 \qquad = 320400$$

$$A'_{22} = 2800 \qquad A'_{31} = 2800 \qquad A'_{32} = 1400$$

Now we get $A'_{11}y_1 + A'_{12}y_2 \le b_1$ or $400y_1 + 800y_2 \le 200$ or $y_1 + 2y_2 \le 1$

$$\Rightarrow 801y_1 + 7y_2 \le 2$$

$$\Rightarrow 28y_1 + 14y_2 \le 5$$

Converting the LP in standard form we have,

$$Maximize\ G(y) = y_1 + y_2 + \frac{1}{2}$$

Subject to $y_1 + 2y_2 \le 1$

$$\Rightarrow 801y_1 + 7y_2 \le 2$$

$$\Rightarrow 28y_1 + 14y_2 \le 5$$

$$y_1, y_2 \ge 0$$

Now we get the following table:

Table 5.1

C_B	C_J	1	1	0	0	0	b
	Basis	y_1	y_2	s_1	s_2	s_3	
0	s_1	1	2	1	0	0	1
0	s_2	**801***	7	0	1	0	2
0	s_3	28	14	0	0	1	5
$\overline{c}_J = E_J - c_J$		-1	-1	0	0	0	$0+0.5$

Table 5.2

C_B	C_J	1	1	0	0	0	b
	Basis	y_1	y_2	s_1	s_2	s_3	
0	s_1	0	1.99	1	–0.0012	0	0.9975
1	y_1	1	**0.0087***	0	0.0012	0	0.0025
0	s_3	0	13.7553	0	–0.0350	1	4.9301
$\overline{c}_J = E_J - c_J$		0	–0.99	0	0	0	$0+0.5$

Table 5.3

C_B	C_J	1	1	0	0	0	b
	Basis	y_1	y_2	s_1	s_2	s_3	
0	s_1	–227.86	0	1	–0.29	0	0.43
1	y_2	114.43	1	0	0.14	0	0.29
0	s_3	–1574	0	0	–2	1	1
$\overline{c}_J = E_J - c_J$		113.43	0	0	0.14	0	$0.29+0.5$

So we have $y_1 = 0$, $y_2 = 0.29$

Now $x_1 = \dfrac{y_1\beta}{1-d_1y_1} = \dfrac{0\times 400}{1-4\times 0} = 0$

$$x_2 = \frac{y_2\beta}{1-d_2y_2} = \frac{0.29\times 400}{1-4\times 0.29} = 276.19$$

Putting this value in the original objective function, we have

$$Max\ (Z) = \frac{3x_1 + 2x_2 + 200}{4x_1 + 2x_2 + 400}$$

$$Max\ (Z) = \frac{3\times 0 + 2\times 276.19 + 200}{4\times 0 + 2\times 276.19 + 400} = 0.7899$$

We solve the above problem by computer program.

Output: *Minimum of Objective Function = 0.789999*

X 1 = 0.000000
X 2 = 276.190476

Example 5.2: (Negative value of $\boldsymbol{\beta}$)

Maximize $Z = \dfrac{4x_1 + 6x_2 + 2}{x_1 + 2x_2 - 1}$

Subject to $x_1 + 4x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$

Solution (Using Bitran & Novaes method**):**

Here $c = (4,6)$ $annd$ $d = (1,2)$

Now $\gamma = (4,6) - \frac{16}{5}(1,2)$

$= \left(\frac{4}{5}, -\frac{2}{5}\right)$

Maximize $L = \frac{4}{5}x_1 - \frac{2}{5}x_2$

Subject to $x_1 + 4x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$

Converting the LP in standard form we have,

Maximize $L = \frac{4}{5}x_1 - \frac{2}{5}x_2$

Subject to $x_1 + 4x_2 + s_1 = 8$

$x_1 + 2x_2 + s_2 = 4$

$x_1, x_2, s_1, s_2 \geq 0$

Now we get the following table:

Table 5.4

C_B	C_J	0.8	–0.4	0	0	b
	Basis	x_1	x_2	s_1	s_2	
0	s_1	1	4	1	0	8
0	s_2	1	2	0	1	4
$\bar{c}_J = E_J - c_J$		–0.8	0.4	0	0	

Table 5.5

C_B	C_J / Basis	0.8 / x_1	–0.4 / x_2	0 / s_1	0 / s_2	b
0	s_1	0	2	1	–1	4
0.8	x_1	1	2	0	1	4
$\overline{c}_J = E_J - c_J$		–0.8	2	0	0.8	

So we have $x_1 = 4$, $x_2 = 0$

Putting this value in the original objective function, we have

$$\text{Maximize} \quad Z = \frac{4\times 4 + 6\times 0 + 2}{4 + 2\times 0 - 1} = \frac{20}{3}$$

Again we have another new linear objective function L' as follows:

$$\text{Minimize} \quad L' = [(4,6) - \frac{20}{3}(1,2)]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= -\frac{8}{3}x_1 - \frac{22}{3}x_2$$

Subject to $x_1 + 4x_2 + s_1 = 8$

$x_1 + 2x_2 + s_2 = 4$

$x_1, x_2, s_1, s_2 \geq 0$

Now we get the following table: **Table 5.6**

C_B	C_J / Basis	–2.67 / x_1	–7.33 / x_2	0 / s_1	0 / s_2	b
0	s_1	1	4	1	0	8
0	s_2	1	2	0	1	4
$\overline{c}_J = E_J - c_J$		2.67	7.33	0	0	

Here all $\overline{c}_J$ is positive in the above table. So we cannot find solution by this method.

Solution (Using proposed method)**:**

Maximize $Z = \dfrac{4x_1 + 6x_2 + 2}{x_1 + 2x_2 - 1}$

Subject to $x_1 + 4x_2 \le 8$

$x_1 + 2x_2 \le 4$

$x_1, x_2 \ge 0$

Maximize $Z = \dfrac{4x_1 + 6x_2 + 2}{x_1 + 2x_2 - 1}$

Maximize $Z^* = \dfrac{Z+1}{Z-1} = \dfrac{(4+1)x_1 + (6+2)x_2 + (2-1)}{(4-1)x_1 + (6-2)x_2 + (2+1)}$

$= \dfrac{5x_1 + 8x_2 + 1}{3x_1 + 4x_2 + 3}$

Subject to $x_1 + 4x_2 \le 8$

$x_1 + 2x_2 \le 4$

$x_1, x_2 \ge 0$

Here we have, $c_1 = 5,\ c_2 = 8\ and\ \alpha = 1$

$d_1 = 3, d_2 = 4\ and\ \beta = 3$

$A_{11} = 1, A_{12} = 4, b_1 = 8$

$A_{21} = 1, A_{22} = 2, b_2 = 4$

Where A_{11}, A_{12} and b_1 is related to the first constraint and A_{21}, A_{22} and b_2 is related to the second constraint. Now we get,

$c_1' = \dfrac{c_1\beta - d_1\alpha}{\beta}$ $= \dfrac{5\times3 - 3\times1}{3}$ $= 4$

$c_2' = \dfrac{c_2\beta - d_2\alpha}{\beta}$ $= \dfrac{8\times3 - 4\times1}{3}$ $= 6.67$

$\alpha' = \dfrac{\alpha}{\beta}$ $= \dfrac{1}{3}$ $= 0.333$

So, we have the new objective function

Maximize $g(y) = c_1' y_1 + c_2' y_2 + \alpha'$

$A'_{11} = A_{11}\beta + b_1 d_1$
$= 1\times3 + 8\times3$
$= 27$

$A'_{12} = A_{12}\beta + b_1 d_2$
$= 4\times3 + 8\times4$
$= 44$

$A'_{21} = A_{21}\beta + b_2 d_1$ $\qquad\qquad$ $A'_{22} = A_{22}\beta + b_2 d_2$

$= 15$ $\qquad\qquad\qquad\qquad$ $= 22$

$$= 4y_1 + 6.67y_2 + 0.333$$

For the constraint,

Now we get $27y_1 + 44y_2 \le 8$

$$\Rightarrow 15y_1 + 22y_2 \le 4$$

Converting the LP in standard form we have,

Maximize $g(y) = 4y_1 + 6.67y_2 + 0.333$

Subject to $27y_1 + 44y_2 \le 8$

$$\Rightarrow 15y_1 + 22y_2 \le 4$$

$$y_1, y_2, s_1, s_2, s_3 \ge 0$$

Now we get the following table:

Table 5.7

C_B	C_J	4	6.67	0	0	b
	Basis	y_1	y_2	s_1	s_2	
0	s_1	27	44*	1	0	8
0	s_2	15	22	0	1	4
$\bar{c}_J = E_J - c_J$		–4	–6.67	0	0	$0 + 0.33$

Table 5.8

C_B	C_J	4	6.67	0	0	b
	Basis	y_1	y_2	s_1	s_2	
6.67	x_2	0.6136	1	0.0227	0	0.1818
0	s_2	1.5	0	–0.5	1	0
$\bar{c}_J = E_J - c_J$		0.093	0	0.1516	0	$1.2127 + 0.33$

So we have $y_1 = 0$, $y_2 = 0.1818$

Now $x_1 = \dfrac{y_1\beta}{1 - d_1 y_1} = \dfrac{0 \times 3}{1 - 3 \times 0} = 0$

$$x_2 = \frac{y_2\beta}{1-d_2y_2} = \frac{0.1818\times 3}{1-4\times 0.1818} = 1.999$$

Putting this value in the original objective function, we have

$$\text{Max} \quad Z = \frac{4x_1+6x_2+2}{x_1+2x_2-1}$$

$$Max(Z) = \frac{4\times 0+6\times 1.999+2}{1\times 0+2\times 1.999-1} = 4.6676$$

We solve the above problem by computer program.

Output:

Minimum of Objective Function = 4.667556
X 1 = 0.000000
X 2 = 1.999266

It is remarked that, this problem cannot be solved by any other method for the presence of negative value of β.

Example 5.3: (Negative values of, $\boldsymbol{\beta}$)

We consider the numerical example

$$\text{Maximize} \quad Z = \frac{5x_1+2x_2-7}{4x_1+2x_2-4}$$

$$\text{Subject to} \quad 4x_1+3x_2 \le 12$$
$$4x_1+x_2 \le 8$$
$$4x_1-x_2 \le 6$$
$$x_1, x_2 \ge 0$$

Solution (Using Bitran & Novaes method)**:**

Here $c = (5,2) \ \ annd \ \ d = (4,2)$

Now $\gamma = (5,2) - \frac{6}{5}(4,2)$

$= \left(\frac{1}{5}, -\frac{2}{5}\right)$

Maximize $L = \frac{1}{5}x_1 - \frac{2}{5}x_2$

Converting the LP in standard form we have,

$$\text{Maximize} \quad L = \frac{1}{5}x_1 - \frac{2}{5}x_2$$

$$\text{Subject to} \quad 4x_1 + 3x_2 + s_1 = 12$$
$$4x_1 + x_2 + s_2 = 8$$
$$4x_1 - x_2 + s_3 = 6$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Now we get the following table:

Table 5.9

C_B	C_J	0.02	–0.40	0	0	0	b
	Basis	x_1	x_2	s_1	s_2	s_3	
0	s_1	4	3	1	0	0	12
0	s_2	4	1	0	1	0	8
0	s_3	4	–1	0	0	1	6
$\bar{c}_J = E_J - c_J$		–0.20	0.40	0	0	0	

Table 5.10

C_B	C_J	0.02	–0.40	0	0	0	b
	Basis	x_1	x_2	s_1	s_2	s_3	
0	s_1	0	4	1	0	–1	6
0	s_2	0	2	0	1	–1	2
0.20	x_1	1	–0.25	0	0	0.25	1.50
$\bar{c}_J = E_J - c_J$		0	0.35	0	0	0.05	

So we have $x_1 = 1.50$ or $\frac{3}{2}$, $x_2 = 0$

Putting this value in the original objective function, we have

$$\text{Maximize} \quad Z = \frac{5 \times \frac{3}{2} + 2 \times 0 - 7}{4 \times \frac{3}{2} + 2 \times 0 - 4} = \frac{1}{4}$$

Again we have another new linear objective function L' as follows:

$$\text{Minimize} \quad L' = [(5,2) - \frac{1}{4}(4,2)]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 4x_1 + \frac{3}{2}x_2$$

$$\text{Subject to} \quad 4x_1 + 3x_2 + s_1 = 12$$
$$4x_1 + x_2 + s_2 = 8$$
$$4x_1 - x_2 + s_3 = 6$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Now we get the following table:

Table 5.11

C_B	C_J	4	$\frac{3}{2}$	0	0	0	b
	Basis	x_1	x_2	s_1	s_2	s_3	
0	s_1	4	3	1	0	0	12
0	s_2	4	1	0	1	0	8
0	s_3	4	–1	0	0	1	6
$\bar{c}_J = E_J - c_J$		–4	–1.50	0	0	0	

After 4 iterations we get the final result which is given below:

Table 5.12

C_B	C_J	0.02	–0.40	0	0	0	b
	Basis	x_1	x_2	s_1	s_2	s_3	
0	s_3	0	0	1	–2	1	2
0	x_2	0	1	0.5	–0.5	0	2
0	x_1	1	0	–0.13	0.38	0	1.5
$\bar{c}_J = E_J - c_J$		0	0	0.25	0.75	0	

So we have $x_1 = 1.50$ or $\frac{3}{2}$, $x_2 = 2$

Putting this value in the original objective function, we have

$$\text{Maximize} \quad Z = \frac{5\times\frac{3}{2}+2\times 2-7}{4\times\frac{3}{2}+2\times 2-4} = \frac{3}{4}$$

Again we have another new linear objective function L' as follows:

$$\text{Minimize} \quad L'' = [(5,2)-\frac{3}{4}(4,2)]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 2x_1 + \frac{1}{2}x_2$$

$$\text{Subject to} \quad 4x_1 + 3x_2 + s_1 = 12$$
$$4x_1 + x_2 + s_2 = 8$$
$$4x_1 - x_2 + s_3 = 6$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Now we get the following table:

Table 5.13

C_B	C_J	2	$\frac{1}{2}$	0	0	0	b
	Basis	x_1	x_2	s_1	s_2	s_3	
0	s_1	4	3	1	0	0	12
0	s_2	4	1	0	1	0	8
0	s_3	4	–1	0	0	1	6
$\bar{c}_J = E_J - c_J$		–2	–0.50	0	0	0	

After 3 iterations we get the final result which is given below:

Table 5.14

C_B	C_J	2	$\frac{1}{2}$	0	0	0	b
	Basis	x_1	x_2	s_1	s_2	s_3	
0	s_1	0	0	1	–2	1	2
0	x_2	0	1	0.5	–0.5	0	1
0	x_1	1	0	0.13	0.13	0	1.75
$\overline{c}_J = E_J - c_J$		0	0	0	0.50	0	

So we have $x_1 = 1.75$ or $\frac{7}{4}$, $x_2 = 2$

We see that

Step 1. $x_1 = \frac{3}{2}$ and $x_2 = 0$

Step 2. $x_1 = \frac{3}{2}$ and $x_2 = 2$

Step 1. $x_1 = \frac{7}{4}$ and $x_2 = 1$

From all of these three cases we observe that the values of x_1 and x_2 are not equal. For this reason we cannot get an optimal solution by using Bitran & Novaes method.

Solution (Using proposed method)**:**

$$\text{Maximize} \quad Z = \frac{5x_1 + 2x_2 - 7}{4x_1 + 2x_2 - 4}$$

$$= \frac{-5x_1 - 2x_2 + 7}{-4x_1 - 2x_2 + 4}$$

$$\text{Subject to} \quad 4x_1 + 3x_2 \le 12$$

$$4x_1 + x_2 \le 8$$

$$4x_1 - x_2 \le 6$$

$$x_1, x_2 \ge 0$$

Here we have, $c_1 = -5,\ c_2 = -2\ and\ \alpha = 7$

$d_1 = -4, d_2 = -2\ and\ \beta = 4$

$A_{11} = 4, A_{12} = 3, b_1 = 12$

$A_{21} = 4, A_{22} = 1, b_2 = 8$

$$A_{31}=4, A_{32}=1, b_3=6$$

Where A_{11}, A_{12} and b_1 is related to the first constraint, A_{21}, A_{22} and b_2 is related to the second constraint and A_{31}, A_{32} and b_2 is related to the third constraint. Now we get,

$$c_1' = \frac{c_1\beta - d_1\alpha}{\beta} = \frac{-5\times 4-(-4)\times 7}{7} = \frac{8}{7}$$

$$c_2' = \frac{c_2\beta - d_2\alpha}{\beta} = \frac{-2\times 4-(-2)\times 7}{7} = \frac{6}{7}$$

$$\alpha' = \frac{\alpha}{\beta} = \frac{7}{4}$$

So, we have the new objective function

$$\text{Maximize } g(y) = c_1' y_1 + c_2' y_2 + \alpha' = \frac{8}{7}y_1 + \frac{6}{7}y_2 + \frac{7}{4}$$

$$\mathrm{A}'_{11} = A_{11}\beta + b_1 d_1 = 4\times 4 + 12\times(-4) = -32$$

$$\mathrm{A}'_{12} = A_{12}\beta + b_1 d_2 = 3\times 4 + 12\times(-2) = -12$$

$$\mathrm{A}'_{21} = A_{21}\beta + b_2 d_1 = 4\times 4 + 8\times(-4) = -16$$

$\mathrm{A}'_{22} = -12$ $\quad$ $\mathrm{A}'_{31} = -8$ $\quad$ $\mathrm{A}'_{32} = -16$

For the constraint,

Now we get $\mathrm{A}'_{11}y_1 + \mathrm{A}'_{12}y_2 \le b_1$ or $-32y_1 - 12y_2 \le 12$ or $-8y_1 - 3y_2 \le 3$

$$\Rightarrow -4y_1 - 3y_2 \le 2$$

$$\Rightarrow -4y_1 - 8y_2 \le 3$$

Converting the LP in standard form we have,

$$\text{Maximize } \mathrm{g}(y) = \frac{8}{7}y_1 + \frac{6}{7}y_2 + \frac{7}{4}$$

Subject to $-8y_1 - 3y_2 + s_1 = 3$

$$-4y_1 - 3y_2 + s_2 = 2$$

$$-4y_1 - 8y_2 + s_3 = 3$$

$$y_1, y_2, s_1, s_2, s_3 \ge 0$$

Now we get the following table:

Table 5.15

C_B	C_J	$\frac{8}{7}$	$\frac{6}{7}$	0	0	0	b
	Basis	y_1	y_2	s_1	s_2	s_3	
0	s_1	–8	–3	1	0	0	3
0	s_2	–4	–3	0	1	0	2
0	s_3	–4	–8	0	0	1	3
$\bar{c}_J = E_J - c_J$		$-\frac{8}{7}$	$-\frac{6}{7}$	0	0	0	

The problem is unbounded.

5.6 Comparison

In this section, we have compared the experimental results of our proposed method with the available methods in the literature.

i. In our method we can solve any type of linear fractional programming problems.

ii. By using some technique we can easily transform the LFP problem into a LP problem.

iii. Its computational steps are so easy that there is no difficulty like other methods.

iv. In this method we need to solve one LP but other method one need to solve more than one LP, which save our valuable time.

v. The final result converges quickly in this method.

vi. In this method there is one restriction that is $\beta \neq 0$

vii. Some cases of denominator and numerator say, $dx + \beta > 0$ and $cx + \alpha < 0 \ \forall x \in X$, where Bitran-Novaes fails and for the negative value of β all other method are also fails but our method can solve the problem very easily.

viii. Using computer program we get the optimal solution of the LFP problem very quickly.

From the above discussion it may conclude that proposed method is comparatively better than existing methods considered for solving FLP.

CHAPTER VI

Discussion and Conclusion

Operations Research (OR) is a science which deals with problem, formulation, solutions and finally appropriate decision making. The tools of operations research are not from any one discipline; rather Mathematics, Statistics, Economics, Engineering, Psychology, etc. have contributed to this newer discipline of knowledge. Today, it has become a professional discipline that deals with the application of scientific methods for decision-making, and especially to the allocation of scare resources.

Now a day Transportation problem with mixed constant and Linear fractional Programming Problems are very well-known in OR to solve real life problem. The More-for-less (MFL) paradox in a TP occurs when it is possible to ship more 'total goods' for less (or equal) 'total cost' while shipping the same amount or more from each origin and to each destination, keeping all shipping costs non-negative. The occurrence of MFL in distribution problems is observed in nature. The mixed constraints TP have extensively been studied by many researchers in the past years. Here we try to introduce a modified VAM method for solving TPs with mixed constraints in MEL paradoxical situation. The optimal MFL solution procedure is illustrated with the help of numerical example. The experimental method is very simple, easy to understand and apply. The MFL situation exists in real life and it could present managers with an opportunity for shipping more units for less or the same cost.

Assuming the positivity of denominator of objective function of linear fractional program (LFP), one can solve LFP problem applying the modified approach of Swarup's primal simplex type method and modified approach of Swarup. If the constraints set of the feasible region X is bounded and denominator of objective function of linear fractional program (LFP) is strictly positive for all $x \in X$, one can solve LFP problem by applying the method of Bitran-Novaes. Bitran-Novaes method is better than the modified approach of Swarup's primal simplex type method because (i) it does not require any modification in the set of constraints, (ii) there is no need to make variable transformations, (iii) no additional constraints or variables are required and (iv) the conventional simplex algorithm can still be used.

In this thesis, we have provided a new technique for solving the linear fractional programming problems. Some cases of denominator and numerator and the negative value of β, all other existing methods are fail to solve linear fractional problem but proposed method able to solve the problem vary efficiently. In the proposed technique, at first we transformed the LFP problems into a LP by some transformation technique and then solve it by using simplex method. For the validity of the proposed method, we have considered some instances along with negative value of β. From the experimental study it is observed that the proposed approach able to solve the LFP including negative value of β efficiently. We have also developed a computer program to implement the proposed method which is included in Appendix.

Finally, we may conclude that the proposed method along the program code is able to solve all linear fractional programming methods for large-scale optimization problem. To do this, one has to build the required mathematical programming model of the problem and required computer program. We further also expect that the proposed concept will be helpful for solving real-life problems involving linear fractional programming problem in agriculture, production planning, financial and corporate planning, health care and hospital planning etc. The proposed program is easy to apply in LFT and which is also effective as well as efficient.

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APPENDIX

Computer Technique (FORTRAN)

we developed a computer program regarding proposed method for solving linear fractional problem in. In real life, the problem is so much large which is difficult to solve by hand calculation. But use this computer technique one can solve the problem very easily.

```
!**********************************************************************
!      ##########          PROGRAME FOR SOLVING LFP           ##########
!*******                        A NEW TECNIQUE                      ******
!**********************************************************************
!*******                    LIST OF MAIN VARIABLES:                 ******
!*******          C:    MAXIMIZE = 1,MINIMIZE = 2                   ******
!*******          N:    NUMBER OF VARIABLES OF ECONOMIC FUNCTION  ****
!*******                     (TO MAXIMIZE OR MINIMIZE).             ******
!*******          M:    NUMBER OF CONSTRAINTS                       ******
!*******         M1:    NUMBER OF <= CONSTRAINTS                    ******
!*******         M2:    NUMBER OF >= CONSTRAINTS                    ******
!*******         M3:    NUMBER OF  = CONSTRAINTS                 ******
!*******         A,M,N,MP,NP,M1,M2,AND M3 ARE INPUT PARAMETERS      ******
!*******         ICASE,IZROV,AND IPOSV ARE OUTPUT PARAMETERS        ******
!************************************************************** *******
  PARAMETER(MP=100,NP=100)
  REAL X(MP,NP),Y(MP,NP),R,P(MP,NP),Q(MP,NP),UP,LO
  INTEGER C,N,M,M1,M2,M3,IPOSV(MP),IZROV(NP)

        PRINT *,' '
        PRINT *,'    ######### LINEAR FRACTIONAL PROGRAMING #########
        PRINT *,'     ************* A NEW TECNIQUE ************'
        PRINT *,' '
        WRITE(*,10,ADVANCE='NO'); READ *,C
```

```
      IF(C.LE.0.OR.C.GE.3) PAUSE ' Bad input.'
      WRITE(*,20,ADVANCE='NO'); READ *,N
      WRITE(*,30,ADVANCE='NO'); READ *,M1
      WRITE(*,40,ADVANCE='NO'); READ *,M2
      WRITE(*,50,ADVANCE='NO'); READ *,M3
      M=M1+M2+M3
      X=0.
      PRINT *,'Input Economic Function:'
      DO i=2,N+1
      WRITE(*,60,ADVANCE='NO') i-1; READ *,X(1,i)
      END DO
      WRITE(*,61,ADVANCE='NO'); READ *,X(1,1)
      DO i=2,N+1
      WRITE(*,60,ADVANCE='NO') i-1; READ *,Y(1,i)
      END DO
      WRITE(*,61,ADVANCE='NO'); READ *,Y(1,1)
      UP=X(1,1)
      LO=Y(1,1)
      DO i=1,M
      WRITE(*,70) i
      DO j=2,N+1
      WRITE(*,60,ADVANCE='NO') j-1; READ *,X(i+1,j)
      END DO
      WRITE(*,61,ADVANCE='NO'); READ *,X(i+1,1)
      END DO
CALL LFP(X,Y,N,M,MP,NP)
      DO i=1,M
      DO j=2,N+1
      X(i+1,j) = -X(i+1,j)
      END DO
      END DO
      IF(C.EQ.2) then
```

```
      DO I=1,N+1
      X(1,I)=-X(1,I)
      END DO
      END IF
      PRINT *,' '
CALL MAIN(X,M,N,MP,NP,M1,M2,M3,ICASE,IZROV,IPOSV)
      PRINT *,' '
      IF(LO.LT.0.AND.UP.GE.0)THEN
      X(1,1)=((X(1,1)+1.0)/(X(1,1)-1.0))
      END IF
      IF(C.EQ.1)THEN
      PRINT *,' Maximum of Objective Function= ',X(1,1)
      ELSE IF(C.EQ.2)THEN
      X(1,1)=-X(1,1)
      PRINT *,' Minimum of Objective Function = ',X(1,1)
      END IF
      DO I=1,N
      DO J=1,M
      IF (IPOSV(J).eq.I) THEN
      Y(J+1,1)=(Y(1,1)*X(J+1,1))/(1-(Y(1,i+1)*X(J+1,1)))
      WRITE(*,110) I,Y(J+1,1)
      GOTO 28
      END IF
      END DO
      WRITE(*,110) I,0.0
28    END DO
      PRINT *,' '
10    FORMAT(' Maximize or Minimize? [MAX=1,MIN=2]..: ')
20    FORMAT(' Number of nonbasic variables: ')
30    FORMAT(' Number of <= inequalities..: ')
40    FORMAT(' Number of >= inequalities..: ')
```

```
 50     FORMAT(' Number of = equalities.....: ')
 60     FORMAT(' Coefficient #',I2,': ')
 61     FORMAT(' Constant term..: ')
 70     FORMAT(' Input constraint #',I2,': ')
110     FORMAT('  X',I2,' = ',F12.6)
        STOP
        END

 SUBROUTINE LFP(X,Y,N,M,MP,NP)
 INTEGER M,N,UP
 REAL X(MP,NP),Y(MP,NP),P(MP,NP)

        IF(Y(1,1).GT.0) THEN
        X(1,1)=X(1,1)/Y(1,1)
        DO i=2,N+1
        X(1,i)=X(1,i)-(Y(1,i)*X(1,1))
        END DO
        ELSE IF(Y(1,1).LT.0.AND.X(1,1).GE.0)THEN
        UP=X(1,1)+Y(1,1)
        Y(1,1)=X(1,1)-Y(1,1)
        X(1,1)=UP/Y(1,1)
        DO I=2,N+1
        P(1,I)=X(1,I)+Y(1,I)
        Y(1,I)=X(1,I)-Y(1,I)
        X(1,I)=P(1,I)-Y(1,I)*X(1,1)
        END DO
        ELSE IF(Y(1,1).LT.0.AND.X(1,1).LT.0)THEN
        UP=-X(1,1)
        Y(1,1)=-Y(1,1)
        X(1,1)=UP/Y(1,1)
        DO I=2,N+1
        P(1,I)=-X(1,I)
        Y(1,I)=-Y(1,I)
        X(1,I)=P(1,I)-Y(1,I)*X(1,1)
```

```
      END DO
      END IF
      DO i=1,M
      X(i+1,1)=(X(i+1,1)/Y(1,1))
      DO j=2,N+1
      X(i+1,j)=X(i+1,j)+(X(i+1,1)*Y(1,j))
      END DO
      END DO
      RETURN
      END
SUBROUTINE MAIN(X,M,N,MP,NP,M1,M2,M3,ICASE,IZROV,IPOSV)
INTEGER M,N,MP,NP,M1,M2,M3,ICASE,IPOSV(M),IZROV(N),MMAX,NMAX
REAL X(MP,NP),EPS
PARAMETER (MMAX=100,NMAX=100,EPS=1.E-6)
INTEGER I,IP,IS,K,KH,KP,NL1,L1(NMAX),L2(MMAX),L3(MMAX)
REAL BMAX,Q1
      IF(M.NE.M1+M2+M3) PAUSE ' Bad input constraint counts in simplex.'
      NL1=N
      DO K=1,N
      L1(K)=K
      IZROV(K)=K
      END DO
      NL2=M
      DO I=1,M
      IF(X(I+1,1).LT.0.) PAUSE ' Bad input tableau in simplex, Constants bi must be
      nonnegative.'
      L2(I)=I
      IPOSV(I)=N+I
      END DO
      DO I=1,M2
      L3(I)=1
      END DO
```

```
      IR=0
      IF(M2+M3.EQ.0) GOTO 30
!********   The origin is a feasible starting solution. Go to phase two.     *******
      IR=1
      DO K=1,N+1
      Q1=0.
      DO I=M1+1,M
      Q1=Q1+X(I+1,K)
      END DO
      X(M+2,K)=-Q1
      END DO
33    CALL SIMP1(X,MP,NP,M+1,L1,NL1,0,KP,BMAX)
      IF(BMAX.LE.EPS.AND.X(M+2,1).LT.-EPS)THEN
      ICASE=-1
      RETURN
      ELSE IF(BMAX.LE.EPS.AND.X(M+2,1).LE.EPS)THEN
      M12=M1+M2+1
      IF (M12.LE.M) THEN
      DO IP=M12,M
      IF(IPOSV(IP).EQ.IP+N)THEN
      CALL SIMP1(X,MP,NP,IP,L1,NL1,1,KP,BMAX)
      IF(BMAX.GT.EPS) GO TO 29
      END IF
      END DO
      END IF
      IR=0
      M12=M12-1
      IF (M1+1.GT.M12) GO TO 30
      DO I=M1+1,M1+M2
      IF(L3(I-M1).EQ.1)THEN
      DO K=1,N+1
```

```
      X(I+1,K)=-X(I+1,K)
      END DO
      END IF
      END DO
      GO TO 30
      END IF
CALL SIMP2(X,M,N,MP,NP,L2,NL2,IP,KP,Q1)
      IF(IP.EQ.0)THEN
      ICASE=-1
      RETURN
      END IF
29    CALL SIMP3(X,MP,NP,M+1,N,IP,KP)
      IF(IPOSV(IP).GE.N+M1+M2+1)THEN
      DO K=1,NL1
      IF(L1(K).EQ.KP) GOTO 31
      END DO
31    NL1=NL1-1
      DO IS=K,NL1
      L1(IS)=L1(IS+1)
      END DO
      ELSE
      IF(IPOSV(IP).LT.N+M1+1) GO TO 32
      KH=IPOSV(IP)-M1-N
      IF(L3(KH).EQ.0) GO TO 32
      L3(KH)=0
      END IF
      X(M+2,KP+1)=X(M+2,KP+1)+1.
      DO I=1,M+2
      X(I,KP+1)=-X(I,KP+1)
      END DO
32    IS=IZROV(KP)
```

```
      IZROV(KP)=IPOSV(IP)
      IPOSV(IP)=IS
      IF (IR.NE.0) GO TO 33
30    CALL SIMP1(X,MP,NP,0,L1,NL1,0,KP,BMAX)
      IF(BMAX.LE.EPS)THEN
      ICASE=0
      RETURN
      END IF
CALL SIMP2(X,M,N,MP,NP,L2,NL2,IP,KP,Q1)
      IF(IP.EQ.0)THEN
      ICASE=1
      RETURN
      END IF
CALL SIMP3(X,MP,NP,M,N,IP,KP)
      GO TO 32
      END
SUBROUTINE SIMP1(X,MP,NP,MM,LL,NLL,IABF,KP,BMAX)
INTEGER MP,NP,MM,LL(NP),NLL,IABF,KP,K
REAL BMAX,X(MP,NP),TEST
      KP=LL(1)
      BMAX=X(MM+1,KP+1)
      IF(NLL.LT.2) RETURN
      DO K=2,NLL
      IF(IABF.EQ.0)THEN
      TEST=X(MM+1,LL(K)+1)-BMAX
      ELSE
      TEST=ABS(X(MM+1,LL(K)+1))-ABS(BMAX)
      END IF
      IF(TEST.GT.0.)THEN
      BMAX=X(MM+1,LL(K)+1)
```

```
      KP=LL(K)
      END IF
      END DO
      RETURN
      END

SUBROUTINE SIMP2(X,M,N,MP,NP,L2,NL2,IP,KP,Q1)
INTEGER M,N,MP,NP,L2(MP),IP,KP,I,K
REAL X(MP,NP),EPS,Q0,Q,Q1,QP
PARAMETER (EPS=1.E-6)

      IP=0
      IF(NL2.LT.1) RETURN
      DO I=1,NL2
      IF(X(I+1,KP+1).LT.-EPS) GO TO 56
      END DO
      RETURN
56    Q1=-X(L2(I)+1,1)/X(L2(I)+1,KP+1)
      IP=L2(I)
      IF(I+1.GT.NL2) RETURN
      DO I=I+1,NL2
      II=L2(I)
      IF(X(II+1,KP+1).LT.-EPS)THEN
      Q=-X(II+1,1)/X(II+1,KP+1)
      IF(Q.LT.Q1)THEN
      IP=II
      Q1=Q
      ELSE IF (Q.EQ.Q1) THEN
      DO K=1,N
      QP=-X(IP+1,K+1)/X(IP+1,KP+1)
      Q0=-X(II+1,K+1)/X(II+1,KP+1)
      IF(Q0.NE.QP)GOTO 57
      END DO
57    IF(Q0.LT.QP) IP=II
```

```
      END IF
      END IF
      END DO
      RETURN
      END

SUBROUTINE SIMP3(X,MP,NP,I1,K1,IP,KP)
INTEGER MP,NP,I1,K1,IP,KP,II,KK
REAL X(mp,np),PIV

      PIV=1./X(IP+1,KP+1)
      IF (I1.GE.0) THEN
      DO II=1,I1+1
      IF(II-1.NE.IP)THEN
      X(II,KP+1)=X(II,KP+1)*PIV
      DO KK=1,K1+1
      IF(KK-1.NE.KP)THEN
      X(II,KK)=X(II,KK)-X(IP+1,KK)*X(II,KP+1)
      END IF
      END DO
      END IF
      END DO
      END IF
      DO KK=1,K1+1
      IF(KK-1.NE.KP) X(IP+1,KK)=-X(IP+1,KK)*PIV
      END DO
      X(IP+1,KP+1)=PIV
      RETURN
      END
```