STUDY ON STANDARD IDEAL AND HOMOMORPHISM THEOREM OF A NEARLATTICE

BY

G.M. JAHID HOSSAIN
Roll No 1551553

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M.Sc. Thesis

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Declaration

This is to certify that the thesis work entitled "**Study on Standard Ideal and Homomorphism Theorem of a Nearlattice**" has been carried out by G.M. JAHID HOSSAIN in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

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At first I express my heart felt gratefulness to Almighty Allah for giving me the ability to perform my M.Sc works.

It is also a great pleasure to express my sincere gratitude to my Supervisor Professor Dr. Md. Zaidur Rahman, Department of Mathematics, Khulna University of Engineering & Technology (KUET), Khulna, for his invariable guidance, generous advice, constructive criticism, inspiration and encouragement conveyed during my thesis work and for the valuable efforts made for the preparation of this thesis.

I express my gratitude to my estimable teacher, Professor Dr. M. M. Touhid Hossain, the honorable Head, Department of Mathematics, Khulna University of Engineering & Technology (KUET), Khulna for providing me all kinds of departmental help. I also express my gratitude to respectable teachers Professor Dr. Md. Bazlar Rahman, Professor Dr. Md. Abul Kalam Azad, for their valuable suggestions and encouragement during thesis work.

I also want to express thanks to my all teacher Depertment of Mathematics, Khulna University of Engineering & Technology, Khulna for their necessary advice, cordial co-operation and authentic information during the period of study.

I am thankful to all my family members especially to my younger sister Zebunnesa Akhter.

Finally, I would like to thanks all of my friends for their cordial encouragement and help.

G. M. Jahid Hossain

Abstract

In this thesis the standard ideal of a nearlattices is presented. By a nearlattice S we will always mean a meet semilattice together with the property that any two elements possessing a common upper bound, have a supremum. Cornish and Hickman [4] referred this property as the upper bound property, and a semilattice of this nature as a semilattice with the upperbound property. Cornish and Noor [5] preferred to call these semilattices as nearlattices, as the behavior of such a semilattice is close to that of a lattice than an ordinary semilattice. Of course a nearlattice with a largest element is a lattice. Since any semilattice satisfying the descending chain condition has the upper bound property, so all finite semilattices are nearlattices. In lattice theory, it is always very difficult to study the non-distributive and non-modular lattices. Geatzer [12] studied the non-distributive lattice by introducing the concept of distributive, standard and neutral elements in lattices. Cornish and Noor [5] extended those concepts for nearlattices to study non-distributive nearlattices. This thesis extend the concept of standard ideal of a nearlattices. We also extend the homomorphism theorem of lattices to nearlattices. Finally we generalize two isomorphism theorems of Gratzer, G. and Schmidt, E. T [14] to nearlattices.

Approval

This is to certify that the thesis work submitted by *G. M. Jahid Hossain* entitled "**Study on Standard Ideal and Homomorphism Theorem of a Nearlattice**" has been approved by the board of examiners for the partial fulfillment of the requirements for the degree of *M.Sc in the Department of Mathematics*, Khulna University of Engineering & Technology, Khulna, Bangladesh in December 2016.

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CHAPTER I

IDEALS AND CONGRUNCES

1.1 Preliminaries

The intention of this section is to outline and fix the notation for some of the concepts of nearlattices which are basic to this thesis. We also formulate some results on arbitrary nearlattices for later use. For the background material in lattice theory we refer the reader to the text of Birkhoff [3] , Gratzer [11], [12] and Davey [8].

By a nearlattice S we always mean a lower (meet) semilattice which has the property that any two elements possessing a common upper bound have a supremum. Cornish and Hickman [4], referred this property as the *upper bound property* and a semilattice of this nature as *a semilattice with the upper bound property*. The behaviour of such a semilattice is closer to that of a lattice than an ordinary semilattice.

Of course, a nearlattice with a largest element is a lattice. Since any semilattice satisfying the descending chain condition has the upper bound property, so all finite semilattices are nearlattices.

Now we give an example of a meet semilattice which is not a nearlattice.
Example: In R^2 let us consider the set, $S = \{(0,0)\} \cup \{(1,0)\} \cup \{(0,1)\} \cup \{(1,y) \mid y > 1\}$ shown in the Figure 1.1

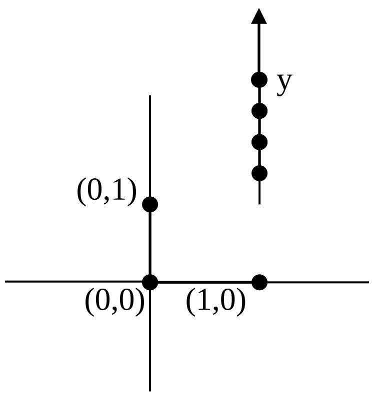

Figure 1.1

Let us define the partial ordering "$\leq$" on S by $(x, y) \leq (x_1, y_1)$ if and only if $x \leq x_1$ and $y \leq y_1$. Clearly, $(S; \leq)$ is a meet semilattice. Both (1,0) and (0,1) have common upper bounds. In fact $\{(1, y) \mid y > 1\}$ are common upper bounds of them. But the supremum of (1,0) and (0,1) does not exist. Therefore $(S; \leq)$ is not a nearlattice.

The upper bound property appears in Gratzer and Lakser [13], while Rozen [24] show that it is the result of placing certain associativity conditions on the partial join operation. Moreover, Evans [9] referred nearlattices as *conditional lattices*. By a conditional lattice he means a lower semilattice S with the condition that for each $x \in S$, $\{y \in S \mid y \leq x\}$ is a lattice; and it is very easy to check that this condition is equivalent to the upper bound property of S. Also Nieminen [19] in his paper refers to nearlattices as *"partial lattices"*. Whenever a nearlattice has a least element we will denote it by 0. If $x_1, x_2, \ldots, x_n$ are elements of a nearlattice then by $x_1 \vee x_2 \vee \cdots \vee x_n$, we mean that the supremum of $x_1, x_2, \ldots, x_n$ exists and $x_1 \vee x_2 \vee \cdots \vee x_n$ symbolizing this supremum.

A non-empty subset K of a nearlattice S is called a *subnearlattice* of S if for any $a, b \in K$, both $a \wedge b$ and $a \vee b$ (whenever it exists in S) belong to K ($\wedge$ and $\vee$ are taken in S), and the $\wedge$ and $\vee$ of K are the restrictions of the $\wedge$ and $\vee$ of S to K. Moreover, a subnearlattice K of a nearlattice S is called a *sublattice* of S if $a \vee b \in K$ for all $a, b \in K$.

A nearlattice S is called *modular* if for any $a, b, c \in S$ with $c \leq a$, $a \wedge (b \vee c) = (a \wedge b) \vee c$ whenever $b \vee c$ exists.

A nearlattice S is called distributive if for any $x, x_1, x_2, \ldots, x_n$, $x \wedge (x_1 \vee x_2 \vee \cdots \vee x_n) = (x \wedge x_1) \vee (x \wedge x_2) \vee \cdots \vee (x \wedge x_n)$ whenever $x_1 \vee x_2 \vee \cdots \vee x_n$ exists. Notice that the right hand expression always exists by the upper bound property of S.

Lemma 1.1.1. *A nearlattice S is distributive (modular) if and only if* $\{y \in S \mid y \leq x\}$ *is a distributive (modular) lattice for each* $x \in S$. ●

Let us consider the following two lattices: pentagonal lattice N_5 and Diamond lattice M_5. Many lattice theorists study on these two lattices and given several results.

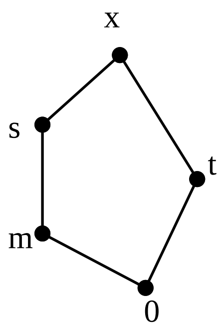

Figure-1.2

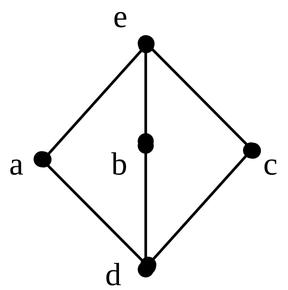

Figure-1.3

Hickman in [10] has given the following extensions of a very fundamental results of lattice theory.

Theorem 1.1.2. *A nearlattice S is distributive if and only if S does not contain a sublattice isomorphic to N_5 or M_5* [in Figure 1.2 and 1.3]. ●

Theorem 1.1.3. *A nearlattice S is modular if and only if S does not contain a sublattice isomorphic to N_5.* ●

In this context it should be mentioned that many lattice theorists have worked with a class of semilattice S which has the property that for each $x, a_1, a_2, \ldots, a_r \in S$, if $a_1 \vee a_2 \vee \ldots \vee a_r$ exists then $(x \wedge a_1) \vee (x \wedge a_2) \vee \ldots \vee (x \wedge a_r)$ exists and equals $x \wedge (a_1 \vee a_2 \vee \ldots \vee a_r)$. Bables [1] called them as prime semilattices while Shum [27] referred them as weakly distributive semilattices.

Hickman in [15] has defined a ternary operation j by $j(x,y,z) = (x \wedge y) \vee (y \wedge z)$, on a nearlattice S (which exists by the upper bound property of S). In fact he has shown, which can also be found in Lyndon [17] Theorem 4, that the resulting algebras of the type $(S; j)$ form a variety, which is referred to as the variety of join algebras and following are its defining identities.

(i) $j(x,x,x) = x$

(ii) $j(x,y,x) = j(y,x,y)$

(iii) $j(j(x,y,x),z,j(x,y,x)) = j(x,j(y,z,y),x)$

(iv) $j(x,y,z) = j(z,y,x)$

(v) $j(j(x,y,z),j(x,y,x),j(x,y,z)) = j(x,y,x)$

(vi) $j(j(x,y,x),y,z) = j(x,y,z)$

(vii) $j(x,y,j(x,z,x)) = j(x,y,x)$

(viii) $j(j(x,y,j(w,y,z)),j(x,y,z),j(x,y,j(x,y,z))) = j(x,y,z)$

We do not elaborate it further as it is beyond the scope of this thesis.

We call a nearlattice S a medial nearlattice if for all $x,y,z \in S$, $m(x,y,z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ exists. For a (lower) semilattice S, if $m(x,y,z)$ exists for all $x,y,z \in S$, then it is not hard to see that S has the upper bound property and hence is a nearlattice. Distributive medial nearlattices were first studied by Sholander [25, 26], and then by Evans [9]. Sholander preferred to call these as *medial semilattices*. He showed that every medial nearlattice S can be characterized by means of an algebra $(S;m)$ of type $\langle 3 \rangle$, known as *medial algebra*, satisfying the following two identities:

(i) $m(a,a,b) = a$

(ii) $m(m(a,b,c),m(a,b,d),e) = m(m(c,d,e),a,b)$.

A nearlattice S is said to have the **three** properties if for any $a,b,c \in S$, $a \vee b \vee c$ exists whenever $a \vee b$, $b \vee c$ and $c \vee a$ exists. Nearlattices with the **three** property were discussed by Evans [9], where he referred it as strong conditional lattices.

The equivalence of (i) and (iii) of the following lemma is trivial, while the proof of (i) < = > (ii) is inductive.

Lemma 1.1.4. {Evans [9]}. *For a nearlattice S the following conditions are equivalent:*

(i) *S has the* ***three*** *property.*

(ii) *Every pair of a finite number* $n\ (\geq 3)$ *of elements of S posses a supremum ensures the existence of the supremum of all the n elements.*

(iii) *S is medial.* ●

A family $\mathcal{A}$ of a subset of a set A is called a closure system on A if

(i) $A \in \mathcal{A}$ and

(ii) $\mathcal{A}$ is closed under arbitrary intersection.

Suppose B is a subfamily of *A*. B is called a directed system if for any $X, Y \in \mathsf{B}$ there exists Z in B such that $X, Y \subseteq Z$.

If $\cup\{X : X \in \mathsf{B}\} \in \mathsf{A}$ for every directed system B contained in the closure system A, then A is called algebraic. When ordered by set inclusion, an algebraic closure system forms an algebraic lattice.

1.2 Ideals of Nearlattices

A non-empty subset I of a nearlattice S is called a down set if for any $x \in S$ and $y \in I$, $x \leq y$ implies $x \in I$.

A non-empty subset I of a nearlattice S is called an ideal if it is a down set and closed under existent finite suprema. We denote the set of all ideals of S by $I(S)$, which is a lattice. If S has a smallest element 0 then $I(S)$ is an algebraic closure system on S and is consequently an algebraic lattice.

However, if S does not possess smallest element then we can only assert that $I(S) \cup \{\Phi\}$ is an algebraic closure system, where Φ is the empty subset of S.

For any subset K of a nearlattice S, $(K]$ denotes the ideal generated by K.

Infimum of two ideals of a nearlattice is their set theoretic intersection. Supremum of two ideals I and J in a lattice L is given by $I \vee J = \{x \in L \mid x \leq i \vee j \text{ for some } i \in I, j \in J\}$. Cornish and Hickman in [4] showed that in a distributive nearlattice S for two ideals I and J, $I \vee J = \{i \vee j \mid i \in I, j \in J \text{ where } i \vee j \text{ exists}\}$. But in a general nearlattice the formula for the supremum of two ideals is not very easy. Let us consider the following lemma which gives the formula for the supremum of two ideals. It is in fact an exercise in Gratzer [11], p-54 for partial lattice.

Theorem 1.2.1. *Let I and J be ideals of a nearlattice S. Let $A_0 = I \cup J$, $A_n = \{x \in S \mid x \leq y \vee z; y \vee z \text{ exists and } y, z \in A_{n-1}\}$ for $n = 1, 2, \ldots$, and $K = \bigcup_{n=0}^{\infty} A_n$. Then $K = I \vee J$.*

Proof: Since $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$, K is an ideal containing I and J. Suppose H is any ideal containing I and J. Of course, $A_0 \subseteq H$. We proceed by induction. Suppose $A_{n-1} \subseteq H$ for some $n \geq 1$ and let $x \in A_n$. Then $x \leq y \vee z$ with $y, z \in A_{n-1}$. Since $A_{n-1} \subseteq H$ and H is an ideal, $y \vee z \in H$ and so $x \in H$. That is $A_n \subseteq H$ for every n. Thus $K = I \vee J$.●

Theorem.1.2.2. *Let K be a non-empty subset of a nearlattice S. Then* $(K] = \bigcup_{n=0}^{\infty}\{A_n \mid n \geq 0\}$, *where* $A_0 = \{t \in S \mid t = j(k_1,t,k_2) \text{ for some } k_1,k_2 \in K\}$ *and* $A_n = \{t \in S \mid t = j(a_1,t,a_2) \text{ for some } a_1,a_2 \in A_{n-1}\}$ *for* $n \geq 1$.

Proof: For any $k \in K$ clearly $k = j(k,k,k)$ and so $K \subseteq A_0$. Similarly, for any $a \in A_{n-1}$, $a = j(a,a,a)$ implies that $A_{n-1} \subseteq A_n$. Thus $K \subseteq A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_{n-1} \subseteq A_n \subseteq \ldots$. Let $t \in \bigcup_{n=0}^{\infty} A_n\,; n = 0,1,2,\ldots$, and $t_1 \in S$ such that $t_1 \leq t$. Then $t \in A_m$ for some $m \geq 0$. Clearly, $t_1 = j(t,t_1,t)$ and so $t_1 \in A_{m+1}$. Thus $\bigcup_{n=0}^{\infty} A_n$ is down set.

Now suppose, $t_1,t_2 \in \bigcup_{n=0}^{\infty} A_n$ and $t_1 \vee t_2$ exists. Let $t_1 \in A_r$ and $t_2 \in A_s$ for some $r,s \geq 0$ with $r \leq s$ (say). Then $t_1,t_2 \in A_s$ and $t_1 \vee t_2 = j(t_1, t_1 \vee t_2, t_2)$ provides $t_1 \vee t_2 \in A_{s+1}$.

Finally, suppose H is an ideal containing K. If $x \in A_0$, then $x = j(k_1,x,k_2) = (k_1 \wedge x) \vee (k_2 \wedge x)$ for some $k_1,k_2 \in K$. As $K \subseteq H$ and H is an ideal, $k_1 \wedge x, k_2 \wedge x \in H$ and so $x \in H$. Thus $A_0 \subset H$. Again we use the induction. Suppose $A_{n-1} \subseteq H$ for some $n \geq 1$. Let $x \in A_n$ so that $x = j(a_1,x,a_2)$ for some $a_1,a_2 \in A_{n-1}$. Then $x \in H$ as $a_1,a_2 \in H$ and $x = (a_1 \wedge x) \vee (a_2 \wedge x)$.●

Theorem 1.2.3. *A non empty subset K of a nearlattice S is an ideal if and only if $x \in K$ whenever $x \in S$ and $x = j(k_1,x,k_2)$ for some $k_1,k_2 \in K$.*●

We now give an alternative formula for the supremum of two ideals in an arbitrary nearlattice.

Theorem 1.2.4. *For any two ideals K_1 and K_2,* $K_1 \vee K_2 = \bigcup_{n=0}^{\infty} B_n$ *where* $B_0 = \{x \in S \mid x = j(k_1,x,k_2), k_i \in K_i\}$ *and* $B_n = \{x \in S \mid x = j(b_1,x,b_2), b_1,b_2 \in B_{n-1}\}$, $n = 1, 2, \ldots$.

Proof : Clearly, $K_1,K_2 \subseteq B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq B_{n-1} \subseteq B_n \subseteq \ldots$. Suppose $b \in \bigcup_{n=0}^{\infty} B_n$ and $b_1 \leq b;\ b_1 \in S$. Then $b \in B_m$ for some $m \geq 0$. Also, $b_1 = j(b,b_1,b)$ and so $b_1 \in B_{m+1}$. Thus $\bigcup_{n=0}^{\infty} B_n$ is a down set. Now suppose $t_1,t_2 \in \bigcup_{n=0}^{\infty} B_n$ such that $t_1 \vee t_2$ exists. Then there exist $r, s \geq 0$ such that $t_1 \in B_r$ and $t_2 \in B_s$. If $r \leq s$ then $t_1, t_2 \in B_s$ and $t_1 \vee t_2 = j(t_1, t_1 \vee t_2, t_2)$ implies that $t_1 \vee t_2 \in B_{s+1}$. Hence , $\overset{\infty}{\underset{n=0}{\cup}} B_n$ is an ideal.

Finally, suppose H is an ideal containing K_1 and K_2. If $x \in B_0$ then $x = j(k_1, x, k_2) = (k_1 \wedge x) \vee (k_2 \wedge x)$ for some $k_1 \in K_1$ and $k_2 \in K_2$. Hence H is an ideal and $K_1, K_2 \subseteq H$, clearly $x \in H$. Then using the induction on n it is very easy to see that $H \supseteq B_n$ for each n. ●

In a lattice L, it is well known that for a convex sublattice C of L. $C = (C] \cap [C)$. The following figure (Fig:1.4) shows that for a convex subnearlattice C in a general nearlattice, this may not be true.

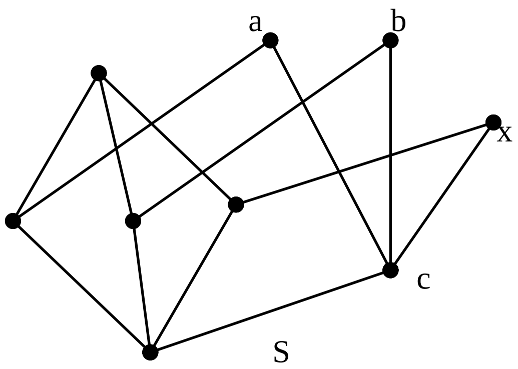

Figure 1.4

Here $C = \{a, b, c\}$ is a convex subnearlattice of S. Observe that $(C] = S$ and $[C) = \{a, b, c. x\}$, hence $(C] \cap [C) \neq C$.

Recently, Shiuly Akhter [28] has proved that for a convex sublattice C of a distributive nearlattice S, $(C] = \{x \in S \mid x = (x \wedge c_1) \vee (x \wedge c_2) \vee \ldots \vee (x \wedge c_n)$ *for some* $c_1, c_2, \ldots, c_n \in C\}$. With the help of this result Rosen [24] have proved that $C = (C] \cap [C)$ when S is distributive. But in a non-distributive nearlattice of S, it is easy to show that $C = (C] \cap [C)$ when S is medial.

Theorem 1.2.5. {Cornish and Hickman [4, Theorem 1.1]}. *The following conditions on a nearlattice S are equivalent:*

(i) S *is distributive.*

(ii) *For any* $H \in H(S)$, $(H] = \{h_1 \vee h_2 \vee \ldots \vee h_n \mid h_1, h_2, \ldots, h_n \in H\}$.

(iii) *For any* $I, J \in I(S)$, $I \vee J = \{a_1 \vee a_2 \vee \ldots \vee a_n \mid a_1, a_2, \ldots, a_n \in I \cup J\}$.

(iv) *$I(S)$ is a distributive lattice.*

(v) *The map $H \to (H]$ is a lattice homomorphism of $H(S)$ onto $I(S)$*

(*which preserves arbitrary suprema*). ●

Observe here that by Theorem1.2.4, (iii) of above could easily be improved to (iii)$'$: For any $I, J \in I(S)$, $I \vee J = \{i \vee j \mid i \in I, j \in J\}$.

Let $I_f(S)$ denote the set of all *finitely generated ideals* of a nearlattice S. Of course $I_f(S)$ is an upper subsemilattice of $I(S)$. Also for any $x_1, x_2, \ldots, x_m \in S$, $(x_1, x_2, \ldots, x_m]$ is clearly equal to $(x_1] \vee (x_2] \vee \cdots \vee (x_m]$. When S is distributive, $(x_1, x_2, \ldots, x_m] \cap (y_1, y_2, \ldots, y_m] = ((x_1] \vee (x_2] \vee \cdots \vee (x_m]) \cap ((y_1] \vee (y_2] \vee \cdots \vee (y_m])$ $= \underset{i\,j}{\vee}(x_i \wedge y_j]$ for any $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n \in S$ and so $I_f(S)$is a distributive sublattice of $I(S)$.

A nearlattice S is said to be *finitely smooth* if the intersection of two finitely generated ideals is itself finitely generated. For example, distributive nearlattices, finite nearlattices, lattices, are finitely smooth. Hickman in [15] exhibited a nearlattice which is not finitely smooth.

From Cornish and Hickman [4], we know that a nearlattice S is distributive if and only if $I(S)$ is so. Our next result shows that the case is not the same with the modularity.

Theorem 1.2.6. *Let S be a nearlattice. If $I(S)$is modular then S is also modular but the converse is not necessarily true.*

Proof: Suppose $I(S)$ is modular. Let $a, b, c \in S$ with $c \le a$ and $b \vee c$ exists. Then $(c] \subseteq (a]$. Since $I(S)$ is modular, so, $(a \wedge (b \vee c)] = (a] \wedge ((b] \vee (c])$ $= ((a] \wedge (b]) \vee (c] = ((a \wedge b) \vee c]$. This implies that $a \wedge (b \vee c) = (a \wedge b) \vee c$, and so S is modular.

Nearlattice S of Figure 1.5 shows that the converse of this result is not true.

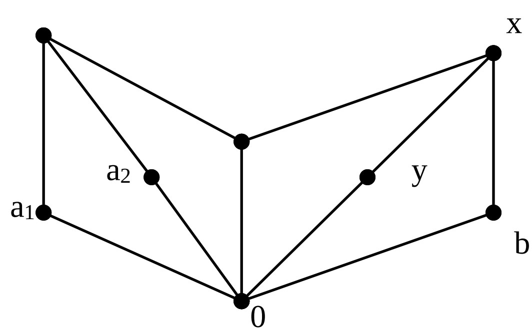

Figure 1.5

Notice that $(r]$ is modular for each $r \in S$. But in $I(S)$, clearly $\{(0],(a_1],(a_1,y],(a_2,b],S\}$ is a pentagonal sublattice. ●

The following theorem is due to Bazlar Rahman [2]

Theorem 1.2.7. {Bazlar Rahman[2]} *Let I and J be two ideals in a distributive nearlattice S. If $I \wedge J$ and $I \vee J$ are principal, then both I and J are principal.* ●

A non empty subset F of a nearlattice S is called an up set if for $x \in S$, $y \in F$ with $x \geq y$ imply $x \in F$.

A non empty subset F of a nearlattice S is called a filter if it is an up set and $f_1 \wedge f_2 \in F$ for all $f_1, f_2 \in F$.

An ideal P in a nearlattice S is called a prime ideal if $P \neq S$ and $x \wedge y \in P$ implies $x \in P$ or $y \in P$.

A filter F is called a prime filter if either $x \in F$ or $y \in F$ whenever $x \vee y$ exists and is in F.

It is not hard to see that a filter F of a nearlattice S is prime if and only if $S - F$ is a prime ideal. The set of all filters of a nearlattice is an upper (join) semilattice ; yet it is not a lattice in general, as there is no guarantee that the intersection of two filters is non empty. The join $F_1 \vee F_2$ of two filters is given by $F_1 \vee F_2 = \{s \in S \mid s \geq f_1 \wedge f_2 \text{ for some } f_1 \in F_1, f_2 \in F_2\}$. The smallest filter containing a subsemilattice H of S is $\{s \in S \mid s \geq h \text{ for some } h \in H\}$ and is denoted by $[H)$. Moreover, the description of the join of filters shows that for all $a,b \in S$, $[a) \vee (b] = [a \wedge b)$.

Following theorem and corollary is due to Noor and Rahman [22] which is an extension of Stone's separation theorem of Gratzer [11] theorem 15, pp74.

Theorem 1.2.8. {Noor and Rahman[22]**}** *Let* S *be a nearlattice. The following conditions are equivalent:*

(i) S *is distributive.*

(ii) *For any ideal* I *and any filter* F *of* S, *such that* $I \cap F = \Phi$, *there exists a prime ideal* $P \supseteq I$ *and disjoint from* F *.*●

Corollary 1.2.9. *A nearlattice* S *is distributive if and only if every ideal is the intersection of all prime ideals containing it.*●

Lemma 1.2.10. *A subset* F *of a nearlattice* S *is a filter if and only if* $S-F$ *is a prime down set.*

Proof: Let $x \in S-F$ and $t \le x$. Then $x \notin F$, and so $t \notin F$, as F is a filter. Hence $t \in S-F$, and so $S-F$ is a down set. Now let $x, y \in S$ such that $x \wedge y \in S-F$. It follows that $x \wedge y \notin F$. This implies either $x \notin F$ or $y \notin F$, as F is a filter. That is, either $x \in S-F$ or $y \in S-F$, and so $S-F$ is a prime down set.

Conversely, suppose $S-F$ is a prime down set. Let $x \in F$ and $t \ge x$. Then $x \notin S-F$ and so $t \notin S-F$ as $S-F$ is a prime down set. Thus $t \in F$ and so F is an upset. Finally let $x, y \in F$. Then $x \notin S-F$, $y \notin S-F$. Since $S-F$ is a prime, so $x \wedge y \notin S-F$. Therefore $x \wedge y \in F$, and so F is a filter. ●

Following result is an easy consequence of above lemma.

Lemma 1.2.11. A subset F *of a nearlattice* S *is a prime filter if and only if* $S-F$ is a prime ideal.●

Now we include a generalization of theorem 1.2.8 in a general nearlattice.

Theorem 1.2.12. *Let* S *be a nearlattice.* F *be a filter and* I *be a down set such that* $I \cap F = \Phi$ *. Then there exists a prime down set* P *containing* I *but disjoint to* F *.*

Proof: Let χ be the collection of all filter containing F *and disjoint to* I . Then χ is non-empty as $F \in \chi$. Suppose C is a chain in χ. Set $M = \cup\{X \mid X \in C\}$. Let $x \in M$ and $y \ge x$. Then $x \in X$ for some $X \in C$. Since X is a filter , so $y \in X$ and hence $y \in M$. Thus M is an upset. Now let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in C$. Since C is a chain , so either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$. This implies $x, y \in Y$, and so

$x \wedge y \in Y$ as Y is a filter. It follows that $x \wedge y \in M$ and hence, M is a filter containing F. Moreover $M \cap I = \phi$. Therefore, M is the largest element of C. Thus by Zorn's lemma, M is a maximal filter containing F. Therefore by Lemma 1.2.10, $L - M$ is a minimal prime down set containing I but disjoint to F.●

Corollary 1.2.13. *Let S be a nearlattice with 0 and F be a proper filter of S. Then there exists a prime down set P such that $F \cap P = \Phi$.*●

The following lemma is very useful in proving many results of distributive nearlattice.

Lemma 1.2.14. *If S_1 is a subnearlattice of a distributive nearlattice S and P_1 is a prime ideal in S_1, then there exists a prime ideal P in S such that $P_1 = S_1 \cap P$.*●

Following theorem is a generalization of Lemma 1.2.14, which will be needed in establishing some results in other chapters.

Theorem 1.2.15. *Let S_1 be a subnearlattice of S. and P_1 be a prime down set of S_1. Then there exists a prime down set P of S such that $P_1 = P \cap S_1$.*
Proof: Let H be a down set generated by P_1 in S. Then $H \cap (S_1 - P_1) = \Phi$. Now $S_1 - P_1$ is an upset in S_1 and $H \cap [S_1 - P_1) = \Phi$ where, $[S_1 - P_1)$ is the filter generated by $S_1 - P_1$ in S. Then by Theorem 1.2.12 , there exists a prime down set $P \supseteq H$ and disjoint to $[S_1 - P_1)$. Now $P_1 \subseteq H \cap S_1 \subseteq P \cap S_1$. Also $P \cap S_1 \subseteq P_1$. Hence, $P_1 = P \cap S_1$.●

1.3 Congruences

An equivalence relation Θ of a nearlattice S is called a congruence relation if $x_i \equiv y_i(\Theta)$ for $i = 1,2$ $(x_i, y_i \in S)$, then

(i) $x_1 \wedge x_2 \equiv y_1 \wedge y_2(\Theta)$, and

(ii) $x_1 \vee x_2 \equiv y_1 \vee y_2(\Theta)$ provided $x_1 \vee x_2$ and $y_1 \vee y_2$ exists.

It can be easily shown that for an equivalence relation Θ on S, the above conditions are equivalent to the conditions that for $x, y \in S$ if $x \equiv y(\Theta)$, then

(i$^/$) $x \wedge t \equiv y \wedge t(\Theta)$ for all $t \in S$ and

(ii$^/$) $x \vee t \equiv y \vee t(\Theta)$ for all $t \in S$ provided both $x \vee t$ and $y \vee t$ exists.

The set $C(S)$ of all congruences on S is an algebraic closure system on $S \times S$ and hence, when ordered by set inclusion, is an algebraic lattice.

Cornish and Hickman [4] showed that for an ideal I of a distributive nearlattice S, the relation $\Theta(I)$, defined by $x \equiv y(\Theta(I))$ if and only if $(x] \vee I = (y] \vee I$, is the smallest congruence containing I as a class. Moreover the equivalence relation $R(I)$, is defined by $x \equiv y(R(I))$ if and only if for any $s \in S$, $s \wedge x \in I$ is equivalent to $s \wedge y \in I$. In fact , this is the largest congruence of S having I as a class.

Suppose S is a distributive nearlattice and $x \in S$ we will use Θ_x as an abbreviation for $\Theta((x])$. Moreover ψ_x denote the congruence, defined by $a \equiv b$ (ψ_x) if and only if $a \wedge x = b \wedge x$.

Cornish and Hickman [4] also showed that for any two elements a,b of a distributive nearlattice S with $a \le b$, the smallest congruence identifying a and b is equal to $\psi_a \cap \Theta_b$ and we denote it by $\Theta(a,b)$. Also in a distributive nearlattice S, they observed that if S has a smallest element 0, then clearly $\Theta_x = \Theta(0,x)$ for any $x \in S$.

Moreover, we see that:

(i) $\Theta_a \vee \psi_a = \tau$, the largest congruence of S.

(ii) $\Theta_a \cap \psi_a = \omega$, the smallest congruence of S and

(iii) $\Theta(a,b)' = \Theta_a \vee \psi_a$ where $a \le b$ and $(')$ denotes the complement.

Now suppose S is an arbitrary nearlattice and $E(S)$ denote the lattice of equivalence relations. For $\Phi_1,\Phi_2 \in E(S)$ with $\Phi_1 \vee \Phi_2$ denoting their supremum $x \equiv y(\Phi_1 \vee \Phi_2)$ if and only if there exist $x = z_0, z_1, \ldots, z_n = y$ such that $z_{i-1} \equiv z_i(\Phi_1 \text{ or } \Phi_2)$ for $i = 1,2,\ldots,n$.

The following result was stated by Gratzer and Lakser in [13] without proof and a proof given below, appeared in Cornish and Hickman [4].

Theorem 1.3.1 *For any nearlattice* S, $C(S)$ *is a distributive (complete) sublattice of* $E(S)$.

Proof: Suppose $\Theta,\Phi \in C(S)$, Define ψ to be the supremum of Θ *and* Φ in the lattice of equivalence relations $E(S)$ on S. Let $x \equiv y(\psi)$. Then there exists $x = z_0, z_1, \ldots, z_n = y$ such that $z_{i-1} \equiv z_i(\Phi_1 \text{ or } \Phi_2)$. Thus, for any $t \in S$, $z_{i-1} \wedge t \equiv z_i \wedge t(\Phi_1 \text{ or } \Phi_2)$ as $\Theta,\Phi \in C(S)$.

Hence $x \wedge t \equiv y \wedge t(\psi)$ and consequently ψ is a semilattice congruence. Then, in particular $x \wedge y \equiv x(\psi)$ and $x \wedge y \equiv y(\psi)$. To show that ψ is a congruence, let $x \equiv y(\psi)$, with $x \le y$, and choose any $t \in S$ such that both $x \vee t$ and $y \vee t$ exists. Then there exists $z_0, z_1, z_2, \ldots, z_n$ such that $x = z_0, z_n = y$ and $z_{i-1} \equiv z_i(\Phi_1 \text{ or } \Phi_2)$. Put $w_i = z_i \wedge y$ for all $i = 0,1,\ldots,n$. Then $x = w_0, w_n = y$, $w_{i-1} \equiv w_i(\Phi_1 \text{ or } \Phi_2)$. Hence by the upper bound property, $w_i \vee t$ exists for all $i = 0,1,\ldots,n$ (as $w_i \vee t \le y \vee t$) and $w_{i-1} \vee t \equiv w_i \vee t(\Phi_1 \text{ or } \Phi_2)$ for all $i = 0,1,\ldots,n$ (as $\Theta,\Phi \in C(S)$), *i.e.* $x \vee t \equiv y \vee t(\psi)$. Then by Cornish and Noor [5] Lemma 2.3 ψ is a congruence on S. Therefore, $C(S)$ is a sublattice of the lattice $E(S)$.

To show the distributivity of $C(S)$, let $x \equiv y(\Theta \cap (\Theta_1 \vee \Theta_2))$. Then $x \wedge y \equiv y(\Theta)$ and $x \wedge y \equiv y(\Theta_1 \vee \Theta_2)$. Also $x \wedge y \equiv x(\Theta)$ and $x \wedge y \equiv x(\Theta_1 \vee \Theta_2)$.

Since $x \wedge y \equiv y(\Theta_1 \vee \Theta_2)$, there exists $t_0, t_1, \ldots, t_n$ such that (as we have seen in the proof of the first part), $x \wedge y = t_0, t_n = y$, $t_{i-1} \equiv t_i(\Theta_1 \text{ or } \Theta_2)$ and $x \wedge y = t_0 \le t_i \le y$ for each $i = 0,1,\ldots,n$. Hence $t_{i-1} \equiv t_i(\Theta)$ for all $i = 0,1,\ldots,n$ and so $t_{i-1} \equiv t_i(\Theta \cap \Theta_1)$ or $t_{i-1} \equiv t_i(\Theta \cap \Theta_2)$. Thus $x \wedge y \equiv y((\Theta \cap \Theta_1) \vee (\Theta \cap \Theta_2))$. By symmetry, $x \wedge y \equiv x((\Theta \cap \Theta_1) \vee (\Theta \cap \Theta_2))$ and the proof completes by transitivity of the congruences.●

In lattice theory it is well known that a lattice is distributive if and only if every ideal is a class of some congruence. Following theorem gives a generalization of this result in case of nearlattices.

This also characterizes the distributivity of a nearlattice, which is an extension of Cornish and Hickman [4] Theorem 3.1.

Thoerem 1.3.2. A nearlattice S *is distributive if and only if every ideal is a class of some congruence*.

Proof: Suppose S is distributive . Then by Cornish and Hickman [4] Theorem 3.1 for each ideal I of S $\Theta(I)$ is the smallest congruence containing I as a congruence class.

To prove the converse, let each ideal of S be a congruence class with respect to some congruence on S. Suppose S is not distributive. Then by Theorem 1.1.2, we have either N_5 (Figure 1.2) or M_5 (Figure 1.3) as a sublattice of S. In both cases consider $I=(a]$ and suppose I is a congruence class with respect to Θ. Since $d \in I$, $d \equiv a(\Theta)$. Now $b=b \wedge c=b \wedge (a \vee c) \equiv b \wedge (d \vee c)=b \wedge c=d(\Theta)$ That is, $b \equiv d(\Theta)$ and this implies $b \in I$, *i.e.* $b \leq a$ which is a contradiction. Thus S is distributive. ●

Following results are due to Noor and Rahman [22].

Theorem 1.3.3.{ Noor and Rahman [22]} *Let* S *be a distributive nearlattice then,*

(i) *For ideals* I *and* J , $\Theta\ (I \cap J)=\Theta(I) \cap \Theta(J)$.

(ii) *For ideals* j_i $i \in A$ *an indexed set,* $\Theta\ (\vee J_i)=\vee\Theta(J_i)$. ●

Theorem 1.3.4.{ Noor and Rahman [22]} *For a distributive nearlattice S, the mapping* $I \to \Theta(I)$ *is an embedding from the lattice of ideals to the lattice of congruences.* ●

CHAPTER II

SOME SPECIAL ELEMENTS IN A NEARLATTICE

2.1 Introduction :

Gratzer and Schmidt [14] introduced the notion of some special elements e.g. distributive, standard and neutral elements to study a larger class of non-distributive lattices. Then Cornish and Noor [5] extended the concepts of standard and neutral elements for nearlattices. They also studied a new type of element known as strongly distributive element.

Recently Talukder and Noor [29] introduced the notion of modular elements in a join semilattice directed below. This notion is also applicable for general lattices.

In this chapter we introduce the concept of modular elements in a nearlattice. We have given several characterization of modular and strongly distributive elements. So therefore, by studying these elements and ideals, we will be able to study a larger class of non-distributive nearlattices.

In a lattice L, an element $m \in L$ is called a *modular element* if for all $x, y \in L$ with $y \leq x$, $x \wedge (m \vee y) = (x \wedge m) \vee y$. Of course, in a modular lattice, every element is a modular element. Moreover, if every element of a lattice is modular, then the lattice itself is a modular lattice.

In the pentagonal lattice of Figure 1.2, observe that *m* is modular but *t* is not. Because, here $m < s$ and $s \wedge (t \vee m) = s$, But $(s \wedge t) \vee m = m$.

Let S be a nearlattice. An element $m \in S$ is called a *modular element* if for all $t, x, y \in S$ with $y \leq x$, $x \wedge [(t \wedge m) \vee (t \wedge y)] = (t \wedge m \wedge x) \vee (t \wedge y)$. Of course, a nearlattice is modular if and only if its every element is modular.

In a lattice L, an element d is called a *distributive element* if for all $x, y \in L$, $d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y)$.

In order to introduce this notion for nearlattices, Cornish and Noor [5] could not give a suitable definition for distributive elements. But they discovered an element $d \in S$, such that $t \wedge d$ is a distributive element in the lattice $(t]$ for every $t \in S$. They found that these elements are also new even in case of lattices, and in fact, they are much stronger than the distributive elements. So they referred them as "*strongly distributive*" elements.

An element d of a nearlattice S is called a *strongly distributive* element if for all $t, x, y \in S$ $(t \wedge d) \vee (t \wedge x \wedge y) = [(t \wedge d) \vee (t \wedge x)] \wedge [(t \wedge d) \vee (t \wedge y)]$.
In other words $t \wedge d$ is distributive in $(t]$ for each $t \in S$.

An element $s \in S$ is called a standard element if for all $t, x, y \in S$, $t \wedge [(x \wedge y) \vee (x \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$.

Due to Zaidur Rahman and Noor [30] we know that $s \in S$ is standard if and only if it is both modular and strongly distributive.

An element $s \in S$ is called neutral if (i) it is standard and (ii) for all $x, y, t \in S$, $s \wedge [(t \wedge x) \vee (t \wedge y)] = (s \wedge t \wedge x) \vee (s \wedge t \wedge y)$.

In this chapter we give several characterizations of modular, strongly distributive, standard and neutral elements of a nearlattice.

2.2 Some special elements in a nearlattice

Theorem 2.2.1 *The definition of modular element in a nearlattice S coincides with the definition of modular element of a lattice, when S is a lattice.*

Proof: Suppose m is a modular element of the lattice S. Let $t, x, y \in S$ with $y \le x$, then $t \wedge y \le t \wedge x$. Since m is modular, so $(t \wedge m \wedge x) \vee (t \wedge y) = (t \wedge x) \wedge [m \vee (t \wedge y)]$ $= x \wedge [t \wedge (m \vee (t \wedge y))] = x \wedge [(t \wedge m) \vee (t \wedge y)]$, which is the definition of modularity of m in a nearlattice.

Conversely, Let m be modular according to the definition given for a nearlattice. Let $x, y \in S$ with $y \le x$.

Choose $t = m \vee y$. Then $x \wedge (m \vee y) = x \wedge ((t \wedge m) \vee (t \wedge y))$

$$= (t \wedge m \wedge x) \vee (t \wedge y)$$

$$= (m \wedge x) \vee y$$

Hence m is modular according to the definition of modular element in a nearlattice. ●

Here is a characterization of modular elements in a lattice.

Theorem 2.2.2 *Let L be a lattice and* $m \in L$. *Then the following conditions are equivalent.*

(i) *m is modular.*

(ii) *For* $y \le x$ *with* $m \vee x = m \vee y$ *and* $m \wedge x = m \wedge y$ *implies* $x = y$.

Proof: $(i) \Rightarrow (ii)$; Suppose m is modular $y \le x$ and $m \vee x = m \vee y$, $m \wedge x = m \wedge y$.

Then $x = x \wedge (m \vee x) = x \wedge (m \vee y) = (x \wedge m) \vee y$ (by modularity of m)

$$= (y \wedge m) \vee y = y \ .$$

$(ii) \Rightarrow (i)$; Suppose (ii) holds.

Let $y \le x$, then $(x \wedge m) \vee y \le x \wedge (m \vee y)$ always holds.

Let $x \wedge (m \vee y) = p$ and $(x \wedge m) \vee y = q$. Then $q \le p$.

Now $p \wedge m = x \wedge m$

Also, $q \wedge m = m \wedge [(x \wedge m) \vee y] = m \wedge [(x \wedge m) \vee (x \wedge y)] = (m \wedge x) \wedge [(x \wedge m) \vee (x \wedge y)] = x \wedge m$.

Thus $p \wedge m = q \wedge m$.

Again, $q \vee m = y \vee m$

$$p \vee m = [x \wedge (m \vee y)] \vee m \leq (m \vee y) \vee m$$
$$= y \vee m = q \vee m \leq p \vee m$$

as $q \leq p$. Thus $p \vee m = q \vee m = y \vee m$.

Hence by (ii) $p = q$, that is $x \wedge (m \vee y) = (x \wedge m) \vee y$ and so m is modular. ●

Now we extend the above result and give a characterization of a modular element m in a nearlattice.

Theorem 2.2.3 *Let* S *be a nearlattice and* $m \in S$. *Then the following conditions are equivalent.*

(i) *m is modular*.

(ii) *For* $t, x, y \in S$ *with* $y \leq x$, $(t \wedge m) \vee (t \wedge x) = (t \wedge m) \vee (t \wedge y)$ *and* $t \wedge m \wedge x = t \wedge m \wedge y$ *implies* $t \wedge x = t \wedge y$.

Proof: $(i) \Rightarrow (ii)$; Suppose m is modular, let $t, x, y \in S$ with $y \leq x$, $(t \wedge m) \vee (t \wedge x) = (t \wedge m) \vee (t \wedge y)$ and $t \wedge m \wedge x = t \wedge m \wedge y$.

Then $t \wedge x = (t \wedge x) \wedge [(t \wedge m) \vee (t \wedge x)] = (t \wedge x) \wedge [(t \wedge m) \vee (t \wedge y)]$

$= (t \wedge m \wedge x) \vee (t \wedge y)$ (by modularity of m)

$= (t \wedge m \wedge y) \vee (t \wedge y) = t \wedge y$.

$(ii) \Rightarrow (i)$; Suppose (ii) holds. Let $t, x, y \in S$ with $y \leq x$

Now $x \wedge [(t \wedge m) \vee (t \wedge y)] \geq (t \wedge m \wedge x) \vee (t \wedge y)$ always holds.

Let $x \wedge [(t \wedge m) \vee (t \wedge y)] = p$ and $(t \wedge m \wedge x) \vee (t \wedge y) = q$. Then $p \geq q$.

Choose $r = (t \wedge m) \vee (t \wedge y)$. Then $r \wedge p = p$ and $r \wedge q = q$.

$$r \wedge m = m \wedge [(t \wedge m) \vee (t \wedge y)] = (t \wedge m) \wedge [(t \wedge m) \vee (t \wedge y)] = t \wedge m.$$

Thus, $(r \wedge m) \vee (r \wedge q) = (t \wedge m) \vee q = (t \wedge m) \vee (t \wedge m \wedge x) \vee (t \wedge y) = (t \wedge m) \vee (t \wedge y) = r$.

Then $(r \wedge m) \vee (r \wedge p) \leq r = (r \wedge m) \vee (r \wedge q) \leq (r \wedge m) \vee (r \wedge p)$ as $q \leq p$

Hence $(r \wedge m) \vee (r \wedge p) = (r \wedge m) \vee (r \wedge q) = r$,

Also, $r \wedge m \wedge p = m \wedge p = m \wedge x \wedge [(t \wedge m) \vee (t \wedge y)] = x \wedge (t \wedge m) \wedge [(t \wedge m) \vee (t \wedge y)] = x \wedge t \wedge m$

and $r \wedge m \wedge q = m \wedge q = m \wedge [(t \wedge m \wedge x) \vee (t \wedge y)] = m \wedge t \wedge x \wedge [(t \wedge m \wedge x) \vee (t \wedge y)] = x \wedge t \wedge m$.

Thus $r \wedge m \wedge p = r \wedge m \wedge p$ and so by (ii) $r \wedge p = r \wedge q$, Hence $p = q$ and so m is modular. ●

Now we include the following result in a nearlattice which is parallel to the characterization theorem for modular elements in a lattice given in Theorem 2.2.2. But this cannot be considered as a definition of a modular element in a nearlattice.

Theorem 2.2.4 *Let S be a nearlattice and* $m \in S$. The *following conditions are equivalent.*

(i) *For all* $x, y \in S$ *with* $y \le x$

$x \wedge (m \vee y) = (x \wedge m) \vee y$ *provided* $m \vee y$ *exists*.

(ii) *For all* $x, y \in S$ *with* $y \le x$ *if* $m \vee x$, $m \vee y$ *exist and* $m \vee x = m \vee y$, $m \wedge x = m \wedge y$, *then* $x = y$.

Proof: $(i) \Leftrightarrow (ii)$ holds by the proof similar to the proof of Theorem.2.1.2 , For the last part, let us consider the following nearlattice.

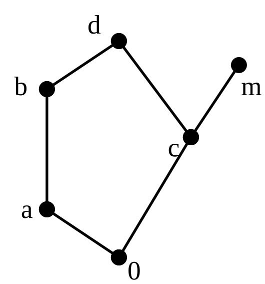

Figure-2.1

It is observed that *m* satisfies the condition of Theorem 2.1.4

Here $a < b$ and $b \wedge [(d \wedge m) \vee (d \wedge a)] = b \wedge (c \vee a) = b \wedge d = b$.

But $(b \wedge d \wedge m) \vee (d \wedge a) = 0 \vee a = a$, so *m* is not modular. ●

Theorem 2.2.5 *In a Lattice, every strongly distributive element is distributive but the converse is not necessarily true.*

Proof. Let *d* be a strongly distributive element of a lattice *L*. Suppose $x, y \in L$ and $t = x \vee y \vee d$.

Then $d \vee (x \wedge y) = (t \wedge d) \vee (t \wedge x \wedge y) = [(t \wedge d) \vee (t \wedge x)] \wedge [(t \wedge d) \vee (t \wedge y)]$

$= (d \vee x) \wedge (d \vee y)$, and so d is distributive.

Now consider the lattice in Figure 2.2.

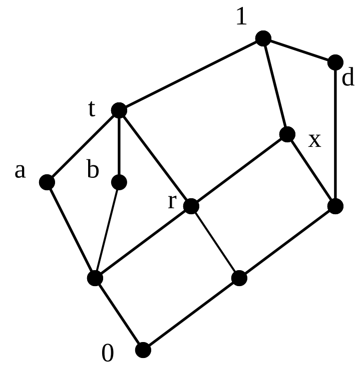

Figure 2.2

Here d is distributive but $(t \wedge d) \vee (t \wedge a \wedge b) = r < t = [(t \wedge d) \vee (t \wedge a)] \wedge [(t \wedge d) \vee (t \wedge b)]$ and so it is not strongly distributive. ●

Following characterization of strongly distributive elements in a nearlattice is due to Cornish and Noor [5].

Theorem 2.2.6 *Let* S *be a nearlattice and* $d \in S$. *Then the following conditions are equivalent.*

(i) *d is strongly distributive.*

(ii) *For all* $x, y, t \in S$, $(x \wedge [(t \wedge y) \vee (t \wedge d)]) \vee (t \wedge d) = (t \wedge x \wedge y) \vee (t \wedge d)$. ●

An element $s \in S$ is called a *standard element* if for all $t, x, y \in S$ $t \wedge [(x \wedge y) \vee (x \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$.

In a distributive nearlattice every element is standard. If every element of S is standard then S is itself a distributive nearlattice.

Theorem 2.2.7 *Every standard element in a nearlattice S is modular but a modular element may not be standard.*

Proof: Let $s \in S$ be standard, let $t, x, y \in S$ with $y \leq x$

$$x \wedge [(t \wedge s) \vee (t \wedge y)] = x \wedge [(t \wedge y) \vee (t \wedge s)]$$
$$= (t \wedge x \wedge y) \vee (t \wedge s \wedge x)$$
$$= (t \wedge s \wedge x) \vee (t \wedge y)$$

So s is modular.

Conversely, consider the lattice of Figure 1.2

Here m is modular

But $\quad s \wedge (m \vee t) = s \wedge x = s$

$(s \wedge m) \vee (s \wedge t) = m \vee 0 = m$

So m is not standard. ●

Theorem 2.2.8 *Every standard element is strongly distributive but the converse may not be true.*

Proof. Suppose s is standard in S. Let $t, a, b \in S$

Then, $\quad [(t \wedge s) \vee (t \wedge a)] \vee [(t \wedge s) \vee (t \wedge b)]$

$= ([(t \wedge s) \vee (t \wedge a)] \wedge (t \wedge s)) \vee ([(t \wedge s) \vee (t \wedge a)] \wedge (t \wedge b))$ (as s is standard.)

$= (s \wedge [(t \wedge a) \vee (t \wedge s)]) \vee (b \wedge [(t \wedge a) \vee (t \wedge s)])$

$= (t \wedge a \wedge s) \vee (t \wedge s) \vee (t \wedge a \wedge b) \vee (t \wedge a \wedge s)$

$= (t \wedge s) \vee (t \wedge a \wedge b)$

so s is strongly distributive.

In Figure 2.2, observes that t is strongly distributive, but it is not standard, because $d \wedge (x \vee t) > (d \wedge x) \vee (d \wedge t)$.●

Remark:

In the pentagonal lattice of Figure 1.2, m is modular and t is strongly distributive . Observe that, $m \leq s$ and $s \wedge (t \vee m) = s \wedge x = s$, but $(s \wedge t) \vee m = 0 \vee m = m$. Thus t is not modular. On the other hand, $(x \wedge m) \vee (x \wedge s \wedge t) = m \vee 0 = m$, but $[(x \wedge m) \vee (x \wedge s)] \wedge [(x \wedge m) \vee (x \wedge t)] = (m \vee s) \wedge (m \vee t) = s \wedge x = s$ implies m is not strongly distributive.

We conclude the section with the following characterization of standard elements in a nearlattice.

Theorem 2.2.9 *Let S be a nearlattice. An element* $s \in S$ *is standard if and only if it is both modular and strongly distributive.*

Proof: If s is standard then by Theorem 2.2.7 and Theorem 2.2.8, s is both modular and strongly distributive. Conversely, suppose s is both modular and strongly distributive. Let $t, x, y \in S$.

Then, $(t \wedge x \wedge y) \vee (t \wedge x \wedge s) = (t \wedge x) \wedge [(x \wedge s) \vee (t \wedge x \wedge y)]$ (as s is modular)

$= (t \wedge x) \wedge [(x \wedge s) \vee (t \wedge x)] \wedge [(x \wedge s) \vee (x \wedge y)]$ (as s is strongly distributive)

$= t \wedge x \wedge [(x \wedge s) \vee (x \wedge y)] = t \wedge [(x \wedge s) \vee (x \wedge y)]$

so s is standard. ●

2.3 Modular ideals in a nearlattice

An ideal M of a nearlattice S is called a modular ideal if it is a modular element of the ideal lattice $I(S)$. That is, M is modular if for all $I, J \in I(S)$ with $J \subseteq I$, $I \cap (M \vee J) = (I \cap M) \vee J$.

An ideal I of a nearlattice S is called a standard ideal if it is standard element of the ideal lattice $I(S)$.

Of course, every standard ideal of a nearlattice (lattice) is modular, but the converse need not be true. In this section we include several characterizations of modular ideals of a nearlattice.

Due to Cornish and Noor [5] we know that the supremum of two ideals in a nearlattice is not very easy to handle.

But due to Talukder and Noor [29], we know that for a standard ideal K of a nearlattice S and for any $J \in I(S)$, $K \vee J = \{k \vee j \mid k \in K, j \in J\}$

But in case of a modular ideal M of a nearlattice, we are unable to give a simple description of $M \vee J$. Even $x \in M \vee J$ does not imply $x \leq m \vee j$ for some $m \in M$ and $j \in J$.

For example, consider the following nearlattice S of Figure 2.3 and ideal lattice $I(S)$ of Figure 2.4.

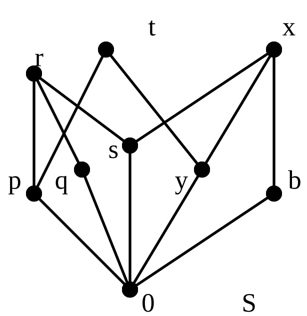

Figure 2.3

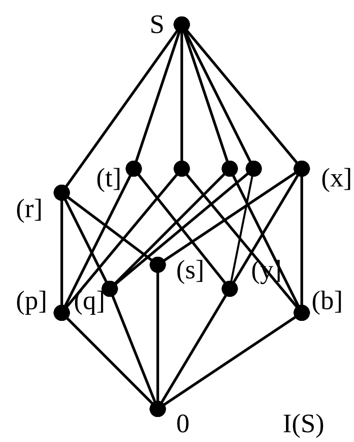

Figure 2.4

Here S is a modular nearlattice by Theorem 2.2.1. In $I(S)$, $(b]$ is modular. Now $q \in (t] \vee (b]$. But $q \nleq p \vee q$ for any $p \in (t]$ and $q \in (b]$.

Theorem 2.3.1 *Let L be a lattice and* $m \in L$, *m is modular if and only if* $(m]$ *is modular in* $I(L)$.

Proof : Suppose *m* is modular in *L*. Suppose $J \subseteq I$. Let $x \in I \cap ((m] \vee J)$.

Then $x \in I$ and $x \in (m] \vee J$.

This implies $x \leq m \vee j$ for some $j \in J$.

So $x \vee j \leq m \vee j$.

Now $j \in J \subseteq I$.

Thus $x \vee j \in I$ and $x \vee j = (x \vee j) \wedge (m \vee j) = ((x \vee j) \wedge m) \vee j$ (as *m* is modular)

$$\in (I \cap (m]) \vee J.$$

Therefore, $x \in (I \cap (m]) \vee J$.

Since the reverse inclusion is trivial, so $I \cap ((m] \vee J) = (I \cap (m]) \vee J$.

Hence $(m]$ is modular in $I(L)$.

Conversely, let $(m]$ be modular in $I(L)$.

Suppose $z \leq x$. Then $(x] \wedge ((m] \vee (z]) = ((x] \wedge (m]) \vee (z]$

That is, $(x \wedge (m \vee z)] = ((x \wedge m) \vee z]$

Therefore, $x \wedge (m \vee z) = (x \wedge m) \vee z$, and so *m* is modular. ●

Our next result shows that in a nearlattice S, Theorem 2.3.1 is not true.

Theorem 2.3.2 *For an element m of a nearlattice S, if* $(m]$ *is modular in* $I(S)$, *then m is modular, but the converse may not be true.*

Proof: Suppose $(m]$ is a modular ideal in *S*. Let $z \leq x$.

Then for all $t \in S$ $t \wedge z \leq t \wedge x \leq x$ implies $(t \wedge z] \subseteq (x]$.

Now $(t \wedge x] \wedge [(t \wedge m] \vee (t \wedge z]] \subseteq (t \wedge x] \wedge [(m] \vee (t \wedge z]]$

$$= ((t \wedge x] \wedge (m]) \vee (t \wedge z] \subseteq (t \wedge x] \wedge [(t \wedge m] \vee (t \wedge z]].$$

So $(t \wedge x] \wedge [(t \wedge m] \vee (t \wedge z]] = ((t \wedge x] \wedge (t \wedge m]) \vee (t \wedge z]$.

This implies $((t \wedge x) \wedge ((t \wedge m) \vee (t \wedge z))] = ((t \wedge x \wedge m) \vee (t \wedge z)]$.

And so, $x \wedge [(t \wedge m) \vee (t \wedge z)] = (t \wedge x) \wedge [(t \wedge m) \vee (t \wedge z)] = (x \wedge t \wedge m) \vee (t \wedge z)$

Therefore, *m* is modular in *S*.

To prove the converse, let us consider the following nearlattice and its ideal lattice.

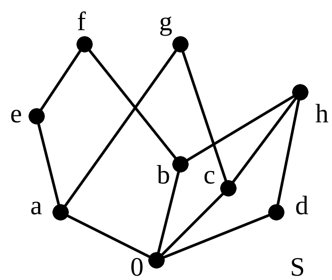

Figure 2.5

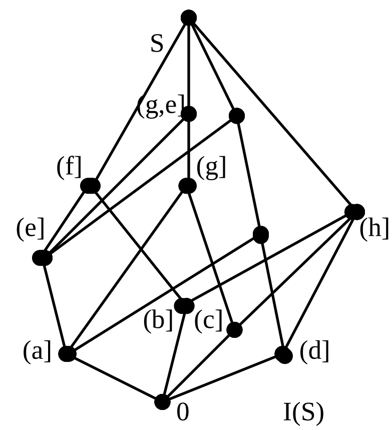

Figure 2.6

Here *d* is modular in *S*. But in $I(S)$ (Figure 2.6), $\{(0],(d],(g],(g,e],S\}$ is a pentagonal sublattice. Hence $(d]$ is not a modular ideal.●

Theorem 2.3.3 *Let S be a nearlattice,* $I, J \in I(S)$ *and* $I, J \in (a]$ *for some* $a \in S$. *Then* $I \vee J = \{x \in S \mid x \leq i \vee j$ *for some* $i \in I, j \in J\}$

Proof: Let $x \in I \vee J$. Then by Theorem 1.2.1, $x \leq i \vee j$ for some $i, j \in A_{n-1}$, where $A_0 = I \cup J$.

Since $i, j \in A_{n-1}$, so $i \leq i_1 \vee j_1$, $j \leq i_2 \vee j_2$ for some $i_1, i_2, j_1, j_2 \in A_{n-2}$.

Then $x \leq i_1 \vee i_2 \vee j_1 \vee j_2$, the supremum exists by the upper bound property of *S* as $i_1, i_2, j_1, j_2 \leq a$. Thus proceding in this way $x \leq (p_1 \vee \cdots \vee p_n) \vee (q_1 \vee \cdots \vee q_n)$ for some $p_i, q_i \in A_0 = I \cup J$, and the supremum exists by the upper bound property again.

Therefore, $x \leq i \vee j$ for some $i \in I$, $j \in J$. ●

Theorem 2.3.4 *Let M be a modular ideal of a nearlattice S and J be an ideal. If* $x \leq m \vee j$ *for some* $m \in M, j \in J$, *then* $x \vee j = m_1 \vee j$ *for some* $m_1 \in M$.

Proof: Let $x \leq m \vee j$, then $x \vee j \leq m \vee j$.

Thus, $x \vee j \in (x \vee j] \cap (M \vee (j]) = ((x \vee j] \cap M) \vee (j]$.

So by Theorem 2.3.3[3], $x \vee j \leq p \vee q$ for some $p \in (x \vee j] \cap M$ and $q \in (j]$.

Since $p \in (x \vee j] \cap M$, so $p \in M$ and $p \leq x \vee j$.

Thus $x \vee j \leq p \vee q \leq p \vee j \leq x \vee j$ implies $x \vee j = p \vee j$, where $p \in M$. ●

Here is a characterizations of modular ideals in a nearlattice.

Theorem 2.3.5 *Let M be an ideal of a nearlattice S with the condition that for all ideals J of S, and* $M \vee J = \{x \in S \mid x \leq m \vee j,\ m \vee j$ *exists for some* $m \in M, j \in J\}$. *Then the following conditions are equivalent.*

(i) *M is modular.*

(ii) $x \in M \vee J$ *implies* $x \vee j = m \vee j$ *for some* $m \in M$, $j \in J$.

Proof: $(i) \Rightarrow (ii)$; Suppose *M* is modular. Let $x \in M \vee J$. Then by the given condition, $x \leq m \vee j$ for some $m \in M$, $j \in J$.

Then by theorem 2.3.4,
$x \vee j = m_1 \vee j$ and so (ii) holds.

$(ii) \Rightarrow (i)$; Suppose (ii) holds.
Let $I, J \in I(S)$ with $J \subseteq I$
Suppose $x \in I \cap (M \vee J)$. Then $x \in I$ and $x \in M \vee J$.
Thus by given condition, $x \vee j = m \vee j$ for some $m \in M$, $j \in J$.
Now, $m \leq x \vee j$ implies $m \in I \cap M$.
Therefore, $x \in (I \cap M) \vee J$, and so $I \cap (M \vee J) \subseteq (I \cap M) \vee J$.
Since the reverse inclusion is trivial. so $I \cap (M \vee J) = (I \cap M) \vee J$.
Hence *M* is modular. ●

In lattices, we know from [29] that an element *m* is modular if and only if for all $b \leq a$ with $a \wedge m = b \wedge m$ & $a \vee m = b \vee m$ imply $a = b$.

We conclude the chapter with the following result which is proved by above characterization of modular elements.

Theorem 2.3.6 *Let M be a modular ideal of a nearlattice S. If* $I \cap M$ *and* $I \vee M$ *are principal, then I is principal.*
Proof: Let $I \vee M = (a]$ and $I \cap M = (b]$.
Then by Theorem 2.3.3, $a \leq i \vee m$ for some $i \in I, m \in M$.
Thus, $(a] = M \vee I \supseteq M \vee (b \vee i] \supseteq M \vee (i] \supseteq (a]$.

This implies $M \vee I = M \vee (b \vee i]$.

Also, $(b] = M \cap I \supseteq M \cap (b \vee i] \supseteq (b]$.

This implies $M \cap I = M \cap (b \vee i]$.

Moreover, $(b \vee i] \subseteq I$.

Therefore, $I = (b \vee i]$ as M is modular.●

CHAPTER III

STANDARD IDEAL

3.1 Introduction: Standard ideal in a lattice have been studied extensively by Gratzer and Schmidt [14] and Jamowitz [16]. Fried and Schmidt [10] and Niemeinen [19] have extended the idea to convex sublattices, also c. f. [6] and [7]. For the background materials on standard ideals we refer the reader to consult the text of Gratzer[11].

Cornish and Noor in [5] have generalized the concept of standard ideals to nearlattices. According to [5], an ideal A of a nearlattice S is called a standard ideal if it is a standard element of the ideal lattice $I(S)$. That is, for any ideal $I, J \in I(S), I \wedge (J \vee A) = (I \wedge J) \vee A$.

An element $s \in S$ is called neutral if

(i) it is standard and

(ii) $\forall x, y, t \in S, s \wedge [(t \wedge x) \vee (t \wedge y)] = (s \wedge t \wedge x) \vee (s \wedge t \wedge y)$.

In this chapter we have given an elaborate description of standard ideals in nearlattices.

In section 1 we have given a characterization of standard ideals which generalize a result of [14]. This is also an extension of result of [5]. We also show that if any standard ideal I both $I \wedge s$ and $I \vee s$ are principal, then I itself is principal.

3.2 Standard ideal and Neutral element

Lemma 3.2.1 *Let s be a standard element of the nearlattice S and a an arbitrary element of S. Then* $a \cap s$ *is a standard element of the principal ideal* $(a]$.

Proof: Any element of the ideal $(a]$ may be written in the form $a \cap x\ (x \in S)$. Hence it is enough to prove that $(x \cap a) \cap [(s \cap a) \cup (y \cap a)] = [(x \cap a) \cap (s \cap a)] \cup [(x \cap a) \cap (y \cap a)]$.

From L. S. of the above

$$(x \cap a) \cap [(s \cap a) \cup (y \cap a)] = (x \cap a) \cap [(s \cup y) \cap a]$$

$$= (x \cap a) \cap (s \cup y)$$

$$= (x \cap a \cap s) \cup (x \cap a \cap y)$$

$$= [(x \cap a) \cap (s \cap a)] \cup [(x \cap a) \cap (y \cap a)]$$

Hence the lemma is proved.●

Theorem 3.2.2 *Let I be an arbitrary ideal and s a standard ideal of the nearlattice S. If* $I \vee s$ *and* $I \wedge s$ *are principal, then I itself is principal.*

Proof: Let $I \vee s = (a]$ and $I \wedge s = (b]$. Then by theorem, $a = x \vee s_1$ for some $x \in I$ and $s_1 \in S$. Since $b \leq a$ and $x \leq a$, So $x \vee b$ exists. By the upper bound property of S, We claim that $I = (x \vee b]$. Of course, $(x \vee b] \subseteq I$. For the reverse inequality, Let $t \in I$. Since $t, x \vee b \leq a$, so again by the upper bound property of S, $w = t \vee x \vee b$ exists and $w \in I$ .Then $(a] \supseteq s \vee (w] \supseteq s \vee (x \vee b] \supseteq s \vee (x] = (a]$ i.e. $s \vee (w] = s \vee (x \vee b]$.Further, $(b] = s \cap I \supseteq s \cup (w] \supseteq s \cap (x \vee b] \supseteq s \cap (b] = (b]$ and so $s \cap (w] = s \cap (x \vee b]$. This two equalities imply that $(w] = (x \vee b]$ as s is standard and so $w = x \vee b \in (x \vee b]$. Since, $t \leq w, t \in (x \vee b]$ and hence $I = (x \vee b]$, this completes the proof.●

Theorem 3.2.3 *Let* s_1 *and* s_2 *be standard elements of the nearlattice S. Then the sub nearlattice* $\{s_1, s_2, x\}$ *of S is distributive for all* $x \in S$.

Proof: Our proof is based upon Th-II [14]. According to this, We have to prove the validity of

$a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$ (1)

$a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ (2)

$(a \cap b) \cup (b \cap c) \cup (c \cap a) = (a \cup b) \cap (b \cup c) \cap (c \cup a)$ (3)

Condition (2) is valid for it asserts the same as (9) of [14]. Since *b* or *c* is standard, as a consequence of condition (i) of δ of Th-1[14], (1) holds if a is standard. Otherwise *b* and *c* are standard. In this case let us start with the right member of (1), apply (9) of [14] for the elements $a, a \cup c$ for the standard element *b* and then for *a*, *b* and the standard element *c*. We get, $(a \cup c) \cap (a \cup b) = [(a \cup c) \cap a] \cup [(a \cup c) \cap b]$

$$= a \cup (a \cap b) \cup (c \cap b)$$

$$= a \cup (c \cap b)$$

Finally, we prove (3), (3) is a symmetric function of its variables, therefore we have to prove it for one permutation of its variables only. Using the assertion of Th-3 [14], according to which $s_1 \cup s_2$ and $s_1 \cap s_2$ are standard, further equality (9) of [14] and condition (i) of δ of Th- 1.

We get, $(s_1 \cap s_2) \cup (s_1 \cap x) \cup (s_2 \cap x) = (s_1 \cap s_2) \cup [(s_1 \cap s_2) \cup x]$

$$= [(s_1 \cap s_2) \cup (s_1 \cap s_2)] \cap [(s_1 \cap s_2) \cup x]$$

$$= (s_1 \cup s_2) \cap [(s_1 \cap s_2) \cup x]$$

$= (s_1 \cup s_2) \cap (s_1 \cup x) \cap (s_2 \cup x)$ and this is just (3).

Thus the proof is completed.●

Theorem 3.2.4 *Let s be a neutral element of (n] and n is neutral in A. Then s is a neutral element of a.*

Proof: By the previous theorem *s* is standard in *A*. To show that *s* is neutral, we need only to show that $s \wedge [(x \wedge y) \vee (x \wedge t)] = (s \wedge x \wedge y) \vee (s \wedge x \wedge t)$ for all $x, y, t \in A$.

Now, $s \wedge [(x \wedge y) \vee (x \wedge t)] = (s \wedge n) \wedge ((x \wedge y) \vee (x \wedge t)) = s \wedge (s \wedge x \wedge n) \vee (x \wedge t \wedge n)$

(as n is neutral.)

$$= (s \wedge x \wedge y \wedge n) \vee (s \wedge x \wedge t \wedge n)$$ (as s is neutral in *(n]*.)

$$= (s \wedge x \wedge y) \vee (s \wedge x \wedge t) .$$

The proof is thus complete.●

Theorem 3.2.5 *Let s and n be elements of a nearlattice A such that n is neutral,* $s \le n$ *and s is standard in (n]. Then s is a standard element of A.*

Proof: Let t, x, y be the elements of A. Then

$$[(x \wedge y) \vee (n \wedge s)] \vee [(x \wedge y) \vee (x \wedge n)]n))$$
$$= ([(x \wedge y) \vee (n \wedge s)] \wedge (x \wedge y)) \vee (((x \wedge y) \vee (n \wedge s)) \wedge (x \wedge n))$$

$$= (x \wedge y) \vee ((x \wedge n) \wedge [(x \wedge y \wedge n) \vee (n \wedge s)])$$ as n is neutral.

$$= (x \wedge y) \vee (x \wedge y \wedge n) \vee (x \wedge n \wedge s)$$ as s is standard in (n]

$$= (x \wedge y) \vee (x \wedge n \wedge s)$$

$$= (x \wedge y) \vee (x \wedge s)$$

Hence using the neutrality of n

$$t \wedge [(x \wedge y) \vee (x \wedge s)]$$

$$= t \wedge [(x \wedge y) \vee (n \wedge s)] \wedge ((x \wedge y) \vee (x \wedge n))$$

$$= ((x \wedge y) \vee (n \wedge s)) \wedge t \wedge ((x \wedge y) \vee (x \wedge n))$$

$$= ((x \wedge y) \vee (n \wedge s)) \wedge ((t \wedge x \wedge y) \vee (t \wedge x \wedge n))$$ as n is neutral.

$$= (t \wedge x \wedge y) \vee [(t \wedge x) \wedge ((x \wedge y \wedge n) \vee (n \wedge s))].$$

$$= (t \wedge x \wedge y) \vee (t \wedge x \wedge n) \wedge ((x \wedge y \wedge n) \vee (n \wedge s))$$

$$= (t \wedge x \wedge y) \vee (t \wedge x \wedge y \wedge n) \vee (t \wedge x \wedge s \wedge n)$$ since s standard in (n].

$$= (t \wedge x \wedge y) \vee (t \wedge x \wedge s) .$$

So s is standard in A.●

Let A be a nearlattice and s be an element of A. Then s is said to standard if $\forall\, x, y, t \in A$, $t \wedge [(x \wedge y) \vee (x \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$. (Notice that both sides exist by the upper bound property).

Obviously, any element of a distributive nearlattice is standard. Now suppose s is a standard element of a lattice L, then $\forall\, x, y, t \in L$,

$$t \wedge [(x \wedge y) \vee (x \wedge s)] = t \wedge [x \wedge (y \vee s)] = (t \wedge x) \wedge (y \vee s) = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$$

This and a part of following proposition show that the two concepts coincide in a lattice.

Proposition 3.2.6 *The following two conditions for an arbitrary element s of a nearlattice A are equivalent.*

(i) *For any $x, y \in A$, $x \wedge (y \wedge s) = (x \wedge y) \vee (x \wedge s)$ where $y \vee s$ exists.*

(ii) *(a) If $x \vee s$ and $y \vee s$ exist for any $x, y \in A$ then $(x \wedge y) \vee s$ exists and $(x \wedge y) \vee s = (x \vee s) \wedge (y \vee s)$.*

(b) For any $x, y \in A$, for which $x \vee s \geq y \vee s$ imply $x \geq y$.

Proof: $(i) \Rightarrow (ii)$

Suppose $x, y \in A$ are such that $x \vee s$ and $y \vee s$ exist. Then, $(x \wedge y) \vee s$ exists because of the upper bound property of A. Due to (i),

$$(x \vee s) \wedge (y \vee s) = [(x \vee s) \wedge y] \vee [(x \vee s) \wedge s] = (x \wedge y) \vee (s \wedge y) \vee s = (x \wedge y) \vee s.$$

Also if $x \wedge s \geq y \wedge s$ and $x \vee s \geq y \vee s$, then

$$x = x \wedge (x \vee s) \geq x \wedge (y \vee s)$$

$$= (x \wedge y) \vee (x \wedge s) \geq (x \wedge y) \vee (y \wedge s)$$

$$= y \wedge (x \vee s) \geq y \wedge (y \vee s) = y$$

$(ii) \Rightarrow (i)$.

Suppose $x, y \in A$ and $y \vee s$ exists. Let $p = x \wedge (y \vee s)$ and $q = (x \wedge y) \vee (x \wedge s)$. Now $p \wedge s = x \wedge s \leq q = (x \wedge y) \vee (x \wedge s) \leq x \wedge (y \vee s) = p$. Hence $p \wedge s \leq q \wedge s \leq p \wedge s$, that is $p \wedge s = q \wedge s$. Observe that as $p, s \leq y \vee s$, $p \vee s$ exists and since

$p = p \wedge (y \vee s), p \vee s = [p \wedge (y \vee s)] \vee s$

$$= (p \vee s) \wedge (y \vee s) \ (by \ (ii)(a))$$

$$= (p \wedge y) \vee s \ \ (by \ (ii)(a))$$

$$= (x \wedge y) \vee s$$

$$= (x \wedge y) \vee (x \wedge s) \vee s = q \vee s$$

Then by (ii) (b) , $p = q$, that is (i) holds.

Now suppose s is standard in A, $x, y \in A$ and $y \vee s$ exists. Then letting $y \vee s = r$ we obtain $x \wedge (y \vee s) = x \wedge [(r \wedge y) \vee (r \wedge s)] = (x \wedge r \wedge y) \vee (x \wedge r \wedge s) = (x \wedge y) \vee (x \wedge s)$, as s is standard, thus (i) and (ii) holds.

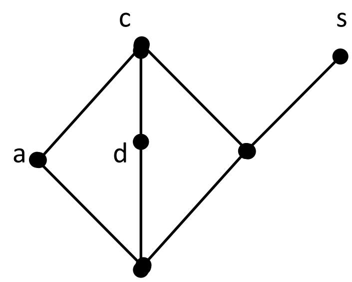

Figure-3.1

Finally, consider the nearlattice A in Fig-3.1. Here, for all $x, y \in A$, the condition (i) holds; but $d \wedge [(c \wedge a) \vee (c \wedge s)] > (d \wedge c \wedge a) \vee (d \wedge c \wedge s)$.●

3.3 Characterization of Standard ideals

We start this section with the following characterization of standard ideal in a nearlattice which is due to [5]. We prefer to include the proof for the convenience of the reader.

Theorem-3.3.*1* *Let K be an ideal in a nearlattice. Then the following conditions are equivalent.*

(i) *K is a standard ideal.*

(ii) *The binary relation* $\Theta(K)$, *defined by* $x \equiv y(\Theta(K))$ *holds if and only if* $x = (x \wedge y) \vee (x \wedge a), y = (x \wedge y) \vee (y \wedge b)$ *for some* $a, b \in K$, *is a nearlattice-congruence.*

(iii) *The binary relation* Φ, *defined by* $x \equiv y(\Phi)$ *holds if and only if for all* $t \in S, (x \wedge t) \vee (t \wedge c) = (y \wedge t) \vee (t \wedge c)$ *for some* $c \in K$, *is a nearlattice-congruence.*

(iv) *For each ideal H,* $K \vee H = \{k \vee h : k \vee h$ *exists and* $k \in K$ *and* $h \in H\}$

Moreover, (ii) and (iii) represent the same nearlattice-congruence, namely $\Theta(K)$, *the smallest join-partial congruence of* A *having* K *as a congruence class.*

Proof: $(i) \Rightarrow (ii)$. Due to condition (i) the relation $J \equiv H(\Theta_k)(J, H \in J(A))$ if and only if $J = (J \cap H) \vee (J \cap K)$ and $H = (J \cap H) \vee (H \cap K)$ is a congruence on $J(A)$. Then $\Theta_{K/A}$ (restriction to A) is a nearlattice-congruence on A and $x \equiv y(\Theta_{K/A})$ if and only if

$(x] = (x \wedge y] \vee ((x] \cap K)$ and $(y] = (x \wedge y] \vee ((y] \cap K)$. Thus to prove (ii), it is sufficient to prove that $(x] = (x \wedge y] \vee ((x] \cap K)$ implies $x = (x \wedge y) \vee (y \wedge a)$ for some $a \in K$. Now $(x \wedge y] \vee ((x] \cap K) = \bigcup_{n=0}^{\infty} A_n$, where $A_0 = (x \wedge y] \cup ((x] \cap K)$ and $A_n = \{t \in S | t \leq p \vee q; p \vee q$ exists and $p, q \in A_{n-1}\}$ for $n = 1,2,3,.....;$ and we show, by induction, that $(x \wedge y] \vee ((x] \cap K) = \{t : t \leq (x \wedge y) \vee (x \wedge a)$ for some $a \in K\}$.

If $t \in A_0$ then $t \in (x \wedge y]$ or $t \in (x] \cap K$. In the first instance, $t \leq x \wedge y \leq (x \wedge y) \vee (x \wedge k)$ and $t \in K$. Thus the result holds for $n = 0$. Suppose the result holds for n-1 for some $n \geq 1$. Let $t \in A_n$. Then $t \leq p \vee q$ with $p, q \in A_{n-1}$. So $p \leq (x \wedge y) \vee (x \wedge k_1)$ and

$q \le (x \wedge y) \vee (x \wedge k_2)$ for some k $k_1 and\, k_2 \in K$ .Then $t \le (x \wedge y) \vee (x \vee k_1) \vee (x \wedge k_2) = (x \wedge y) \vee (x \wedge k)$ for some $k \in K$ (Since $(x \wedge k_1) \vee (x \wedge k_2) \le x$ and is in *K*, it is of the form $x \wedge k$ for some $k \in K$). Thus we have $(x \wedge y) \vee ((x] \cap k) = \{t : t \le (x \wedge y) \vee (x \wedge k)$ for some $k \in K$ } in effect , $x \le (x \wedge y) \vee (x \wedge a)$ for some $a \in K$ and so $x = (x \wedge y) \vee (x \wedge a)$,as required.

$(i) \Rightarrow (ii)$. Let $x \equiv y(\Theta(K))$ .Since $\Theta(K)$ is a congruence, $x \wedge y \equiv y \wedge t(\Theta(K))$ for any $t \in A$ }, So $x \wedge t = (x \wedge y \wedge t) \vee (x \wedge t \wedge a)$ and $y \wedge t = (x \wedge y \wedge t) \vee (y \wedge t \wedge b)$ for some $a, b \in K$. Then $(x \wedge t) \vee (t \wedge [(t \wedge a) \vee (t \wedge b)]) = (x \wedge t) \vee (t \wedge a) \vee (t \wedge b)$ $= (x \wedge y \wedge t) \vee (t \wedge a) \vee (t \wedge b) = (y \wedge t) \vee (t \wedge [(t \wedge a) \vee (t \wedge b)])$ .Observe that $(t \wedge a) \vee (t \wedge b) \in K$. Thus $x \equiv y(\Phi)$. Conversely, if $x \equiv y(\Phi)$ then for any $t \in A, (x \wedge t) \vee (t \wedge c) = (y \wedge t) \vee (t \wedge c)$ for some $c \in S$. Choosing $t = x$ and $t = y$, we have $x = (x \wedge y) \vee (x \wedge c)$ and $y = (x \wedge y) \vee (y \wedge c)$ respectively. Thus, $x \equiv y(\Theta(S))$ and Φ is the congruence $\Theta(S)$.

$(iii) \Rightarrow (iv)$. Let $T = \{s \vee k : s \vee k exists and s \in S and k \in K\}$. Suppose $x \le s \vee k, s \in S, k \in K$. Clearly $s \vee k \equiv k(\Theta(S))$ and so $x = x \wedge (s \vee k) \equiv (x \wedge k)(\Theta(S))$. Hence for all $t \in A, (x \wedge t) \vee (t \wedge c) = (x \wedge k \wedge t) \vee (t \wedge c)$ for some $c \in S$.

Choosing $t = x$, we obtain $x = (x \wedge k) \vee (x \wedge c)$ and so $x \in T$. But T is closed under existent finite suprema. It follows that *T* is an ideal of *A* and $T = S \vee K$.

$(iv) \Rightarrow (i)$. Let $x \in J \cap (K \vee H)$ then $x \in J$ and $x \in K \vee H$. So $x = k \vee h$ for suitable $k \in K$ and $h \in H$. Then $x = (x \wedge k) \vee (x \wedge h)$ and so $x \in (J \cap K) \vee (J \vee H)$. The reverse inclusion is obvious. Thus $J \cap (K \vee K) = (J \cap K) \vee (J \cap H)$; *K* is a standard ideal. The final assertion is clear in view of the proof of $(ii) \Rightarrow (iii)$.●

Theorem 3.3.2 *For an ideal A of a nearlattice S, the following conditions are equivalent;*

(i) *A is a standard ideal.*

(ii) *The equality* $I \cap (A \vee K) = (I \cap A) \vee (I \cap K)$ *holds if I and K are principal ideals.*

(iii) *If for the principal ideals I and J the inequality* $J \subseteq (A \vee I)$ *holds, then* $J = (J \cap A) \vee (J \cap I)$.

(iv) *The relation* $\Theta(A)$ *of* S *defined by* $x \equiv y(\Theta[A])$ *if and only if* $x=(x\wedge y)\vee(x\wedge a)$ *and* $y=(x\wedge y)\vee(y\wedge b)$ *for some* $a,b\in A$, *is a congruence relation.*

Proof:

$(i)\Rightarrow(ii)$ is obvious, from the definition of the standard ideal.
$(ii)\Rightarrow(iii)$ is clear.

$(iii)\Rightarrow(iv)$. Obviously the relation is an equivalence relation.

Let $x\le y$ and $x\equiv y\Theta(A)$ then $y=x\vee(y\wedge b)$ for some $b\in A$, suppose for some $t\in S, y\vee t$ exists.

Then $x\vee t$ exists.

Hence, $y\vee t=(x\vee t)\vee(y\wedge b)\le(x\vee t)\vee((y\vee t)\wedge b)\le y\vee t$

Thus, $y\vee t=(x\vee t)\vee((y\vee t)\wedge b)$

So $x\vee t\equiv y\vee t(\Theta(A))$.

Now, $y\wedge t\le x\vee(y\wedge b)\in(x]\vee A$, So $(y\wedge t]\subseteq(x]\vee A$.

Then by (iii), $(y\wedge t]=(x\wedge y\wedge t)\vee(A\wedge(y\wedge t])=(x\wedge t]\vee(A\wedge(y\wedge t])$.

Then a similar proof of (i) implies (ii) of theorem 4.1.1 shows that $y\wedge t=(x\wedge t)\vee(y\wedge t\wedge a)$ for some $a\in A$.

Thus by lemma 2.1.3, $\Theta(A)$ is a congruence relation.

$(iv)\Rightarrow(i)$ holds by theorem-3.3.1.●

An element s is called an upper element of a nearlattice S if $s\vee x$ exists for all $x\in S$.

Central element: An element $s\in S$ is called a central element if

(i) S is upper and neutral and

(ii) S is complemented in each interval containing it.

Theorem-3.3.3: *The following conditions on an element z of a nearlattice S coincide.*

(i) *z is central .*

(ii) *z is neutral and upper, and complemented in each interval, which contain it.*

(iii) *z is standard and upper, and complemented in each interval, which contains it.*

Proof:

$(i) \Rightarrow (ii)$. Suppose (i) holds. Due to Lemma-3.2[18], z is neutral and the remainder is not hard to obtain.

$(ii) \Rightarrow (i)$. As z is upper, standard and distribuant, proposition-2.2 and Lemma-3.1 [5] imply that the map $\Phi : A \to (z] \times (z]$, where [$z$) is the subnearlattice $\{t \in A : t \geq z\}$, is a neaelattice homomorphism. Also Φ is one-to-one. But Φ is onto as $(a,b) \in (z] \times (z]$ says that $a \leq z \leq b$, and so $(a,b) = \Phi(c)$, where c is the relative complement of z in the interval [a, b]. Thus Φ is an isomorphism and it does the required thing for z. Hence z is central.

$(i) \Rightarrow (iii)$. is obvious.

$(iii) \Rightarrow (ii)$. Suppose (iii) holds and $x, y, t \in S$. Consider the interval $[(x \wedge y \wedge z) \vee (t \wedge x \wedge z), z \vee (x \wedge y) \vee (x \wedge t)]$, which certainly contains z. Let r be the complement of z in this interval. From proposition-2.2 [18], we infer that,
$x \wedge y = (x \wedge y) \vee [z \vee (x \wedge y) \vee (x \wedge z)] = (x \wedge y) \vee (r \vee z) = (x \wedge y \wedge r) \vee (x \wedge y \wedge z)$.

Similarly, $x \wedge t = (x \wedge t \wedge r) \vee (x \wedge t \wedge z)$.

Thus $(x \wedge y) \vee (x \wedge t) = (x \wedge y \wedge r) \vee (x \wedge t \wedge r) \vee (x \wedge y \wedge r) \vee (x \wedge t \wedge z)$

$$= (x \wedge y \wedge r) \vee (x \wedge t \wedge r) \vee (r \wedge z) \leq r$$

and so $z \wedge [(x \wedge y) \vee (x \wedge t)] \leq r \wedge z = (x \wedge y \wedge z) \vee (t \wedge x \wedge z)$.

Therefore, $x \wedge [(x \wedge y) \vee (x \wedge t)] = (z \wedge x \wedge y) \vee (z \wedge x \wedge t)$,

Which says that z is also distribuant and, therefore, neutral.

From the above theorem, it is clear that a nearlattice is a relatively complemented distributive lattice if and only if each of its elements is central.●

Proposition 3.3.4 The following conditions upon an element a of a nearlattice S are equivalent.

(i) a is standard.

(ii) The relation Θ_a, defined by $x \equiv y(\Theta_a)$ if and only if $x \vee y = (x \vee y) \vee a_1$ for some $a_1 \leq a$, is a congruence relation.

(iii) A is a distributive element, that is $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$ for any $x, y \in S$, and $b = c$ whenever $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$.

(iv) For each ideal K, $(a] \vee K = \{a_1 \vee k : a_1 \leq a, k \in K\}$.

(v) *(a]* is a standard element of the ideal nearlattice of *S*.●

CHAPTER IV

HOMOMORPHISM AND STANDARD IDEALS

4.1 Introduction: Gratzer and Schmidt in [14] proved many results on homomorphism kernels and standard ideals of a lattice. Their main aim was to translate several theorems of group theory to lattice theory. In this chapter we have generalized some of their results to nearlattices. We have also given the characterizations of those nearlattices whose all congruences are standard (neutral) which are generalizations of two recent papers [6] and [7].

A congruence Φ of a nearlattice A is called a standard (neutral) if $\Phi = \Theta(S)$ for some standard (neutral) ideal S of A.

For any two nearlattices A_1 and A_2 a map $\Phi : A_1 \to A_2$ is called an isotone if for any $x, y \in A_1$ with $x \le y$ implies $\Phi(x) \le \Phi(y)$ in A_2. Φ is called a meet homomorphism if for all $x, y \in A_1, \Phi(x \wedge y) = \Phi(x) \wedge \Phi(y)$, clearly every meet homomorphism is an isotone.

A meet homomorphism $\Phi : A_1 \to A_2$ is called nearlattice homomorphism if $\Phi(x \wedge y) = \Phi(x) \wedge \Phi(y)$ when $x \wedge y$ exists in A_1. Since Φ is isotone $\Phi(x), \Phi(y) \le \Phi(x \vee y)$. Therefore $\Phi(x) \vee \Phi(y)$ exists by the upper bound property of A_2.

In section-2, we have shown that if s is a standard ideal of a nearlattice A, then Θ_S is the extension of $\Theta(S)$ to $I(A)$ and $\Theta(S)$ is the restriction of Θ_S to the nearlattices A. Then we have shown that in a sectionally complemented nearlattices all congruences are standard. We also show that in a relatively complemented nearlattice A with 0, if every standard ideal of A is generated by a finite number of standard elements, then the congruence lattice $C(A)$ is Boolean. Finally we have generalized two results of [6] and [7] regarding nearlattices all of whose congruences are standard (neutral).

In section-3, we have given homomorphism theorem for nearlattices. Then we have generalized two isomorphism theorems of [14] for nearlattices.

4.2 Homomorphism kernels and standard ideals

By [5, corollary-2.7], we know that the set of all standard ideals of a nearlattice A is a subnearlattice of $I(A)$. Also the congruence Θ_S, where S is standard form a sublattice of $\Theta(I(A))$, and $S \to \Theta_S$ is an isomorphism. Suppose Θ is a congruence relation , Θ defines in the natural way a homomorphism of $I(A)$ under which $I \equiv J(I, J \in I(A))$ if and only if to any $x \in I$, there exists a $y \in J$ such that $x \equiv y(\Theta)$ and conversely. We call this congruence relation Φ of $I(A)$ includes a congruence relation of A under which $x \equiv y$ if and only if $(x] \equiv (y](\Phi)$.This is called the restriction of Φ to *A*. We now give the following result which is a generalization of [4, Lemma 5].

Theorem 4.2.1 *Let S be a standard ideal. Then* Θ_S *is the extension of* $\Theta(S)$ *to* $I(A)$ *and* $\Theta(S)$ *is the restriction of* Θ_S *to the nearlattice A.*

Proof: Let $\overline{\Theta}(S)$ be the extension of $\Theta(S)$ to $I(A)$ and $I \equiv J(\overline{\Theta}(S))$. We suppose $I \subseteq J$. Choosing a $y \in J$ we can find an $x \in I(y \geq x)$ with $x \equiv y(\Theta(S))$ and so there exists an S_{xy} with $y = x \vee (y \wedge S_{xy})$. The ideal $S^{'}$ generated by the $y \wedge S_{xy}$ satisfies $S^{'} \subseteq S$ and $I \vee S^{'} = J$ hence $I \equiv J(\Theta)$. On the other hand, if $I \equiv J(\Theta_S)$ then $I \vee S^{'} = J$ with a suitable $S^{'} \subseteq S$. Then for any $y \in J$ it follows that $y \in I \vee S$ and so $y = x \vee s = x \vee (y \wedge s)$ for some $s \in S$ as S is standard. Thus, $x \equiv y(\Theta(S))$ by [Th-3.1.1], and hence $\overline{\Theta}(S) = \Theta_S$.

To prove the 2[nd] assertion, suppose $(a] \equiv (a \wedge b](\Theta_S)$.

Then $(a] \equiv (a] \wedge (b]\Theta_S = (a \wedge b](\Theta_S)$ and hence $(a] = (a \wedge b] \vee S^{'}$ for suitable $S^{'} \subseteq S$. Then $a \in (a \wedge b] \vee S$ and since S is standard so by Theorem-3.1.1, $a = (a \wedge b) \vee (a \wedge s_1)$ for some $s_1 \in S$. Similarly, we can show that $b = (a \wedge b) \vee (b \wedge s_2)$ for some $s_2 \in S$.

Thus $a \equiv b\Theta(S)$. Hence $\Theta(S)$ is the restriction of Θ_S to A.●

Corollary 4.2.2 (Nasima Akhter [18]).*The correspondence* $\Theta(S) \to \Theta_S$ *is an isomorphism between the lattice of all standard congruence relations of A and the lattice of all principal standard congruence relations of* $I(A)$.●

Theorem 4.2.3 *Let S be a sectionally complemented nearlattice. Then every homomorphism kernel of S is a standard ideal is the kernel of precisely one congruence-relation.*

Proof: Suppose the ideal I of S is homomorphism kernel induced by the congruence relation Θ. Let $a \equiv b(\Theta), a, b \in S$ then $a \wedge b \equiv a(\Theta)$ and $0 \leq a \wedge b \leq a$. Since S is sectionally complemented, So there exists c; Such that $a \wedge b \wedge c = 0$ and $(a \wedge b) \vee c = a$. This implies $0 = (a \wedge b) \wedge c = a \wedge c \equiv c(\Theta)$.

Since I is a homomorphism kernel.
So $c \in I$, Moreover, $a = (a \wedge b) \vee c = (a \wedge b) \vee (a \wedge c)$
Similarly, We can show that, $b = (a \wedge b) \vee (a \wedge d)$ for $d \in I$.

Therefore, I is a standard ideal. At the same time, We have proved that if I is the kernel of homomorphism induced by Θ, then $\Theta = I(\Theta)$.

Hence, every standard ideal is the kernel of precisely one congruence relation.●

Theorem 4.2.4. (Nasima Akhter [18]) *Let A be a relatively complemented neaelattice with 0. If every standard ideal of A is generated by a finite number of standard elements then C(A), the congruence lattice is Boolean. Moreover, the converse of this is not true.*●

Theorem 4.2.5 *$\Theta : s_1 \to s_2$ is an onto homomorphism, where S_1, S_2 are nearlattices and $0'$ is least element of S_2, then $\ker\Theta$ is an ideal of S.*

Proof: Since Θ is onto, $0' \in S_2$ thus $\ker\Theta \neq \Phi$ as pre-image of $0'$ exists in S_1.

Now $a, b \in \ker\Theta \Rightarrow \Theta(a) = 0' = \Theta(b)$
$\Theta(a \vee b) = \Theta(a) \vee \Theta(b) = 0' \vee 0' \Rightarrow a \vee b \in \ker\Theta$.
Again $a \in \ker\Theta, s \in S$ gives $\Theta(a) = 0'$.
Also $\Theta(a \wedge s) = \Theta(a) \wedge \Theta(s) = 0' \wedge s_1 = 0' \Rightarrow a \wedge s \in \ker\Theta$, where $s_1 \in S_2$
Hence $\ker\Theta$ is an ideal of S.●

Theorem 4.2.6 *Let A be a nearlattice. Then the following conditions are equivalent,*

(i) All congruence of A are standard.
(ii) A has a zero and for all $x, y \in A$ there exists $a \in A$ such that $x = (x \wedge y) \vee (x \wedge a), a \equiv 0\Theta(x \wedge y, x)$.

Proof: $(i) \Rightarrow (ii)$. Since the smallest congruence ω of A is standard, A must have a zero.

Let $x, y \in A$ then $\Theta(x \wedge y, x) = \Theta(I)$, for some standard ideal I .i.e. $x \equiv x \wedge y\Theta(I)$, where I is standard, hence $x = (x \wedge y) \vee (x \wedge a)$, for some $a \in I$. Hence $a \equiv 0\Theta(x \wedge y, x)$.

$(ii) \Rightarrow (i)$. Let Φ be a congruence and $I = [0]\Phi$. Suppose $x \equiv y(\Phi)$. Then by (ii) there exists $a \in A$ such that $x = (x \wedge y) \vee (x \wedge a)$ and $a \equiv 0(\Theta(x \wedge y, x))$. Since $\Theta(x \wedge y, x) \leq \Phi$, so $a \equiv 0(\Theta)$ and hence $a \in I$. Similarly, $y = (x \wedge y) \vee (y \wedge b)$, for some $b \in I$. Thus by [Theorem-3.1.1] I is a standard ideal and $\Phi = \Theta(I)$, and so (i) holds.●

4.3 Isomorphism Theorem

In [14] Gratzer and Schmidt have proved isomorphism theorems for standard ideals in lattices. In their paper they have translated several theorems of group theory to lattice theory using ideal, standard ideal, factor lattice and join operation for subgroup, invariant subgroup, factor subgroup and group operation respectively. In this section we generalize two isomorphism theorems for standard ideals of nearlattices.

For any congruence Θ on A, A/Θ denotes, the set of all congruence classes of A. We define $\wedge$ on A/Θ by $[a]\Theta \wedge [b]\Theta = [a \wedge b]\Theta.$ If for any $a, b \in A, a \vee b$ exists, then we define $[a]\Theta \vee [b]\Theta = [a \vee b]\Theta.$

Theorem 4.3.1 A/Θ *is a nearlattice.*

Proof: Of course A/Θ is a meet semilattice. We need to show that it has the upper bound property.

Let $[a]\Theta, [b]\Theta \le [c]\Theta$, then $[a]\Theta = [a]\Theta \wedge [c]\Theta = [a \wedge c]\Theta$

$[b]\Theta = [b]\Theta \wedge [c]\Theta = [b \wedge c]\Theta$

Now, $(a \wedge c) \vee (b \wedge c)$ exists by the upper bound property of A. Hence,

$[a \wedge c]\Theta \vee [b \wedge c]\Theta = [(a \wedge c) \vee (b \wedge c)]\Theta$ and so $[a]\Theta \vee [b]\Theta$ exists. Therefore A/Θ is a nearlattice.

If Θ is a congruence of a nearlattice A then the map $\Phi : A \to A/\Theta$ defined by $\Theta(a) = [a]\Theta$ is the natural homomorphism induced by Θ. For a standard ideal S of A we denote the quotient nearlattice $A/\Theta(S)$, simply by A/S.

Now we give the homomorphism theorem for nearlattices which is a generalization of [11, Th-11, P-26].●

Theorem 4.3.2 [Nasima Akhter[18], Th-2.2] *Every homomorphic image of a nearlattice A is isomorphic to a suitable quotient nearlattice A. In fact* $\Phi : A \to A_1$ *is a homomorphism of* A *onto* A_1 *and if* Θ *is the congruence relation of* A *defined by* $x \equiv y(\Theta)$ *if and only if* $\Phi(x) = \Phi(y)$ *then* $A/\Theta \cong A_1$*; an isomorphism is given by* $\Psi : [x]\Theta \to \Phi(x), x \in A$.●

Lemma 4.3.3 *Let S be a nearlattice and A a standard ideal of S. Then* $[I \cap A, I] \cong [A, I \cup A]$ *for all* $I \in I(S)$. *An isomorphism is given by the correspondence* $X \to X \cup A(X \in [I \cap A, I])$. *The inverse correspondence is* $Y \to Y \cap I(Y \in [A, I \cup A])$.

Proof: From [14, cond. $(\sigma^{'})(i)of$ Th-2], we get that $X \to X \cup A(X \in [I \cap A, I])$ is a homomorphism. If $X_1, X_2 \in [I \cap A, I]$ then $A \cap X_1 = A \cap X_2$(for $X_1, X_2 \subseteq I$ and so $A \cap X_i = A \cap X_1 \cap I = A \cap I, i = 1,2)$.

Thus from [14, cond. $(\sigma^{'})(ii)$ Th-2.], we get $A \cup X_1 \neq A \cup X_2$

Therefore, $X \to X \cup A$ is an isomorphism of $[I \cap A, I]$ into $[A, I \cup A]$, we prove that, $(Y \cap I) \cup A = Y(Y \in [A, I \cup A])$ and this will prove that, $Y \to Y \cap I(Y \in [A, I \cup A])$ is the inverse of $X \to X \cup A(X \in [I \cap A, I])$ and the latter correspondence maps$[I \cap A, I]$ onto $[A, I \cup A]$. Indeed using [14, cond. $(\sigma^{'})(i)of$ Th-2], we get $(Y \cap I) \cup A = (Y \cup A) \cap (I \cup A) = Y \cap (I \cup S) = Y$ and this completes the proof of the Lemma .●

Theorem 4.3.*4* *(First isomorphism theorem for standard ideals) : Let S be a nearlattice, A be a standard ideal and I an arbitrary ideal of S. Then* $A \cap I$ *is a standard ideal of I and* $(I \cup A)/A \cong I/(A \cap I)$.

Proof : Corollary of Lemma-9 of standard ideals in lattice by Gratzer and Schmidt [14] is just the first assertion of our theorem. The simplest mean to prove the isomorphism statement is the use of the first general isomorphism theorem of REDEI [23], (Chapter-1). We have only to prove that every congruence class of the nearlattice $I \cup A$ may be represented by an element of *I*. Indeed, any element *x* of $I \cup A$ is of the form $y \vee a$ where $a \in A$ and $y \in I$

[Gratzer and Schmidt[14], cond. $(\sigma^{'})(ii)$ Th-2.],

Further, $x = y \cup a \equiv y(\Theta[A])$, and so the congruence class that contains *x* may be represented by $y \in I$.

According to Gratzer and Schmidt [14] Th-4, The isomorphism theorem is equivalent to the isomorphism to the intervals $[A, I \cup A]$ and$[I \cap A, I]$ of $I(S)$. We can add to the isomorphism statement of Lemma 4.3.3

In the last proof of Lemma we have got a new proof of the isomorphism theorem.●

Theorem 4.3.5 *(2nd isomorphism theorem of standard ideals): Let S be a nearlattice and A be an ideal and T a standard ideal of S. $A \supseteq T$ .Then A is standard if and only if A/T is standard in S/T , and in this case $S/A \cong (S/T)/(A/T)$.*

Proof: If A is standard, then from Lemma-6 [14], We get, that, A/T is standard in S/T .Conversely, suppose A/T is standard in S/T . We show Gratzer and Schmidt [14], cond. $(\gamma^{"}\ ) of Th-2$ holds for *A*.

We have seen in the proof of Th-1[14, $"(\beta) implies (\gamma)"$], that it is enough to prove that $x \equiv y(\Theta[A])$ and $x \geqq y$ imply $x \cap u \equiv y \cap u(\Theta[A])$ for all $u \in S$. (Here $\Theta[A]$ denotes the relation defined in cond. $(\gamma^{"}\ )$*of* Th-2 of [14]. We denote by $[\ a]$ the image of the element *a* under the homomorphism $S \sim S/T$. Then we have $\bar{x} \equiv \bar{y}(\Theta[A/T])$, and since A/T is standard in S/T , therefore, with a suitable $\bar{s} \in A/T$ we get, $\bar{x} \cap \bar{u} = (\bar{y} \cap \bar{u}) \cup \bar{s}$.

Further, since T is standard in S, we can find a $t \in T$ such that $x \cap u = [(y \cap u) \cup s] \cup t$; we put $s_1 = s \cup t$ and get $x \cap u = [(y \cap u) \cap s_1, s_1 \in A$.

This proves A is standard.

We remark that during the proof we have made effective use of the fact that the congruence classes of S/T under $\Theta[A/T]$ are the homomorphic image of those of *S* under $\Theta[A]$.●

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