# A STUDY ON WEAKLY COMPLEMENTED NEARLATTICE

by

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A thesis submitted in partial fulfillment of the requirements for the degree of
Master of Science
in Department of Mathematics.

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This is to certify that the thesis work entitled " A study on Weakly Complemented Nearlattice" has been carried out by Mahfuza Rahman in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

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*DEDICATED MY PARENTS WHO HAVE PROFOUNDLY INFLUENCE MY LIFE*

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Mahfuza Rahman
(Mahfuza Rahman)

**Abstract**

In this thesis study of the nature of the weakly complemented nearlattice is presented. By a nearlattice $S$ we will always mean a meet semilattice together with the property that any two elements possessing a common upper bound, have a supremum. Cornish and Hickman [7] referred this property as the upper bound property, and a semilattice of this nature as a semilattice with the upperbound property. Cornish and Noor [8] preferred to call these semilattices as nearlattices, as the behaviour of such a semilattice is close to that of a lattice than an ordinary semilattice. Of course a nearlattice with a largest element is a lattice. Since any semilattice satisfying the descending chain condition has the upper bound property, so all finite semilattices are nearlattices. In lattice theory, it is always very difficult to study the non-distributive and non-modular lattices. Gratzer [12] studied the non-distributive lattices by introducing the concept of distributive, standard and neutral elements in lattices. Cornish and Noor [8] extended those concepts for nearlattices to study non-distributive nearlattices. On the other hand, J.C Varlet [33] studied another class of non-distributive lattices with 0 by introducing the concept of 0-distributivity. In fact this concept also generalizes the idea of pseudocomplement in a general lattice. In this thesis we have extended the concept of weakly complemented nearlattice in terms of homomorphism theorem

# Approval

This is to certify that the thesis work submitted by *Mahfuza Rahaman* entitled " ***A Study on Weakly Complemented Nearlattice***" has been approved by the board of examiners for the partial fulfillment of the requirements for the degree of *M.Sc* in the Department of *Mathematics*, Khulna University of Engineering & Technology, Khulna, Bangladesh in July 2016.

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# Contents


| | | PAGE |
|---|---|---|
| Title Page | | i |
| Declaration | | ii |
| Acknowledgement | | iii |
| Abstract | | iv |
| Approval | | v |
| Contents | | vi |
| **CHAPTER I** | **1. IDEALS, CONGRUCES AND RELATIVE ANNIHILATORS IN A NEARLATTICE** | 1 |
| | **1.1 Preliminaries** | 1 |
| | **1.2 Ideals of Nearlattices** | 6 |
| | **1.3 Congruences** | 14 |
| | **1.4 Relative annihilators** | 18 |
| **CHAPTER II** | **2. DISJUNCTIVE NEARLATTICE AND SEMIBOOLEAN ALGEBRA** | 23 |
| | **2.1 Disjunctive Nearlattice and Semiboolean Algebras** | 23 |
| **CHAPTER III** | **3. 0-DISTRIBUTIVE NEARLATTICE AND SEMIPRIME IDEALS IN A NEARLATTICE** | 31 |
| | **3.1 Introduction** | 31 |
| | **3.2 0-distributive nearlattice** | 34 |
| | **3.3 Semi prime ideals in a Nearlattice** | 48 |
| **CHAPTER IV** | **4. WEAKLY COMPLEMENTED NEARLATTICE** | 56 |
| | **4.1 Introduction** | 56 |
| | **4.2 Homomorphism and semi-prime ideals** | 58 |
| | **4.3 Weakly complemented nearlattice** | 61 |
| **References** | | 63 |

# CHAPTER I

# IDEALS, CONGRUNCES AND RELATIVE ANNIHILATORS IN A NEARLATTICE

## 1. 1 Preliminaries

The intention of this section is to outline and fix the notation for some of the concepts of nearlattices which are basic to this thesis. We also formulate some results on arbitrary nearlattices for later use. For the background material in lattice theory we refer the reader to the text of Birkhoff [4] , Gratzer [12], [13] and Davey [10].

By a nearlattice $S$ we will always mean a lower (meet) semilattice which has the property that any two elements possessing a common upper bound have a supremum. Cornish and Hickman [7], referred this property as the *upper bound property* and a semilattice of this nature as *a semilattice with the upper bound property*. The behaviour of such a semilattice is closer to that of a lattice than an ordinary semilattice.

Of course, a nearlattice with a largest element is a lattice. Since any semilattice satisfying the descending chain condition has the upper bound property, so all finite semilattices are nearlattices.

Now we give an example of a meet semilattice which is not a nearlattice.

Example: In $R^2$ let us consider the set, $S = \{(0,0)\} \cup \{(1,0)\} \cup \{(0,1)\} \cup \{(1, y) \mid y > 1\}$ shown in the Figure 1.1

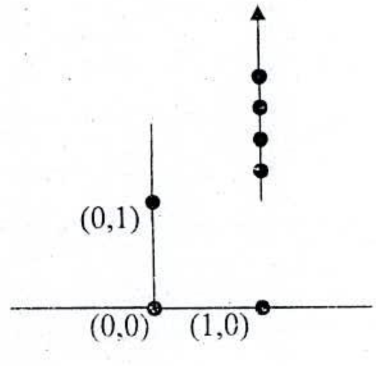


Figure 1.1

Let us define the partial ordering "$\leq$" on $S$ by $(x,y)\leq(x_1,y_1)$ if and only if $x\leq x_1$ and $y\leq y_1$. Clearly, $(S;\leq)$ is a meet semilattice. Both (1,0) and (0,1) have common upper bounds. In fact $\{(1,y) \mid y>1\}$ are common upper bounds of them. But the supremum of (1,0) and (0,1) does not exist. Therefore $(S;\leq)$ is not a nearlattice.

The upper bound property appears in Gratzer and Lakser [14], while Rozen [28] show that it is the result of placing certain associativity conditions on the partial join operation. Moreover, Evans [11] referred nearlattices as *conditional lattices*. By a conditional lattice he means a lower semilattice $S$ with the condition that for each $x \in S$, $\{y \in S \mid y \leq x\}$ is a lattice; and it is very easy to check that this condition is equivalent to the upper bound property of $S$. Also Nieminen [20] in his paper refers to nearlattices as *"partial lattices"*. Whenever a nearlattice has a least element we will denote it by 0. If $x_1, x_2, \cdots, x_n$ are elements of a nearlattice then by $x_1 \vee x_2 \vee \cdots \vee x_n$, we mean that the supremum of $x_1, x_2, \cdots, x_n$ exists and $x_1 \vee x_2 \vee \cdots \vee x_n$ symbolizing this supremum.

A non-empty subset $K$ of a nearlattice $S$ is called a *subnearlattice* of $S$ if for any $a,b \in K$, both $a \wedge b$ and $a \vee b$ (whenever it exists in $S$) belong to $K$ ($\wedge$ and $\vee$ are taken in $S$), and the $\wedge$ and $\vee$ of $K$ are the restrictions of the $\wedge$ and $\vee$ of $S$ to $K$. Moreover, a subnearlattice $K$ of a nearlattice $S$ is called a *sublattice* of $S$ if $a \vee b \in K$ for all $a,b \in K$.

A nearlattice $S$ is called *modular* if for any $a,b,c \in S$ with $c \leq a$, $a \wedge (b \vee c) = (a \wedge b) \vee c$ whenever $b \vee c$ exists.

A nearlattice $S$ is called distributive if for any $x_1, x_2, \cdots, x_n$, $x \wedge (x_1 \vee x_2 \vee \cdots \vee x_n) = (x \wedge x_1) \vee (x \wedge x_2) \vee \cdots \vee (x \wedge x_n)$ whenever $x_1 \vee x_2 \vee \cdots \vee x_n$ exists. Notice that the right hand expression always exists by the upper bound property of $S$.

**Lemma 1. 1. 1:** *A nearlattice S is distributive (modular) if and only if* $\{y \in S \mid y \le x\}$ *is a distributive (modular) lattice for each* $x \in S$. •

Let us consider the following two lattices: pentagonal lattice $N_5$ and Diamond lattice $M_5$. Many lattice theorists study on these two lattices and given several results.

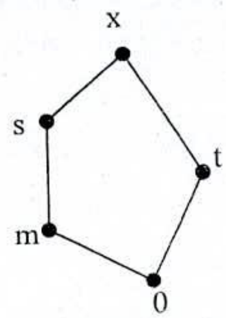


Figure-1.2

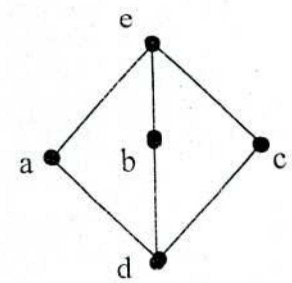


Figure-1.3

Hickman in [15] has given the following extensions of very fundamental results of lattice theory.

**Theorem 1. 1. 2:** *A nearlattice S is distributive if and only if S does not contain a sublattice isomorphic to* $N_5$ *or* $M_5$ [in Figure 1. 2 and 1. 3]. •

**Theorem 1. 1. 3:** *A nearlattice S is modular if and only if S does not contain a sublattice isomorphic to* $N_5$. •

In this context it should be mentioned that many lattice theorists (e.g. R. Bables [2], J. C. Varlet[33], R. C. Hickman[15] and K. P. Shum[31]) have worked with a class of semilattice $S$ which has the property that for each $x, a_1, a_2, \cdots, a_r \in S$, if $a_1 \vee a_2 \vee \cdots \vee a_r$ exists then $(x \wedge a_1) \vee (x \wedge a_2) \vee \cdots \vee (x \wedge a_r)$ exists and equals $x \wedge (a_1 \vee a_2 \vee \cdots \vee a_r)$. Bables [2] called them as prime semilattices while Shum [31] referred them as weakly distributive semilattices.

Hickman in [15] has defined a ternary operation $j$ by $j(x, y, z) = (x \wedge y) \vee (y \wedge z)$, on a nearlattice $S$ (which exists by the upper bound property of $S$). In fact he has shown, which can also be found in Lyndon [18] Theorem 4, that the resulting algebras of the type

$(S; j)$ form a variety, which is referred to as the variety of join algebras and following are its defining identities.

(i) $j(x,x,x) = x$

(ii) $j(x,y,x) = j(y,x,y)$

(iii) $j(j(x,y,x),z,j(x,y,x)) = j(x,j(y,z,y),x)$

(iv) $j(x,y,z) = j(z,y,x)$

(v) $j(j(x,y,z),j(x,y,x),j(x,y,z)) = j(x,y,x)$

(vi) $j(j(x,y,x),y,z) = j(x,y,z)$

(vii) $j(x,y,j(x,z,x)) = j(x,y,x)$

(viii) $j(j(x,y,j(w,y,z)),j(x,y,z),j(x,y,j(x,y,z))) = j(x,y,z)$

We do not elaborate it further as it is beyond the scope of this thesis.

We call a nearlattice $S$ a medial nearlattice if for all $x,y,z \in S$, $m(x,y,z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ exists. For a (lower) semilattice $S$, if $m(x,y,z)$ exists for all $x,y,z \in S$, then it is not hard to see that $S$ has the upper bound property and hence is a nearlattice. Distributive medial nearlattices were first studied by Sholander [29, 30], and then by Evans [11]. Sholander preferred to call these as *medial semilattices*. He showed that every medial nearlattice $S$ can be characterized by means of an algebra $(\mathrm{S};\mathrm{m})$ of type $\langle 3 \rangle$, known as *medial algebra*, satisfying the following two identities:

(i) $m(a,a,b) = a$

(ii) $m(m(a,b,c),m(a,b,d),e) = m(m(c,d,e),a,b)$.

A nearlattice $S$ is said to have the **three properties** if for any $\mathrm{a,b,c} \in \mathrm{S}$, $\mathrm{a} \vee \mathrm{b} \vee \mathrm{c}$ exists whenever $\mathrm{a} \vee \mathrm{b}$, $\mathrm{b} \vee \mathrm{c}$ and $\mathrm{c} \vee \mathrm{a}$ exists. Nearlattices with the **three properties** were discussed by Evans [11], where he referred it as strong conditional lattices.

The equivalence of (i) and (iii) of the following lemma is trivial, while the proof of (i) <=> (ii) is inductive.

**Lemma 1. 1. 4:** {Evans [11]}. *For a nearlattice $S$ the following conditions are equivalent:*

*(i)* *$S$ has the **three properties**.*

*(ii)* *Every pair of a finite number $n\ (\geq 3)$ of elements of $S$ posses a supremum ensures the existence of the supremum of all the n elements.*

*(iii)* *$S$ is medial.* •

A family $\mathcal{A}$ of a subset of a set $A$ is called a closure system on $A$ if

(i) $A \in \mathcal{A}$ and

(ii) $\mathcal{A}$ is closed under arbitrary intersection.

Suppose $\mathcal{B}$ is a subfamily of $A$. $\mathcal{B}$ is called a directed system if for any $X, Y \in \mathcal{B}$ there exists $Z$ in $\mathcal{B}$ such that $X, Y \subseteq Z$.

If $\cup\{X : X \in \mathcal{B}\} \in \mathcal{A}$ for every directed system $\mathcal{B}$ contained in the closure system $\mathcal{A}$, then $\mathcal{A}$ is called algebraic. When it is ordered by set inclution, an algebraic closure system forms an algebraic lattice.

## 1. 2 Ideals of Nearlattices

A non-empty subset $I$ of a nearlattice $S$ is called a down set if for any $x \in S$ and $y \in I$, $x \leq y$ implies $x \in I$.

A non-empty subset $I$ of a nearlattice $S$ is called an ideal if it is a down set and closed under existent finite suprema. We denote the set of all ideals of $S$ by $I(S)$, which is a lattice. If S has a smallest element 0 then $I(S)$ is an algebraic closure system on $S$ and is consequently an algebraic lattice.

However, if $S$ does not possess smallest element then we can only assert that $I(S) \cup \{\Phi\}$ is an algebraic closure system, where $\Phi$ is the empty subset of $S$.

For any subset $K$ of a nearlattice $S$, $(K]$ denotes the ideal generated by $K$.

Infimum of two ideals of a nearlattice is their set theoretic intersection. Supremum of two ideals $I$ and $J$ in a lattice $L$ is given by $I \vee J = \{x \in L \mid x \leq i \vee j \ \textit{for some } i \in I, j \in J\}$. Cornish and Hickman in [7] showed that in a distributive nearlattice $S$ for two ideals $I$ and $J$, $I \vee J = \{i \vee j \mid \ i \in I, j \in J \textit{ where } i \vee j \textit{ exists}\}$. But in a general nearlattice the fomula for the supremum of two ideals is not very easy. Let us consider the following lemma which gives the formula for the supremum of two ideals. It is in fact an exercise in Gratzer [12], p-54 for partial lattice.

**Theorem 1. 2. 1:** *Let $I$ and $J$ be ideals of a nearlattice $S$. Let $A_0 = I \cup J$, $A_n = \{x \in S \mid x \leq y \vee z;\, y \vee z \ \textit{exists and } y, z \in A_{n-1}\}$ for $n = 1, 2, \cdots$, and $K = \bigcup\limits_{n=0}^{\infty} A_n$. Then $K = I \vee J$.*

**Proof:** Since $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$, $K$ is an ideal containing $I$ and $J$. Suppose $H$ is any ideal containing $I$ and $J$. Of course, $A_0 \subseteq H$. We proceed by induction. Suppose $A_{n-1} \subseteq H$ for some $n \geq 1$ and let $x \in A_n$. Then $x \leq y \vee z$ with $y, z \in A_{n-1}$. Since

$A_{n-1} \subseteq H$ and $H$ is an ideal, $y \vee z \in H$ and so $x \in H$. That is $A_n \subseteq H$ for every n. Thus $K = I \vee J$.•

**Theorem 1. 2. 2:** *Let $K$ be a non-empty subset of a nearlattice $S$. Then $(K] = \bigcup_{n=0}^{\infty}\{A_n \mid n \geq 0\}$, where $A_0 = \{t \in S \mid t = j(k_1, t, k_2)$ for some $k_1, k_2 \in K\}$ and $A_n = \{t \in S \mid t = j(a_1, t, a_2)$ for some $a_1, a_2 \in A_{n-1}\}$ for $n \geq 1$.*

**Proof:** For any $k \in K$ clearly $k = j(k,k,k)$ and so $K \subseteq A_0$. Similarly, for any $a \in A_{n-1}$, $a = j(a,a,a)$ implies that $A_{n-1} \subseteq A_n$. Thus $K \subseteq A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{n-1} \subseteq A_n \subseteq \cdots$.

Let $t \in \bigcup_{n=0}^{\infty} A_n; n = 0,1,2,\cdots$, and $t_1 \in S$ such that $t_1 \leq t$. Then $t \in A_m$ for some $m \geq 0$.

Clearly, $t_1 = j(t, t_1, t)$ and so $t_1 \in A_{m+1}$. Thus $\bigcup_{n=0}^{\infty} A_n$ is down set.

Now suppose, $t_1, t_2 \in \bigcup_{n=0}^{\infty} A_n$ and $t_1 \vee t_2$ exists. Let $t_1 \in A_r$ and $t_2 \in A_s$ for some $r, s \geq 0$ with $r \leq s$ (say). Then $t_1, t_2 \in A_s$ and $t_1 \vee t_2 = j(t_1, t_1 \vee t_2, t_2)$ provides $t_1 \vee t_2 \in A_{s+1}$.

Finally, suppose $H$ is an ideal containing $K$. If $x \in A_0$, then $x = j(k_1, x, k_2) = (k_1 \wedge x) \vee (k_2 \wedge x)$ for some $k_1, k_2 \in K$. As $K \subseteq H$ and $H$ is an ideal, $k_1 \wedge x, k_2 \wedge x \in H$ and so $x \in H$. Thus $A_0 \subset H$. Again we use the induction. Suppose $A_{n-1} \subseteq H$ for some $n \geq 1$. Let $x \in A_n$ so that $x = j(a_1, x, a_2)$ for some $a_1, a_2 \in A_{n-1}$. Then $x \in H$ as $a_1, a_2 \in H$ and $x = (a_1 \wedge x) \vee (a_2 \wedge x)$.•

**Theorem 1. 2. 3:** *A non empty subset $K$ of a nearlattice $S$ is an ideal if and only if $x \in K$ whenever $x \in S$ and $x = j(k_1, x, k_2)$ for some $k_1, k_2 \in K$.*•

We now give an alternative formula for the supremum of two ideals in an arbitrary nearlattice.

**Theorem 1. 2. 4:** *For any two ideals* $K_1$ *and* $K_2$, $K_1 \vee K_2 = \bigcup_{n=0}^{\infty} B_n$ *where* $B_0 = \{x \in S \mid x = j(k_1, x, k_2), k_i \in K_i\}$ *and* $B_n = \{x \in S \mid x = j(b_1, x, b_2), b_1, b_2 \in B_{n-1}\}$, $n = 1, 2, \cdots$.

**Proof** : Clearly, $K_1, K_2 \subseteq B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_{n-1} \subseteq B_n \subseteq \cdots$. Suppose $b \in \bigcup_{n=0}^{\infty} B_n$ and $b_1 \leq b;\ b_1 \in S$. Then $b \in B_m$ for some $m \geq 0$. Also, $b_1 = j(b, b_1, b)$ and so $b_1 \in B_{m+1}$. Thus $\bigcup_{n=0}^{\infty} B_n$ is a down set. Now suppose $t_1, t_2 \in \bigcup_{n=0}^{\infty} B_n$ such that $t_1 \vee t_2$ exists. Then there exist $r, s \geq 0$ such that $t_1 \in B_r$ and $t_2 \in B_s$. If $r \leq s$ then $t_1, t_2 \in B_s$ and $t_1 \vee t_2 = j(t_1, t_1 \vee t_2, t_2)$ implies that $t_1 \vee t_2 \in B_{s+1}$. Hence , $\bigcup_{n=0}^{\infty} B_n$ is an ideal.

Finally, suppose $H$ is an ideal containing $K_1$ and $K_2$. If $x \in B_0$ then $x = j(k_1, x, k_2) = (k_1 \wedge x) \vee (k_2 \wedge x)$ for some $k_1 \in K_1$ and $k_2 \in K_2$. Hence $H$ is an ideal and $K_1, K_2 \subseteq H$, clearly $x \in H$. Then using the induction on n it is very easy to see that $H \supseteq B_n$ for each n. •

In a lattice $L$, it is well known that for a convex sublattice $C$ of $L$. $C = (C] \cap [C)$. The following figure (Fig:1.4) shows that for a convex subnearlattice $C$ in a general nearlattice, this may not be true.

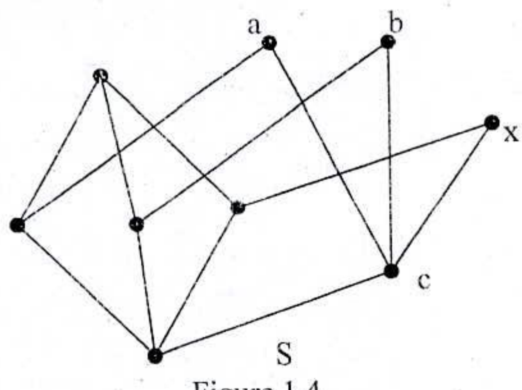

Figure 1.4

Here $C = \{a, b, c\}$ is a convex subnearlattice of $S$. Observe that $(C] = S$ and $[C) = \{a, b, c, x\}$, hence $(C] \cap [C) \neq C$.

Recently, Shiuly Akter [32] has proved that for a convex sublattice $C$ of a distributive nearlattice $S$, $(C]=\{x \in S \mid x=(x \wedge c_1) \vee (x \wedge c_2) \vee \cdots \vee (x \wedge c_n)$ *for some* $c_1, c_2, \cdots, c_n \in C\}$. With the help of this result Rosen [28] have proved that $C=(C] \cap [C)$ when $S$ is distributive. But in a non-distributive nearlattice of $S$, it is easy to show that $C=(C] \cap [C)$ when $S$ is medial.

**Theorem 1. 2. 5:** {Cornish and Hickman [7], Theorem 1. 1}. *The following conditions on a nearlattice $S$ are equivalent:*

*(i)* $S$ *is distributive.*

*(ii)* *For any* $H \in H(S)$, $(H]=\{h_1 \vee h_2 \vee \cdots \vee h_n \mid h_1, h_2, \cdots, h_n \in H\}$.

*(iii)* *For any* $I, J \in I(S)$, $I \vee J=\{a_1 \vee a_2 \vee \cdots \vee a_n \mid a_1, a_2, \cdots, a_n \in I \cup J\}$.

*(iv)* $I(S)$ *is a distributive lattice.*

(v) *The map* $H \to (H]$ *is a lattice homomorphism of* $H(S)$ *onto* $I(S)$ *(which preserves arbitrary suprema).* ●

Observe here that by Theorem1. 2. 5, (iii) of above could easily be improved to (iii)$'$: For any $I, J \in I(S)$, $I \vee J=\{i \vee j \mid i \in I, j \in J\}$.

Let $I_f(S)$ denote the set of all *finitely generated ideals* of a nearlattice $S$. Of course $I_f(S)$ is an upper subsemilattice of $I(S)$. Also for any $x_1, x_2, \cdots, x_m \in S$, $(x_1, x_2, \cdots, x_m]$ is clearly equal to $(x_1] \vee (x_2] \vee \cdots \vee (x_m]$. When $S$ is distributive, $(x_1, x_2, \cdots, x_m] \cap (y_1, y_2, \cdots, y_m]=((x_1] \vee (x_2] \vee \cdots \vee (x_m]) \cap ((y_1] \vee (y_2] \vee \cdots \vee (y_m])$ $=\vee_{ij}(x_i \wedge y_j]$ for any $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n \in S$ and so $I_f(S)$ is a distributive sublattice of $I(S)$.

A nearlattice $S$ is said to be *finitely smooth* if the intersection of two finitely generated ideals is itself finitely generated. For example, distributive nearlattices, finite nearlattices, lattices, are finitely smooth. Hickman in [15] exhibited a nearlattice which is not finitely smooth.

From Cornish and Hickman [7], we know that a nearlattice $S$ is distributive if and only if $I(S)$ is so. Our next result shows that the case is not the same with the modularity.

**Theorem 1. 2. 6:** *Let $S$ be a nearlattice. If $I(S)$ is modular then $S$ is also modular but the converse is not necessarily true.*

Proof: Suppose $I(S)$ is modular. Let $a, b, c \in S$ with $c \leq a$ and $b \vee c$ exists. Then $(c] \subseteq (a]$. Since $I(S)$ is modular, so, $(a \wedge (b \vee c)] = (a] \wedge ((b] \vee (c])$ $= ((a] \wedge (b]) \vee (c] = ((a \wedge b) \vee c]$. This implies that $a \wedge (b \vee c) = (a \wedge b) \vee c$, and so $S$ is modular.

Nearlattice $S$ of Figure 1.5 shows that the converse of this result is not true.

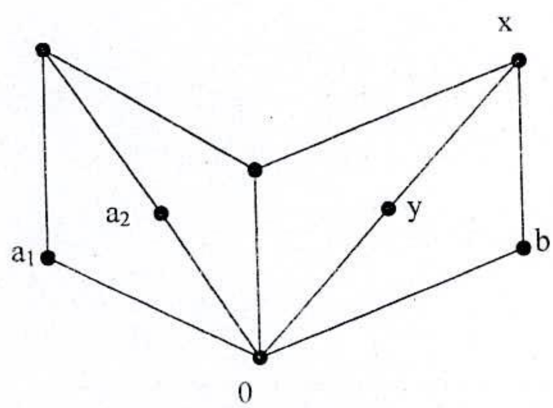


Figure 1.5

Notice that $(r]$ is modular for each $r \in S$. But in $I(S)$, clearly $\{(0], (a_1], (a_1, y], (a_2, b], S\}$ is a pentagonal sublattice. •

The following theorem is due to Bazlar Rahman [3]

**Theorem 1. 2. 7:** {Bazlar Rahman [3]} *Let $I$ and $J$ be two ideals in a distributive nearlattice S. If $I \wedge J$ and $I \vee J$ are principal, then both $I$ and $J$ are principal.* •

A non empty subset $F$ of a nearlattice $S$ is called an up set if for $x \in S$, $y \in F$ with $x \geq y$ imply $x \in F$.

A non empty subset $F$ of a nearlattice $S$ is called a filter if it is an up set and $f_1 \wedge f_2 \in F$ for all $f_1, f_2 \in F$.

An ideal $P$ in a nearlattice $S$ is called a prime ideal if $P \neq S$ and $x \wedge y \in P$ implies $x \in P$ or $y \in P$.

A filter $F$ is called a prime filter if either $x \in F$ or $y \in F$ whenever $x \vee y$ exists and is in $F$.

It is not hard to see that a filter $F$ of a nearlattice $S$ is prime if and only if $S - F$ is a prime ideal. The set of all filters of a nearlattice is an upper (join) semilattice ; yet it is not a lattice in general, as there is no guarantee that the intersection of two filters is non empty. The join $F_1 \vee F_2$ of two filters is given by $F_1 \vee F_2 = \{s \in S \mid s \geq f_1 \wedge f_2 \ for\ some\ f_1 \in F_1, f_2 \in F_2\}$. The smallest filter containing a subsemilattice $H$ of $S$ is $\{s \in S \mid s \geq h \ for\ some\ h \in H\}$ and is denoted by $[H)$. Moreover, the description of the join of filters shows that for all $a, b \in S$, $[a) \vee (b] = [a \wedge b)$.

Following theorem and corollary is due to Noor and Rahman [21] which is an extension of Stone's separation theorem of Gratzer [12] theorem 15, pp74.

**Theorem 1. 2. 8:** {Noor and Rahman[21]} *Let $S$ be a nearlattice. The following conditions are equivalent:*

*(i)* *$S$ is distributive.*

(ii) *For any ideal I and any filter $F$ of $S$, such that $I \cap F = \Phi$, there exists a prime ideal $P \supseteq I$ and disjoint from $F$.*●

**Corollary 1. 2. 9:** *A nearlattice $S$ is distributive if and only if every ideal is the intersection of all prime ideals containing it.*●

**Lemma 1. 2. 10:** *A subset $F$ of a nearlattice $S$ is a filter if and only if $S-F$ is a prime down set.*

**Proof:** Let $x \in S-F$ and $t \leq x$. Then $x \notin F$, and so $t \notin F$, as $F$ is a filter. Hence $t \in S-F$ , and so $S-F$ is a down set. Now let $x, y \in S$ such that $x \wedge y \in S-F$. It follows that $x \wedge y \notin F$. This implies either $x \notin F$ or $y \notin F$, as $F$ is a filter. That is, either $x \in S-F$ or $y \in S-F$, and so $S-F$ is a prime down set.

Conversely, suppose $S-F$ is a prime down set. Let $x \in F$ and $t \geq x$. Then $x \notin S-F$ and so $t \notin S-F$ as $S-F$ is a prime down set. Thus $t \in F$ and so $F$ is an upset. Finally let $x, y \in F$. Then $x \notin S-F$, $y \notin S-F$. Since $S-F$ is a prime, so $x \wedge y \notin S-F$. Therefore $x \wedge y \in F$, and so ,$F$ is a filter. •

Following result is an easy consequence of above lemma.

**Lemma 1. 2. 11:** A subset *$F$ of a nearlattice $S$ is a prime filter if and only if $S-F$* is a prime ideal. •

Now we include a generalization of theorem 1. 2. 8 in a general nearlattice.

**Theorem 1. 2. 12:** *Let $S$ be a nearlattice. $F$ be a filter and $I$ be a down set such that $I \cap F = \Phi$. Then there exists a prime down set $P$ containing $I$ but disjoint to $F$.*

**Proof:** Let $\chi$ be the collection of all filter containing $F$ *and disjoint to* $I$. Then $\chi$ is non-empty as $F \in \chi$. Suppose $C$ is a chain in $\chi$. Set $M = \cup\{X \mid X \in C\}$. Let $x \in M$ and $y \geq x$. Then $x \in X$ for some $X \in C$. Since $X$ is a filter , so $y \in X$ and hence $y \in M$. Thus $M$ is an upset. Now let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in C$. Since $C$ is a chain , so either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$. This implies $x, y \in Y$ , and so $x \wedge y \in Y$ as $Y$ is a filter. It follows that $x \wedge y \in M$ and hence, $M$ is a filter containing $F$. Moreover $M \cap I = \phi$. Therefore, $M$ is the largest element of $C$. Thus by Zorn's lemma, $M$ is a maximal filter containing $F$. Therefore by Lemma 1.2.10, $L-M$ is a minimal prime down set containing $I$ but disjoint to $F$ .•

**Corollary 1. 2. 13:** *Let $S$ be a nearlattice with 0 and $F$ be a proper filter of $S$ . Then there exists a prime down set $P$ such that $F \cap P = \Phi$ .*•

The following lemma is very useful in proving many results of distributive nearlattice.

**Lemma 1. 2. 14:** *If $S_1$ is a subnearlattice of a distributive nearlattice S and $P_1$ is a prime ideal in $S_1$, then there exists a prime ideal P in S such that $P_1 = S_1 \cap P$ .*•

Following theorem is a generalization of Lemma 1.2.14, which will be needed in establishing some results in other chapters.

**Theorem 1. 2. 15:** *Let $S_1$ be a subnearlattice of $S$. and $P_1$ be a prime down set of $S_1$. Then there exists a prime down set $P$ of $S$ such that $P_1 = P \cap S_1$.*

**Proof:** Let $H$ be a down set generated by $P_1$ in $S$. Then $H \cap (S_1 - P_1) = \Phi$. Now $S_1 - P_1$ is an upset in $S_1$ and $H \cap [S_1 - P_1) = \Phi$ where, $[S_1 - P_1)$ is the filter generated by $S_1 - P_1$ in $S$. Then by Theorem 1.2.12 , there exists a prime down set $P \supseteq H$ and disjoint to $[S_1 - P_1)$. Now $P_1 \subseteq H \cap S_1 \subseteq P \cap S_1$. Also $P \cap S_1 \subseteq P_1$. Hence, $P_1 = P \cap S_1$.•

## 1. 3 Congruences

An equivalence relation $\Theta$ of a nearlattice $S$ is called a congruence relation if $x_i \equiv y_i(\Theta)$ for $i = 1,2$ $(x_i, y_i \in S)$, then

(i) $x_1 \wedge x_2 \equiv y_1 \wedge y_2(\Theta)$, and

(ii) $x_1 \vee x_2 \equiv y_1 \vee y_2(\Theta)$ provided $x_1 \vee x_2$ and $y_1 \vee y_2$ exists.

It can be easily shown that for an equivalence relation $\Theta$ on $S$, the above conditions are equivalent to the conditions that for $x, y \in S$ if $x \equiv y(\Theta)$, then

(i$'$) $x \wedge t \equiv y \wedge t(\Theta)$ for all $t \in S$ and

(ii$'$) $x \vee t \equiv y \vee t(\Theta)$ for all $t \in S$ provided both $x \vee t$ and $y \vee t$ exists.

The set $C(S)$ of all congruences on S is an algebraic closure system on $S \times S$ and hence, when ordered by set inclusion, is an algebraic lattice.

Cornish and Hickman [7] showed that for an ideal $I$ of a distributive nearlattice S, the relation $\Theta(I)$ , defined by $x \equiv y(\Theta(I))$ if and only if $(x] \vee I = (y] \vee I$, is the smallest congruence containing $I$ as a class. Moreover the equivalence relation $R(I)$, is defined by $x \equiv y(R(I))$ if and only if for any $s \in S$, $s \wedge x \in I$ is equivalent to $s \wedge y \in I$. In fact , this is the largest congruence of $S$ having $I$ as a class.

Suppose $S$ is a distributive nearlattice and $x \in S$ we will use $\Theta_x$ as an abbreviation for $\Theta((x])$. Moreover $\psi_x$ denote the congruence, defined by $a \equiv b\ (\psi_x)$ if and only if $a \wedge x = b \wedge x$.

Cornish and Hickman [7] also showed that for any two elements $a,b$ of a distributive nearlattice $S$ with $a \leq b$, the smallest congruence identifying $a$ and $b$ is equal to $\psi_a \cap \Theta_b$ and we denote it by $\Theta(a,b)$. Also in a distributive nearlattice $S$, they observed that if $S$ has a smallest element 0, then clearly $\Theta_x = \Theta(0,x)$ for any $x \in S$.

Moreover, we see that:

(i) $\Theta_a \vee \psi_a = \tau$, the largest congruence of $S$.

(ii) $\Theta_a \cap \psi_a = \omega$, the smallest congruence of $S$ and

(iii) $\Theta(a,b)' = \Theta_a \vee \psi_a$ where $a \le b$ and $(')$ denotes the complement.

Now suppose $S$ is an arbitrary nearlattice and $E(S)$ denote the lattice of equivalence relations. For $\Phi_1, \Phi_2 \in E(S)$ with $\Phi_1 \vee \Phi_2$ denoting their supremum $x \equiv y(\Phi_1 \vee \Phi_2)$ if and only if there exist $x = z_0, z_1, \cdots, z_n = y$ such that $z_{i-1} \equiv z_i(\Phi_1 \text{ or } \Phi_2)$ for $i = 1,2,\cdots,n$.

The following result was stated by Gratzer and Lakser in [14] without proof and a proof given below, appeared in Cornish and Hickman [7].

**Theorem 1. 3. 1:** *For any nearlattice $S$, $C(S)$ is a distributive (complete) sublattice of $E(S)$.*

**Proof:** Suppose $\Theta, \Phi \in C(S)$. Define $\psi$ to be the supremum of $\Theta$ *and* $\Phi$ in the lattice of equivalence relations $E(S)$ on $S$. Let $x \equiv y(\psi)$. Then there exists $x = z_0, z_1, \cdots, z_n = y$ such that $z_{i-1} \equiv z_i(\Phi_1 \text{ or } \Phi_2)$. Thus, for any $t \in S$, $z_{i-1} \wedge t \equiv z_i \wedge t(\Phi_1 \text{ or } \Phi_2)$ as $\Theta, \Phi \in C(S)$.

Hence $x \wedge t \equiv y \wedge t(\psi)$ and consequently $\psi$ is a semilattice congruence. Then, in particular $x \wedge y \equiv x(\psi)$ and $x \wedge y \equiv y(\psi)$. To show that $\psi$ is a congruence, let $x \equiv y(\psi)$, with $x \le y$, and choose any $t \in S$ such that both $x \vee t$ and $y \vee t$ exists. Then there exists $z_0, z_1, z_2, \cdots, z_n$, such that $x = z_0, z_n = y$ and $z_{i-1} \equiv z_i(\Phi_1 \text{ or } \Phi_2)$. Put $w_i = z_i \wedge y$ for all $i = 0,1,\cdots,n$. Then $x = w_0, w_n = y$, $w_{i-1} \equiv w_i(\Phi_1 \text{ or } \Phi_2)$. Hence by the upper bound property, $w_i \vee t$ exists for all $i = 0,1,\cdots,n$ (as $w_i \vee t \le y \vee t$) and $w_{i-1} \vee t \equiv w_i \vee t(\Phi_1 \text{ or } \Phi_2)$ for all $i = 0,1,\cdots,n$ ( as $\Theta, \Phi \in C(S)$), *i.e.* $x \vee t \equiv y \vee t(\psi)$. Then by Cornish and Noor [8] Lemma 2. 3 $\psi$ is a congruence on $S$. Therefore, $C(S)$ is a sublattice of the lattice $E(S)$.

To show the distributivity of $C(S)$, let $x \equiv y(\Theta \cap (\Theta_1 \vee \Theta_2))$. Then $x \wedge y \equiv y(\Theta)$ and $x \wedge y \equiv y(\Theta_1 \vee \Theta_2)$. Also $x \wedge y \equiv x(\Theta)$ and $x \wedge y \equiv x(\Theta_1 \vee \Theta_2)$.

Since $x \wedge y \equiv y(\Theta_1 \vee \Theta_2)$, there exists $t_0, t_1, \cdots, t_n$ such that (as we have seen in the proof of the first part), $x \wedge y = t_0, t_n = y$, $t_{i-1} \equiv t_i(\Theta_1 \ or \ \Theta_2)$ and $x \wedge y = t_0 \leq t_i \leq y$ for each $i = 0,1,\cdots,n$. Hence $t_{i-1} \equiv t_i(\Theta)$ for all $i = 0,1,\cdots,n$ and so $t_{i-1} \equiv t_i(\Theta \cap \Theta_1)$ or $t_{i-1} \equiv t_i(\Theta \cap \Theta_2)$. Thus $x \wedge y \equiv y((\Theta \cap \Theta_1) \vee (\Theta \cap \Theta_2))$. By symmetry, $x \wedge y \equiv x((\Theta \cap \Theta_1) \vee (\Theta \cap \Theta_2))$ and the proof completes by transitivity of the congruences.●

In lattice theory it is well known that a lattice is distributive if and only if every ideal is a class of some congruence. Following theorem gives a generalization of this result in case of nearlattices.

This also characterizes the distributivity of a nearlattice, which is an extension of Cornish and Hickman [7] Theorem 3. 1.

**Thoerem 1. 3. 2:** A nearlattice $S$ *is distributive if and only if every ideal is a class of some congruence.*

**Proof:** Suppose $S$ is distributive . Then by Cornish and Hickman [7] Theorem 3.1 for each ideal $I$ of $S$ $\Theta(I)$ is the smallest congruence containing $I$ as a congruence class.

To prove the converse, let each ideal of $S$ be a congruence class with respect to some congruence on $S$. Suppose $S$ is not distributive. Then by Theorem 1. 1. 2, we have either $N_5$ (Figure 1.2) or $M_5$ (Figure 1.3) as a sublattice of $S$. In both cases consider $I = (a]$ and suppose $I$ is a congruence class with respect to $\Theta$. Since $d \in I, d \equiv a(\Theta)$. Now $b = b \wedge c = b \wedge (a \vee c) \equiv b \wedge (d \vee c) = b \wedge c = d(\Theta)$ , that is, $b \equiv d(\Theta)$ and this implies $b \in I$ , *i.e.* $b \leq a$ which is a contradiction. Thus $S$ is distributive. ●

Following results are due to Bazlar Rahman [3].

**Theorem 1. 3. 3:**{Bazlar Rahman [3], theorem 3. 4} *Let* $S$ *be a distributive nearlattice then,*

(i) *For ideals* $I$ *and* $J$, $\Theta\ (I \cap J) = \Theta(I) \cap \Theta(J)$.

*(ii)* *For ideals* $j_i$ $i \in A$ *an indexed set,* $\Theta\left(\vee J_i\right) = \vee\Theta(J_i)$. •

**Theorem 1. 3. 4:**{Bazlar Rahman [3], corollary 3. 5} *For a distributive nearlattice S, the mapping* $I \to \Theta(I)$ *is an embedding from the lattice of ideals to the lattice of congruences.* •

## 1. 4 Relative Annihilators:

Recall that a nearlattice $S$ is distributive if for all $x, y, z \in S$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ provided $y \vee z$ exists. Since for all $x, y, z \in S$, $(x \wedge y) \vee (x \wedge z)$ always exists by the upper bound property, we give an alternative definition of distributivity of $S$ by the following lemma.

**Lemma 1. 4. 1:** *A nearlattice $S$ is distributive if and only if for all $t, x, y, z \in S$,* $t \wedge ((x \wedge y) \vee (x \wedge z)) = (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$.

**Proof:** Suppose $S$ is distributive. Then obviously,

$t \wedge ((x \wedge y) \vee (x \wedge z)) = (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$.

Conversely, suppose $S$ has the given property. Let $a, b, c \in S$ with $b \vee c$ exists. Set $t = b \vee c$. Then

$a \wedge (b \vee c) = a \wedge ((t \wedge b) \vee (t \wedge c)) = (a \wedge t \wedge b) \vee (a \wedge t \wedge c) = (a \wedge b) \vee (a \wedge c)$. Therefore $S$ is distributive •

Recall that a nearlattice $S$ is modular if for all $x, y, z \in S$ with $z \leq x$ and whenever $y \vee z$ exists then $x \wedge (y \vee z) = (x \wedge y) \vee z$. Like lemma 1.4.1, we can also easily characterize modular nearlattice by the following result.

**Lemma 1. 4. 2:** *A nearlattice $S$ is modular if and only if for all $t, x, y, z \in S$, with $z \leq x$,* $x \wedge ((t \wedge y) \vee (t \wedge z)) = (x \wedge t \wedge y) \vee (t \wedge z)$.

**Proof:** Suppose $S$ is modular. Then obviously,

$x \wedge ((t \wedge y) \vee (t \wedge z)) = (x \wedge t \wedge y) \vee (t \wedge z)$.

Conversely, suppose $S$ has the given property. Let $a, b, c \in S$ with $c \leq a$ and whenever $b \vee c$ exists. Set $t = b \vee c$, then

$a \wedge (b \vee c) = a \wedge ((t \wedge b) \vee (t \wedge c)) = (a \wedge t \wedge b) \vee (a \wedge t \wedge c) = (a \wedge t \wedge b) \vee (t \wedge c) = (a \wedge b) \vee c$. Therefore $S$ is modular. •

Now we generalized Theorem 1 and Theorem 2 of Katrinak, [17].

**Theorem 1. 4. 3:** *For a nearlatice S the following conditions are equivalent:*

*(i)* $S$ *is distributive.*

*(ii)* $\langle a,b\rangle$ *is an ideal for all* $a,b \in S$.

*(iii)* $\langle a,b\rangle$ *is an ideal whenever* $b \le a$.

**Proof:** Since (i) implies (ii) and (ii) implies (iii) are trivial, we shall prove only (iii) implies (i).

Suppose (iii) holds. Let $t,x,y,z \in S$. Then

$(t \wedge x \wedge y) \vee (t \wedge x \wedge z) \le x$ implies $\langle x,(t \wedge x \wedge y) \vee (t \wedge x \wedge z)\rangle$ is an ideal. Again $(t \wedge x \wedge y) \le (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$ implies $t \wedge y \in \langle x,(t \wedge x \wedge y) \vee (t \wedge x \wedge z)\rangle$.

Similarly, $t \wedge z \in \langle x,(t \wedge x \wedge y) \vee (t \wedge x \wedge z)\rangle$.

Hence $(t \wedge y) \vee (t \wedge z) \in \langle x,(t \wedge x \wedge y) \vee (t \wedge x \wedge z)\rangle$.

Thus, $x \wedge ((t \wedge y) \vee (t \wedge z)) \le (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$. Since the reverse inequality is trivial, so $x \wedge ((t \wedge y) \vee (t \wedge z)) = (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$.

Therefore by lemma 1. 4. 1, $S$ is distributive. ∘

**Theorem 1. 4. 4:** *A nearlattice* $S$ *is modular if and only if whenever* $b \le a$, *if* $t \wedge x \in (b]$ *and* $t \wedge y \in \langle a,b\rangle$ *for any* $t \in S$, *then* $(t \wedge x) \vee (t \wedge y) \in \langle a,b\rangle$.

**Proof:** Suppose $S$ is modular. Since $t \wedge y \in \langle a,b\rangle$, so $a \wedge t \wedge y \le b$. Also $t \wedge x \le b \le a$.

Thus by modularity of $S$, $a \wedge ((t \wedge x) \vee (t \wedge y)) = (a \wedge t \wedge y) \vee (t \wedge x) \le b$, and so $(t \wedge x) \vee (t \wedge y) \in \langle a,b\rangle$.

Conversely, let the given condition holds, suppose $t, x, y, z \in S$, with $z \leq x$. Then $(t \wedge z) \vee (t \wedge x \wedge y) \leq x$ and $t \wedge z \in ((t \wedge z) \vee (t \wedge x \wedge y)]$. Also, $t \wedge x \wedge y \leq (t \wedge z) \vee (t \wedge x \wedge y)$ implies $t \wedge y \in \langle x, (t \wedge z) \vee (t \wedge x \wedge y) \rangle$. Then by hypothesis $(t \wedge z) \vee (t \wedge y) \in \langle x, (t \wedge z) \vee (t \wedge x \wedge y) \rangle$.

This implies $x \wedge ((t \wedge y) \vee (t \wedge z)) \leq (t \wedge x \wedge y) \vee (t \wedge z)$. Since the reverse inequality is trivial, so by lemma 1. 4. 2, $S$ is modular. •

Following result is a generalization of a lemma of Katrinak [17] in section 3.

**Lemma 1. 4. 5:** *In any distributive nearlattice $S$, each of the following conditions on a given filter $F$ implies the next.*

*(i)* *For all $a, b \in S$, there exists an element $x \in F$ such that $a \wedge x$ and $b \wedge x$ are comparable.*

*(ii)* *The filters containing $F$ form a chain.*

*(iii)* *$F$ is prime.*

*(iv)* *$F$ contains a prime filter.*

**Proof:** (i) implies (ii): Suppose (i) holds. If (ii) fails then there exists non-comparable filters $G$ and $H$ containing $F$. Choose elements $a \in G - H$ and $b \in G - H$. Then by (i) there exists $x \in F$ such that $a \wedge x$ and $b \wedge x$ are comparable. Suppose $a \wedge x \leq b \wedge x$. Since $x \in F - G$, so $a \wedge x \in G$. Then $a \wedge x \leq b$ implies $b \in G$, which gives a contradiction. Therefore (ii) holds.

(ii) implies (iii): Suppose (ii) holds. Let $a, b \in S$ with $a \vee b$ exists and $a \vee b \in F$. Let $G = F \vee [a)$ and $H = F \vee [b)$. By (ii), either $G \subseteq H$ or $H \subseteq G$. Suppose $G \subseteq H$. Then $a \in H$, and so $a = x \wedge b$ for some $x \in F$. Since $x, a \vee b \in F$, so $x \wedge (a \vee b) \in F$. Thus by distributivity of $S$, $(x \wedge a) \vee (x \wedge b) = (x \wedge a) \vee a = a \in F$. Therefore $F$ is prime.

(iii) implies (iv) is trivial. •

For a lattice $L$, the identity $\langle a,b\rangle \vee \langle b,a\rangle = L$ for all $a,b \in L$ is well known in lattice theory. This identity in fact, characterizes relatively Stone and relatively normal lattice; c.f. [17] and [5].

**Theorem 1. 4. 6:** *For a distributive nearlattice $S$ the identity $\langle a,b\rangle \vee \langle b,a\rangle = S$ for all $a,b \in S$ bolds if and only if all the conditions of lemma 1.4.5 are equivalent.*

**Proof:** Suppose the identity holds. We need only to show that (iv) implies (i) of lemma 1.4.5. Let $a,b \in S$ . Suppose $P$ is a prime filter contained in $F$ . Choose $z \in P$ . Since $\langle a,b\rangle \vee \langle b,a\rangle = S$, so $z = x \vee y$ for some $x \in \langle a,b\rangle$ and $y \in \langle b,a\rangle$ . Since $P$ is prime , either $x \in P$ or $y \in P$. Suppose $x \in P$. Then $x \in F$, and $x \in \langle a,b\rangle$ implies $a \wedge x \leq b$ and so $a \wedge x \leq b \wedge x$. Therefore (i) holds.

Conversely, suppose all the conditions of the lemma 1.4.5 are equivalent. Let there exists $a,b \in S$ such that $I = \langle a,b\rangle \vee \langle b,a\rangle$ is proper ideal of S. Then by theorem 1. 2. 7, there exists a prime filter $P$ disjoint from $I$. Then by (iii) implies (i), there exists $x \in P$, such that $a \wedge x$ and $b \wedge x$ are comparable. Suppose $a \wedge x \leq b \wedge x$. Then $a \wedge x \leq b$ implies $x \in \langle a,b\rangle$ which is a contradiction as $P \cap I = \Phi$. Therefore $\langle a,b\rangle \vee \langle b,a\rangle = S$. •

We conclude this section with the following generalization of Katrinak [17], Theorem 4.

**Theorem 1. 4. 7:** *For any distributive nearlattice $S$, the following conditions are equivalent:*

(i) *For all $a,b \in S$, $\langle a,b\rangle \vee \langle b,a\rangle = S$.*

(ii) *For any prime filter* $P$ *and for any* $a, b \in S$. *there exists* $x \in P$. *such that* $a \wedge x$ *and* $b \wedge x$ *are comparable.*

(iii) *The filters containing any given prime filter form a chain.*

**Proof:** (i) implies (ii) easily follows from the proof of first part of Theorem 1.4.6; while (ii) implies (iii) holds by lemma 1.4.5.

(iii) implies (i): Suppose (iii) holds. Let for $a, b \in S$, $I = \langle a,b \rangle \vee \langle b,a \rangle$ be a proper ideal of $S$. Then by stones representation theorem there exists a prime filter $P$ disjoint from $I$. Let $G = P \vee [a)$ and $H = P \vee [b)$. By (iii) either $G \subseteq H$ or $H \subseteq G$. Suppose $G \subseteq H$. Then $a \in P \vee [b)$ implies $a = x \wedge b$ for some $x \in P$. Then $x \in \langle b,a \rangle$, which is a contradiction as $P \cap I = \Phi$. Therefore $\langle a,b \rangle \vee \langle b,a \rangle = S$.●

# DISJUNCTIVE NEARLATTICES AND SEMIBOOLEAN ALGEBRAS

A distributive nearlattice $S$ with 0 is called *disjunctive nearlattice* if $0 \le a < b$ implies there is an element $x \in S$ such that $x \wedge a = 0$ where $0 < x \le b$. A subset $A$ of a complete lattice L is said to be *join-dense* if $L = \{\vee R \mid R \subseteq A\}$.

A non empty subset $T$ of a nearlattice $S$ is called *large* if $x \wedge t = y \wedge t$ for all $t \in T$, $(x, y \in S)$ imply $x = y$, while $T$ is called join-dense if each $z \in S$ is the join of its predecessors in $T$. Following result shows that two concepts coincide when $T$ is a convex subsemilattice of a distributive nearlattice and hence an ideal of a nearlattice is large if and only if it is join-dense.

## 2. 1 Disjunctive Nearlattice and Semiboolean Algebras

**Lemma 2. 1. 1:** *A convex subsemilattice $J$ of a distributive nearlattice $S$ is large if and only if it is join- dense in $S$.*

**Proof:** Obviously, every join-dense subset of $S$ is large in $S$. Thus, let $J$ be large in $S$. Suppose $x \in S$ and $\{j_i\}$ are its predecessors in $J$. Let t be an upper bound of $\{j_i\}$. Clearly, for any $j \in J$, $j_i \wedge j \le x \wedge j \le j$ and so $x \wedge j \in J$ by the convexity of $J$. Thus, $x \wedge j = j_k$ for some $k$. Hence, $x \wedge j \le t$ for all $j \in J$; it follows that $x \wedge j = x \wedge t \wedge j$ for all $j \in J$. Since $J$ is large, $x \wedge t = x$, i.e., $x \le t$. This implies that $x$ is the supremum of $\{j_i\}$.•

Now we give a characterization of join dense ideals in terms of skeletal congruences.

**Lemma 2. 1. 2:** *An ideal $J$ of a distributive nearlattice is $S$ join-dense if and only if $\Theta(J)$ is dense in $C(S)$, that is $\Theta(J)^{\perp} = \omega$, the smallest element of $C(S)$.*

**Proof:** Suppose $J$ is join-dense. Then by lemma 2.1, $J$ is large. Let $x \equiv y(\Theta(J^{\perp}))$, then $x \wedge j = y \wedge j$ for all $j \in J$. This implies $x = y$ as $J$ is large. So $\Theta(J)^{\perp} = \omega$. That is, $\Theta(J)$ is dense.

Conversely, let $\Theta(J)^{\perp} = \omega$. Suppose $x \wedge j = y \wedge j$ for all $j \in J$. Then according to Bazlar Rahman [3], $x \equiv y\ \Theta(J)^{\perp}\ (= \omega)$ and so $x = y$. This implies $J$ is large and so by lemma 2. 1. 1, it is join-dense. •

We know that for an ideal $I$ of a distributive nearlattice $S$, the relation $R(I)$ defined by $x \equiv y\, R(I)$ if and only if for all $r \in S$, $x \wedge r \in I$ is equivalent to $y \wedge r \in I$ is a congruence of $S$. Moreover, it is the largest congruence of $S$ containing $I$ as a class.

**Proposition 2. 1. 3:** *For an ideal $I$ of a distributive nearlattice $S$, $S / R(I)$ is disjunctive.*

**Proof:** If $I$ is a prime ideal, then $S / R(I)$ is a two element chain $\{I, S - I\}$ and so it is disjunctive (in fact, Boolean). Suppose $I$ is not prime, consider the interval $I \subseteq [x] \subset [y]$ in $S / R(I)$, where $x, y \in S$. We claim that there exists at least one $t \notin I$, such that $t \wedge x \in I$. If not, then for all $t \notin I, x \wedge t \notin I$ and since $[x \wedge t] \subseteq [y \wedge t]$, so $y \wedge t \notin I$. This implies that $x \equiv y\, R(I)$ and so $[x] = [y]$, which is a contradiction. Moreover, there exists a $t \notin I$ such that $x \wedge t \in I$ but $y \wedge t \notin I$. For otherwise $x \equiv y\, R(I)$ would lead to another contradiction. Put $s = y \wedge t$. Then $I \subset [S] \subseteq [y]$ and $[x] \wedge [s] = [x] \wedge [y \wedge t] = [x \wedge y \wedge t] = I$ and this implies that $S / R(I)$ is distributive. •

Following theorem gives characterizations of distributive nearlattices.

**Theorem 2. 1. 4:** *For a distributive nearlattice $S$ with $0$, the following conditions are equivalent:*

*(i)* *$S$ is disjunctive.*

*(ii)* *For all $a \in S$, $(a] = (a]^{\perp\perp}$.*

*(iii)* $R((0]) = \omega$.

**Proof:** (i) implies (ii): Suppose $S$ is disjunctive. For any $a \in S$, obviously, $(a] \subseteq (a]^{\perp\perp}$. To prove the reverse, let $x \in (a]^{\perp\perp}$. If $x \notin (a]$, then $x \not\leq a$ i.e. , $x \neq x \wedge a$. Then $0 \leq x \wedge a < x$. Since $S$ is disjunctive there exists $t$ with $0 < t \leq x$ such that $t \wedge x \wedge a = 0$ i.e. , $t \wedge a = 0$. This implies $t \in (a]^{\perp}$. Since $x \in (a]^{\perp\perp}$, so $x \wedge t = 0$, i.e. , t = 0, which gives a contradiction. Hence $x \in (a]$. In other words $(a] = (a]^{\perp\perp}$ For all $a \in S$.

(ii) implies (iii): Suppose (ii) holds and $x \equiv y$ $R((0])$ for some $x, y \in S$. If $x \neq y$, then either $x \wedge y < y$ or $x \wedge y < x$. Suppose $x \wedge y < y$. Then $y^{\perp} \subset (x \wedge y]^{\perp}$. Since $(a] = (a]^{\perp\perp}$ for all $a \in S$, $(y]^{\perp} \neq (x \wedge y]^{\perp}$. Thus, $(y]^{\perp} \subset (x \wedge y]^{\perp}$. So there exists $t \in (x \wedge y]^{\perp}$, such that $t \notin (y]^{\perp}$. Then $t \wedge x \wedge y = 0$ but $t \wedge y \neq 0$, which implies $x \wedge y \not\equiv y$ $R((0])$, and so $x \not\equiv y$ $R((0])$, which is a contradiction. Therefore, $R((0]) = \omega$.

(iii) implies (i): Suppose $R((0]) = \omega$. Let $0 \leq x < y$ $(x, y \in S)$. Since $R((0]) = \omega$, there exists $t \in S$ such that $t \wedge x = 0$ but $t \wedge y \neq 0$. For otherwise $x \equiv y$ $R((0])$, which implies $x = y$ and there is a contradiction to our assumption. Thus we have $0 < t \wedge y \leq y$, such that $x \wedge t \wedge y = 0$, and so $S$ is disjunctive. •

In chapter I, we have already denoted the set of all finitely generated ideals of a nearlattice $S$ by $I_f(S)$. Of course $I_f(S)$ is a join semilattice of $I(S)$. In [15], Hickman exhibited a nearlattice $S$ for which $I_f(S)$ is a meet semilattice. But in [7], Cornish and Hickman have shown that if $S$ is distributive then $I_f(S)$ is a distributive sublattice of $I(S)$, the lattice of ideals.

**Lemma 2. 1. 5:** *A distributive nearlattice $S$ with $0$ is disjunctive if and only if $I_f(S)$ is disjunctive.*

**Proof :** Let $S$ be disjunctive and $(a_1, \ldots\ldots\ldots, a_r] \subset (b_1, \ldots\ldots\ldots, b_t]$ in $I_f(S)$. Choose $x \in (b_1, \ldots\ldots\ldots, b_t] - (a_1, \ldots\ldots\ldots, a_r]$. Then $(a_1 \wedge x, \ldots\ldots\ldots, a_r \wedge x] = (a_1, \ldots\ldots\ldots, a_r] \cap (x] \subset (x]$. Now, by the upper bound property of $S$, $(a_1 \wedge x) \vee \ldots\ldots\ldots \vee (a_r \wedge x) = e$ (say) exists and

$0 \le e < x$. Since $S$ is disjunctive, there exists $d \in S$ such that $0 = d \wedge e$ and $0 < d \le x$. Thus $(d] \cap (e] = (0]$ and so $(d] \cap (a_1, \ldots\ldots\ldots, a_r] \cap (x] = (0]$. This implies that $(d] \cap (a_1, \ldots\ldots\ldots, a_r] = (0]$. Of course, $(0] \ne (d] \subseteq (x] \subseteq (b_1, \ldots\ldots\ldots\ldots, b_t]$ and hence, $I_f(S)$ is disjunctive.

Conversely, let $I_f(S)$ be disjunctive and suppose $0 \le c < d$; $c, d \in S$. Then, $(0] \subseteq (c] \subseteq (d]$. Since $I_f(S)$ is disjunctive, there exists $(a_1, \ldots\ldots\ldots, a_r]$ in $I_f(S)$ such that $(c] \cap (a_1, \ldots\ldots\ldots, a_r] = (0]$, where $(0] \ne (a_1, \ldots\ldots\ldots\ldots, a_r] \subseteq (d]$. Now, by the upper bound property of $S$, $a_1 \vee \ldots\ldots\ldots \vee a_r = f$ (say) exists. Thus, we have $c \wedge f = 0$ and $0 < f \le d$,

and which proves that $S$ is disjunctive. •

**Theorem 2. 1. 6:** *In a distributive nearlattice $S$ with $0$, the following conditions are equivalent:*

(i) *$S$ is disjunctive.*

(ii) *Each dense ideal $J$ (i.e. $J^{\perp} = (0]$) is join-dense.*

(iii) *For each dense ideal $J$, $\Theta(J^{\perp}) = \Theta(J)^{\perp}$.*

(iv) *For each dense ideal $J$, $\Theta(J^{\perp\perp}) = \Theta(J)^{\perp\perp}$.*

**Proof :** Since $J^{\perp} = (0]$ if and only if $J^{\perp\perp} = S$ and $J$ is join-dense if and only if $\Theta(J)^{\perp} = \omega$, obviously (ii), (iii) and (iv) are equivalent.

(i) implies (ii): Suppose $J$ is dense ideal and $x \wedge j = y \wedge j$ $(x, y \in S)$ for all $j \in J$. If $x \ne y$, then either $x \wedge y < x$ or $x \wedge y < y$. Without loss of generality suppose $x \wedge y < x$. Since $S$ is disjunctive, there exists $a(\ne 0) \in S$, $a \le x$ such that $a \wedge x \wedge y = 0$. Then, $0 = a \wedge x \wedge y \wedge j = a \wedge x \wedge j$ for all $j \in J$. Hence, $a \wedge x = 0$ as $J$ is dense ; i.e. , a = 0 which is a contradiction. Thus $J$ is join-dense.

(ii) implies (i): For any $a \in S$, $(a] \vee (a]^{\perp}$ is always a dense ideal. Thus, with holding (ii), $(a] \vee (a]^{\perp}$ is join-dense. Then by lemma 2.1.1,

$\omega = \Theta\left((a] \vee (a]^{\perp}\right)^{\perp} = \left(\Theta(a] \vee \Theta(a]^{\perp}\right)^{\perp} = \Theta((a])^{\perp} \cap \Theta\left((a]^{\perp}\right)^{\perp}$. Thus, $\Theta\left((a]^{\perp}\right)^{\perp} \subseteq \Theta((a])^{\perp\perp} = \Theta_a$. Taking the kernel on both sides we have $(a]^{\perp\perp} \subseteq (a]$ by using theorem 2.2.3(ii), due to Bazlar Rahman [3]. It follows that $(a] = (a]^{\perp\perp}$ and hence $S$ is disjunctive. •

Next theorem is an extension of 2.2 of Cornish [6]. We omit the proof as this can be proved exactly in a similar way the corresponding result of [6] was proved.

**Theorem 2. 1. 7:** *For a distribuiive nearlattice $S$ with $0$, the following conditions are equivalent:*

*(i)* *$S$ is disjunctive.*

*(ii)* *For each congruence $\Phi$, $\Phi^{\perp} = \Theta(\ker \Phi)^{\perp}$.*

*(iii)* *For each ideal $J$, $R(J)^{\perp} = \Theta(J)^{\perp}$.*

*(iv)* *For each congruence $\Phi$, $\ker(\Phi^{\perp}) = (\ker \Phi)^{\perp}$.*

*(v)* *For each congruence $\Phi$, $\ker(\Phi^{\perp\perp}) = (\ker \Phi)^{\perp\perp}$.*

*(vi)* *The kernel of each skeletal congruence is an annihilator ideal.* •

Due to Bazlar Rahman [3], a nearlattice $S$ with $0$ is called semiboolean if it is distributive and $[0,x]$ is complemented for all $x \in S$. By 1.4.5 of Bazlar Rahman [3], we know that the lattice of all ideals of a nearlattice is isomorphic to the lattice of congruences if and only if $S$ is semiboolean. Using this result we get the following theorem, which is an extension of 2.3 of [6].

**Theorem 2. 1. 8:** *For a distributive nearlattice $S$ with $0$, the following conditions are equivalent:*

*(i)* *$S$ is semiboolean.*

*(ii)* *For each congruence $\Phi$, $\Phi^{\perp} = \Theta(\ker \Phi^{\perp})$.*

*(iii)* *For each ideal $J$, $\Theta(J^{\perp}) = \Theta(J)^{\perp}$.*

(iv) *For each ideal $J$, $\Theta(J^{\perp\perp}) = \Theta(J)^{\perp\perp}$.*

**Proof :** (i) implies (ii): Suppose $S$ is semiboolean. Then by 1.4.5 of Bazlar Rahman [3], *I(S)* is isomorphic to *C(S)*. Hence for any congruence $\Psi$, $\Psi = \Theta(\ker\Psi)$. Taking $\Psi = \Phi^{\perp}$, we see that (i) implies (ii).

(ii) implies (iii) follows from theorem 2.2.3 of Bazlar Rahman [3], (ii) and (iii)$\Rightarrow$(iv) is obvious.

(iv) implies (i): Suppose (iv) holds. Put $J = (a]\vee(a]^{\perp}$. Then $J^{\perp} = (0]$ and so $J^{\perp\perp} = S$. Then by (iv), $\Theta\left(\left((a]\vee(a]^{\perp}\right)^{\perp\perp}\right) = \tau$. It follows that $\Theta((a])^{\perp}\cap\Theta\left((a]^{\perp}\right)^{\perp} = \omega$ and so $\Theta\left((a]^{\perp}\right)^{\perp} \subseteq \Theta((a])^{\perp\perp} = \Theta_a^{\ \perp\perp} = \Theta_a$. Since ker $\Psi_a = (a]^{\perp}$, we have $\Theta\left((a]^{\perp}\right) \subseteq \Psi_a = \Theta_a^{\ \perp\perp}$ and so $\Theta_a = \Theta_a^{\ \perp\perp} \subseteq \Theta\left((a]^{\perp}\right)^{\perp}$. Thus $\Theta\left((a]^{\perp}\right)^{\perp} = \Theta_a$. But $(a]^{\perp} = (a]^{\perp\perp\perp}$. Now, by (iv), $\Theta\left((a]^{\perp}\right)^{\perp\perp} = \Theta\left((a]^{\perp\perp\perp}\right) = \Theta\left((a]^{\perp}\right)$. But $\Theta_a^{\ \perp} = \Theta\left((a]^{\perp}\right)^{\perp\perp}$, and so $\Theta\left((a]^{\perp}\right) = \Theta_a^{\ \perp} = \Psi_a$. Now if $0 \le a \le b$, then $a \equiv b \ (\Psi_a)$ and so $a \equiv b \left(\Theta\left((a]^{\perp}\right)\right)$. Then $(a]\vee(a]^{\perp} = (b]\vee(a]^{\perp}$ and so $b = a\vee j$ for some $j \in (a]^{\perp}$. Then $j\wedge a = 0$, and so $[0,b]$ is complemented. Hence $S$ is semiboolean. •

The skeleton $Sc(S)=\{\Theta\in c(S) \mid \Theta = \Phi^{\perp} \textit{ for some } \Phi\in c(S)\} = \{\Theta\in c(S) \mid \Theta = \Theta^{\perp\perp}\}$ s a complete Boolean lattice. The meet of a set $\{\Theta_i\}\subseteq Sc(S)$ is $\cap\Theta_i$ | as in c(S), while the join is given by $\underline{\vee}\Theta_i = (\vee\Theta_i)^{\perp\perp} = \left(\cap\Theta_i^{\ \perp}\right)^{\perp}$ and the complement of $\Theta\in Sc(S)$ is $\Theta^{\perp}$. The fact that Sc(S) is complete follows from the fact that Sc(S) is precisely the set of closed elements associated with the closure operation $\Theta\to\Theta^{\perp\perp}$ on the complete lattice C(S) and Sc(S) is Boolean because of Glivenko's theorem, c.f. Gratzer [12],theorem 4, p.58.

The set $KSc(S) = \{Ker\Theta \mid \Theta\in Sc(S)\}$ is closed under arbitrary set-theoretic intersections and hence is a complete lattice. We will use the symbol $\underline{\vee}$ to denote the join in $Sc(S)$ and in $KSc(S)$. We also denote $A(S) = \{J \mid J\in I(S); J = J^{\perp\perp}\}$, which is a complete Boolean lattice.

The following theorems are extensions of 2.4 and 2.5 of Cornish [6] to nearlattices.

**Theorem 2. 1. 9:** *For a distributive nearlattice $S$ with $0$, the following conditions are equivalent:*

*(i)* *$S$ is disjunctive.*

*(ii)* *The map $\Theta \to Ker\Theta$ of $Sc(S)$ onto $KSc(S)$ is one-to-one.*

*(iii)* *The map $\Theta \to Ker\Theta$ of $Sc(S)$ onto $KSc(S)$.*

*(iv)* *The map $\Theta \to Ker\Theta$ is a lattice isomorphism of $Sc(S)$ onto $KSc(S)$, whose inverse is the map $J \to \Theta(J)^{\perp\perp}$.*

**Proof:** (i) implies (iv) . Suppose $S$ is disjunctive. Then by theorem 2. 1. 7(vi) $KSc(S) = A(S)$. By 2.1.7 (ii) , $\Phi = \Phi^{\perp\perp} = \Theta(Ker\Phi)^{\perp\perp}$ for any $\Phi \in Sc(S)$. Thus, the map $\Theta \to Ker\Theta$ is one-to-one. Clearly it preserves meet.

Now using 2.1.7(iv), for $\Theta$, $\Phi \in Sc(S)$, $Ker(\Theta \underline{\vee} \Phi)$ $= Ker\left(\left(\Theta^{\perp} \cap \Phi^{\perp}\right)^{\perp}\right) = \left(Ker\left(\Theta^{\perp} \cap \Phi^{\perp}\right)\right)^{\perp} = \left(Ker\Theta^{\perp} \cap Ker\Phi^{\perp}\right)^{\perp} = \left((Ker\Theta)^{\perp} \cap (Ker\Phi)^{\perp}\right)^{\perp} =$ $Ker\Theta \underline{\vee} Ker\Phi$ as KSc(S) = A(S). Thus $\Theta \to Ker\Theta$ is a lattice isomorphism. Moreover, by 2.1.7, $Ker\left(\Theta(J)^{\perp\perp}\right) = \left(Ker\Theta(J)\right)^{\perp\perp} = J^{\perp\perp} = J$ for all $J \in A(S) = KSc(S)$, while $\Theta(Ker\Phi)^{\perp\perp} = \Phi^{\perp\perp} = \Phi$ for all $\Phi \in Sc(S)$. Therefore $J \to \Theta(J)^{\perp\perp}$ is the inverse of $\Theta \to Ker\Theta$.

(iv) implies (ii) is trivial.

(ii) implies (iii): If $\Theta \to Ker\Theta$ is one-to-one, then it is a meet isomorphism of the lattice $Sc(S)$ onto the lattice $KSc(S)$, then of course it is a lattice isomorphism and so (iii) holds.

Finally we shall show that (iii) implies (i). If (iii) holds, then of course $\Theta \to Ker\Theta$ is a lattice homomorphism of $Sc(S)$ onto $KSc(S)$. Hence $KSc(S)$ must be Boolean. Since for all $a \in S$, $(a] = Ker(\Theta_a)$, the map $a \to (a]$ embeds $S$, as a join-dense subnearlattice, into the complete Boolean lattice $KSc(S)$. Therefore S must be disjunctive. . •

We conclude this chapter with the following theorem which is also a generalization of Cornish [6], Theorem 2.5.

**Theorem 2. 1. 10:** *A distributive nearlattice $S$ is semiboolean if and only if the map $\Theta \to Ker\Theta$ is a lattice isomorphism of $Sc(S)$ onto $KSc(S)$ , whose inverse is the map $J \to \Theta(J)$.*

**Proof :** If $S$ is semiboolean, then of course it is disjunctive and so by Theorem 2.1.9, the inverse of $\Theta \to Ker\Theta$ is $J \to \Theta(J)^{\perp\perp}$. Now by Theorem 2.1.8, $\Theta(J)^{\perp\perp} = \Theta(J^{\perp\perp})$ for any $J \in KSc(S)$. Since by theorem 2.1.7, $J \in A(S)$ so $J = J^{\perp\perp}$. Thus $J \to \Theta(J)$ is the inverse.

Conversely, suppose $J \to \Theta(J)$ is the inverse of $\Theta \to Ker\Theta$. Then by Theorem 2.1.9, $S$ is disjunctive and so $Ker(\Theta(K)^{\perp\perp}) = (Ker\Theta(K))^{\perp\perp} = K^{\perp\perp}$ for any ideal K. This implies $K^{\perp\perp} \in KSc(S)$. Then using the description of the inverse, $\Theta(K^{\perp\perp}) = \Theta(Ker(\Theta(K)^{\perp\perp})) = \Theta(K)^{\perp\perp}$. Hence by Theorem 2.1.8, $S$ is semiboolean.

# 0-DISTRIBUTIVE NEARLATTICE AND SEMI-PRIME IDEALS IN A NEARLATTICE

## 3. 1 Introduction

J.C. Varlet [33] has given the definition of a 0-distributive lattice to generalize the notion of pseudocomplemented lattice. According to him a lattice $L$ with 0 is called a *0-distributive lattice* if for all $a,b,c \in L$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$. In other words, a lattice with 0 is 0-distributive if and only if for each $a \in L$, the set of elements disjoint to a is an ideal of $L$. Of course, every distributive lattice with 0 is 0-distributive. Also, every pseudocomplemented lattice is 0-distributive. In fact, in a pseudocomplemented lattice L, the set of all elements disjoint to $a \in L$, is a principal ideal $(a^*]$. Many authors including Balasubramani and Venkatanarasimhan [1], Jayaram [16] and Pawar and Thakare [25] studied the 0-distributive and 0-modular properties in lattices and meet semilattices. In fact, Jayaram [16] has referred the condition of 0-distributive nearlattice given in this chapter as weakly 0-distributive semilattice in a general meet semilattice.

Recently, Rav [26] has generalized the concept of 0-distributivity and gave the definition of semi-prime ideals in a lattice. An ideal I of a lattice $L$ is called a *semi-prime ideal* if for all $x,y,z \in L$, $x \wedge y \in I$ and $x \wedge z \in I$ imply $x \wedge (y \vee z) \in I$. Thus, for lattice $L$ with 0, $L$ is called *0-distributive* if and only if $(0]$ is a semi-prime ideal. In a distributive lattice $L$, every ideal is a semi-prime ideal. Moreover, every prime ideal is semi prime. In a pentagonal lattice (Figure 3.1) $(0]$ is semi-prime but not prime. Here (b] and (c] are prime, but $(a]$ is not even semi-prime. Again in Figure 3.2, $(0]$, $(a]$, $(b]$, $(c]$ are not semi-prime.

In this chapter we will provide a number of characterization of 0-distributive nearlattices. We also extend the concept of 0-distributivty and give the notion of semi-prime ideals in nearlattice. Then we include a number of separation properties in a general nearlattice with respect to the annihilator ideals. Moreover, by studying a congruence related to Glivenko congruence we give a separation theorem related to separation properties in distributive nearlattices given by Noor and Bazlar Rahman [21].

Let us define a 0-distributive nearlattice as follows: A nearlattice S with 0 is called 0-distributive if for all $x, y, z \in S$ with $x \wedge y = 0 = x \wedge z$ and $y \vee z$ exists imply $x \wedge (y \vee z) = 0$.

It can easily be shown that it has the following alternative definition:

$S$ is 0-distributive if for all $x, y, z, t \in S$ with $x \wedge y = 0 = x \wedge z$ imply $x \wedge ((t \wedge y) \vee (t \wedge z)) = 0$; $(t \wedge y) \vee (t \wedge z)$ exists by the upper bound property of $S$. Of course, every distributive nearlattice $S$ with 0 is 0-distributive. Figure 3.1 is an example of a non-modular nearlattice which is 0-distributive, while Figure 3.2 gives a modular nearlattice which is not 0-distributive.

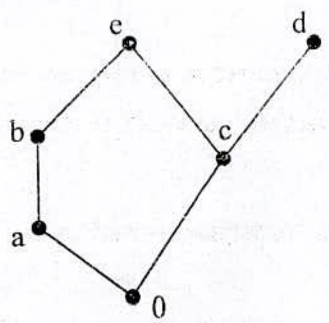


Figure 3.1

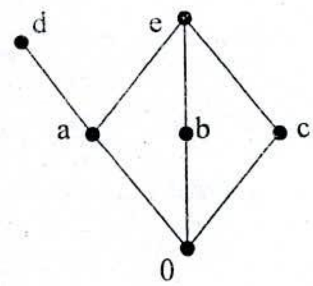


Figure 3.2

A proper filter $M$ of a nearlattice $S$ is called *maximal* if for any filter $Q$ with $Q \supseteq M$ implies either $Q = M$ or $Q = S$. Dually, we define a *minimal prime ideal* (*down set*)

Let $L$ be a lattice with 0. An element $a^*$ is called the *pseudocomplement* of $a$ if $a \wedge a^* = 0$ and if $a \wedge x = 0$ for some $x \in L$, then $x \leq a^*$. A lattice $L$ with 0 and 1 is called *pseudocomplemented* if its every element has a pseudocomplement. Since a nearlattice S with 1 is a lattice, so the concept of pseudocomplementation is not possible in a general nearlattice. A nearlattice $S$ with 0 is called sectionally pseudocomplemented if the interval $[0, x]$ for each

$x \in S$ is pseudocomplemented. For $A \subseteq S$, we denote $A^{\perp} = \{x \in S \mid x \wedge a = 0 \text{ for all } a \in A\}$. If $S$ is distributive then clearly $A^{\perp}$ is an ideal of $S$.

Moreover, $A^{\perp} = \bigcap_{a \in A} \{\{a\}^{\perp}\}$. If A is an ideal, then obviously $A^{\perp}$ is the pseudocomplement of A in $I(S)$ and we denote it by $A^{*}$. Therefore, for a distributive nearlattice $S$ with 0, *I(S)* is pseudocomplemented.

## 3. 2 0-Distributive Nearlattice

**Theorem 3. 2. 1:** *If a nearlattice $S$ with 0 is sectionally pseudocomplemented, then $I(S)$ is pseudocomplemented.*

**Proof:** Suppose $S$ is sectionally pseudocomplemented. Let $I \in I(S)$. $I^{\perp} = \{x \in S \mid x \wedge i = 0$ for all $i \in I\}$. Suppose $x \in I^{\perp}$ and $t \leq x$. Then $x \wedge i = 0$ for all $i \in I$ and so $t \wedge i = 0$ for all $i \in I$. Hence $t \in I^{\perp}$. Now let $x, y \in I^{\perp}$ and $x \vee y$ exists. Let $r = x \vee y$. Then $0 \leq x, y, r \wedge i \leq r$ for all $i$, and $x \wedge (r \wedge i) = 0 = y \wedge (r \wedge i)$. Since $[0, r]$ is pseudocomplemented, $x, y \leq (r \wedge i)^{+}$ for all $i \in I$, where $(r \wedge i)^{+}$ is the relative pseudocomplement of $r \wedge i$ in $[0, r]$. Then $x \vee y \leq (r \wedge i)^{+}$, and so $r \wedge i \wedge (x \vee y) = 0$. That is $i \wedge (x \vee y) = 0$ for all $i \in I$. This implies $x \vee y \in I^{\perp}$. Therefore, $I^{\perp}$ is an ideal. Clearly $I^{\perp}$ is the pseudocomplement of I in $I(S)$. Hence $I(S)$ is pseudocomplemented.●

Following example (Figure 3.3) shows that $I(S)$ can be pseudocomplemented but $S$ is not sectionally pseudocomplemented.

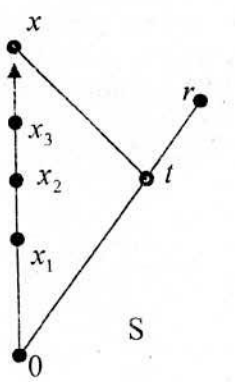


Figure 3.3

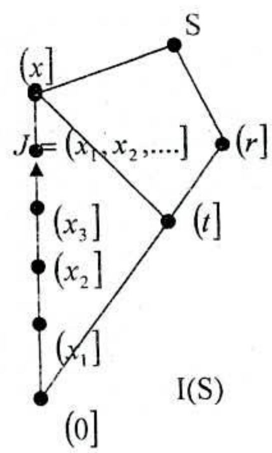


Figure 3.4

In $S$, observe that $t$ has no sectionally pseudocomplement in $[0, x]$. But $I(S)$ is pseudocomplement and the ideal $J$ is the pseudocomplement of both $(t]$ and $(r]$. Again, Figure 3.1 gives a non-distributive nearlattice $S$ where $I(S)$ is pseudocomplemented.

**Theorem 3. 2. 2:** *If the intersection of all prime ideals of a nearlattice S with 0 is* $\{0\}$, *then S is 0-distributive.*

**Proof:** Let $a,b,c \in S$ such that $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists. Let $P$ be any prime ideal of $S$. If $a \in P$, then $a \wedge (b \vee c) \leq a$ implies that $a \wedge (b \vee c) \in P$. If $a \notin P$, then by the primeness of $P$, $b,c \in P$, and so $b \vee c \in P$. This implies $a \wedge (b \vee c) \in P$. Thus $a \wedge (b \vee c)$ is in every prime ideal $P$ of $S$, and hence $a \wedge (b \vee c) = 0$, proving that $S$ is 0-distributive. •

From Bazlar Rahman [3] we know that a nearlattice $S$ is distributive if and only if $I(S)$ is distributive, which is also equivalent to that $D(S)$, the lattice of filters of $S$ is distributive. Thus if S is a nearlattice with 0 such that $I(S)$ (similarly $D(S)$) is distributive, then $S$ is 0-distributive.

Following lemma are needed for further development of the thesis.

**Lemma 3. 2. 3:** *Every proper filter of a nearlattice with 0 is contained in a maximal filter.*

**Proof:** Let $F$ be a proper filter in $S$ with 0.Let $\mathcal{F}$ be the set of all proper filters containing $F$. Then $\mathcal{F}$ is non-empty as $F \in \mathcal{F}$. Let $C$ be a chain in $\mathcal{F}$ and let $M = \cup\{X \mid X \in C\}$. We claim that $M$ is a filter with $F \subseteq M$. Let $x \in M$ and $y \geq x$. Then $x \in X$ for some $X \in C$. Hence $y \in X$ as $X$ is a filter. Therefore, $y \in M$. Let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in C$. Since $C$ is a chain, either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$. So $x, y \in Y$. Then $x \wedge y \in Y$ as $Y$ is a filter. Hence $x \wedge y \in M$. Moreover $M$ contains $F$. So $M$ is a maximum element of $C$. Then by Zorn's lemma $\mathcal{F}$ has a maximal element, say $Q$ with $F \subseteq Q$.•

**Lemma 3. 2. 4:** *Let S be a nearlattice with 0. A proper filter M in S is maximal if and only if for any element* $a \notin M$, *there exists an element* $b \in M$ *with* $a \wedge b = 0$.

**Proof:** Suppose $M$ is maximal and $a \notin M$. Let $a \wedge b \neq 0$ for all $b \in M$. Consider $M_1 = \{y \in S \mid y \geq a \wedge b, \textit{for some } b \in M\}$. Clearly $M_1$ is a filter and is proper as $0 \notin M$. For every $b \in M$ we have $b \geq a \wedge b$ and so $b \in M_1$. Thus $M \subseteq M_1$. Also $a \notin M$ but $a \in M_1$.

So $M \subset M_1$, which contradicts the maximality of $M$. Hence there must exist some $b \in M$ such that $a \wedge b = 0$.

Conversely, if the proper filter $M$ is not maximal, then as $0 \in S$, there exists a maximal filter $N$ such that $M \subset N$. For any element $a \in N - M$ there exists an element $b \in M$ such that $a \wedge b = 0$. Hence $a, b \in N$ imply $0 = a \wedge b \in N$, which is a contradiction. Thus $M$ must be a maximal filter. •

Following result gives several nice characterizations of 0-distributive nearlattice.

**Theorem 3. 2. 5:** *For a nearlattice S with 0, the following conditions are equivalent:*

*(i)* *S is 0-distributive.*

*(ii)* $\{a\}^{\perp}$ *is an ideal for all* $a \in S$.

*(iii)* $A^{\perp}$ *is an ideal for all* $A \subseteq S$.

*(iv)* $I(S)$ *is pseudocomplemented.*

*(v)* $I(S)$ *is 0-distributive.*

*(vi)* *Every maximal filter is prime.*

**Proof:** $(i)$ implies $(ii)$ implies $(iii)$ are trivial.

$(iii)$ implies $(iv)$: For any ideal $I$ of S, $I^{\perp}$ is clearly the pseudocomplement of $I$ in $I(S)$ if $I^{\perp} \in I(S)$, and so $(iv)$ holds.

$(iv)$ implies $(v)$: Since every pseudocomplemented lattice is 0-distributive, so $(iv) \Rightarrow (v)$.

$(v)$ implies $(vi)$: Let $I(S)$ be 0-distributive and $F$ be a maximal filter. Suppose $f, g \notin F$ with $f \vee g$ exists. By Lemma 3.2.4, there exist $a, b \in F$ such that $a \wedge f = 0 = b \wedge g$. Hence $(f] \wedge (a \wedge b] = (0]$ and $(g] \wedge (a \wedge b] = (0]$. Then $(f \vee g] \wedge (a \wedge b] = ((f] \vee (g]) \wedge (a \wedge b] = (0]$, by 0-distributivity of $I(S)$. Hence $(f \vee g) \wedge (a \wedge b) = 0$. Since $F$ is maximal, $0 \notin F$. Therefore $f \vee g \notin F$, and so $F$ is prime.

$(vi)$ implies $(i)$: Let $(vi)$ holds. Suppose $a,b,c \in S$ such that $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists. If $a \wedge (b \vee c) \neq 0$, then by Lemma 3.2.3, $a \wedge (b \vee c) \in F$ for some maximal filter $F$ of $S$. Then $a \in F$ and $b \vee c \in F$. As $F$ is prime, by assumption, so either $a \in F$ and $b \in F$ or $c \in F$. That is, either $a \wedge b \in F$ or $a \wedge c \in F$. This implies $0 \in F$, which gives a contradiction and hence $a \wedge (b \vee c) = 0$. In other words, $S$ is 0-distributive. •

**Corollary 3. 2. 6:** *In a 0-distributive nearlattice, every proper filter is contained in a prime filter.* •

**Theorem 3. 2. 7:** *Every prime down set of a nearlattice contains a minimal prime down set.*

**Proof:** Let $P$ be a prime down set of $L$ and let $\chi$ be the set of all prime down sets $J$ such that $J \subseteq P$. Then $P$ is non-empty since $P \in \chi$. Let $C$ be a chain in $\chi$ and let $M = \cap\{X : X \in C\}$.

We claim that $M$ is a prime down set. $M$ is non-empty as $0 \in M$. Let $a \in M$ and $b \leq a$. Then $a \in X$ for all $X \in C$. Hence $b \in X$ for all $X \in C$ as $X$ is a down set. Then $b \in M$. Now let $x \wedge y \in M$ for some $x, y \in S$. Then $x \wedge y \in X$ for all $X \in C$. As $X$ is a prime down set, so either $x \in X$ or $y \in X$. Thus either $M = \cap\{X : x \in X\}$ or $M = \cap\{X : y \in X\}$, proving that either $x \in M$ or $y \in M$. Thus $M$ is a prime down set. Thus by applying the dual form of Zorn's Lemma, we conclude the existence of a minimal member of $P$. •

**Theorem 3. 2. 8:** *In a 0-distributive nearlattice S, if $\{0\} \neq A$ is the intersection of all non-zero ideals of S, then $A^{\perp} = \{x \in S \mid \{x\}^{\perp} \neq \{0\}\}$.*

**Proof:** Let $x \in A^{\perp}$. Then $x \wedge a = 0$ for all $a \in A$. Since $A \neq \{0\}$, so $\{x\}^{\perp} \neq \{0\}$. Thus $x \in \{x \in S \mid \{x\}^{\perp} \neq \{0\}\}$. That is $A^{\perp} \subseteq \{x \in S \mid \{x\}^{\perp} \neq \{0\}\}$.

Conversely, let $x \in \{x \in S \mid \{x\}^{\perp} \neq \{0\}\}$. Since S is 0-distributive, so $\{x\}^{\perp}$ is a non-zero ideal of $S$. Then $A \subseteq \{x\}^{\perp}$ and so $A^{\perp} \supseteq \{x\}^{\perp\perp}$. This implies $x \in A^{\perp}$,which completes the proof. •

**Theorem 3. 2. 9:** *Let S be a nearlattice with 0. S is 0-distributive if and only if for any filter F disjoint with* $\{x\}^{\perp}$; $x \in S$, *there exist a prime filter containing F and disjoint with* $\{x\}^{\perp}$ .

**Proof.** Let S be 0-distributive. Consider the set $\mathcal{F}$ of all filters of $S$ containing $F$ and disjoint with $\{x\}^{\perp}$ . Clearly $\mathcal{F}$ is non-empty as $F \in \mathcal{F}$. Then using Zorn's lemma, there exists a maximal element $Q$ in $\mathcal{F}$ . Now we claim that $x \in Q$ . If not, then $Q \vee [x) \supset Q$. So by the maximality of $Q$, $\{Q \vee [x)\} \cap \{x\}^{\perp} \neq \phi$. Then there exists $t \in Q \vee [x)$ and $t \in \{x\}^{\perp}$. Then $t \geq q \wedge x$ for some $q \in Q$ and $t \wedge x = 0$. Thus, $0 = t \wedge x \geq q \wedge x$, and so $q \wedge x = 0$. This implies $q \in \{x\}^{\perp}$, which contradicts the fact that $Q \cap \{x\}^{\perp} = \phi$. Therefore $x \in Q$. Finally, let $z \notin Q$. Then $\{Q \vee [z)\} \cap \{x\}^{\perp} \neq \phi$. Let $y \in \{Q \vee [z)\} \cap \{x\}^{\perp}$. Then $y \wedge x = 0$ and $y \geq q \wedge z$ for some $q \in Q$. Thus $0 = y \wedge x \geq q \wedge x \wedge z$, which implies $q \wedge x \wedge z = 0$. Now $x \in Q$ implies $q \wedge x \in Q$, and $z \wedge (q \wedge x) = 0$. Hence by Lemma 3.2.4, $Q$ is a maximal filter of $S$, and so by Theorem 3. 2. 5, $Q$ is prime.

Conversely, let $x \wedge y = 0 = x \wedge z$ and $y \vee z$ exists. If $x \wedge (y \vee z) \neq 0$. Then $y \vee z \notin \{x\}^{\perp}$. Thus $[y \vee z) \cap \{x\}^{\perp} = \phi$. So, there exists a prime filter $Q$ containing $[y \vee z)$ and disjoint with $\{x\}^{\perp}$. As $y, z \in \{x\}^{\perp}$, so $y, z \notin Q$. Thus $y \vee z \notin Q$, as $Q$ is prime. This implies $[y \vee z) \not\subset Q$, a contradiction. Hence $x \wedge (y \vee z) = 0$ and so $S$ is 0-distributive. •

Pawar and Thakare [25] have mentioned as a corollary to the above result that for distinct elements $a, b \in S$ for which $a \wedge b \neq 0$ are separated by a prime filter in a 0-distributive semilattice, which is not true. For example, Figure 3.1 is an example of a 0-distributive nearlattice, where $a, b$ are distinct and $a \wedge b \neq 0$. But there does not exist any prime filter containing $b$ but not containing $a$.

Now we give few more characterizations for 0-distributive nearlattices.

**Theorem 3. 2. 10:** *Let $S$ be a nearlattice with $0$ . Then the following conditions are equivalent:*

*(i)* *$S$ is 0-distributive.*

*(ii)* *Every maximal filter of $S$ is prime.*

*(iii)* *Every minimal prime down set of $S$ is a minimal prime ideal.*

*(iv)* *Every proper filter of $S$ is disjoint from a minimal prime ideal.*

*(v)* *For each non-zero element $a \in S$, there is a minimal prime ideal not containing $a$ .*

*(vi)* *Each non-zero element $a \in S$ is contained in a prime filter.*

**Proof** $(ii)$ $(ii)$ implies $(i)$: follows from Theorem 3. 2. 5.

$(ii)$ implies $(iii)$: Let $A$ be a minimal prime down set. Then $S$-$A$ is a maximal filter. Then by (ii), $S$-$A$ is a prime filter, and so $A$ is an ideal. That is, $A$ is a minimal prime ideal.

$(iii)$ implies $(ii)$: Let $F$ be a maximal filter of $S$ . Then $S$-$F$ is a minimal prime down set . Thus by $(iii)$ $S$-$F$ is a minimal prime ideal and so $F$ is a prime filter.

$(i)$ implies $(iv)$: Let $F$ be a proper filter of $S$. Then by Corollary 3.2.6, there is a prime filter $Q \supseteq F$ . Then $S$-$Q$ is a minimal prime ideal disjoint from $F$.

$(v)$: Let $a \in S$ and $a \neq 0$ . Then $[a)$is a proper filter. Then by (iv) there exists a minimal prime ideal $A$ such that $A \cap [a) = \phi$ . Thus $a \notin A$ .

$(v)$ implies $(iv)$: Let $a \in S$ and $a \neq 0$ . Then by (v) there is a minimal prime ideal $P$ such that $a \notin P$ . Thus $a \in L - P$ and $L$-$P$ is a prime filter.

$(iv)$ implies $(i)$: Let $S$ be not 0-distributive. Then there exist $a,b,c \in S$ such that $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists but $a \wedge (b \vee c) \neq 0$. Then by (vi) there exists a prime filter $Q$ such that $a \wedge (b \vee c) \in Q$. Let $F = [a \wedge (b \vee c))$. This is proper as $0 \notin F$ and $F \subseteq Q$. Now, $a \wedge (b \vee c) \in Q$ implies $a \in Q$ and $b \vee c \in Q$. Since $a \wedge b = 0 = a \wedge c$, so $b,c \notin Q$ as $0 \notin Q$, but $b \vee c \in Q$, which contradicts that $Q$ is prime. Hence $a \wedge (b \vee c) = 0$ and so $S$ is 0-distributive. •

**Theorem 3. 2. 11:** *Let $S$ be a 0-distributive nearlattice and $x \in S$. Then a prime ideal $P$ containing $\{x\}^{\perp}$ is a minimal prime ideal containing $\{x\}^{\perp}$ if and only if for $p \in P$ there is $q \in S - P$ such that $p \wedge q \in \{x\}^{\perp}$.*

**Proof:** Let $P$ be a prime ideal of $S$ containing $\{x\}^{\perp}$ such that the given condition holds. Let $K$ be a prime ideal containing $\{x\}^{\perp}$ such that $K \subseteq P$. Let $p \in P$. Then there is $q \in S - P$ such that $p \wedge q \in \{x\}^{\perp}$. Hence $p \wedge q \in K$. Since $K$ is prime and $q \notin K$, so $p \in K$. Thus , $P \subseteq K$ and so $K = P$. Therefore, $P$ must be a minimal prime ideal containing $\{x\}^{\perp}$.

Conversely, let $P$ be a minimal prime ideal containing $\{x\}^{\perp}$. Let $p \in P$. Suppose for all $q \in S - P$, $p \wedge q \notin \{x\}^{\perp}$. Set $D = (S - P) \vee [p)$. We claim that $\{x\}^{\perp} \cap D = \varphi$. If not , let $y \in \{x\}^{\perp} \cap D$. Then $y \geq r \wedge p$ for some $r \in S - P$. Thus , $p \wedge r \leq y \in \{x\}^{\perp}$, which is a contradiction to the assumption. Then by Theorem 3.2.9, there exists a maximal (prime) filter $Q \supseteq D$ and disjoint with $\{x\}^{\perp}$. By the proof of Theorem 3.1.9, $x \in Q$. Let M = S-Q. Then $M$ is prime ideal. Since $x \in Q$, so $x \notin M$. Let $t \in \{x\}^{\perp}$. Then $t \wedge x = 0 \in M$ implies $t \in M$ as $M$ is prime. Thus $\{x\}^{\perp} \subseteq M$.

Now $M \cap D = \phi$. Therefore, $M \cap (S - P) = \phi$, and hence $M \subseteq P$. Also $M \neq P$, because $p \in D$ implies $p \notin M$ but $p \in P$. Hence $M$ is a prime ideal containing $\{x\}^{\perp}$ which is properly contained in $P$. This gives a contradiction to the minimal property of $P$. Therefore, the given condition holds. ∘

Now we refer the reader about a conjecture made by Noor and Bazlar Rahman [21] that whether the well known Stone's separation property holds in a 0-distributive nearlattice. Separation theorem for distributive nearlattices is given in [21]. Unfortunately this does not hold even in case of a 0-distributive lattice. Consider the pentagonal lattice $\{0, a, b, c, 1;\ 0 < a < b < 1,\ 0 < c < 1\}$, which is 0-distributive. Consider $I = (a]$ and $F = [b)$. Here $I \cap F = \phi$ and there does not exist any prime filter $Q$ containing $F$ and disjoint with $I$.

But in a 0-distributive nearlattice, instead of a general ideal, we can give a separation theorem for an annihilator ideal $I = J^{\perp}$ when J is a subset of S. An ideal $I$ in a nearlattice $S$ with 0 is called an annihilator ideal if $I = J^{\perp}$ for some $J \subseteq S$.

Recently, Zaidur Rahman, Bazlar Rahman and Noor [34] have studied the semi-prime ideals in a nearlattice. This concept was given by Rav [26] in a general lattice. An ideal $I$ of a nearlattice S is called a *semi-prime ideal* if for all $x, y, z \in S$, $x \wedge y \in I$ and $x \wedge z \in I$ imply $x \wedge (y \vee z) \in I$ provided $y \vee z$ exists. Thus, for nearlattice $S$ with 0, S is called *0-distributive* if and only if (0] is a semi-prime ideal in $S$. In a distributive nearlattice $S$, every ideal is a semi-prime ideal. Moreover, every prime ideal is semi-prime. From [34], it is known that for any subset $A$ of a nearlattice $S$, $A^{\perp}$ is a semi-prime ideal if $S$ is 0-distributive. Here we give a separation theorem by using the semi-prime ideals.

**Theorem 3. 2. 12:** *(The Separation Theorem) A nearlattice S is 0-distributive if and only if for a proper filter F and an annihilator $I = J^{\perp}$, where J is a non empty subset of S, with $F \cap I = \phi$, there exists a prime filter Q containing F such that $Q \cap I = \phi$.*

**Proof:** Suppose $S$ is 0-distributive and $I = J^{\perp}$ for some non-empty subset $J$ of $S$. Let $\mathcal{F}$ be the set of all filters containing $F$, and disjoint with $I$. Then using Zorn's lemma, there exists a maximal filter $Q$ containing $F$ and disjoint with $I$. Since by Theorem 5 of [34] $I$ is semi-prime, so by Theorem 10 of [34], $Q$ is prime.

Conversely, suppose the condition holds. Suppose $S$ is not 0-distributive. Then there exist $a, b, c \in S$ such that $a \wedge b = 0$, $a \wedge c = 0$ and $a \wedge (b \vee c) \neq 0$, $b \vee c$ exists. Then

$b \vee c \notin \{a\}^{\perp}$. Let $F = [b \vee c)$. Since $0 \notin F$, $F$ is proper. Then proceeding according to the proof of converse part of Theorem 3. 2. 9, we find that $a \wedge (b \vee c) = 0$, and so $S$ is 0-distributive.•

A nearlattice $S$ with 0 is called weakly complemented if for any pair of distinct elements *a, b* of *S*, there exists an element c disjoint from one of these elements but not from the other.

**Theorem 3. 2. 13:** *S is weakly complemented if and only if R is an equality relation and hence is a nearlattice congruence .*

**Proof:** Suppose $S$ is weakly complemented. Let $a \equiv b(R)$. Suppose $a \neq b$. Then there exists $c$ such that $a \wedge c = 0$ but $b \wedge c \neq 0$. This implies $a \not\equiv b(R)$, which is a contradiction . Hence $a = b$. So, $R$ is an equality relation. That is, $R$ is a nearlattice congruence.

Suppose $R$ is equality. We need to prove $S$ is weakly complemented . Let $a, b \in S$ and $a \neq b$. Then $a \not\equiv b(R)$. This implies there exists $c \in S$, such that $a \wedge c = 0$ but $b \wedge c \neq 0$. Hence $S$ is weakly complemented. •

In the following nearlattice *S, R* is a nearlattice congruence. Here the classes are *{0}, {a}, {b}, {1}, {c, d, e}*. But *S* is neither 0-distributive nor weakly complemented.

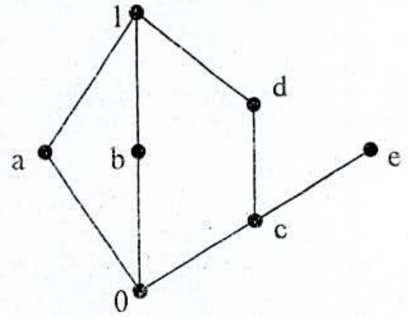


Figure 3.5

**Theorem 3. 2. 14:** *For any nearlattice S, the quotient lattice* $\frac{S}{R}$ *is weakly complemented . Furthermore, a nearlattice S with 0 is 0-distributive if and only if* $\frac{S}{R}$ *is a distributive nearlattice and R is a nearlattice congruence.*

**Proof:** Let $A$ and $B$ be two classes in $\frac{S}{R}$ such that $A< B$. Then there exists $a \in A$ and $b \in B$ such that a< b in S . So, by the definition of $R$ there is an element $c \in S$, such that $a \wedge c = 0$ but $b \wedge c \neq 0$. Suppose $x \in [0]$. Then $x \equiv 0(R)$ and so $0 \wedge x = 0$ which implies $x \wedge x = x = 0$. So $[0] = \{0\}$. This implies $A \wedge C = [a] \wedge [c] = \{0\}$ but $B \wedge C \neq \{0\}$. Hence $\frac{S}{R}$ is weakly complemented.

Now let $S$ be a nearlattice for which $R$ is a nearlattice congruence and $\frac{S}{R}$ is distributive. Let $a,b,c \in S$ with $a \wedge b = 0 = a \wedge c$ such that $b \vee c$ exists. Then $[a] \wedge ([b] \vee [c]) = ([a] \wedge [b]) \vee ([a] \wedge [c]) = [0] \vee [0] = [0]$. This implies $[a \wedge (b \vee c)] = [0]$. Since $[0] = \{0\}$, so $a \wedge (b \vee c) = 0$. Hence $S$ is 0-distributive .

Conversely, let $S$ be 0-distributive. Then by Theorem 3.2.13, $R$ is a nearlattice congruence. Let $[a],[b],[c] \in \frac{S}{R}$. We need to prove $[a] \wedge ([b] \vee [c]) = ([a] \wedge [b]) \vee ([a] \wedge [c])$ provided $[b] \vee [c]$ exists. Suppose $[b] \vee [c] = [d]$. Then $[b] = [b] \wedge [d] = [b \wedge d]$, $c = [c] \wedge [d] = [c \wedge d]$, and so $[b] \vee [c] = [(b \wedge d) \vee (c \wedge d)]$. So we need to prove that $[a \wedge ((b \wedge d) \vee (c \wedge d))] = [(a \wedge b \wedge d) \vee (a \wedge c \wedge d)]$. Let $a \wedge ((b \wedge d) \vee (c \wedge d)) \wedge x = 0$. Since $(a \wedge b \wedge d) \vee (a \wedge c \wedge d) \leq a \wedge ((b \wedge d) \vee (c \wedge d))$ so, $((a \wedge b \wedge d) \vee (a \wedge c \wedge d)) \wedge x = 0$. On the other hand, if $((a \wedge b \wedge d) \vee (a \wedge c \wedge d)) \wedge x = 0$, then $a \wedge b \wedge d \wedge x = 0 = a \wedge c \wedge d \wedge x$ and by 0-distributivity of $S$, $a \wedge ((b \wedge d) \vee (c \wedge d)) \wedge x = 0$.

Thus $a \wedge ((b \wedge d) \vee (c \wedge d)) \equiv (a \wedge (b \wedge d)) \vee (a \wedge (c \wedge d))(R)$ and hence $[a] \wedge ([b] \vee [c]) = ([a] \wedge [b]) \vee ([a] \wedge [c])$. •

**Theorem 3. 2. 15:** *If a 0-distributive nearlattice $S$ is weakly complemented then $S$ is distributive*

**Proof:** If $S$ is weakly complemented. Then by Theorem 3.2.15 of Zaidur Rahman [34], $R$ is an equality relation and so by above theorem $S \cong \frac{S}{R}$ implies $S$ is distributive .●

A nearlattice $S$ with 0 is called Sectionally complemented if the intervals [0,x] are complemented for each $x \in S$. A nearlattice which is sectionally complemented and distributive is called a Semi Boolean nearlattice.

**Corollary 3. 2. 16:** *If a 0-distributive nearlattice S is sectionally complemented and weakly complemented , then S is semi Boolean.* ●

**Theorem 3. 2. 17:** *Suppose S is sectionally complemented and in every interval [0, x] , every element has a unique relative complement . Then S is semi Boolean if and only if it is 0-distributive.*

**Proof:** Let $S$ be 0-distributive and for every $x \in S$, the interval *[0,x]* is unicomplemented. Let $x, y \in S$ with $x \neq y$. If they are comparable, without loss of generality, suppose $x < y$. Then $0 \leq x < y$. Then there exists a unique $t \in [0, y]$ such that $t \wedge x = 0$ and $t \vee x = y$. Thus $t \wedge x = 0$ but $t \wedge y = t \neq 0$. If x, y are not comparable, then $0 \leq x \wedge y < x$ and $0 \leq x \wedge y < y$. Then there exist $s, t \in S$ such that $x \wedge y \wedge s = 0$, $(x \wedge y) \vee s = x$, $x \wedge y \wedge t = 0$ and $(x \wedge y) \vee t = y$. Now $s \wedge t \leq x \wedge y$ implies $s \wedge t \leq x \wedge y \wedge s = 0$, which implies $s \wedge t = 0$. Now $s \wedge t = 0$ and $s \wedge x \wedge y = 0$ implies $0 = s \wedge ((x \wedge y) \vee t) = s \wedge y$ as $S$ is 0-distributive, but $s \wedge x \neq 0$. Therefore, $S$ is weakly complemented and so by above corollary, $S$ is semi Boolean. Since the reverse implication always holds in a Semi-Boolean nearlattice, this completes the proof. ●

There is another characterization of 0-distributive nearlattices.

**Theorem 3. 2. 18:** *Let $S$ be a nearlattice with 0. Then $S$ is 0-distributive if and only if $[0,x]$ is a 0-distributive lattice for every $x \in S$.*

**Proof:** Let $S$ is a nearlattice with 0 then $S$ *is* 0-distributive. Then trivially $[0,x]$ is also 0-distributive.

Conversely, suppose $[0,x]$ for all $x \in S$. Let $a,b,c \in S$ with $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists. Let $a \wedge (b \vee c) = t$ Consider the interval $[0,b \vee c]$. Then $t \in [0,b \vee c]$. Also $b,c \in [0,b \vee c]$

Now $\quad t \wedge b = a \wedge (b \vee c) \wedge b = a \wedge b = 0$

$\quad t \wedge c = a \wedge (b \vee c) \wedge c = a \wedge c = 0$

Since $[0,b \vee c]$ is 0-distributive, so, $t \wedge (b \vee c) = 0$.

So, $0 = t \wedge (b \vee c) = a \wedge (b \vee c) \wedge (b \vee c) = a \wedge (b \vee c)$

Hence, $S$ is 0-distributive. •

A nearlattice $S$ with 0 is called Sectionally complemented if the intervals [0,x] are complemented for each $x \in S$. A nearlattice which is sectionally complemented and distributive is called a Semi Boolean nearlattice.

**Corollary 3. 2. 19:** *If a 0-distributive nearlattice S is sectionally complemented and weakly complemented, then S is semi Boolean.* •

**Theorem 3. 2. 20:** *Suppose S is sectionally complemented and in every interval* [0,x] , *every element has a unique relative complement . Then S is semi Boolean if and only if it is 0-distributive.*

**Proof:** Let $S$ be 0-distributive and for every $x \in S$, the interval [0,x] is unicomplemented. Let $x, y \in S$ with $x \neq y$. If they are comparable, without loss of generality, suppose $x < y$. Then $0 \leq x < y$. Then there exists a unique $t \in [0,y]$ such that $t \wedge x = 0$ and $t \vee x = y$. Thus $t \wedge x = 0$ but $t \wedge y = t \neq 0$. If x, y are not comparable, then $0 \leq x \wedge y < x$ and $0 \leq x \wedge y < y$.

Then there exist $s,t \in S$ such that $x \wedge y \wedge s = 0$, $(x \wedge y) \vee s = x$, $x \wedge y \wedge t = 0$ and $(x \wedge y) \vee t = y$. Now $s \wedge t \leq x \wedge y$ implies $s \wedge t \leq x \wedge y \wedge s = 0$, which implies $s \wedge t = 0$. Now $s \wedge t = 0$ and $s \wedge x \wedge y = 0$ implies $0 = s \wedge ((x \wedge y) \vee t) = s \wedge y$ as $S$ is 0-distributive, but $s \wedge x \neq 0$. Therefore, $S$ is weakly complemented and so by above corollary, $S$ is semi Boolean. Since the reverse implication always holds in a Semi-Boolean nearlattice, this completes the proof. •

There is another characterization of 0-distributive nearlattices.

**Theorem 3. 2. 21:** *Let $S$ be a nearlattice with 0. Then $S$ is 0-distributive if and only if $[0,x]$ is a 0-distributive lattice for every $x \in S$.*

**Proof:** Let $S$ is a nearlattice with 0 then $S$ *is* 0-distributive. Then trivially $[0,x]$ is also 0-distributive.

Conversely, suppose $[0,x]$ for all $x \in S$. Let $a,b,c \in S$ with $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists. Let $a \wedge (b \vee c) = t$ Consider the interval $[0,b \vee c]$. Then $t \in [0,b \vee c]$. Also $b,c \in [0,b \vee c]$

Now $\quad t \wedge b = a \wedge (b \vee c) \wedge b = a \wedge b = 0$

$\quad t \wedge c = a \wedge (b \vee c) \wedge c = a \wedge c = 0$

Since $[0,b \vee c]$ is 0-distributive, so, $t \wedge (b \vee c) = 0$.

So, $0 = t \wedge (b \vee c) = a \wedge (b \vee c) \wedge (b \vee c) = a \wedge (b \vee c)$

Hence, $S$ is 0-distributive. •

Now we give a generalization of theorem 1.4.1. of Zaidur Rahman [35].

**Theorem 3. 2. 22:** *Let $S$ be a 0-distributive nearlattice and $[0,x]$ be 1-distributive for every $x \in S$, then the following conditions are equivalent.*

(i) *S is sectionally complemented.*

(ii) $(x] \vee (x]^{\perp} = (x] \vee (x]^{*} = S$ *for every* $x \in S$

$(iii)$ *The prime ideals of* $[0,x]$ *are unordered for each* $x \in S$.

**Proof:** $(i)$ implies $(ii)$ : Suppose $S$ is sectionally complemented. Then for every $x \in S$, $[0,x]$ is complemented. If $(ii)$ does not holds, then there exist elements $s,t \in S$ such that $s \notin (t] \vee (t]^*$. Now $0 \leq s \wedge t \leq s$. Then by $(i)$, there exists $r \in [0,s]$ such that $r \wedge s \wedge t = r \wedge t = 0$ and $r \vee (s \wedge t) = s$. Thus $r \in (t]^*$ and so $s = r \vee (s \wedge t) \in (t]^* \vee (t]$ gives a contradiction. Therefore, $(ii)$ must holds.

$(ii)$ implies $(iii)$ : Suppose $(ii)$ holds but $(iii)$ does not. Then there exist prime ideal $P,Q$ of some $[0,x]$, $x \in S$ such that $P \subset Q$. Thus there exists $y \in Q - P$. Since $Q$ is a prime ideal of $[0,x]$, $x \notin Q$. By $(ii)$ $(y] \vee (y]^* = S$ Thus $x \in (y] \vee (y]^*$. Then $x \leq p \vee q$ for some $p \in (y]$ and $q \in (y]^*$. Then $q \wedge y = 0 \in P$. Since $y \notin P$ and $P$ is prime, so $q \in P \subset Q$. Also $p \leq y$ implies $p \in Q$. Therefore, $x \leq p \vee q$ implies $x \in Q$ gives a contradiction. Hence the prime ideals of $[0,x]$ for each $s \in S$ are unordered.

$(iii)$ implies $(i)$ : Since here every $[0,x]$ is both a 0-distributive and 1-distributive lattice, so by Razia Sultana [27], $[0,x]$ must be complemented.

### 3. 3 Semi-prime ideals in a Nearlattice

An ideal $I$ of a nearlattice $S$ is called a *semi-prime ideal* if for all $x, y, z \in S$, $x \wedge y \in I$ and $x \wedge z \in I$ imply $x \wedge (y \vee z) \in I$ provided $y \vee z$ exists. Thus, for a nearlattice $S$ with 0, $S$ is called *0-distributive* if and only if (0] is a semi-prime ideal. In a distributive nearlattice $S$, every ideal is a semi-prime ideal. Moreover, every prime ideal is semi-prime. Of course every nearlattice $S$ with 0 itself is semi-prime. In the nearlattice of Figure 3.1, (b] and (d] are prime, (c] is not prime but semi-prime and $(a]$ is not even semi-prime. Again in Figure 3.2, (0], $(a]$, $(b]$, $(c]$ and (d] are not semi-prime.

**Lemma 3. 3. 1:** *Non empty intersection of all prime (semi prime) ideals of a nearlattice is a semi-prime ideal.*

**Proof:** Let $a, b, c \in S$ and $I = \bigcap\{P : P$ is a prime ideal $\}$ and $I$ is nonempty. Let $a \wedge b \in I$ and $a \wedge c \in I$. Then $a \wedge b \in P$ and $a \wedge c \in P$ for all $P$. Since each $P$ is prime (semi-prime), so $a \wedge (b \vee c) \in P$ for all $P$. Hence $a \wedge (b \vee c) \in I$, and so $I$ is semi-prime. •

**Corollary 3. 3. 2:** *Intersection of two prime(semi prime) ideals is a semi-prime ideal.* •

**Lemma 3. 3. 3:** *Every filter disjoint from an ideal $I$ is contained in a maximal filter disjoint from $I$.*

**Proof:** Let $F$ be a filter in $L$ disjoint from $I$. Let $\mathcal{F}$ be the set of all filters containing $F$ and disjoint from $I$. Then $\mathcal{F}$ is nonempty as $F \in \mathcal{F}$. Let $C$ be a chain in $\mathcal{F}$ and let $M = \bigcup(X : X \in C)$. We claim that $M$ is a filter. Let $x \in M$ and $y \geq x$. Then $x \in X$ for some $X \in C$. Hence $y \in X$ as $X$ is a filter. Therefore, $y \in M$. Let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in C$. Since $C$ is a chain, either $X \subseteq Y$ or $Y \subseteq X$. Without loss of generality suppose $X \subseteq Y$. So $x, y \in Y$. Then $x \wedge y \in Y$ and so $x \wedge y \in M$. Moreover, $M \supseteq F$. So $M$ is a maximum element of $C$. Then by Zorn's Lemma, $\mathcal{F}$ has a maximal element, say $Q \supseteq F$. •

**Lemma 3. 3. 4:** *Let $I$ be an ideal of a nearlattice S. A filter $M$ disjoint from $I$ is a maximal filter disjoint from $I$ if and only if for all $a \notin M$, there exists $b \in M$ such that $a \wedge b \in I$.*

**Proof:** Let $M$ be maximal and disjoint from $I$ and $a \notin M$. Let $a \wedge b \notin I$ for $b \in M$. Consider $M_1 = \{y \in L : y \geq a \wedge b, \quad b \in M\}$. Clearly $M_1$ is a filter. For any $b \in M$, $b \geq a \wedge b$ implies $b \in M_1$. So $M_1 \supseteq M$. Also $M_1 \cap I = \phi$. For if not, let $x \in M_1 \cap I$. This implies $x \in I$ and $x \geq a \wedge b$ for some $b \in M$. Hence $a \wedge b \in I$, which is a contradiction. Hence $M_1 \cap I \neq \phi$. Now $M \subset M_1$ because $a \notin M$ but $a \in M_1$. This contradicts the maximality of $M$. Hence there exists $b \in M$ such that $a \wedge b \in I$.

Conversely, if $M$ is not maximal disjoint from $I$, then there exists a filter $N \supset M$ and disjoint with $I$. For any $a \in N - M$, there exists $b \in M$ such that $a \wedge b \in I$. Hence, $a, b \in N$ implies $a \wedge b \in I \cap N$, which is a contradiction. Hence $M$ must be a maximal filter disjoint with $I$. •

**Theorem 3. 3. 5:** *Let $S$ be a 0-distributive nearlattice. Then for $A \subseteq S$, $A^{\perp} = \{x \in S : x \wedge a = 0$ for all $a \in A\}$ is a semi-prime ideal.*

**Proof:** We have already mentioned that $A^{\perp}$ is a down set of S. Let $x, y \in A^{\perp}$ and $x \vee y$ exists. Then $x \wedge a = 0 = y \wedge a$ for all $a \in A$. Since S is 0-distributive, so $a \wedge (x \vee y) = 0$ for all $a \in A$. This implies $x \vee y \in A^{\perp}$ and so $A^{\perp}$ is an ideal.

Now let $x \wedge y \in A^{\perp}$ and $x \wedge z \in A^{\perp}$ and $y \vee z$ exists. Then $x \wedge y \wedge a = 0 = x \wedge z \wedge a$ for all $a \in A$. This implies $(x \wedge a) \wedge y = 0 = (x \wedge a) \wedge z$ and so by 0-distributivity again, $x \wedge a \wedge (y \vee z) = 0$ for all $a \in A$. Hence $x \wedge (y \vee z) \in A^{\perp}$ and so $A^{\perp}$ is a semi-prime ideal. •

Let $A \subseteq S$ and $J$ be an ideal of $S$. We define $A^{\perp_J} = \{x \in S : x \wedge a \in J$ *for all* $a \in A\}$. This is clearly a down set containing $J$. In presence of distributivity, this is an ideal. $A^{\perp_J}$ is called an annihilator of $A$ relative to $J$. We denote $I_J(S)$, by the set of all ideals containing $J$. Of course, $I_J(S)$ is a bounded lattice with $J$ and $S$ as the smallest and the largest elements. If $A \in I_J(S)$, and $A^{\perp_J}$ is an ideal, then $A^{\perp_J}$ is called an annihilator ideal and it is the pseudocomplement of $A$ in $I_J(S)$.

**Theorem 3. 3. 6:** *Let A be a non-empty subset of a nearlattice S and J be an ideal of S. Then* $A^{\perp_J} = \bigcap(P : P \textit{ is minimal prime down set containing J but not containing } A)$.

**Proof:** Suppose $X = \bigcap(P : A \not\subset P, P \textit{ is a minimal prime down set})$. Let $x \in A^{\perp_J}$. Then $x \wedge a \in J$ for all $a \in A$. Choose any $P$ of right hand expression. Since $A \not\subset P$, there exists $z \in A$ but $z \notin P$. Then $x \wedge z \in J \subseteq P$. So $x \in P$, as $P$ is prime. Hence $x \in X$.

Conversely, let $x \in X$. If $x \notin A^{\perp_J}$, then $x \wedge b \notin J$ for some $b \in A$. Let D = $[x \wedge b)$. Hence $D$ is a filter disjoint from J. Then by Lemma 3.2.3, there is a maximal filter $M \supseteq D$ but disjoint from *J*. Then *L-M* is a minimal prime down set containing *J.* Now $x \notin S - M$ as $x \in D$ implies $x \in M$. Moreover, $A \not\subseteq S - M$ as $b \in A$, but $b \in M$ implies $b \notin S - M$, which is a contradiction to $x \in X$. Hence $x \in A^{\perp_J}$. ●

Following Theorem gives some nice characterization of semi-prime ideals.

**Theorem 3. 3. 7:** *Let S be a nearlattice and* $J$ *be an ideal of S. The following conditions are equivalent.*

*(i)* $J$ *is semi-prime.*

*(ii)* $\{a\}^{\perp_J} = \{x \in S : x \wedge a \in J\}$ *is a semi-prime ideal containing* $J$.

*(iii)* $A^{\perp_J} = \{x \in S : x \wedge a \in J \textit{ for all } a \in A\}$ *is a semi prime ideal containing* $J$.

*(iv)* $I_J(S)$ *is pseudocomplemented*

*(v)* $I_J(S)$ *is a* $0$*-distributive lattice.*

*(vi) Every maximal filter disjoint from* $J$ *is prime.*

**Proof:** $(i)$ implies $(ii)$: $\{a\}^{\perp_J}$ is clearly a down set containing $J$. Now let $x, y \in \{a\}^{\perp_J}$ and $x \vee y$ exists. Then $x \wedge a \in J$, $y \wedge a \in J$. Since $J$ is semi prime, so $a \wedge (x \vee y) \in J$. This implies $x \vee y \in \{a\}^{\perp_J}$, and so it is an ideal containing $J$. Now let $x \wedge y \in \{a\}^{\perp_J}$ and

$x \wedge z \in \{a\}^{\perp_J}$ with $y \vee z$ exists. Then $x \wedge y \wedge a \in J$ and $x \wedge z \wedge a \in J$. Thus, $(x \wedge a) \wedge y \in J$ and $(x \wedge a) \wedge z \in J$. Then $(x \wedge a) \wedge (y \vee z) \in J$, as $J$ is semi-prime. This implies $x \wedge (y \vee z) \in \{a\}^{\perp_J}$, and so $\{a\}^{\perp_J}$ is semi-prime.

$(ii)$ implies $(iii)$ : This is trivial by Lemma 3.2.1, as $A^{\perp_J} = \bigcap(\{a\}^{\perp_J}; a \in A)$.

$(iii)$ implies $(iv)$ : Since for any $A \in I_J(S)$, $A^{\perp_J}$ is an ideal, it is the pseudocomplement of $A$ in $I_J(S)$, so $I_J(S)$ is pseudocomplemented.

$(iv)$ implies $(v)$ : This is trivial as every pseudocomplemented lattice is 0-distributive.

$(v)$ implies $(vi)$ : Let $I_J(S)$ is 0-distributive. Suppose $F$ is a maximal filter disjoint from $J$. Suppose $f, g \notin F$ and $f \vee g$ exists. By Lemma 3.2.4, there exist $a, b \in F$ such that $a \wedge f \in J, b \wedge g \in J$. Then $f \wedge a \wedge b \in J, g \wedge a \wedge b \in J$. Hence $(f] \wedge (a \wedge b] \subseteq J$ and $(g] \wedge (a \wedge b] \subseteq J$. Then $(f \vee g] \wedge (a \wedge b] = ((f] \vee (g]) \wedge (a \wedge b] \subseteq J$, by the 0-distributive property of $I_J(S)$. Hence, $(f \vee g) \wedge a \wedge b \in J$. This implies $f \vee g \notin F$ as $F \cap J = \phi$, and so $F$ is prime.

$(vi)$ implies $(i)$: Let (vi) holds. Suppose $a, b, c \in S$ with $a \wedge b \in J, a \wedge c \in J$ with $b \vee c$ exists. If $a \wedge (b \vee c) \notin J$, then $[a \wedge (b \vee c)) \cap J = \phi$. Then by Lemma 3.2.3, there exists a maximal filter $F \supseteq [a \wedge (b \vee c))$ and disjoint from $J$. Then $a \in F, b \vee c \in F$. By $(vi)$ $F$ is prime. Hence either $a \wedge b \in F$ or $a \wedge c \in F$. In any case $J \cap F \neq \phi$, which gives a contradiction. Hence $a \wedge (b \vee c) \in J$, and so $J$ is semi-prime. •

**Corollary 3. 3. 8:** *In a nearlattice S, every filter disjoint to a semi-prime ideal $J$ is contained in a prime filter.* •

**Theorem 3. 3. 9:** *If $J$ is a semi-prime ideal of a nearlattice S and $J \neq A = \bigcap\{J_\lambda : J_\lambda$ is an ideal containing $J\}$, Then $A^{\perp_J} = \{x \in S : \{x\}^{\perp_J} \neq J\}$.*

**Proof:** Let $x \in A^{\perp_J}$. Then $x \wedge a \in J$ for all $a \in A$. So $a \in \{x\}^{\perp_J}$ for all $a \in A$. Then $A \subseteq \{x\}^{\perp_J}$ and so $\{x\}^{\perp_J} \neq J$ .Conversely, let $x \in S$ such that $\{x\}^{\perp_J} \neq J$. Since $J$ is semi-prime, so $\{x\}^{\perp_J}$ is an ideal containing $J$. Then $A \subseteq \{x\}^{\perp_J}$, and so $A^{\perp_J} \supseteq \{x\}^{\perp_J \perp_J}$. This implies $x \in A^{\perp_J}$, which completes the proof. •

Rav have provided a series of characterizations of 0-distributive lattices in [26]. Here we give some results on semi-prime ideals related to their results for nearlattices.

**Theorem 3. 3. 10:** *Let S be a nearlattice and J be an ideal. Then the following conditions are equivalent.*

- *(i)* *J is semi-prime.*
- *(ii)* *Every maximal filter of S disjoint with J is prime.*
- *(iii)* *Every minimal prime down set containing J is a minimal prime ideal containing J.*
- *(iv)* *Every filter disjoint with J is disjoint from a minimal prime ideal containing J.*
- *(v)* *For each element $a \notin J$, there is a minimal prime ideal containing J but not containing a.*
- *(vi)* *Each $a \notin J$ is contained in a prime filter disjoint to J.*

**Proof.** $(ii)$ $(ii)$ implies $(i)$: Follows from Theorem 3.3.7.

$(ii)$ implies $(iii)$: Let $A$ be a minimal prime down set containing $J$. Then $S$-$A$ is a maximal filter disjoint with $J$. Then by (ii) $S$-$A$ is prime and so $A$ is a minimal prime ideal.

$(iii)$ implies $(ii)$: Let $F$ be a maximal filter disjoint with $J$. Then $S$-$F$ is a minimal prime down set containing $J$. Thus by (iii), $S$-$F$ is a minimal prime ideal and so $F$ is a prime filter.

$(i)$ implies $(iv)$: Let $F$ a filter of $S$ disjoint from $J$. Then by Corollary 3.3.8, there is a prime filter $Q \supseteq F$ and disjoint from $F$.

$(iv)$ implies $(v)$: Let $a \in S$, $a \notin J$. Then $[a) \cap J = \varphi$. Then by (iv) there exists a minimal prime ideal $A$ disjoint from $[a)$. Thus $a \notin A$.

$(v)$ implies $(vi)$: Let $a \in S$, $a \notin J$. Then by (v) there exists a minimal prime ideal $P$ such that $a \notin P$, which implies $a \in S - P$ and $S$-$P$ is a prime filter.

$(vi)$ implies $(i)$: Suppose $J$ is not semi-prime . Then there exists $a,b,c \in L$ such that $a \wedge b \in J$, $a \wedge c \in J$ and $b \vee c$ exists, but $a \wedge (b \vee c) \notin J$. Then by (vi) there exists a prime filter $Q$ disjoint from J and $a \wedge (b \vee c) \in Q$. Let $F = [a \wedge (b \vee c))$. Then $J \cap F = \varphi$ and $F \subseteq Q$. Now $a \wedge (b \vee c) \in Q$ implies $a \in Q$, $b \vee c \in Q$. Since $Q$ is prime so either $a \wedge b \in Q$ or $a \wedge c \in Q$ . This gives a contradiction to the fact that $Q \cap J = \varphi$ . Therefore, $a \wedge (b \vee c) \in J$ and so $J$ is semi-prime. •

Now we give another characterization of semi-prime ideals with the help of Prime Separation Theorem using annihilator ideals.

**Theorem 3. 3. 11:** *Let $J$ be an ideal in a nearlattice S. $J$ is semi- prime if and only if for all filter $F$ disjoint to $\{x\}^{\perp_J}$, there is a prime filter containing $F$ disjoint to $\{x\}^{\perp_J}$.*

**Proof:** Using Zorn's Lemma we can easily find a maximal filter $Q$ containing $F$ and disjoint to $\{x\}^{\perp_J}$. We claim that $x \in Q$. If not, then $Q \vee [x) \supset Q$. By maximality of $Q$, $(Q \vee [x)) \cap \{x^{\perp_J}\} \neq \phi$. If $t \in (Q \vee [x)) \cap \{x\}^{\perp_J}$, then $t \geq q \wedge x$ for some $q \in Q$ and $t \wedge x \in J$. This implies $q \wedge x \in J$ and so $q \in \{x\}^{\perp_J}$ gives a contradiction. Hence $x \in Q$.

Now let $z \notin Q$. Then $(Q \vee [z)) \cap \{x\}^{\perp_J} \neq \phi$. Suppose $y \in (Q \vee [z)) \cap \{x\}^{\perp_J}$ then $y \geq q_1 \wedge z$ and $y \wedge z \in J$ for some $q_1 \in Q$. This implies $q_1 \wedge x \wedge z \in J$ and $q_1 \wedge x \in Q$. Hence by Lemma 3. 3. 4, $Q$ is a maximal filter disjoint to $\{x\}^{\perp_J}$. Then by Theorem 3.3.7, $Q$ is prime.

Conversely, let $x \wedge y \in J, x \wedge z \in J$ and $y \vee z$ exists. If $x \wedge (y \vee z) \notin J$, then $y \vee z \notin \{x\}^{\perp_J}$. Thus $[y \vee z) \cap \{x\}^{\perp_J} = \varphi$. So there exists a prime filter $Q$ containing $[y \vee z)$ and disjoint from $\{x\}^{\perp_J}$. As $y, z \in \{x\}^{\perp_J}$, *so* $y, z \notin Q$. Thus $y \vee z \notin Q$, as $Q$ is prime. This implies $[y \vee z) \not\subset Q$, a contradiction. Hence $x \wedge (y \vee z) \in J$, and so $J$ is semi-prime. •

Here is another characterization of semi-prime ideals.

**Theorem 3. 3. 12:** *Let $J$ be a semi-prime ideal of a nearlattice $S$ and $x \in S$. Then a prime ideal $P$ containing $\{x\}^{\perp_J}$ is a minimal prime ideal containing $\{x\}^{\perp_J}$ if and only if for $p \in P$, there exists $q \in S - P$ such that $p \wedge q \in \{x\}^{\perp_J}$.*

**Proof:** Let $P$ be a prime ideal containing $\{x\}^{\perp_J}$ such that the given condition holds. Let $K$ be a prime ideal containing $\{x\}^{\perp_J}$ such that $K \subseteq P$. Let $p \in P$. Then there is $q \in S - P$ such that $p \wedge q \in \{x\}^{\perp_J}$. Hence $p \wedge q \in K$. Since $K$ is prime and $q \notin K$, so $p \in K$. Thus, $P \subseteq K$ and so $K = P$. Therefore, $P$ must be a minimal prime ideal containing $\{x\}^{\perp_J}$.

Conversely, let $P$ be a minimal prime ideal containing $\{x\}^{\perp_J}$. Let $p \in P$. Suppose for all $q \in S - P$, $p \wedge q \notin \{x\}^{\perp_J}$. Let $D = (S - P) \vee [p)$. We claim that $\{x\}^{\perp_J} \cap D = \varphi$. If not, let $y \in \{x\}^{\perp_J} \cap D$. Then $p \wedge q \leq y \in \{x\}^{\perp_J}$, which is a contradiction to the assumption. Then by Theorem 3.3.11, there exists a maximal (prime) filter $Q \supseteq D$ and disjoint to $\{x\}^{\perp_J}$. By the proof of Theorem 3.3.11, $x \in Q$. Let $M = S - Q$. Then $M$ is a prime ideal. Since $x \in Q$, so $t \wedge x \in J \subseteq M$ implies $t \in M$ as $M$ is prime. Thus $\{x\}^{\perp_J} \subseteq M$. Now $M \cap D = \varphi$. This implies $M \cap (S - P) = \varphi$ and hence $M \subseteq P$. Also $M \neq P$, because $p \in D$ implies $p \notin M$ but $p \in P$. Hence $M$ is a prime ideal containing $\{x\}^{\perp_J}$ which is properly contained in $P$. This gives a contradiction to the minimal property of $P$. Therefore the given condition holds. •

Observe that by Theorem 3.3.7 we can easily give a Separation theorem in a 0-distributive nearlattice for $A^{\perp}$, when $A$ is a finite subset of $S$. But now we are in a position to give a proof of the theorem for any subset $A$.

**Theorem 3. 3. 13:** *Let $F$ be a filter of a 0-distributive nearlattice $S$ such that $F \cap A^{\perp} = \phi$ for any non-empty subset $A$ of $S$. Then there exists a prime filter $Q \supseteq F$ such that $Q \cap A^{\perp} = \phi$.*

**Proof:** By Theorem 3. 2. 5, $A^{\perp}$ is a semi-prime . Thus by Lemma 3. 3. 3, there exists a maximal filter $Q \supseteq F$ *such that* $Q \cap A^{\perp} = \phi$. Since $A^{\perp}$ is semi-prime, so by Theorem 3. 3. 7, $Q$ is prime. ●

# WEAKLY COMPLEMENTED NEARLATTICE

## 4. 1 Introduction

In this chapter we will study the homomorphism on nearlattices. Then we include homomorphism theorem for nearlattices. We establish some results on homomorphic images of semi prime ideals. We also show that in a 0-distributive semi lattice, a map $f: S \to \{\{a\}^{\perp\perp} : a \in S\}$ is a semi lattice homomorphism if and only if $f(\{a\}^{\perp}) = \{f(a)\}^{\perp}$. Finally, we included some characterizations of weakly complemented nearlattices relative to $J$.

Varlet [33] first introduced the concept of 0-distributive lattices. Then many authors including [1, 19, 22, 23, 24, 25] studied them for lattices and semi lattices. A nearlattice $S$ with 0 is called 0-distributive if for all $a,b,c \in S$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge d = 0$ for some $d \geq b$, c Chakraborty [9]. The concept of semi-prime ideals of a lattice is introduced in [26]. Recently, Begum and Noor [22] have extended the concept for meet semi lattices. An ideal $J$ of a nearlattice $S$ is called a semi-prime ideal if for all $a,b,c \in S$ with $a \wedge b \in J$, $a \wedge c \in J$, imply $a \wedge d \in J$ for some $d \geq b,c$. Hence a nearlattice $S$ with 0 is called 0-distributive if $(0]$ is a semi prime ideal of $S$. A meet semi lattice $S$ is called directed above if for all $a,b \in S$, there exists $c \in S$ such that $c \geq a,b$. We know that every modular and distributive semi lattice have the directed above property. Moreover Chakraborty[9] have shown that every 0-distributive meet semi lattice is directed above.

Let $S$ and $T$ be two nearlattices. A map $f: S \to T$ is said to be a homomorphism if $f$ is a meet preserving map. That is, for all $a,b \in S$, $f(a \wedge b) = f(a) \wedge f(b)$. A homomorphism is called 0-homomorphism if $f(0) = 0$. A one-to-one homomorphism is called a monomorphism or an embedding. A onto homomorphism is called an epimorphism. If $f: A \to B$ is an epimorphism, we say that $B$ is a homomorphic image of $A$. An epimorphism is called an isomorphism if it is a one-to-one map. A homomorphism $f: A \to A$ is called an endomorphism, and an isomorphism $f: A \to A$ is called an automorphism. The nearlattice $S$ and $T$ are isomorphic if there exists an isomorphism $f$ from $S$ to $T$. We denote it symbolically by $S \cong T$.

## 4. 2 Homomorphism and semi prime ideals

Let $A \subseteq S$ and $J$ be an ideal of $S$. We define $A^{\perp_J} = \{x \in S : x \wedge a \in J$ *for all* $a \in A\}$, then $A^{\perp_J}$ is called an annihilator of $A$ relative to $J$ which is clearly a down set containing $J$. If it is an ideal, then it is called an annihilator ideal relative to $J$. By [9, 25] we know that, for any $a \in A, \{a\}^{\perp_J}$ is an ideal if and only if S is 0-distributive.

The following result is due to Noor and Begum[24].

**Lemma 4. 2. 1:** *Let $J$ be an ideal of a nearlattice $S$. Suppose $A, B \subseteq S$ and $a, b \in S$ then the following hold:*

*(i)* *If* $A \cap B = J$, *then* $B \subseteq A^{\perp_J}$.

*(ii)* $A \cap A^{\perp_J} = J$.

*(iii)* $A \subseteq B$ *implies that* $B^{\perp_J} \subseteq A^{\perp_J}$.

*(iv)* $a \leq b$ *implies that* $\{b\}^{\perp_J} \subseteq \{a\}^{\perp_J}$ *and* $\{a\}^{\perp_J \perp_J} \subseteq \{b\}^{\perp_J \perp_J}$.

*(v)* $\{a\}^{\perp_J} \cap \{a\}^{\perp_J \perp_J} = J$.

*(vi)* $\{a \wedge b\}^{\perp_J \perp_J} = \{a\}^{\perp_J \perp_J} \cap \{b\}^{\perp_J \perp_J}$.

*(vii)* $A \subseteq A^{\perp_J \perp_J}$.

*(viii)* $A^{\perp_J \perp_J \perp_J} = A^{\perp_J}$. •

Homomorphism theorem for lattice can be found in Gratzer [12], theorem 11. In a similar way, we can easily state the following homomorphism theorem for nearlattices. We prefer to omit the proof as it is almost similar to the proof of homomorphism theorem for lattices.

**Theorem 4. 2. 2:** *(Homomorphism theorem for nearlattices) Every homomorphic image of a nearlattice $S$ is isomorphic to a suitable quotient nearlattice of $S$. In fact, if $\Phi : S \to T$ is a homomorphism of $S$ onto $T$ and $\Theta$ is a congruence relation of $S$ defined by $x \equiv y(\Theta)$ if and only if $\Phi(x) = \Phi(y)$. Then $S / \Theta \cong T$.* •

**Theorem 4. 2. 3:** *Let $S$ and $T$ be two nearlattices. $I$ is an ideal of $S$. $f: S \to T$ is a homomorphism and onto such that $f^{-1}(f(I)) = I$. Then $I$ is semi-prime in $S$ implies $f(I)$ is semi-prime in $T$.*

**Proof:** Suppose $I$ is semi-prime. Let $x, y, z \in T$ with $x \wedge y \in f(I)$ and $x \wedge z \in f(I)$. Then there exists $a, b, c \in S$ such that $x = f(a), y = f(b), z = f(c)$.

Now $f(a) \wedge f(b) = f(a \wedge b) \in f(I)$.

$f(a) \wedge f(c) = f(a \wedge c) \in f(I)$. This implies $a \wedge b, a \wedge c \in I$. Since $I$ is semi-prime, so there exists $d \in S, d \geq b, c$ such that $a \wedge d \in I$. Let $t = f(d)$. Then $t = f(d) \geq f(b), f(c)$. That is, $t \geq y, z$. Also $f(a) \wedge f(d) = f(a \wedge d) \in f(I)$. Thus $x \wedge t \in f(I)$, and so $f(I)$ is semi-prime. •

Since $S$ is 0-disrtibutive if and only if $(0]$ is a semi prime ideal so the following corollary immediately follows by above theorem.

**Corollary 4. 2. 4:** *Let $S$ and $T$ be two nearlattices with $0$. $f: S \to T$ is 0-homomorphism, onto and $f^{-1}(0) = 0$. Then $T$ is 0-distributive if $S$ is 0-distributive.* •

**Lemma 4. 2. 5:** *Let $J$ be a semi-prime ideal of a nearlattise $S$. $f: S \to \{\{a\}^{\perp_J \perp_J} : a \in S\}$ given by $f(a) = \{a\}^{\perp_J \perp_J}$. Then the following results holds:*

*(i)* *$f$ is a meet homomoephism.*

*(ii)* *For $a \in S, f(a) = J$ if and only if $a \in J$.*

*(iii)* $f(\{a\}^{\perp_J}) = \{f(a)\}^{\perp_J}$

**Proof:** $(i)$ Let $a, b \in S$. Now

$$f(a \wedge b) = \{a \wedge b\}^{\perp_J \perp_J}$$

$$= \{a\}^{\perp_J \perp_J} \cap \{b\}^{\perp_J \perp_J}$$

$$= f(a) \cap f(b)$$

$$= f(a) \wedge f(b)$$

Hence the map is a meet homomorphism.

$(ii)$ If $f(a) = J$, then $\{a\}^{\perp_J \perp_J} = J$. Thus $\{a\}^{\perp_J} = \{a\}^{\perp_J \perp_J \perp_J} = S$ and so $a \in \{a\}^{\perp_J}$. This implies $a = a \wedge a \in J$.

Conversely, if $a \in J$, then $f(a) = \{a\}^{\perp_J \perp_J} = S^{\perp_J} = J$.

$(iii)$ $f(\{a\}^{\perp_J}) = \{\{b\}^{\perp_J \perp_J} \mid b \in \{a\}^{\perp_J}\}$

$$f(\{a\}^{\perp_J}) = \{\{b\}^{\perp_J \perp_J} \mid a \wedge b \in J\}$$

$$= \{\{b\}^{\perp_J \perp_J} \mid f(a \wedge b) \in J\}$$

$$= \{\{b\}^{\perp_J \perp_J} \mid f(a) \wedge f(b) \in J\}$$

$$= \{f(a)\}^{\perp_J}$$

Hence the proof is completed. •

**Corollary 4. 2. 6:** *Let $S$ be a 0-distributive nearlattice $S$. $f : S \to \{\{a\}^{\perp\perp} : a \in S\}$ given by $f(a) = \{a\}^{\perp\perp}$. Then the following results hold:*

*(i)* *$f$ is a meet homomorphism.*

*(ii)* *For $a \in S, f(a) = \{0\}$ if and only if $a = 0$.*

*(iii)* *$f(\{a\}^{\perp}) = \{f(a)\}^{\perp}$*

**Note:** Observe that lemma 4. 2. 5 is also true for an ordinary ideal $J$ of $S$. But we have consider semi primeness of $J$ as $\{a\}^{\perp_J}$ and $\{a\}^{\perp_J \perp_J}$ are ideals only when $J$ is semi-prime. Similarly, in a semi lattice with $0$, $\{a\}^{\perp}$ or $\{a\}^{\perp\perp}$ are ideals only when $S$ is 0-distributive.

## 4. 3 Weakly complemented Nearlattices

Let $S$ be a nearlattice with $0$. $S$ is called weakly complemented if for any pair of distinct elements $a,b \in S$, there exists an element $c$ such that only one of $a \wedge c$ and $b \wedge c$ is equal to $0$.

Similarly, for an ideal $J$ of a nearlattice $S$, we call $S$ is weakly complemented with respect to $J$ if for any pair of distinct elements $a,b \in S$, there exists an element $c$ such that only one of $a \wedge c$ and $b \wedge c$ belongs to $J$. In particular, if $a < b$, then there exists $c \in S$ such that $a \wedge c \in J$ but $b \wedge c \notin J$.

Note that the definition of weakly complemented semi lattice relative to ideal $J$ can also be given in the following way:

For an ideal $J$ of a nearlattice $S$, $S$ is called weakly complemented relative to $J$ if for all $a,b \in S$, $a \neq b$ implies that either $\{a\}^{\perp_J} - \{b\}^{\perp_J} \neq \Phi$ or $\{b\}^{\perp_J} - \{a\}^{\perp_J} \neq \Phi$. These semi lattices are also known as disjunctive semi lattices relative to $J$.

**Theorem 4. 3. 1:** *Let $S$ be a nearlattice and $J$ be a semi-prime ideal of $S$. Then the following are equivalent:*

(i) $f : S \to \{\{a\}^{\perp_J \perp_J} \mid a \in S\}$ *defined by* $f(a) = \{a\}^{\perp_J \perp_J}$ *is isomorphism.*

(ii) $\{a\}^{\perp_J} = \{b\}^{\perp_J} \in I_J(S)$ *implies that* $a = b$ *for all* $a,b \in S$.

(iii) $S$ *is weakly complemented relative to* $J$.

**Proof:** $(i)$ implies $(ii)$: Let $\{a\}^{\perp_J} = \{b\}^{\perp_J}$ and $a \neq b$. Then as $f$ is an isomorphism, we have, $f(a) \neq f(b)$ which implies that $\{a\}^{\perp_J \perp_J} \neq \{b\}^{\perp_J \perp_J}$. Then there exists $x \in \{a\}^{\perp_J \perp_J}$, such that $x \notin \{b\}^{\perp_J \perp_J}$ which implies that $x \wedge z \notin J$ for some $z \in \{b\}^{\perp_J}$. Since $\{a\}^{\perp_J} = \{b\}^{\perp_J}$ then we have $x \wedge z \notin J$ for some $z \in \{a\}^{\perp_J}$ which implies $x \notin \{a\}^{\perp_J \perp_J}$. This gives is a contradiction. Hence $\{a\}^{\perp_J} = \{b\}^{\perp_J}$ implies $a = b$.

$(ii)$ implies $(iii)$: Let $a < b$. Then by lemma 4.2.1 and (ii), we have $\{a\}^{\perp_J} \supset \{b\}^{\perp_J}$. Hence there exists $x \in \{a\}^{\perp_J}$ such that $x \notin \{b\}^{\perp_J}$, which implies that $S$ is weakly complemented relative to $J$.

$(iii)$ implies $(ii)$: Let $a \neq b$ then either $a \wedge b < a$ or $a \wedge b < b$. Assume that $a \wedge b < a$. As $S$ is weakly complemented, so there exists $x \in \{a \wedge b\}^{\perp_J}$ such that $x \wedge a \notin J$.

Thus we have $x \wedge (a \wedge b) \in J$. This implies $(x \wedge a) \wedge b \in J$, and so $(x \wedge a) \in \{b\}^{\perp_J}$ and $x \wedge a \notin \{a\}^{\perp_J}$. Hence $\{a\}^{\perp_J} \neq \{b\}^{\perp_J}$, and so (ii) holds.

$(ii)$ implies $(i)$: To prove $f$ is an isomorphism. For all $a, b \in S$, $a = b$

$$\Leftrightarrow \{a\}^{\perp_J} = \{b\}^{\perp_J}$$

$$\Leftrightarrow \{a\}^{\perp_J \perp_J} = \{b\}^{\perp_J \perp_J}$$

$$\Leftrightarrow f(a) = f(b)$$

This implies f is well defined and one to one.

Obviously, the mapping is onto.

Moreover, by lemma 4. 2. 5, $f$ is a $\wedge$ preserving map. Therefore, $f$ is an isomorphism. •

We conclude the thesis with the following result as $(0]$ is semi-prime if and only if $S$ is 0-distributive.

**Corollary 4. 3. 2:** *Let $S$ be a 0-distributive nearlattice. Then the following are equivalent:*

*(i)* $f : S \to \{\{a\}^{\perp\perp} \mid a \in S\}$ *defined by* $f(a) = \{a\}^{\perp\perp}$ *is an isomorphism.*

*(ii)* $\{a\}^{\perp} = \{b\}^{\perp} \in I(S)$ *implies that* $a = b$ *for all* $a, b \in S$.

*(iii)* $S$ is weakly complemented. •

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