Analytical Solutions of Second Order Nonlinear Ordinary Differential Systems with High Order Nonlinearity

by

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Dedicated to My

Beloved Parents

Declaration

This is to certify that the thesis work entitled "**Analytical Solutions of Second Order Nonlinear Ordinary Differential Systems with High Order Nonlinearity**" has been carried out by Deepa Rani Ghosh in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna 9203, Bangladesh. The above thesis work or any part of the thesis work has not been submitted anywhere for the award of any degree or diploma.

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Approval

This is to certify that the thesis work submitted by Deepa Rani Ghosh, Roll No. 1451564 entitled "**Analytical Solutions of Second Order Nonlinear Ordinary Differential Systems with High Order Nonlinearity**" has been approved by the board of examiners for the partial fulfillment of the requirements for the degree of Master of Science in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna-9203, Bangladesh in February 2016.

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Abstract

Nonlinear oscillator models have been widely used in many areas of physics and engineering and are of significant importance in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of motion.

The aim of the present study is to solve the second order autonomous nonlinear differential systems with strong and high $((9th))$ order nonlinearity in presence of small damping by combing He's homotopy perturbation and the extended form of the KBM methods. The results obtained by the presented method are compared with those solutions obtained by the fourth order Runge-Kutta method in graphically.

Publication

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CHAPTER I

Introduction

Most of real systems are modeled by nonlinear differential equations. Obtaining exact solution for these nonlinear problems is difficult and time consuming for researchers, thus scientists are tried to find new approaches to overcome this difficulties.

The subject of differential equations constitutes a large and very important branch of modern mathematics. Numerous physical, mechanical, chemical, biological, mechanics in which we want to describe the motion of the body (automobile, electron, or satellite) under the action of a given force, and many other relations appear mathematically in the form of differential equations that are linear or nonlinear, autonomous or non-autonomous. Also, in ecology and economics the differential equations are vastly used. Basically, many differential equations involving physical phenomena are nonlinear such as spring-mass systems, resistor-capacitor-inductor circuits, bending of beams, chemical reactions, the motion of a pendulum, the motion of the rotating mass around another body, population model etc. In mathematics and physics, linear generally means "simple" and nonlinear means "complicated". The theory for solving linear equations is very well developed because linear equations are simple enough to be solvable. Nonlinear equations can usually not be solved exactly and are the subject of much on-going research. In such situations, mathematicians, physicists and engineers convert the nonlinear equations into linear equations i.e., they linearize them by imposing some special conditions. Small oscillations are well-known example of the linearization for the physical problems. But, such a linearization is not always possible and when it is not, then the original nonlinear equation itself must be used. The study of nonlinear equations is generally confined to a variety of rather special cases, and one must resort to various methods of approximations.

At first van der Pol [1] paid attention to the new (self-excitations) oscillations and indicated that their existence is inherent in the nonlinearity of the differential equations characterizing the process. Thus, this nonlinearity appears as the very essence of these phenomena and by linearizing the differential equations in the sense of small oscillations, one simply eliminates the possibility of investigating such problems. Thus, it is necessary to deal with the nonlinear differential equations directly instead of evading them by dropping the nonlinear terms. To solve nonlinear differential equations there exist some methods such as perturbation method [2-57], homotopy perturbation [58-67] method, variational iterative method [68], harmonic balance method [69] etc. Among the methods, the method of perturbations, i.e., asymptotic expansions in terms of a small parameter are first and foremost.

A perturbation method known as "the asymptotic averaging method" in the theory of nonlinear oscillations was first introduced by Krylov and Bogoliubov (KB) [2] in 1947. Primarily, the method was developed only for obtaining the periodic solutions of second order weakly conservative nonlinear differential systems. Later, the method of KB has been improved and justified by Bogoliubov and Mitropolskii [3] in 1961. In literature, this method is known as the Krylov-Bogoliubov-Mitropolskii (KBM) [2, 3] method.

A perturbation method is based on the following aspects: the equations to be solved are sufficiently "smooth" or sufficiently differentiable a number of times in the required regions of variables and parameters. The KBM [2, 3] method was developed for obtaining only the periodic solutions of second order weakly nonlinear differential equations without damping. Now a days, this method is used for obtaining the solutions of second, third and fourth order weakly nonlinear differential systems for oscillatory, damped oscillatory, over damped, critically damped and more critically damped cases by imposing some special restrictions with quadratic and cubic nonlinearities.

Several authors [5-57] have investigated and developed many significant results concerning the solutions of the weakly nonlinear differential systems. Extensive uses have been made and some important works are done by several authors [5-57] based on the **KBM** method.

The method of KB [2] is an asymptotic method in the sense that $\varepsilon \to 0$. An asymptotic series itself may not be convergent, but for a fixed number of terms, the approximate solution tends to the exact solution as $\varepsilon \to 0$. It may be noted that the term asymptotic is frequently used in the theory of oscillations in the sense that

$\varepsilon \to 0$. But, in this case, the mathematical method is quite different. It is an important approach to the study of such nonlinear oscillations in the small parameter expansion. Two widely spread methods in this theory are mainly used in the literature; one is averaging asymptotic KBM method and the other is multiple-time scale method. The KBM method is particularly convenient and extensively used technique for determining the approximate solutions among the methods used for studying the weakly nonlinear differential systems with small nonlinearity. The KBM method starts with the solution of linear equation (sometimes called the generating solution of the linear equation) assuming that in the nonlinear case, the amplitude and phase variables in the solution of the linear differential equation are time dependent functions instead of constants. This method introduces an additional condition on the first derivative of the assumed solution for determining the solution of second order nonlinear differential systems. The KBM method demands that the asymptotic solutions are free from secular terms. These assumptions are mainly valid for second and third order equations. But, for the fourth order differential equations, the correction terms sometimes contain secular terms, although the solution is generated by the classical KBM asymptotic method. For this reason, the traditional solutions fail to explain the proper situation of the systems. To remove the presence of secular terms for obtaining the desired results, one needs to impose some special conditions.

Ji-Huan He [58-61] has developed a homotopy perturbation technique for solving second order strongly nonlinear differential systems without damping effects. Uddin *et al.* [62-64] have presented an approximate technique for solving second order strongly nonlinear oscillatory differential systems with quadratic and cubic nonlinearity in presence of small damping by combing the He's [58-61] homotopy perturbation and the extended form of the KBM [2-4] methods.

The KBM [2, 3] method is failed to tackle the strongly and weakly nonlinear differential systems with high order nonlinearity. Also He's [58-61] homotopy perturbation technique is failed to tackle both the strongly and weakly nonlinear differential systems in presence of small damping. In this thesis, He's homotopy perturbation method (HPM) has been extended for obtaining the analytical approximate solutions of second order strongly and weakly nonlinear differential systems with high order nonlinearity in presence of small damping based on the extended form of the KBM method. The results may be used in mechanics, physics, chemistry, plasma physics, circuit and control theory, population dynamics, economics, etc.

In Chapter II, the review of literature is presented. In **Chapter III**, an approximate analytical technique has been extended for solving second order strongly and weakly nonlinear differential system with high order nonlinearity in presence of small damping. Finally, **in Chapter IV**, the conclusions are given.

CHAPTER II

Literature Review

The nonlinear differential equations are generally difficult to solve and their exact solutions are difficult to obtain. But, mathematical formulations of many physical problems often results in differential equations that are nonlinear. In many situations, linear differential equation is substituted for a nonlinear differential equation, which approximates the original equation closely enough to give expected results. In many cases such a linearization is not possible and when it is not, the original nonlinear differential equation must be considered directly. During last several decades in the $20th$ century, some famous Russian scientists like Krylov and Bogoliubov [2], Bogoliubov and Mitropolskii [3], Mitropolskii [4], have investigated the nonlinear dynamics. To solve nonlinear differential equations there exist some methods. Among the methods, the method of perturbations, i.e., an asymptotic expansion in terms of small parameter is foremost. Firstly, Krylov and Bogoliubov (KB) [2] considered the equation of the form

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}, t, \varepsilon), \tag{2.1}$$

where $\ddot{x}$ denotes the second order derivative with respect to t, ε is a small positive parameter and f is a power series in ε, whose coefficients are polynomials in $x, \dot{x}, \sin t$ and $\cos t$ and the proposed solution procedure proposed by Krylov and Bogoliubov [2] is known as KB method. In general, f does not contain either ε or t explicitly. In literature, the method presented [2, 3] is known as Krylov-Bogoliubov-Mitropolskii (KBM) method. To describe the behavior of nonlinear oscillations by the solutions obtained by the perturbation method. Poincare [5] discussed only periodic solutions, transient were not considered.

The KBM [2, 3] method started with the solution of the linear equation, assuming that in the nonlinear systems, the amplitude and phase variables in the solution of the linear equations are time dependent functions rather than constants. This procedure introduces an additional condition on the first derivative of the assumed solution for determining the desired results. Some mirthful works are done and elaborative uses have been made by Stoker [6], McLachlan [7], Minorsky [8] and Bellman [9].

Duffing [10] has investigated many significant results for the periodic solutions of the following damped nonlinear differential system

$$\ddot{x} + 2k\dot{x} + \omega^2 x = -\varepsilon x^3. \tag{2.2}$$

Sometimes, different types of nonlinear phenomena occur, when the amplitude of a dynamic system is less than or greater than unity. The damping is negative when the amplitude is less than unity and the damping is positive when the amplitude is greater than unity. The governing equation having these phenomena is

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0. \tag{2.3}$$

In literature, this equation is known as van der Pol [1] equation and is used in electrical circuit theory. Kruskal [11] has extended the KB [2] method to solve the fully nonlinear differential equation of the following form

$$\ddot{x} = F(x, \dot{x}, \varepsilon). \tag{2.4a}$$

Cap [12] has studied nonlinear system of the form

$$\ddot{x} + \omega^2 x = \varepsilon F(x, \dot{x}). \tag{2.4b}$$

Generally, F does not contain ε or t explicitly, thus the equation (2.1) becomes

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}). \tag{2.5}$$

In the treatment of nonlinear oscillations by the perturbation method, only periodic solutions are discussed, transients are not considered by different investigators, where as KB [2] have discussed transient response.

When $\varepsilon = 0$, the equation (2.5) reduces to linear equation and its solution can be obtained as

$$x = a\cos(\omega t + \varphi). \tag{2.6}$$

where a and φ are arbitrary constants and the values of a and φ are determined by using the given initial conditions.

When $\varepsilon \neq 0$, but is sufficiently small, then KB [2] have assumed that the solution of equation (2.5) is still given by equation (2.6) together with the derivative of the form

$$\dot{x} = -a\omega\sin(\omega t + \varphi). \tag{2.7}$$

where a and φ are functions of t, rather than being constants. In this case, the solution of equation (2.5) is

$$x = a(t)\cos(\omega t + \varphi(t)) \tag{2.8}$$

and the derivative of the solution is

$$\dot{x} = -a(t)\omega\sin(\omega t + \varphi(t)). \tag{2.9}$$

Differentiating the assumed solution equation (2.8) with respect to time t, we obtain

$$\dot{x} = \dot{a}\cos\psi - a\omega\sin\psi - a\dot{\varphi}\sin\psi,\ \ \psi = \omega t + \varphi(t). \tag{2.10}$$

Using the equations (2.7) and (2.10), we get

$$\dot{a}\cos\psi = a\dot{\varphi}\sin\psi. \tag{2.11}$$

Again, differentiating equation (2.9) with respect to t, we have

$$\ddot{x} = -\dot{a}\omega\sin\psi - a\omega^2\cos\psi - a\omega\dot{\varphi}\cos\psi. \tag{2.12}$$

Putting the value of $\ddot{x}$ from equation (2.12) into the equation (2.5) and using equations (2.8) and (2.9), we obtain

$$\dot{a}\omega\sin\psi + a\omega\dot{\varphi}\cos\psi = -\varepsilon f(a\cos\psi, -a\omega\sin\psi). \tag{2.13}$$

Solving equations (2.11) and (2.13), we have

$$\dot{a} = -\frac{\varepsilon}{\omega}\sin\psi\, f(a\cos\psi, -a\omega\sin\psi), \tag{2.14}$$

$$\dot{\varphi} = -\frac{\varepsilon}{a\omega}\cos\psi\, f(a\cos\psi, -a\omega\sin\psi). \tag{2.15}$$

It is observed that, a basic differential equation (2.5) of the second order in the unknown x, reduces to two first order differential equations (2.14) and (2.15) in the unknowns a and φ.

Moreover, $\dot{a}$ and $\dot{\varphi}$ are proportional to ε; a and φ are slowly varying functions of the time period $T = \frac{2\pi}{\omega}$. It is noted that these first order equations are now written in terms of the amplitude a and phase φ as dependent variables. Therefore, the right sides of equations (2.14) and (2.15) show that both $\dot{a}$ and $\dot{\varphi}$ are periodic functions of period T. In this case, the right-hand terms of these equations contain a small parameter ε and also contain both a and φ, which are slowly varying functions of the time t with period $T = \frac{2\pi}{\omega}$. We can transform the equations (2.14) and (2.15) into more convenient form. Now, expanding $\sin\psi\, f(a\cos\psi, -a\omega\sin\psi)$ and $\cos\psi\, f(a\cos\psi, -a\omega\sin\psi)$ in Fourier series with phase ψ, the first approximate solution of equation (2.5) by averaging equations (2.14) and (2.15) with period $T = \frac{2\pi}{\omega}$, is

$$\langle \dot{a} \rangle = -\frac{\varepsilon}{2\pi\,\omega}\int_0^{2\pi} \sin\psi\, f(a\cos\psi, -a\omega\,\sin\psi)\,d\psi,$$
$$\langle \dot{\varphi} \rangle = -\frac{\varepsilon}{2\pi\,\omega\,a}\int_0^{2\pi} \cos\psi\, f(a\cos\psi, -a\,\omega\,\sin\psi)\,d\psi, \qquad (2.16)$$

where a and φ are independent of time t under the integrals. KB [2] have called their method asymptotic in the sense that $\varepsilon \to 0$. An asymptotic series itself is not convergent, but for a fixed number of terms the approximate solution tends to the exact solution as $\varepsilon \to 0$. Later, this technique has been extended mathematically by Bogoliubov and Mitropolskii [3], and has extended to non-stationary vibrations by Mitropolskii [4]. They have assumed the solution of equation (2.5) in the following form

$$x = a\cos\psi + \varepsilon\, u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \ldots\ldots\ldots\ldots + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}), \qquad (2.17)$$

where u_k, $(k = 1, 2, \ldots\ldots., n)$ are periodic functions of ψ with a period 2π, and the terms a and ψ are functions of time t and the following set of first order ordinary differential equations are satisfied by a and ψ

$$\dot{a} = \varepsilon\, A_1(a) + \varepsilon^2 A_2(a) + \ldots\ldots\ldots\ldots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}),$$
$$\dot{\psi} = \omega + \varepsilon\, B_1(a) + \varepsilon^2 B_2(a) + \ldots\ldots. + \varepsilon^n B_n(a) + O(\varepsilon^{n+1}). \qquad (2.18\text{ a, b})$$

The functions u_k, A_k and B_k, $(k = 1, 2, \ldots\ldots., n)$ are to be chosen in such a way that the equation (2.17), after replacing a and ψ by the functions defined in equation (2.18), is a solution of equation (2.5). Since there are no restrictions in choosing functions A_k and B_k, it generates the arbitrariness in the definitions of the functions u_k (Bogoliubov and Mitropolskii [3]). To remove this arbitrariness, the following additional conditions are imposed

$$\int_0^{2\pi} u_k(a,\psi)\cos\psi\, d\psi = 0,$$
$$\int_0^{2\pi} u_k(a,\psi)\sin\psi\, d\psi = 0. \qquad (2.19\text{a, b})$$

Secular terms are removed by using these conditions in all successive approximations. Differentiating equation (2.17) two times with respect to t, substituting the values of, $\ddot{x}, \dot{x}$ and x into equation (2.5), and using the relations equation (2.18) and equating the coefficients of ε^k, $(k = 1, 2, \ldots\ldots., n)$, leads to

$$\omega^2((u_k)_{\psi\psi} + u_k) = f^{(k-1)}(a,\psi) + 2\omega(a\, B_k \cos\psi + A_k \sin\psi), \qquad (2.20)$$

where $(u_k)_\psi$ denotes partial derivatives with respect to ψ,

$$
\begin{aligned}
&f^{(0)}(a,\psi) = f(a\cos\psi, -a\omega\sin\psi),\\
&f^{(1)}(a,\psi) = u_1 f_x(a\cos\psi, -a\omega\sin\psi) + (A_1\cos\psi - aB_1\sin\psi + \omega(u_1)_\psi)\times\\
&f_x(\cos\psi, -a\omega\sin\psi) + (aB_1^2 - A_1\frac{dA_1}{da})\cos\psi + (2A_1B_1 - aA_1\frac{dB_1}{da})\sin\psi -\\
&2\omega(A_1(u_1)_{a\psi} + B_1(u_1)_{\psi\psi}).
\end{aligned}
\qquad (2.21a,\ b)
$$

Here $f^{(k-1)}$ is a periodic function of ψ with period 2π which depends also on the amplitude a. Therefore, $f^{(k-1)}$ and u_k can be expanded in a Fourier series as

$$
\begin{aligned}
&f^{(k-1)}(a,\psi) = g_0^{(k-1)}(a) + \sum_{n=1}^{\alpha}(g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi),\\
&u_k(a,\psi) = v_0^{(k-1)}(a) + \sum_{n=1}^{\alpha}(v_n^{(k-1)}(a)\cos n\psi + \omega_n^{(k-1)}(a)\sin n\psi),
\end{aligned}
\qquad (2.22a,\ b)
$$

where

$$
g_0^{(k-1)}(a) = \frac{1}{2\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\,d\psi. \qquad (2.23)
$$

Here, $v_1^{(k-1)} = \omega_1^{(k-1)} = 0$ for all values of k, since both integrals of equation (2.19) are vanished. Substituting these values into the equation (2.20), we obtain

$$
\begin{aligned}
&\omega^2 v_0^{(k-1)}(a) + \sum_{n=2}^{\alpha}\omega^2(1-n^2)[v_n^{(k-1)}(a)\cos n\psi + \omega_n^{(k-1)}(a)\sin n\psi]\\
&= g_0^{(k-1)}(a) + (g_1^{(k-1)}(a) + 2\omega a B_k)\cos n\psi + (h_1^{(k-1)}(a) + 2\omega A_k)\sin\psi\\
&+ \sum_{n=2}^{\alpha}[g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi].
\end{aligned}
\qquad (2.24)
$$

Now, equating the coefficients of the harmonics of the same order, yield

$$
\begin{aligned}
&g_1^{(k-1)}(a) + 2\omega a B_k = 0, \quad h_1^{(k-1)}(a) + 2\omega A_k = 0, \quad v_0^{(k-1)}(a) = \frac{g_0^{(k-1)}(a)}{\omega^2},\\
&v_n^{(k-1)}(a) = \frac{g_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad \omega_n^{(k-1)}(a) = \frac{h_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad n \ge 1.
\end{aligned}
\qquad (2.25)
$$

These are the sufficient conditions to obtain the desired order of approximation. For the first order approximation, we have

$$
\begin{aligned}
&A_1 = -\frac{h_1^{(0)}(a)}{2\omega} = -\frac{1}{2\pi\omega}\int_0^{2\pi} f(a\cos t\psi, -a\omega\sin\psi)\sin\psi\,d\psi,\\
&B_1 = -\frac{g_1^{(0)}(a)}{2a\omega} = -\frac{1}{2\pi a\omega}\int_0^{2\pi} f(a\cos t\psi, -a\omega\sin\psi)\cos\psi\,d\psi.
\end{aligned}
\qquad (2.26a,\ b)
$$

Thus, the variational equations in equation (2.18) become

$$\dot{a} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\, d\psi,$$

$$\dot{\psi} = \omega - \frac{\varepsilon}{2\pi\, a\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi\, d\psi. \qquad (2.27a, b)$$

It is seen that, the equation (2.27) are similar to the equation (2.16). Thus, the first approximate solution obtained by Bogoliubov and Mitropolskii [3] is identical to the original solution obtained by KB [2]. The correction term u_1 is obtained from equation (2.22) by using equation (2.25) as

$$u_1 = \frac{g_0^{(0)}(a)}{\omega^2} + \sum_{n=2}^{\alpha} \frac{g_n^{(0)}(a)\cos n\psi + h_n^{(0)}(a)\sin n\psi}{\omega^2(1-n^2)} \qquad (2.28)$$

The solution equation (2.17) together with u_1 is known as the first order improved solution in which a and ψ are obtained from equation (2.27). If the values of the functions A_1 and B_1 are substituted from equation (2.26) into the second relation of equation (2.21b), the function $f^{(1)}$ and in the similar way, the functions A_2, B_2 and u_2 can be found. Therefore, the determination of the second order approximation is completed. The KB [2] method is very similar to that of van der Pol [1] and related to it. van der Pol has applied the method of variation of constants to the basic solution $x = a\cos\omega t + b\sin\omega t$ of $\ddot{x} + \omega^2 x = 0$, on the other hand KB [2] has applied the same method to the basic solution $x = a\cos(\omega t + \varphi)$ of the same equation. Thus, in the KB [2] method the varied constants are a and φ, while in the van der Pol's method the constants are a and b. The method of KB [2] seems more interesting from the point of view of applications, since it deals directly with the amplitude and phase of the quasi-harmonic oscillation. The solution of the equation (2.4a) is based on recurrent relations and is given as the power series of the small parameter. Cap [16] has solved the equation (2.4b) by using elliptical functions in the sense of KB [2]. The KB [2] method has been extended by Popov [13] to damped nonlinear differential systems represented by the following equation

$$\ddot{x} + 2k\dot{x} + \omega^2 x = \varepsilon f(\dot{x}, x), \qquad (2.29)$$

where $2k\dot{x}$ is the linear damping force and $0 < k < \omega$. It is noteworthy that, because of the importance of the Popov's method in the physical systems, involving damping force, Mendelson [14] and Bojadziev [15] have retrieved Popov's [13] results. In case of damped nonlinear differential systems, the first equation of equation (2.18a) has been replaced by

$$\dot{a} = -k\,a + \varepsilon\, A_1(a) + \varepsilon^2 A_2(a) + ... + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}). \tag{2.18a}$$

Murty and Deekshatulu [16] have developed a simple analytical method to obtain the time response of second order nonlinear over damped systems with small nonlinearity represented by the equation (2.29), based on the KB [2] method of variation of parameters. In accordance to the KBM [2, 3] method, Murty *et al.* [17] have found a hyperbolic type asymptotic solution of an over damped system represented by the nonlinear differential equation (2.29), i.e., in the case $k > \omega$. They have used hyperbolic functions, $\cosh\varphi$ and $\sinh\varphi$ instead of their circular counterpart, which are used by KBM [2, 3], Popov [13] and Mendelson [14]. Murty [18] has presented a unified KBM method for solving the nonlinear systems represented by the equation (2.29), which cover the undamped, damped and over-damped cases. Bojadziev and Edwards [19] have investigated solutions of oscillatory and non-oscillatory systems represented by equation (2.29) when k and ω are slowly varying functions of time t. Initial conditions may be used arbitrarily for the case of oscillatory or damped oscillatory process. But, in case of non-oscillatory systems $\cosh\varphi$ or $\sinh\varphi$ should be used depending on the given set of initial conditions (Murty *et al.* [17], Murty [18], Bojadziev and Edwards [19]). Arya and Bojadziev [20, 21] have examined damped oscillatory systems and time dependent oscillating systems with slowly varying parameters and delay. Sattar [22] has developed an asymptotic method to solve a second order critically damped nonlinear system represented by equation (2.29). He has found the asymptotic solution of the equation (2.29) in the following form

$$x = a(1+\psi) + \varepsilon\, u_1(a,\psi) + ... + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}), \tag{2.30}$$

where a is defined by the equation (2.18a) and ψ is defined by

$$\dot{\psi} = 1 + \varepsilon C_1(a) + \varepsilon^2 C_2(a) + ... + \varepsilon^n C_n(a) + O(\varepsilon^{n+1}) \tag{2.18b}$$

Also Sattar [23] has extended the KBM asymptotic method for three dimensional over damped nonlinear systems

Osiniskii [24] has extended the KBM method to the following third order nonlinear differential equation

$$\dddot{x} + c_1\,\ddot{x} + c_2\,\dot{x} + c_3\,x = \varepsilon\, f(\ddot{x},\dot{x},x), \tag{2.31}$$

KBM method to over damped nonlinear differential systems. Also, Alam *et al.* [41] have extended the KBM method to certain non-oscillatory nonlinear systems with slowly varying coefficients. Alam and Sattar [42] have studied time dependent third order oscillating systems with damping based on the extended form of the KBM method. Alam [43] has presented perturbation theory based on the KBM method to find the approximate solutions of second order nonlinear systems with large damping. Later, Alam [44, 45] has extended the KBM method for solving *nth*, $(n \geq 2,3)$ order nonlinear differential systems. Alam [46] has also presented a unified KBM method, which is not the formal form of the original KBM method for solving *nth*, $(n \geq 2,3)$ order nonlinear systems. The solution contains some unusual variables, yet this solution is very important. Alam [47] has extended the KBM method presented in [38] to find the approximate solutions of critically damped nonlinear systems in presence of different damping forces by considering different sets of variational equations. Alam [48] has also extended the KBM method to a third order over damped system when two of the eigen values are almost equal (i.e., the system is near to the critically damped) and the rest is small. Alam [49] has presented an asymptotic method for certain third order non-oscillatory nonlinear system, which gives desired results when the damping force is near to the critically damping force. Alam [50] has developed a simple method to obtain the time response of second order over damped nonlinear systems under some special conditions. Alam [51] has investigated a unified KBM method for solving *nth* order nonlinear differential equation with varying coefficients. Alam and Hossain [52] have extended the method presented in [50] to obtain the time response of *nth* order $(n \geq 2)$, over damped systems. Alam and Sattar [53] have developed an asymptotic method for third order nonlinear systems with slowly varying coefficients. Nayfeh [54, 55] and Murdock [56] have developed perturbation methods and theory for obtaining the solutions of weakly nonlinear differential systems. Sachs *et al.* [57] have developed a simple ODE model of tumor growth and anti-angiogenic or radiation treatment.

The HPM was first proposed by the Chinese mathematician Ji Huan He [58]. The essential idea of this method is to introduce a homotopy parameter, say p, which varies from 0 to 1. At $p=0$, the system of equations usually has been reduced to a simplified form which normally admits a rather simple solution. As p gradually increases continuously toward 1, the system goes through a sequence of deformations, and the solution at each stage is close to that at the previous stage of the deformation.

Eventually at $p=1$ the system takes the original form of the equation and the final stage of the deformation give the desired solution.

He [58] has investigated a novel homotopy perturbation technique to find a periodic solution of a general nonlinear oscillator for conservative systems. He [58] has considered the following nonlinear differential equation in the form

$$A(u)-f(r)=0,\ r\in\Omega, \tag{2.34}$$

with the boundary conditions

$$B(u,\frac{\partial u}{\partial t})=0,\quad r\in\Gamma, \tag{2.35}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function, Γ is the boundary of the domain Ω. Then He [58] has written Eq. (2.34) in the following form

$$L(u)+N(u)-f(r)=0, \tag{2.36}$$

where L is linear part, while N is nonlinear part. He [58] has constructed a homotopy $v(r,p):\Omega\times[0,1]\rightarrow\Re$ which satisfies

$$\mathrm{H}(v,p)=(1-p)[L(v)-L(u_0)]+p[A(u)-f(r)]=0,\ p\in[0,1],\ r\in\Omega \tag{2.37a}$$

or

$$\mathrm{H}(v,p)=L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0, \tag{2.37b}$$

where $p\in[0,1]$ is an embedding parameter, u_0 is an initial approximation of equation (2.34), which satisfies the boundary conditions. Obviously, from equation (2.37), it becomes

$$\mathrm{H}(v,0)=L(v)-L(u_0)=0, \tag{2.38}$$

$$\mathrm{H}(v,1)=A(v)-f(r)=0. \tag{2.39}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. He [58] has assumed the solution of Eq. (2.37) as a power series of p in the following form

$$v=v_0+pv_1+p^2v_2+\cdots. \tag{2.40}$$

The approximate solution of Eq. (2.34) is given by setting $p=1$ in the form

$$u=v_0+v_1+v_2+\cdots. \tag{2.41}$$

The series (2.41) is convergent for most of the cases, and also the rate of convergence depends on how one choose $A(u)$.

He [59] has obtained the approximate solution of nonlinear differential equation with convolution product nonlinearities. He [60] has developed some new approaches to Duffing equation with strongly and high order nonlinearity without damping. Also, He [61] has presented a new interpretation of homotopy perturbation method. Uddin *et al.* [62] and Uddin and Sattar [63, 64] have presented an approximate technique for solving second order strongly nonlinear differential systems with damping by combing the He's [58-61] homotopy perturbation and the extended form of the KBM [2-4] methods. Uddin *et al.* [65] have also developed an analytical approximate technique for solving a certain type of fourth order strongly nonlinear oscillatory differential system with small damping and cubic nonlinearity by combining He's homotopy perturbation [58-61] and the extended form of the KBM [2-4] methods. Recently, Uddin *et al.* [66] have also developed an approximate analytical technique for solving second order strongly nonlinear generalized Duffing equation with small damping. Belendez *et al.* [67] have applied He's homotopy perturbation method to Duffing harmonic oscillator. He [68] have presented a variational iteration method for solving nonlinear differential systems. Ghadimi and Kaliji [69] have presented an application of the harmonic balance method on nonlinear equation. Ganji *et al.* [70] have presented an approximate solutions to van der Pol damped nonlinear oscillators by means of He's energy balance method.

CHAPTER III

An Approximate Technique for Solving Second Order Strongly Nonlinear Differential Systems with High Order Nonlinearity in Presence of Small Damping

3.1. Introduction

Most of the phenomena in the real world are essentially nonlinear and described by nonlinear differential systems. So, the study of nonlinear differential systems is very important in all areas of applied mathematics, physics, engineering, medical science, economics and other disciplines. In general, it is too much difficult to handle nonlinear problems and it is often very difficult to get an analytical solution than a numerical one.

Common methods for constructing approximate analytical solutions to the nonlinear differential equations are the perturbation techniques. Some well known perturbation techniques are the Krylov-Bogoliubov-Mitropolskii (KBM)[2, 3] method, the Lindstedt-Poincare (LP) method [55, 56], and the method of multiple time scales[55]. Almost all perturbation methods are based on an assumption that small parameters must exist in the equations, which is too strict to find wide application of the classical perturbation techniques. It determines not only the accuracy of the perturbation approximations, but also the validity of the perturbation methods itself. However, in science and engineering, there exist many nonlinear problems which do not contain any small parameter, especially those appear in nature with strongly and high order nonlinearity in presence of small damping. Therefore, many new techniques have been proposed to eliminate the "small parameter" assumption, such as the homotopy perturbation method (HPM) [58-61], variational iteration method [68], energy balance method [70], etc. Arya and Bojadziev [21] have presented time depended oscillating systems with small damping, slowly varying parameters and delay. Sachs et al. [57] have presented simple ODE models of tumor growth and anti-angiogenic or radiation treatment. He [59] has obtained the approximate solution of nonlinear differential equation with convolution product nonlinearities. In recent years, He [60] has developed some new approaches to Duffing equation with strongly and high order non-linearity without damping. He [61] has presented a new interpretation of homotopy perturbation method. He [68] has presented the variational iteration method for strongly nonlinear differential systems without damping. Uddin *et al.* [62-64] have presented an approximate technique for solving strongly cubic and quadratic nonlinear differential systems with damping effects. Recently Uddin *et al.* [66] have developed an approximate analytical technique for solving second order strongly nonlinear generalized Duffing equation with small damping. Belendez *et al.* [67] have presented the application of He's homotopy perturbation method to Duffing harmonic oscillator without damping. From our study, it has been seen that the most of the authors [58-61, 67] have studied non-linear differential systems with small nonlinearity and without considering damping effects. But most of the physical and engineering problems occur in nature in the form of nonlinear differential systems with small damping effects. In this thesis, we are interested to present a coupling technique for solving second order strongly nonlinear differential systems with high order nonlinearity in presence of small damping based on the He's homotopy perturbation and the extended form of the KBM methods but the classical perturbation techniques are unable to tackle this situation. The presented method transforms a difficult problem under simplification, into a simple problem which is easy to solve, especially with high order non-linearity. The advantage of the presented method is that the first approximate solutions show a good agreement with the corresponding numerical solutions.

3.2. The method

Let us consider the second order ordinary differential systems modeling with high order nonlinearity in presence of small damping in the following form:

$$\ddot{x} + 2k(\tau)\dot{x} + \upsilon^2 x = -\varepsilon_1 f(x,\dot{x}), \tag{3.1}$$

where the over dots denote differentiation with respect to time t, υ is a constant, ε_1 $(\varepsilon_1 = 1.0)$ is a parameter which is not necessarily small, $k \geq 0, 2k$, is the linear damping coefficient, $\tau = \varepsilon t$, is the slowly varying time, ε is a small positive parameter and the coefficients in equation (3.1) are varying slowly in that their time dervatives are proportional to ε, $f(x,\dot{x})$ is a given high order nonlinear function which satisfies the following condition:

$$f(-x,-\dot{x}) = -f(x,\dot{x}). \tag{3.2}$$

We are going to use the following transformation to change the dependent variable

$$x = y(t)e^{-kt}. \tag{3.3}$$

Now differentiating equation (3.3) twice with respect to time t and substituting the values of $\ddot{x}, \dot{x}$ and x into equation (3.1) and then simplifying we obtain

$$\ddot{y} + (\upsilon^2 - k^2)y = -\varepsilon_1 e^{kt} f(ye^{-kt}, (\dot{y} - ky)e^{-kt}). \quad (3.4)$$

According to the homotopy perturbation method [58-67] Eq. (3.4) can be re-written as

$$\ddot{y} + \omega^2 y = \lambda y - \varepsilon_1 e^{kt} f(ye^{-kt}, (\dot{y} - ky)e^{-kt}), \quad (3.5)$$

where

$$\omega^2 = \upsilon^2 - k^2 + \lambda. \quad (3.6)$$

Here ω is known as the angular frequency of the nonlinear differential systems and is a constant for undamped nonlinear oscillators. But for the damped nonlinear differential systems, ω is a time dependent function and it varies slowly with time t and λ is an unknown function which can be evaluated by eliminating the secular terms from particular solution. To handle this situation, we are interested to use the extended form of the KBM [2, 3] method by Mitropolskii [4]. According to this method, the solution of equation (3.5) can be chosen in the following form:

$$y = a\cos\varphi, \quad (3.7)$$

where a and φ vary slowly with time t. In physical problems, a and φ are known as the amplitude and phase variables respectively and they keep an important role to the nonlinear physical systems. The following first order differential equations are satisfied by amplitude a and phase variable φ:

$$\begin{aligned} \dot{a} &= \varepsilon A_1(a,\tau) + \varepsilon^2 A_2(a,\tau) + \cdots, \\ \dot{\varphi} &= \omega(\tau) + \varepsilon B_1(a,\tau) + \varepsilon^2 B_2(a,\tau) + \cdots, \end{aligned} \quad (3.8a, b)$$

where ε is a small positive parameter, A_j and B_j, $j = 1,2,3\cdots$ are unknown functions. Now differentiating equation (3.7) twice with respect to time t with the help of equation (3.8) and substituting the values of $\ddot{y}, \dot{y}, y$ into equation (3.5) and then equating the coefficients of $\sin\varphi$ and $\cos\varphi$, we obtain

$$A_1 = -a\omega'/(2\omega),\ B_1 = 0, \quad (3.9)$$

where prime denotes differentiation with respect to τ. Now inserting equation (3.7) into equation (3.3) and equation (3.9) into equation (3.8a, b), we obtain the following equations:

$$x = a e^{-kt} \cos\varphi, \tag{3.10}$$

$$\dot{a} = -\varepsilon \omega' a / 2\omega, \qquad \dot{\varphi} = \omega(\tau). \tag{3.11a, b}$$

First approximate solution of equation (3.1) is given by equation (3.10) with help of equation (3.11a, b) by the presented method. Usually the integration of equation (3.11) is performed by the well-known techniques of calculus [54-56], but sometimes they are calculated by a numerical procedure [39-66]. Thus, the first approximate solution of equation (3.1) is determined.

3.3. Example

As an example of the above procedure, let us consider the second strongly nonlinear differential systems with high ($9th$) order [59, 60] nonlinearity in presence of small damping as the following form:

$$\ddot{x} + 2k(\tau)\dot{x} + \upsilon^2 x = -\varepsilon_1 x^9, \tag{3.12}$$

where $f(x,\dot{x}) = x^9$. Now using the transformation equation (3.3) into equation (3.12) and then simplifying them, we obtained

$$\ddot{y} + (\upsilon^2 - k^2)\, y = -\varepsilon_1 y^9 e^{-8kt}. \tag{3.13}$$

According to the homotopy perturbation [58-66] technique, equation (3.13) can be written as

$$\ddot{y} + \omega^2 y = \lambda y - \varepsilon y^9 e^{-8kt}, \tag{3.14}$$

where ω is calculated from equation (3.6). According to the extended form of the KBM [2-4] method, the solution of equation (3.14) is obtained from equation (3.7).

From the trigonometric identity, we obtain

$$\cos^9\varphi = (\cos 9\varphi + 9\cos 7\varphi + 36\cos 5\varphi + 84\cos 3\varphi + 126\cos\varphi)/256. \tag{3.15}$$

For avoiding the secular terms in particular solution of equation (3.14), we need to impose that the coefficient of the $\cos\varphi$ term is zero. Setting this term to zero, we obtain,

$$\lambda a - \frac{126\varepsilon_1 a^9 e^{-8kt}}{256} = 0, \tag{3.16}$$

which leads to

$$\lambda = \frac{63\varepsilon_1 a^8 e^{-8kt}}{128}. \tag{3.17}$$

Putting the value of λ from equation (3.17) into equation (3.6), we obtain the following frequency equation:

$$\omega^2 = \upsilon^2 - k^2 + \frac{63\varepsilon_1 a^8 e^{-8kt}}{128}. \tag{3.18}$$

From equation (3.18), it is clear that the frequency of the damped nonlinear physical systems depends on both amplitude a and time t. When $t \to 0$ then equation (3.18) yields

$$\omega_0 = \omega(0) = \sqrt{\upsilon^2 - k^2 + \frac{63\varepsilon_1 a_0^8}{128}}, \tag{3.19}$$

where ω_0, a_0 are known as the initial frequency and amplitude of the nonlinear physical systems.

Integrating the first equation of equation (3.11 a), we get

$$a = a_0\sqrt{\frac{\omega_0}{\omega}}. \tag{3.20}$$

Now putting equation (3.20) into equation (3.18), we obtain a six degree polynomial in ω in the following form:

$$\omega^6 + p\omega^2 + r = 0, \tag{3.21}$$

where

$$p = k^2 - \upsilon^2, r = -\frac{63\varepsilon a_0^8 \omega_0^4 e^{-8kt}}{128}. \tag{3.22}$$

Finaly, the first order analytical approximate solution of equation (3.12) is obtained as follows:

$$x = a e^{-kt}\cos\varphi \tag{3.23}$$

$$a = a_0\sqrt{\frac{\omega_0}{\omega}}, \qquad \varphi = \varphi_0 + \int_0^t \omega(\tau)dt, \tag{3.24 a, b}$$

where ω_0 is obtained by equation (3.19) ; ω is calculated from equation (3.21) by using the well-known Newton–Raphson method and a and φ are given by equation (3.24 a, b).

3.4 Results and Discussions

In this thesis, we have extended He's homotopy perturbation method for solving the second order typical [59, 60] strongly nonlinear differential systems with high order nonlinearity in presence of small damping. It is almost impossible to solve the strongly nonlinear differential systems, especially with high order nonlinearity by the classical perturbation methods [2-4, 21, 54-57]. But the suggested method has been successfully applied to solve the second order strongly nonlinear differential systems with high ($9th$) order nonlinearity in presence small damping. The first order approximate solutions of equation (3.12) is computed with small damping and high order nonlinearity by equation (3.23) and the corresponding numerical solutions are obtained by using fourth order Runge-Kutta method. The variational equations of the amplitude and phase variables appeared in a set of first order differential equations. The integration of these variational equations is performed by the well-known techniques of calculus [54-56]. In the lack of analytical solutions, numerical procedure [21, 54-67] is applied to solve them. The amplitude and phase variables change slowly with time t. The behavior of amplitude and phase variables characterizes the oscillating processes and amplitude tends to zero in presence of small damping as $t \to \infty$. Presented technique can take full advantage of the classical perturbation method. It is also noticed that the presented method is also capable to handle the typical second order weakly $(\varepsilon_1 = 0.1)$ nonlinear differential systems with high order nonlinearity in presence of small damping. Comparison is made between the solutions obtained by the presented technique and those obtained by the numerical procedure in **Figs. 3.1-3.2** for both strongly $(\varepsilon_1 = 1.0)$ and weakly $(\varepsilon_1 = 0.1)$ nonlinear differential systems high order nonlinearity in presence of with small damping for small amplitude. In **Figs.3.1-3.2,** it is seen that the solutions obtained by the presented method show a good agreement with those solution obtained by the numerical procedure with several small damping effects.

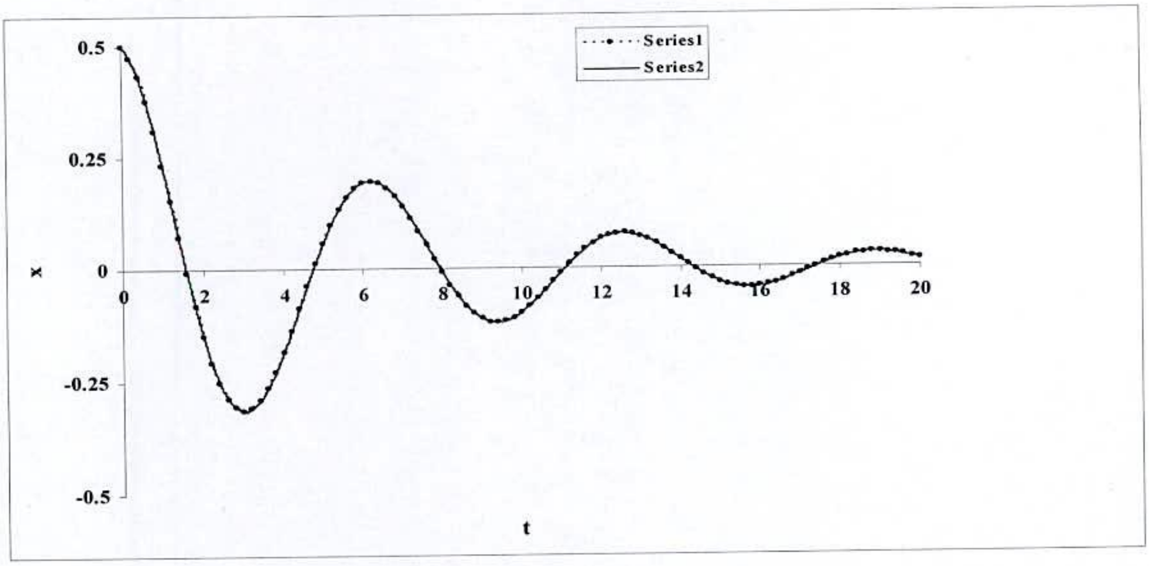

Fig.3.1(a). First approximate solution of equation (3.12) is denoted by dotted lines $(-\bullet-)$ by obtained the presented analytical technique with the initial conditions $a_0 = 0.5, \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.07853]$ with $\nu = 1.0, k = 0.15, \varepsilon_1 = 1.0,\ \varepsilon = 0.1$ and $f = x^9$ and the corresponding numerical solution is denoted by solid line $(-)$.

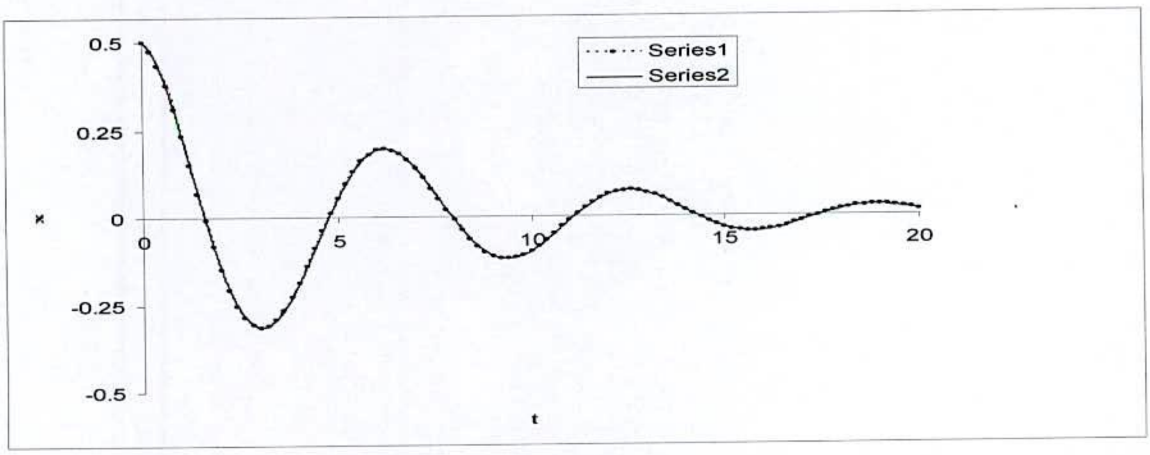

Fig.3.1(b). First approximate solution of equation (3.12) is denoted by dotted lines $(-\bullet-)$ obtained the presented analytical technique with the initial conditions $a_0 = 0.5, \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.07535]$ with $\nu = 1.0, k = 0.15, \varepsilon_1 = 0.1,\ \varepsilon = 0.1$ and $f = x^9$ and the corresponding numerical solution is denoted by solid line $(-)$.

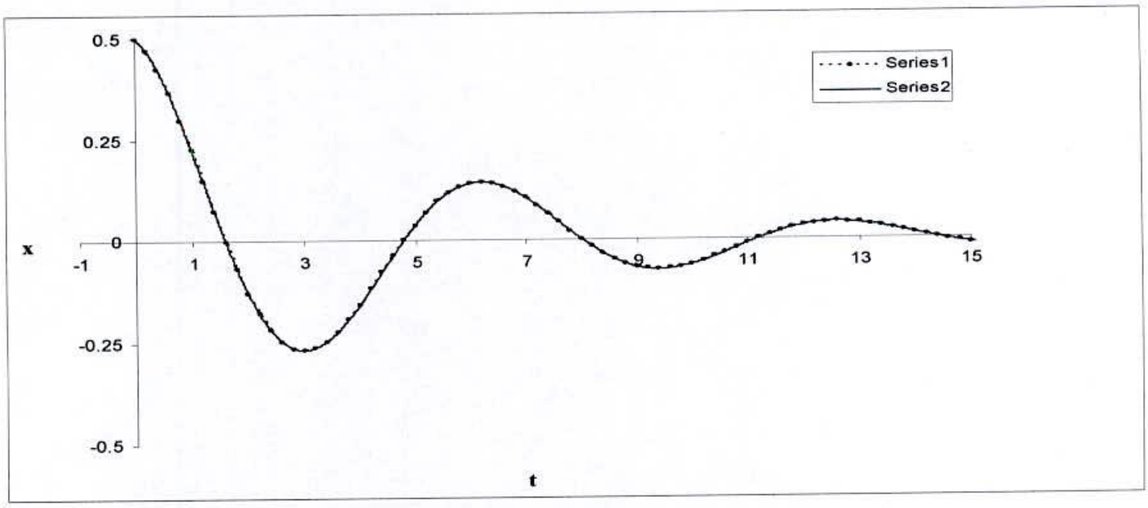

Fig.3.2(a). First approximate solution of equation (3.12) is denoted by dotted lines $(-\bullet-)$ obtained the presented analytical technique with the initial conditions $a_0 = 0.5, \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.10360]$ with $\nu = 1.0, k = 0.2, \varepsilon_1 = 1.0, \varepsilon = 0.1$ and $f = x^9$ and the corresponding numerical solution is denoted by solid line $(-)$.

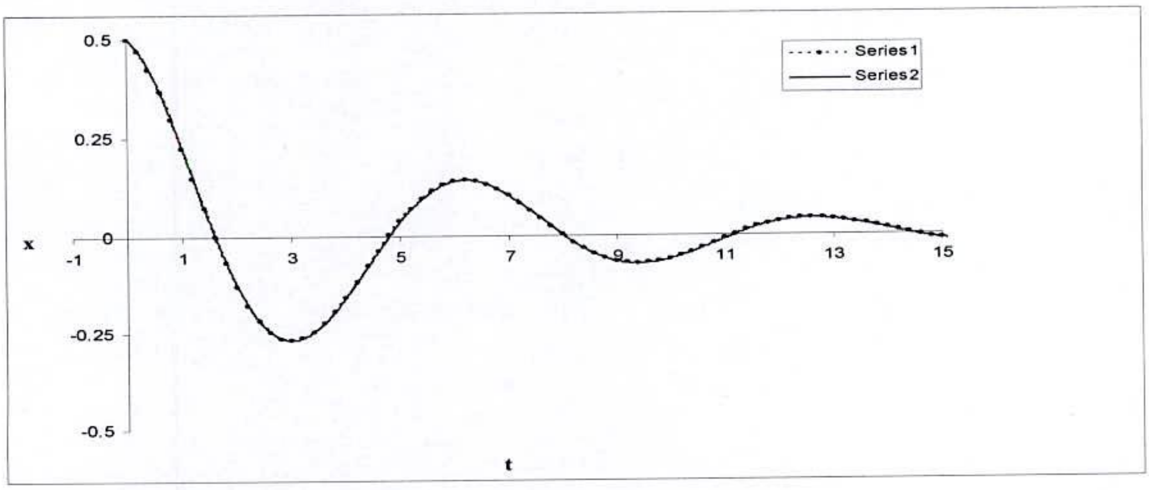

Fig.3.2(b). First approximate solution of equation (3.12) is denoted by dotted lines $(-\bullet-)$ obtained the presented analytical technique with the initial conditions $a_0 = 0.5, \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.10036]$ with $\nu = 1.0, k = 0.2, \varepsilon_1 = 0.1, \varepsilon = 0.1$ and $f = x^9$ and the corresponding numerical solution is denoted by solid line $(-)$.

CHAPTER IV

Conclusions

The determination of amplitude and phase variables is important for both strongly and weakly nonlinear differential systems in presence of small damping and they play very important role for physical problems. The advantage of the presented approximate technique is that it is able to give the position of the physical objects at any time as well as amplitudes and phases. The amplitude and phase variables characterize the oscillatory processes. In presence of damping, amplitude $a \to 0$ as $t \to \infty$ (i.e., for large time t).

It is also mentioned that, the classical KBM method is failed to tackle the second order strongly and weakly nonlinear differential systems with high order nonlinearity and in presence of damping and He's homotopy perturbation method is failed to handle nonlinear systems in presence of damping. Some limitations of He's homotopy perturbation (without damping) technique and the KBM method (weak nonlinearity) have been overcome by the presented method.

The presented method does not require a small parameter in the equation like the classical one. The method has been successfully implemented to illustrate the effectiveness and convenience of the suggested procedure and shown that the first approximate solutions show a good agreement with those solutions obtained by the numerical procedure with high order nonlinearity in presence of several small damping for both strongly $(\varepsilon_1 = 1.0)$ and weakly $(\varepsilon_1 = 0.1)$ nonlinear physical systems. The graphical representations show good agreement (**Figs. 3.1- 3.2**) between the first approximate analytical solutions and the corresponding numerical solutions for second order strongly and weakly nonlinear differential systems with high order nonlinearity.

REFERENCES

[1] van der Pol, B., 1926, "On Relaxation Oscillations", Philosophical Magazine,7-th series, Vol. 2.

[2] Krylov, N. N. and Bogoliubov, N. N., 1947, "Introduction to Nonlinear Mechanics", Princeton University Press, New Jersey.

[3] Bogoliubov, N. N. and Mitropolskii, Yu. A., 1961, "Asymptotic Methods in the Theory of Nonlinear Oscillation", Gordan and Breach, New York.

[4] Mitropolskii, Yu. A., 1964, "Problems on Asymptotic Methods of Non-stationary Oscillations (in Russian)", Izdat, Nauka, Moscow.

[5] Poincare, H., 1892, "Les Methods Nouvelles de la Mecanique Celeste", Paris.

[6] Stoker, J. J., 1950, "Nonlinear Vibrations in Mechanical and Electrical Systems", Interscience, New York.

[7] McLachlan, N. W., 1950, "Ordinary Nonlinear differential Equations in Engineering and Physical Science", Clarendon Press, Oxford.

[8] Minorsky, N., 1962, "Nonlinear oscillations", Van Nostrand, Princeton, New Jersey, pp. 375.

[9] Bellman, R., 1966, "Perturbation Techniques in Mathematics, Physics and Engineering", Holt, Rinehart and Winston, New York.

[10] Duffing, G., 1918, "Erzwungene Schwingungen bei Veranderlicher Eigen Frequenz und Ihre Technische Bedeutung", Ph. D. Thesis (Sammlung Vieweg, Braunchweig).

[11] Kruskal, M., 1962, "Asymptotic Theory of Hamiltonian and Other Systems with all Situations Nearly Periodic", Journal of Mathematical Physics, Vol. 3, pp. 806-828.

[12] Cap, F. F., 1974, "Averaging Method for the Solution of Nonlinear Differential Equations with Periodic Non-harmonic Solutions", International Journal of Nonlinear Mechanics, Vol. 9, pp. 441-450.

[13] Popov, I. P., 1956, "A Generalization of the Bogoliubov Asymptotic Method in the Theory of Nonlinear Oscillations (in Russian)", Doklady Akademii USSR, Vol. 3, pp. 308-310.

[14] Mendelson, K. S., 1970, "Perturbation Theory for Damped Nonlinear Oscillations", Journal of Mathematical Physics, Vol. 2, pp. 413-415.

[15] Bojadziev, G. N., 1980, "Damped Oscillating Processes in Biological and Biochemical Systems", Bulletin of Mathematical Biology, Vol. 42, pp. 701-717.

[16] Murty, I. S. N., Deekshatulu, B. L., 1969, "Method of Variation of Parameters for Over-damped Nonlinear Systems", International Journal of Control, Vol. 9, no. 3, pp. 259-266.

[17] Murty, I. S. N., Deekshatulu, B. L. and Krishna, G., 1969, "On an Asymptotic Method of Krylov-Bogoliubov for Over-damped Nonlinear Systems", Journal of the Franklin Institute, Vol. 288, pp. 49-65.

[18] Murty, I. S. N., 1971, "A Unified Krylov-Bogoliubov method for Solving Second Order Nonlinear Systems", International Journal of Nonlinear Mechanics, Vol. **6**, pp. 45-53.

[19] Bojadziev, G. N. and Edwards J., 1981, "On Some Method for Non-oscillatory and Oscillatory Processes", Journal of Nonlinear Vibration Problems., Vol. 20, pp. 69-79.

[20] Arya, J. C. and Bojadziev, G. N., 1980, "Damped Oscillating Systems Modeled by Hyperbolic Differential Equations with Slowly Varying Coefficients", Acta Mechanica, Vol. 35, pp. 215-221.

[21] Arya, J. C. and Bojadziev, G. N., 1981, "Time Dependent Oscillating Systems with Damping, Slowly Varying Parameters and Delay", Acta Mechanica, Vol. 41, pp. 109-119.

[22] Sattar, M. A., 1986, "An Asymptotic Method for Second Order Critically Damped Nonlinear Equations", Journal of the Franklin Institute, Vol. 321, pp. 109-113.

[23] Sattar, M. A., 1993, "An Asymptotic Method for Three Dimensional Over-damped Nonlinear Systems", Ganit: Journal of Bangladesh Mathematical Society, Vol. 13, pp. 1-8.

[24] Osiniskii, Z., 1962, "Longitudinal, Torsional and Bending Vibrations of a Uniform Bar with Nonlinear Internal Friction and Relaxation", Journal of Nonlinear Vibration Problems, Vol. 4, pp. 159-166.

[25] Osiniskii, Z., 1963, "Vibration of a One Degree Freedom System with Nonlinear Internal Friction and Relaxation", Proceedings of International Symposium of Nonlinear Vibrations, Kiev, Izadt, Akad, Nauk USSR. Vol. 111, pp. 314-325.

[26] Lin, J. and Khan, P. B., 1974, "Averaging Methods in Prey-Pridictor Systems and Related Biological models", Journal of Theoretical Biology, Vol. 57, pp. 73-102.

[27] Proskurjakov, A. P., 1964, "Comparison of the Periodic Solutions of Quassi-linear Systems Constructed by the Method of Poincare and Krylov-Bogoliubov (in Russian)", Journal of Applied Mathematics and Mechanics, Vol. 28, pp-765-770.

[28] Bojadziev, G. N., 1972, "On Asymptotic Solutions of Nonlinear Differential Equations with Time Lag, Delay and Functional Differential Equations and Their Applications (edited by K. Schmit)", New York and London Academic Press, pp. 299-307.

[29] Bojadziev, G. N., Lardner, R. W. and Arya, J. C., 1973, "On the Periodic Solutions of Differential Equations Obtained by the Method of Poincare and Krylov-Bogoliubov", Journal of Utilitas Mathematica, Vol. 3, pp. 49-64.

[30] Bojadziev, G. N. and Lardner, R. W., 1973, "Monofrequent Oscillations in Mechanical Systems Governed by Hyperbolic Differential Equation with Small Nonlinearities", International Journal of Nonlinear Mechanics, Vol. 8, pp. 289-302.

[31] Bojadziev, G. N. and Lardner, R. W., 1974, "Second Order Hyperbolic Equations with Small Nonlinearities in the Case of Internal Resonance", International Journal of Nonlinear Mechanics, Vol. 9, pp. 397-407.

[32] Bojadziev, G. N. and Lardner, R. W., 1974, "Asymptotic Solution of a Nonlinear Second Order Hyperbolic Differential Equation with Large Time Delay", Journal of Institute of Mathematics and its Applications, Vol. 14, pp. 203-210.

[33] Bojadziev, G. N., 1976, "Damped Forced Nonlinear Vibrations of Systems with Delay", Journal of Sound and Vibrations, Vol. 46, pp. 113-120.

[34] Bojadziev, G. N., 1978, "The Krylov-Bogoliubov-Mitropolskii Method Applied to Models of Population Dynamics", Bulletin of Mathematical Biology, Vol. 40, pp. 335-345.

[35] Mulholland, R. J., 1971, "Nonlinear Oscillations of Third Order Differential Equation", International Journal of Nonlinear Mechanics, Vol. 6, pp. 279-294.

[36] Lardner, R. W. and Bojadziev, G. N., 1979, "Asymptotic Solutions for Third Order Partial Differential Equations with Small Nonlinearities", Meccanica, Vol. 14, pp. 249-256.

[37] Raymond, P. Vitro and Cabak, G., 1979, "The Effects of Internal Resonance on Impulsively Forced Nonlinear Systems with Two Degree of Freedom", International Journal of Nonlinear Mechanics, Vol. 14, pp. 93-99.

[38] Alam, M. S. and Sattar, M. A., 1996, "An Asymptotic Method for Third Order Critically Damped Nonlinear Equations", Journal of Mathematical and Physical Sciences, Vol. 30, pp. 291-298.

[39] Alam, M. S. and Sattar, M. A., 1997, "A Unified Krylov-Bogoliubov-Mitropolskii Method for Solving Third Order Nonlinear Systems", Indian Journal of Pure And Applied Mathematics, Vol. 28, pp. 151-167.

[40] Alam, M. S., 2001, "Asymptotic Methods for Second Order Over-Damped and Critically Damped Nonlinear Systems", Soochow Journal of Mathematics, Vol. 27, pp. 187-200.

[41] Alam, M. S., Alam, M. F. and Shanta, S. S., 2001, "Approximate Solution of Non-Oscillatory Systems with Slowly Varying Coefficients", Ganit: Journal of Bangladesh Mathematical Society, Vol. 21, pp. 55-59.

[42] Alam, M. S. and Sattar, M. A., 2001, "Time Dependent Third-order Oscillating Systems with Damping", Acta Ciencia Indica, Vol. 27, pp. 463-466.

[43] Alam, M. S., 2001, "Perturbation Theory for Nonlinear Systems with Large Damping", Indian Journal of Pure and Applied Mathematics, Vol. 32, pp. 453-461.

[44] Alam, M. S., 2002, "Perturbation Theory for nth Order Nonlinear Systems with Large Damping", Indian Journal of Pure and Applied Mathematics, Vol. 33, pp. 1677-1684.

[45] Alam, M. S., 2002, "Method of Solution to the *nth* Order Over-damped Nonlinear Systems Under Some Special Conditions", Bulletin of Calcutta Mathematical Society, Vol. 94, pp. 437-440.

[46] Alam, M. S., 2002, "A Unified Krylov-Bogoliubov-Mitropolskii Method for Solving nth Order Nonlinear Systems", Journal of the Franklin Institute, Vol. 339, pp. 239-248.

[47] Alam, M. S., 2002, "Bogoliubov's Method for Third Order Critically Damped Nonlinear Systems", Soochow Journal of mathematics, Vol. 28, pp. 65-80.

[48] Alam, M. S., 2002, "Asymptotic Method for Non-oscillatory Nonlinear Systems", Far East Journal of Applied Mathematics, Vol. 7, pp. 119-128.

[49] Alam, M. S., 2003, "Asymptotic Method for Certain Third Order Non-oscillatory Nonlinear Systems", Journal of Bangladesh Academy of Sciences, Vol. 27, pp. 141-148.

[50] Alam, M. S., 2003, "On Some Special Conditions of Over-damped Nonlinear Systems", Soochow Journal of mathematics, Vol. 29, pp. 181-190.

[51] Alam, M. S., 2003, "A Unified KBM Method for Solving *nth* Order Nonlinear Differential Equation with Varying Coefficients", Journal of Sound and Vibration, Vol. 265, pp. 987-1002.

[52] Alam, M. S. and Hossain, M. B., 2003 "On Some Special Conditions of *nth* Order Non-oscillatory Nonlinear Systems", Communication of Korean Mathematical Society, Vol. 18, pp. 755-765.

[53] Alam, M. S. and Sattar, M. A., 2004, "Asymptotic Method for Third Order Nonlinear Systems with Slowly Varying Coefficients", Journal of Southeast Asian Bulletin of Mathematics, Vol. 28, pp. 979-987.

[54] Nayfeh, A. H., 1973, "Perturbation Methods", John Wiley and Sons, New York.

[55] Nayfeh, A. H., 1981, "Introduction to Perturbation Techniques", John Wiley and Sons, New York.

[56] Murdock, J. A., 1991, "Perturbations: Theory and Methods", Wiley, New York.

[57] Sachs, R. K., Hlatky, L. R. and Hahnfeldt, P., 2001, "Simple ODE Models of Tumor Growth and Anti-angiogenic or Radiation Treatment", Journal of Mathematical and Computer Modeling, Vol. 33, pp. 1297-1305.

[58] He, Ji-Huan, Homotopy Perturbation Technique, J. Computer Methods in Applied Mechanics and Engineering, Vol. 178, pp. 257-262, 1999.

[59] He, J. H., 1998, "Approximate Solution of Nonlinear Differential Equations with Convolution Product Nonlinearities", Journal of Computer Methods in Applied Mechanics and Engineering, Vol. 167, pp. 69-73.

[60] He, J. H., 1999, "Some New Approaches to Duffing Equation with Strongly and High Order Nonlinearity (I) Linearized Perturbation Method", Journal of

Communications in Nonlinear Science & Numerical Simulation, Vol. 4, No. 1, pp. 78-80.

[61] He, J. H., 2006, "New Interpretation of Homotopy Perturbation Method", International Journal of Modern Physics B, Vol.20, No. 18, pp. 2561-2568.

[62] Uddin, M. A., Sattar, M. A. and Alam, M. S., 2011, "An Approximate Technique for Solving Strongly Nonlinear Differential Systems with Damping Effects", Indian Journal of Mathematics, Vol. 53, No.1, pp. 83-98.

[63] Uddin, M. A. and Sattar, M. A., 2011, "An Approximate Technique For Solving Strongly Nonlinear Biological Systems with Small Damping Effects", Journal of the Calcutta Mathematical Society, Vol. 7, No.1, pp. 51-62.

[64] Uddin, M. A. and Sattar, M. A., 2011, "An Approximate Technique to Duffing Equation with Small Damping and Slowly Varying Coefficients", Journal of Mechanics of Continua and Mathematical Sciences, Vol. 5, No. 2, pp. 627-642.

[65] Uddin, M. A., Alom, M. A. and Ullah, M. W., 2012, "An Analytical Approximate Technique for Solving a Certain Type of Fourth Order Strongly Nonlinear Oscillatory Differential System with Small Damping", Far East Journal of Mathematical Sciences, Vol. 67, No.1, pp.59-72.

[66] Uddin, M. A., Ullah, M. W. and Bipasha, R. S., 2015, "An Approximate Analytical Technique for Solving Second Order Strongly Nonlinear Generalized Duffing Equation with Small Damping", Journal of Bangladesh Academy of Sciences, Vol. 39, No. 1, pp. 103-114.

[67] Belendez, A., Hernandez, A., Belendez, T., Fernandez, E., Alvarez, M. L. and Neipp, C., 2007, "Application of He's Homotopy Perturbation Method to Duffing Harmonic Oscillator", International Journal of Nonlinear Science and Numerical Simulation, Vol. 8, No. 1, pp. 78-88.

[68] He, J.H., 1999, "Variational Iteration Method: a Kind of Nonlinear Analytical Technique: Some Examples", International Journal of Nonlinear Mechanics, Vol.34, No. 4, pp. 699-704.

[69] Ghadimi, M. and Kaliji, H.D., 2013, "Application of the Harmonic Balance Method on Nonlinear Equation", World Applied Sciences Journal, Vol. 22, No. 4, pp.532-537.

[70] Ganji, D.D., Esmaeilpour, M. and Soleimani, S., 2009, "Approximate Solutions to van der Pol Damped Nonlinear Oscillators by Means of He's Energy Balance Method", International Journal of Computer Mathematics, Vol.87, No. 9, pp.2014-2023