# Approximate Solution of the Quadratic Nonlinear Oscillator by Iteration Procedure

By

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**Dedicated to My Parents**

**Rokeya Begum & Late Md. Nizamuddin**

and

**Younger Uncle Md. Hafizur Rahman.**

Who have desired underprivileged life to continue my smile.

## Declaration

This is to certify that the thesis work entitled "Approximate Solution of Quadratic Nonlinear Oscillator by Iteration Procedure" has been carried out by Md. Rakib Hossain, in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part work of this work has not been submitted anywhere for the award of any degree or diploma.

Signature of Supervisor

Signature of Student

## Approval

This is to certify that the thesis work submitted by Md. Rakib Hossain entitled "Approximate Solution of the Quadratic Nonlinear Oscillator by Iteration Procedure" has been approved by the board of examiners for the partial fulfillment of the requirements for the degree of Master of Science in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh in December 2015.

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## Abstract

An analytical technique has been developed based on an iteration method to determine higher-order approximate periodic solutions for a nonlinear oscillator with discontinuity for which the elastic force term anti-symmetric and quadratic. Usually, a set of nonlinear algebraic equations is solved with this method. However, analytical solutions of these algebraic equations are not always possible, especially in the case of large oscillations. A modified approximate analytic solution of the quadratic nonlinear oscillator "$\ddot{x}+x^2\operatorname{sgn}(x)=0$" has been obtained based on an iteration procedure. Here we have used the truncated Fourier series in each iterative step. The approximate frequencies obtained by the technique shows a good agreement with the exact frequency and periodic solutions with the exact ones. The percentage of error between exact frequency and fourth approximate frequency of the method is as low as 0.00003%.The method is mainly illustrated by the quadratic nonlinear oscillator but it is also useful for many other nonlinear problems.

## Publication

The following paper has been extracted from this thesis:

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# CHAPTER 1

## Introduction

Nonlinear oscillation is a topic to intensive research for many years in the field of physics, mathematics and engineering. A large variety of approximate techniques have been developed to determine periodic solutions of nonlinear oscillatory systems. Dynamical systems are mathematical objects used to model physical phenomena whose state (or instantaneous description) changes over time. These models are used in financial and economic forecasting, environmental modeling, medical diagnosis, industrial equipment diagnosis, and in many other applications. Systems of nonlinear equations arise in many fields of practical importance such as engineering, medical science, chemistry, and robotics. Nonlinear equations have also demonstrated their usefulness in ecology, business cycle and biology. A large number of problems in engineering and science can be formulated in the form of differential equations. Nonlinear systems are also important in the field of music, game etc. Such as, in music, all sound comes in waves which are a nonlinear equation. So it helps to know, how those waves interact with each other when recording music, or when at a live event and setting up the speakers. Generally speaking, dynamics is a concise term referring to the study of time-evolving processes, and the corresponding system of equations, which describes this evolution, is called a dynamical system. Therefore the solution of such problems lies essentially in solving the corresponding differential equations. The differential equations may be linear or nonlinear, autonomous or non-autonomous. The world around us is inherently nonlinear. A vast body of scientific knowledge has developed over a long period of time, devoted to a description of natural phenomena. Ultimately, many differential equations involving physical phenomena are nonlinear. Differential equations, which are linear, are comparatively easy to solve and nonlinear are laborious and in some cases it is impossible to solve them analytically. In many cases it is possible to replace such a nonlinear equation by a related linear equation, which approximates the actual non-linear equation closely enough to give useful results. The method of small oscillations is a well-known example of the linearization. But such a linearization is not always possible and when it is not possible, then the original nonlinear equation itself must be used. The study of nonlinear equations is generally confined to a variety of rather special cases, and one must resort to various method of approximation.

Van der Pol first paid attention to the new (self-excited) oscillations and indicated that their existence is inherent in the nonlinearity of the differential equations characterizing the process. This nonlinearity appears, thus, as the very essence of these phenomena and by linearizing the differential equation in the sense of the method of small oscillations, one simply eliminates the possibility of investigating such problems. Thus it is necessary to deal with the nonlinear problems directly instead of evading them by dropping the nonlinear terms. Many methods exist for constructing analytical approximations to the solution to the oscillatory system. Such as perturbation methods, harmonic balance method, iteration method etc. Among the methods, the method of perturbation, i. e., asymptotic expansions in terms of a small parameter are foremost.

Analytical solutions for nonlinear differential equations or linear differential equations with variable coefficients play an important role in the study of nonlinear dynamical systems, but sometimes it is difficult to find these solutions, especially for nonlinear problems with strong nonlinearity. Perturbation method is used for small nonlinear problems. One the other hand iteration method is used for small as well as large amplitude of oscillations.

The perturbation methods are, in principle, for solving problems with a small parameter. In this case, the solution is analytically expanded in power series of the parameter. However, there exist many nonlinear problems in which parameters are not small and other methods such as harmonic balance method are able to provide analytical approximations valid for large value of amplitudes.

Harmonic balance method is a technique for determining analytic approximations to the periodic solutions of differential equations by using a truncated Fourier series representation. An important advantage of this method is that it can be applied to nonlinear oscillatory problems where the nonlinear terms are not "small" i.e., no perturbation parameter need to exist. A disadvantage of harmonic balance method is that it is a priori difficult to predict whether a first order harmonic balance calculation will provide a sufficiently accurate approximation to periodic solution for a given nonlinear differential equation or not.

The iteration method [25] was proposed by Mickens in 1987. The method introduces a reliable and efficient process for wide variety of scientific and engineering application for the case of nonlinear systems. Two important advantages of iteration method are as follows:

i. Only linear, inhomogeneous differential equations are required to be solved at each level of the calculation.
ii. The coefficients of the higher harmonics, for a given values of the iteration index, decrease rapidly with increasing harmonic number. This implies that higher order solutions may not be required.

In computational mathematics, an iterative method is a mathematical procedure that generates a sequence of improving approximate solutions for a class of problems. A specific implementation of an iterative method, including the termination criteria, is an algorithm of the iterative method. An iterative method is called convergent if the corresponding sequence converges for given initial approximations. A mathematically rigorous convergence analysis of an iterative method is usually performed; however, heuristic-based iterative methods are also common.

In the problems of finding the root of an equation (or a solution of a system of equations), an iterative method uses an initial guess to generate successive approximations to a solution. In contrast, direct methods attempt to solve the problem by a finite sequence of operations. In the absence of rounding errors, direct methods would deliver an exact solution (like solving a linear system of equations $Ax = b$ by Gaussian elimination). Iterative methods are often the only choice for nonlinear equations. However, iterative methods are often useful even for linear problems involving a large number of variables (sometimes of the order of millions), where direct methods would be prohibitively expensive (and in some cases impossible) even with the best available computing power.

It is worthy to note that the majority of scientists have not been led to their discoveries by a process of deduction from general advances, or general principles, but rather by a thorough examination of properly chosen particular case. The generalizations have come later, because it is far easier to generalize an established result than to discover a new line of argument. Generalization is the temptation of a lot of researchers working now with nonlinear dynamical systems.

The important development of the theory of nonlinear dynamical systems, during these centuries, has essentially its origins in the studies if the "natural effects" encountered in these systems, and the rejection of non-essential generalizations, i.e. the study of concrete nonlinear systems have been possible due to the foundation of results from the theory or nonlinear dynamical system field.

In this thesis, a quadratic nonlinear oscillator, describing a dynamical system, has been considered. The main object of this thesis is to determine the new approximate frequencies and corresponding analytical solutions of 'Quadratic Nonlinear Oscillator' by iterative procedure and compare them to some existing results.

# CHAPTER 2

## Literature Review

Most phenomena in our world are essentially nonlinear and are described by nonlinear equations. Nonlinear processes are one of the biggest challenges and are not easy to control because the nonlinear characteristic of the system abruptly changes due to some small changes of valid parameters including time. Thus, the issue becomes more complicated and hence needs ultimate solution. Thus, the studies of approximate solutions of nonlinear differential equations play a vital role in understanding the internal mechanism of nonlinear phenomena. Advanced nonlinear techniques are significant to solve inherent nonlinear problems, particularly those involving dynamical systems and related areas. In recent years, both mathematicians and physicists have made significant improvement in finding a new mathematical tool related to nonlinear dynamical systems whose understanding will rely not only on analytic techniques but also on numerical and asymptotic methods. Many effective and powerful methods have been established in handling nonlinear dynamical systems. The study of nonlinear problems is of crucial importance not only in all areas of mathematics but also in engineering and other disciplines, since most phenomena in the world are essentially nonlinear and are described by nonlinear differential equations. It is not easy to solve nonlinear problems and in general it is often more difficult to get an analytic approximation than a numerical one for given nonlinear problems. In a lot of situations, linear differential equation is used for a nonlinear differential equation, which approximates the former equation close enough to give expected results. In many cases linearization is not possible and when it is not, the original differential equations must be tackled directly. There are many analytical techniques to solving nonlinear differential equations, such as: Perturbation method [1-8], Harmonic Balance (HB) method [9-19], Homotopy Perturbation method [20-23], Iterative method [24-34] etc.

The perturbation method is the most widely used method in which the nonlinear term is small. The perturbation method aims getting a periodic solution in the form of a power series with respect to small parameter $\varepsilon$. This method introduced by Poisson [1], was at first applied formally, without any theoretical justification. Nevertheless, it has been successfully used to obtain some effective solution especially in celestial mechanics. However, the main contribution of the perturbation method is due to Poincare [35] who elaborated in 1892 its theoretical grounds and made possible its systematic application to various nonlinear problems. The method of Lindstedt-Poincare (LP) [1, 36, 37], Homotopy method [38-40], and Differential Transform method [41-43] are most important among all perturbation methods.

The method of Lindstedt-Poincare [1, 30] is an introductory method to solve the following second order nonlinear differential equation

$$\ddot{x}+\omega_0^2 x+\varepsilon f(x,\dot{x})=0, \tag{2.1}$$

where $\omega_0$ is the unperturbed frequency and $\varepsilon$ is a small parameter.

The fundamental idea in Lindstedt's technique is based on the observation that the nonlinearities alter the frequency of the system from the linear one $\omega_0$ and $\omega(\varepsilon)$.To account for this change in frequency, he introduced a new variable $\tau=\omega t$ and expand $\omega$ and $x$ in powers of $\varepsilon$ as

$$\begin{aligned} x &= x_0(\tau)+\varepsilon x_1(\tau)+\varepsilon^2 x_2(\tau)+..., \\ \omega &= \omega_0+\varepsilon\omega_1+\varepsilon^2\omega_2+... \end{aligned} \tag{2.2}$$

where $\omega_i$, $i=0,1,2,...$ are unknown constants to be determined.

Substituting equation (2.2) into equation (2.1) and equating the coefficients of the various powers of $\varepsilon$, the following equations are obtained

$$\begin{aligned} &\ddot{x}_0+x_0=0 \\ &\ddot{x}_1+x_1=-2\omega_1\ddot{x}_0-f(x_0,\dot{x}_0) \\ &\ddot{x}_2+x_2=-2\omega_1\ddot{x}_1-(\omega_1^2+2\omega_2)\ddot{x}_0-f_x(x_0,\dot{x}_0)x_1+f_{\dot{x}}(x_0,\dot{x}_0)(\omega_1\dot{x}_0+\dot{x}_1) \\ &\qquad \cdots \quad \cdots \quad \cdots \\ &\ddot{x}_m+x_m=g_m(x_0,\, x_1,\, x_2,....,\, x_{m-1};\dot{x}_0,\, \dot{x}_1,...,\, \dot{x}_{m-1})\,. \end{aligned} \tag{2.3}$$

where over dots represent the differentiation with respect to $\tau$. Clearly equation (2.3) is a linear system and it is solved by the elementary techniques. This method is used only for finding the periodic solution, but the method cannot discuss transient cases. An important aspect of various perturbation methods is their relationship with each other. Among them, Krylov and Bogoliubov are certainly to be found most active. In most treatments of nonlinear oscillations by perturbation methods only periodic oscillations are treated, transients are not considered. Krylov and Bogoliubov [36] have introduced a new perturbation technique to discuss transients in the equation

$$\ddot{x}+\omega^2 x=\varepsilon f(x,\dot{x},t,\varepsilon). \qquad (2.4)$$

where over dots denote ordinary derivative with respect to time $t$, $\varepsilon$ is a small positive parameter, $f$ is a power series in $\varepsilon$, whose coefficients are polynomials in $x$, $\dot{x}$, $\sin t$ and $\cos t$. The method of Krylov and Bogoliubov start with the solution of the linear equation, assuming that in the nonlinear case, the amplitude and phase in solution of the linear equation are time dependent functions rather than constants. This procedure introduces an additional condition on the first derivative of the assumed solution for determining the desired results.

The method of Krylov and Bogoliubov is an asymptotic method. Generally an asymptotic series itself may not be convergent, but up to a fixed number of terms, the approximate solution tends to the exact solution because of $\varepsilon \to 0$. It is noted that the asymptotic term is frequently used in the field of oscillations also in the sense that $\varepsilon \to \infty$. But in this case the mathematical treatment is quite different. Some vital works are done and the elaborative uses have been made by Stoker [44], Minorsky [45], Nayfeh [1, 3, 6], Bellman [46]. Duffing [47] has investigated many significant results about the periodic solutions of the following nonlinear damped differential equation.

$$\ddot{x}+2k\,\dot{x}+\omega^2 x=-\varepsilon x^3. \qquad (2.5)$$

When the amplitude of the dependent variable of the dynamic system is less than or greater than unity then different types of nonlinear phenomena occur. The damping is negative when the amplitude is less then unity and the damping is positive when the amplitude is greater than unity. The governing nonlinear differential equation having these phenomena is

$$\ddot{x}-\varepsilon(1-x^2)\dot{x}+x=0. \qquad (2.6)$$

The equation (2.6) was introduced by Van-der-Pol [48] and in literature. The equation has very extensive field of application in connection with self-excited oscillations in electron-tube circuits.

Harmonic balance (HB) method is another technique for finding the periodic solutions of a nonlinear system. Harmonic balance method is originated by Mickens [24]. The method of harmonic balance provides a general technique for calculating approximations to the periodic solutions of differential equations. It corresponds to a truncated Fourier series and allows for the systematic determination of the coefficients to the various harmonics and the angular frequency. The mathematical foundations of harmonic balancing have been investigated by several researchers. However, in order to avoid solving an infinite system of algebraic equations, it is better to approximate the solution by a suitable finite sum of trigonometric function. This is the main task of harmonic balance method. Thus approximate solutions of an oscillator are obtained by harmonic balance method using a suitable truncation of Fourier series. The significance of the method is that, it may be applied to differential equations for which the nonlinear terms are not small.

A vast amount of literature on the method of HB is available. Selected lists of the articles that study and apply this method to a variety of differential equations are given in the references Mickens [9, 10, 50], Lim and Wu [11], Twari et al. [12], Gottlieb [13], Alam et al. [14], Leung and Zhongjin [17], Ghadimi and Kaliji [18], Wu et al. [49]. The formulation of the method of harmonic balance focuses primarily by Mickens [9] and further work has been done by Mickens [7, 27], Gottlieb [13], Hosen [19], Lim and Wu [26], Hu [51, 53], Hu and Tang [52], Wu et al. [59] and so on, for solving the strong nonlinear problems. However, it should be indicated that various generalizations of the method of harmonic balance have been made by several investigators such as, an intrinsic method of harmonic analysis by Huseyin and Lin [54], the use of Jacobi elliptic functions by Garcia et al. [55] and two time scale harmonic balance by Summers and Savage [56]. Recently combining the method of averaging and harmonic balance, Lim and Lai [57] presented an analytic technique to obtain first approximate perturbation solution; their solutions gives desired results for some non-conservative systems when the damping force is very small. Another technique is developed by Yamgoue and Kofane [58] to determine approximate solutions of nonlinear problems with strong damping effect; more than two harmonic terms are involved in their solution. Further work has been done by various researches. Some of them who handled strong nonlinearities are Wu et al. [49, 59], Belendez et al. [60], Gottlieb [13, 61] and Mickens [30].

The following methodology of direct harmonic balance method was given by Mickens [30].

He considered the equation for all Truly Nonlinear (TNL) oscillators as:

$$F(x,\dot{x},\ddot{x})=0, \tag{2.7}$$

where $F(x,\dot{x},\ddot{x})$ is of odd-parity, i.e.

$$F(-x,-\dot{x},-\ddot{x})=-F(x,\dot{x},\ddot{x}). \tag{2.8}$$

A major consequence of this property is that the corresponding Fourier expansions of the periodic solutions only contain odd harmonics (Mickens [67]), i. e.,

$$x(t)=\sum_{k=1}^{\infty}\{A_k\cos[(2k-1)\Omega t]+B_k\sin[(2k-1)\Omega t]\}. \tag{2.9}$$

The $N$-th order harmonic balance approximation to $x(t)$ is the expression

$$x_N(t)=\sum_{k=1}^{N}\{\overline{A}_k^N\cos[(2k-1)\overline{\Omega}_N t]+\overline{B}_k^N\sin[(2k-1)\overline{\Omega}_N t]\}, \tag{2.10}$$

where $\overline{A}_k^N,\overline{B}_k^N,\overline{\Omega}_N$ are approximations to $A_k,B_k,\Omega$ for $k=1, 2, 3, \ldots\ldots, N$.

For the case of a conservative oscillator, equation (2.7) generally takes the form

$$\ddot{x}+f(x,\lambda)=0, \tag{2.11}$$

where $\lambda$ denotes the various parameters appearing in $f(x,\lambda)$ and $f(-x,\lambda)=-f(x,\lambda)$.The following initial conditions are selected

$$x(0)=A,\quad \dot{x}(0)=0 \tag{2.12}$$

And this has the consequence that only the cosine terms are needed in the Fourier expansions, and therefore we have

$$x_N(t)=\sum_{k=1}^{N}\overline{A}_k^N\cos[(2k-1)\overline{\Omega}_N t] \tag{2.13}$$

Observe that $x_N(t)$ has $(N+1)$ unknowns, the $N$ coefficients, $(\overline{A}_1^N, \overline{A}_2^N, .......\overline{A}_N^N)$ and $\Omega_N$, the angular frequency. These quantities may be calculated by carrying out the following steps:

**Step-1:** Substitute equation (2.13) into equation (2.11) and expand the resulting form into an expression that has the following structure

$$\sum_{k=1}^{N} H_k \cos[(2k-1)\Omega_N t] + HOH \cong 0, \text{ HOH= Higher Order Harmonic} \tag{2.14}$$

where they $H_k$ are functions of the coefficients, the angular frequency, and the parameters, i.e., $H_k = H_k(\overline{A}_1^N, \overline{A}_2^N, .......\overline{A}_N^N, \Omega_N, \lambda)$.

Herein equation (2.14), we only retain as many harmonics in our expansion as initially occur in the assumed approximation to the periodic solution.

**Step-2:** Set the functions $H_k$ to zero, i.e.,

$$H_k = 0, \quad k = 1, 2, ......., N. \tag{2.15}$$

The action is justified since the cosine functions are linearly independent, as a result any linear sum of them that is equal to zero must have the property that the coefficient are all zero.

**Step-3:** Solve the $N$ equations, in equation (2.15), for $(\overline{A}_2^N, \overline{A}_3^N, .......\overline{A}_N^N)$ and $\Omega_N$, in terms of $\overline{A}_1^N$.

Using the initial conditions, equation (2.12), we have for $\overline{A}_1^N$ the relation

$$x_N(0) = A = \overline{A}_1^N + \sum_{k=2}^{N} \overline{A}_k^N(\overline{A}_1^N, \lambda). \tag{2.16}$$

An important point is that equation (2.15) will have many distinct solutions and the "one" selected for a particular oscillator equation is that one for which we have known a priori restrictions on the behavior of the approximations to the coefficients. However, as the worked examples in the next section demonstrate, in general, no essential difficulties arise.

For the case of non-conservative oscillators, where $\dot{x}$ appears to an "odd power" the calculation of approximations to periodic solutions follows a procedure modified for the case of conservative oscillators presented above. Many of these equations take the form

$$\ddot{x}+f(x,\lambda_1)=g(x,\dot{x},\lambda_2)\dot{x}, \tag{2.17}$$

where

$$f(-x,\lambda_1)=-f(x,\lambda_1),\ \ g(-x,-\dot{x},\lambda_2)=-g(x,\dot{x},\lambda_2), \tag{2.18}$$

and $(\lambda_1,\lambda_2)$ denote the parameters appearing in $f$ and $g$. For this type of differential equation, a limit-cycle may exist and the initial conditions cannot, in general, be a priori specified.

Harmonic balancing, for systems where limit-cycles may exist, uses the following procedures:

**Step-1:** Take the $N$-th order approximation to the periodic solution to be

$$x_N(t)=\bar{A}_1^N\cos(\bar{\Omega}_N t)+\sum_{k=2}^{N}\{\bar{A}_k^N\cos[(2k-1)\bar{\Omega}_N t]+\bar{B}_k^N\sin[(2k-1)\bar{\Omega}_N t]\}, \tag{2.19}$$

where the $2N$ unknowns $\bar{A}_1^N,\bar{A}_2^N,......,\bar{A}_N^N;\ \bar{\Omega}_N,\bar{B}_2^N,......,\bar{B}_N^N$ and $\bar{\Omega}_N$ are to be determined.

**Step-2:** Substitute equation (2.19) into equation (2.17) and write the result as

$$\sum_{k=1}^{N}\{H_k\cos[(2k-1)\Omega_N t]+L_k\sin[(2k-1)\Omega_N t]\}+HOH\cong 0, \tag{2.20}$$

where the $\{H_k\}$ and $\{L_k\}$, $k=1$ to $N$, are functions of the $2N$ unknowns which are mentioned above.

**Step-3:** Next equate the $2N$ functions $\{H_k\}$ and $\{L_k\}$ to zero and solve them for the $(2N-1)$ amplitudes and the angular frequency. If a "valid" solution exists, then it corresponds to a limit-cycle. In general, the amplitudes and angular frequency will be expressed in terms of the parameters $\lambda_1$ and $\lambda_2$.

Mickens [50] has presented the following example:

Let us consider the differential equation given by

$$\ddot{x}+\frac{1}{x}=0,\ x(0)=A,\ \dot{x}(0)=0 \tag{2.21}$$

The exact period can be calculated and its value is

$$T_e(A)=2\sqrt{2}A\int_0^1\frac{ds}{\sqrt{\ln(\frac{1}{s})}}=2\sqrt{2\pi}A, \tag{2.22}$$

where the value of the integral is available in Gradshteyn and Ryzhik [62]. The corresponding exact angular frequency is

$$\Omega_{exact}(A)=\frac{2\sqrt{\pi}}{2A}=\frac{1.2533141}{A}. \tag{2.23}$$

For the first-order harmonic balance, the solution is $x_1(t)=A\cos\theta,\ \theta=\Omega_1 t$. This calculation is best achieved if equation (2.21) is written to the form

$$x\ddot{x}+1=0. \tag{2.24}$$

Substituting $x_1(t)$ into this equation gives

$$\begin{gathered}(A\cos\theta)(-\Omega_1^2 A\cos\theta)+1+HOH\cong 0,\\ or\\ [-(\frac{\Omega_1^2A^2}{2})+1]+HOH\cong 0.\end{gathered} \tag{2.25}$$

Therefore, in lowest order, the angular frequency is

$$\Omega_1(A)=\frac{\sqrt{2}}{A}=\frac{1.4142}{A}. \tag{2.26}$$

The second order harmonic balance approximation is

$$x_2(t)=A_1\cos\theta+A_2\cos3\theta,\ \ \theta=\Omega_2 t. \tag{2.27}$$

Substituting this expression into equation (2.24) gives

$$(A_1\cos\theta+A_2\cos3\theta)[-\Omega_2^2(A_1\cos\theta+9A_2\cos3\theta)]+1\cong 0. \tag{2.28}$$

Performing the required expansions, we obtain

$$[-\Omega_2^2(\frac{A_1^2+9A_2^2}{2})+1]-\Omega_2^2(\frac{A_1^2+10A_1A_2}{2})\cos 2\theta + HOH \cong 0. \tag{2.29}$$

Setting the constant term and the coefficient of $\cos 2\theta$ to zero gives

$$-\Omega_2^2(\frac{A_1^2+9A_2^2}{2})+1=0, \quad A_1^2+10A_1A_2=0. \tag{2.30}$$

with the solutions

$$A_2=-(\frac{A_1}{10}), \qquad \Omega_2^2=\frac{200}{109A_1^2}. \tag{2.31}$$

Therefore

$$x_2(t)=A_1[\cos(\Omega_2 t)-(\frac{1}{10})\cos(3\Omega_2 t)]. \tag{2.32}$$

and requiring

$$x_2(0)=A=(\frac{9}{10})A_1 \text{ or } A_1=(\frac{10}{9})A. \tag{2.33}$$

gives

$$x_2(t)=(\frac{10}{9})A[\cos(\Omega_2 t)-(\frac{1}{10})\cos(3\Omega_2 t)]. \tag{2.34}$$

with

$$\Omega_2^2(A)=\frac{200}{109A_1^2}=(\frac{162}{109})\frac{1}{A^2} \text{ or } \Omega_2(A)=\frac{1.2191138}{A}. \tag{2.35}$$

Recently, some authors have an iteration procedure [25, 30, 31, 32, 33, 34] which is valid for small together with large amplitude of oscillation, to attain the approximate frequency and the harmonious periodic solution of such nonlinear problems. Beside this method, there are some methods (Matko and Šafarič [63], Matko [64], Matko and Milanović [65]) which are used to find approximate solution in the case of large amplitude of oscillations.

Iterative technique is also used as a technique for calculating approximate periodic solutions and corresponding frequencies of truly nonlinear oscillators for small and as well as large amplitude of oscillation. In the paper Xu and Cang [66] provided a general basis for iteration method as they are currently used to calculate the approximate periodic solutions of various nonlinear oscillatory systems successfully. Further, Mickens [30] used the iterative technique to calculate a higher-order approximation to the periodic solutions of a conservative oscillator.

The general methodology of iteration procedure by Mickens [25] is as follows:

He considered the differential equation of interest is

$$F(\ddot{x},x)=0,\ \ x(0)=A,\ \ \dot{x}(0)=0, \tag{2.36}$$

where over dots denote differentiation with respect to time, $t$.

Further he assume that equation (2.36) can be rewritten the nonlinear oscillator modeled by the equation

$$\ddot{x}+f(\ddot{x},x)=0\,,x(0)=A,\ \dot{x}(0)=0. \tag{2.37}$$

In general, the equation (2.37) is of odd parity. i.e., $f(-\dot{x},-\ddot{x})=-f(\dot{x},\ddot{x})$.

He chooses the natural frequency $\Omega$ of this system. Then adding $\Omega^2 x$ to both sides of equation (2.37), gives

$$\ddot{x}+\Omega^2 x=\Omega^2 x-f(\ddot{x},x)\equiv G(x,\ddot{x}), \tag{2.38}$$

where $\Omega^2 x$ is currently unknown.

He formulated the iteration scheme of equation (2.38) in the following way

$$\ddot{x}_{k+1}+\Omega_k^2 x_{k+1}=G(x_k,\ddot{x}_k)\ ;\ k=0,1,2,3,... \tag{2.39}$$

together with initial condition

$$x_0(t)=A\,cos(\Omega_0 t) \tag{2.40}$$

Hence $x_{k+1}$ satisfies the initial conditions

$$x_{k+1}(0)=A,\ \ \dot{x}_{k+1}(0)=0. \tag{2.41}$$

At each stage of the iteration, $\Omega_k$ is determined by the requirement that secular terms should not occur in the full solution of $x_{k+1}(t)$.

The above procedure gives the sequence of solutions: $x_0(t),\ x_1(t),\ x_2(t),...,$ Since all solutions are obtained from solving linear equations, they are, in principle, easy to calculate. The only difficulty might be the algebraic intensity required to complete the calculations. Though the equation (2.37) is of odd parity, the solution will only contain odd multiples of the angular frequency (Mickens [67]).

# CHAPTER 3

## An analytic investigation of the quadratic nonlinear oscillator by an iteration method

### 3.1 Introduction

The latter half of twentieth century saw remarkable advances in our understanding of physical systems governed by nonlinear equations. Nonlinear problems have many important applications in several aspects of mathematical-physical sciences as well as other natural and applied sciences. Most of the phenomena in our world are essentially nonlinear and are described by nonlinear equations. Again we know that most of the natural systems are nonlinear and not nice. Although it is possible in many cases when they are formulated by differential equations, to replace the nonlinear differential equation by a corresponding linear differential equation which approximates the original equation, such linearization is not always feasible or possible. In such situation several methods have been devised to find approximate solutions to nonlinear problems, such as the Perturbation method [1-8] where the nonlinear term is small, and Harmonic balance method [9-19] is another technique for determining analytical approximate periodic solution by using a truncated Fourier series representation. This method can be applied when the nonlinear terms are not small and no perturbation parameter is required. The iteration method have been studied earlier by some authors like [24-30, 67] etc. Haque et al. [31, 33], Haque [32, 34] have also studied some important and more complicated nonlinear oscillators by direct and extended iteration procedure.

The main purpose of this thesis is to improve the solution presented by Hosen [19], Mickens and Ramadhani [68] and Belendez et al. [69]. We have utilized the truncated Fourier series to expand the nonlinear terms in 'Cosine series'. Approximations from first to the fourth (in a particular case the third approximation) have been presented and compared to the existing solutions.

### 3.2 The method

Let us consider the initial value problem governing by the nonlinear oscillator

$$\ddot{x}+f(\ddot{x},x)=0,\ x(0)=A,\ \dot{x}(0)=0, \tag{3.1}$$

where over dots denote differentiation with respect to time, $t$.

We choose the natural frequency $\Omega$ of this system. Then adding $\Omega^2 x$ to both sides of equation (3.1) and rearranging, we obtain

$$\ddot{x}+\ \Omega^2\text{x}=\Omega^2\text{x-}f(\ddot{x},x)\equiv G(x,\ddot{x}). \tag{3.2}$$

Now, we formulate the iteration scheme as

$$\ddot{x}_{k+1}+\Omega_k^2 x_{k+1}=G(x_k,\ddot{x}_k);\ k=0,1,2,3,... \tag{3.3}$$

together with initial condition

$$x_0(t)=A\cos(\Omega_0 t). \tag{3.4}$$

Hence $x_{k+1}$ satisfies the initial conditions

$$x_{k+1}(0)=A,\quad \dot{x}_{k+1}(0)=0. \tag{3.5}$$

At each stage of the iteration, $\Omega_k$ is determined by the requirement that secular terms should not occur in the full solution of $x_{k+1}(t)$.

The above procedure gives the sequence of solutions: $x_0(t),\ x_1(t),\ x_2(t),\cdots$.

The method can be proceed to any order of approximation; but due to growing algebraic complexity the solution is confined to a lower order usually the second [25].

At this point, the following observations should be noted:

(a) The solution for $x_{k+1}(t)$ depends on having the solutions for $k$ less than $(k+1)$.

(b) The linear differential equation for $x_{k+1}(t)$ allows the determination of $\Omega_k$ by the requirement that secular terms be absent. Therefore, the angular frequency, "$\Omega$" appearing on the right-hand side of equation (2.3) in the function $x_k(t)$, is $\Omega_k$.

### 3.3 Example

Let us consider the differential equation representing the quadratic nonlinear oscillator

$$\ddot{x}+x^2\,\mathrm{sgn}(x)=0, \tag{3.6}$$

where,

$$\mathrm{sgn}(x)=\begin{cases}1, & x>0,\\ -1, & x<0.\end{cases} \tag{3.7}$$

We choose the case $x>0$, therefore equation (3.6) becomes

$$\ddot{x}=-x^2 \tag{3.8}$$

It is noted that, if we consider the case $x<0$ then the solution cannot be convergent for this technique.

Now adding $\Omega^2 x$ to both sides of equation (3.8), we obtain

$$\ddot{x}+\Omega^2 x=\Omega^2 x-x^2 \tag{3.9}$$

Let us formulate the iteration scheme in the following way:

$$\ddot{x}_{k+1}+\Omega_k^2 x_{k+1}=\Omega_k^2 x_k-x_k^2 \tag{3.10}$$

The initial condition is

$$x_0(t)=A\,cos\theta, \tag{3.11}$$

For the first iteration (i. e. when $k=0$ and $\theta=\Omega_0\,t$), substituting equation (3.11) into equation (3.10) and using truncated Fourier series the equation (3.10) becomes

$$\begin{aligned}\ddot{x}_1+\Omega_0^2 x_1 &=\Omega_0^2 x_0-x_0^2\\ &=\Omega_0^2 A\cos\theta-A^2\cos^2\theta\\ &=\Omega_0^2 A\cos\theta-A^2(0.848826\cos\theta+0.169765\cos 3\theta-0.0242522\cos 5\theta\\ &\quad+0.00808406\cos 7\theta-0.00367457\cos 9\theta+0.00197862\cos 11\theta).\end{aligned} \tag{3.12}$$

To avoid secular term in the solution for $x_1$, we require setting the coefficient of $\cos\theta$ to zero on the right hand side,

i. e,

$$\Omega_0^2 A-0.848826A^2=0$$

so $\Omega_0 = 0.921318\sqrt{A}$ (3.13)

Now we obtain from equation (3.12) after dropping the secular terms

$$\ddot{x}_1 + \Omega_0^2 x_1 = -A^2(0.169765\cos 3\theta - 0.0242522\cos 5\theta + 0.00808406\cos 7\theta - 0.00367457\cos 9\theta + 0.00197862\cos 11\theta) \tag{3.14}$$

Here the particular solution of equation (3.14) is

$$\begin{aligned} x_1^p &= \frac{-A^2}{D^2+\Omega_0^2}(0.169765\ cos3\theta - 0.0242522\ cos5\theta + 0.00808406\ cos7\theta \\ &\quad - 0.00367457\ cos9\theta + 0.00197862\ cos11\theta) \\ &= -A^2(\frac{0.169765}{-8\Omega_0^2}\cos 3\theta - \frac{0.02425221}{-24\Omega_0^2}cos5\theta + \frac{0.00808406}{-48\Omega_0^2}cos7\theta \\ &\quad - \frac{0.00367457}{-80\Omega_0^2}cos9\theta + \frac{0.00197862}{-120\Omega_0^2}cos11\theta) \\ &= A(0.025\ cos3\theta - 0.00119048\ cos5\theta + 0.000198413\ cos7\theta \\ &\quad - 0.0000541126\ cos9\theta + 0.000019425\ cos11\theta) \end{aligned} \tag{3.15}$$

The complementary solution of equation (3.14) is

$$x_1^c = B\cos\theta, \tag{3.16}$$

where B is any arbitrary constant.

Hence the general solution of equation (3.14) is

$$\begin{aligned} x_1(t) &= x_1^c + x_1^p \\ &= B\cos\theta + A(0.025\ cos3\theta - 0.00119048\ cos5\theta + 0.000198413\ cos7\theta \\ &\quad - 0.0000541126\ cos9\theta + 0.000019425\ cos11\theta) \end{aligned} \tag{3.17}$$

Applying initial condition $x_1(0) = A$ into the above equation, we have,

$$B = 0.976027A$$

Putting this value of B into equation (3.17), we obtain,

$$\begin{aligned} x_1(t) &= A(0.976027cos\theta + 0.025\ cos3\theta - 0.00119048\ cos5\theta \\ &\quad + 0.000198413\ cos7\theta - 0.0000541126\ cos9\theta \\ &\quad + 0.000019425\ cos11\theta) \end{aligned} \tag{3.18}$$

This is the first approximate solution of equation (3.9).

Now for $k = 1$ and $\theta = \Omega_1(t)$ we obtain from equation (3.10)

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2 x_1 - x_1^2 \tag{3.19}$$

Substituting $x_1(t)$ from equation (3.18) into the right hand side of equation (3.19),we obtain,

$$\begin{aligned}\ddot{x}_2 + \Omega_1^2 x_2 = {} & \Omega_1^2 A(0.976027 cos\theta + 0.025\ cos3\theta - 0.00119048\ cos5\theta \\ & + 0.000198413\ cos7\theta - 0.0000541126\ cos9\theta \\ & + 0.000019425\ cos11\theta) - A^2(0.976027\, cos\theta \\ & + 0.025\ cos3\theta - 0.00119048\ cos5\theta + 0.000198413 cos7\theta \\ & - 0.0000541126 cos9\theta + 0.000019425 cos11\theta)^2\end{aligned} \tag{3.20}$$

which after multiplication yields, with the corresponding truncated Fourier series,

$$\begin{aligned}\ddot{x}_2 + \Omega_1^2 x_2 = {} & \Omega_1^2 A(0.976027 cos\theta + 0.025\ cos3\theta - 0.00119048\ cos5\theta \\ & + 0.000198413\ cos7\theta - 0.0000541126\ cos9\theta + 0.000019425\ cos11\theta) \\ & - A^2(0.817358 cos\theta + 0.192994\ cos3\theta - 0.0143718\ cos5\theta \\ & + 0.00581554\ cos7\theta - 0.00276284\ cos9\theta + 0.00152151\ cos11\theta),\end{aligned} \tag{3.21}$$

Secular term will not appear in the solution for $x_2$ if the coefficient of the $\cos\theta$ term is zero, i. e,

$$0.976027 A\Omega_1^2 = 0.817359 A^2$$

$$\Omega_1 = 0.915115\sqrt{A} \tag{3.22}$$

Now we obtain from equation (3.21) after removing secular term

$$\begin{aligned}\ddot{x}_2 + \Omega_1^2 x_2 = {} & \Omega_1^2 A(0.025\ cos3\theta - 0.00119048\ cos5\theta \\ & + 0.000198413\ cos7\theta - 0.0000541126\ cos9\theta \\ & + 0.000019425\ cos11\theta) - A^2(0.192994\ cos3\theta \\ & - 0.0143718\ cos5\theta + 0.00581554\ cos7\theta \\ & - 0.00276284\ cos9\theta + 0.00152151\ cos11\theta)\end{aligned} \tag{3.23}$$

Here the particular solution of equation (3.23) is

$$\begin{aligned}x_2^p = {} & \frac{1}{D^2 + \Omega_1^2}\{\Omega_1^2 A(0.025\ cos3\theta - 0.00119048\ cos5\theta + 0.000198413\ cos7\theta \\ & - 0.0000541126\ cos9\theta + 0.000019425\ cos11\theta) \\ & - A^2(0.192994\ cos3\theta - 0.0143718\ cos5\theta + 0.00581554\ cos7\theta \\ & - 0.00276284\ cos9\theta + 0.00152151\ cos11\theta)\}\end{aligned}$$

$$= \Omega_1^2 A(\frac{0.025}{-8\Omega_1^2} cos3\theta - \frac{0.00119048}{-24\Omega_1^2} cos5\theta + \frac{0.000198413}{-48\Omega_1^2} cos7\theta - \frac{0.0000541126}{-80\Omega_1^2} cos9\theta$$
$$+ \frac{0.000019425}{-120\Omega_1^2} cos11\theta) - A^2(\frac{0.192994}{-8\Omega_1^2} cos3\theta - \frac{0.0143718}{-24\Omega_1^2} cos5\theta$$
$$+ \frac{0.00581554}{-48\Omega_1^2} cos7\theta - \frac{0.00276284}{-80\Omega_1^2} cos9\theta + \frac{0.00152151}{-120\Omega_1^2} cos11\theta)$$
$$= A(0.0256823 \cos 3\theta - 0.000665468 \cos 5\theta + 0.000140543 \cos 7\theta - 0.0000405632 \cos 9\theta + 0.0000149787 \cos 11\theta) \quad (3.24)$$

The complementary solution of equation (3.23)

$$x_2^c = M \cos\theta, \quad (3.25)$$

where $M$ is any arbitrary constant.

The general solution of equation (3.23) is

$$x_2(t) = x_2^c + x_2^p$$
$$= M\cos\theta + A(0.0256823 \cos 3\theta - 0.000665468 \cos 5\theta + 0.000140543 \cos 7\theta - 0.0000405632 \cos 9\theta + 0.0000149787 \cos 11\theta) \quad (3.26)$$

Applying initial condition $x_2(0) = A$ into the above equation, we have,

$$M = 0.974868A$$

Putting this value of $M$ into equation (3.26), we obtain,

$$x_2(t) = A(0.974868 \cos\theta + 0.0256823\ c0s3\theta - 0.000665468 \cos 5\theta + 0.000140543 \cos 7\theta - 0.0000405632 \cos 9\theta + 0.0000149787 \cos 11\theta \quad (3.27)$$

The third approximation $x_3$ and the value of $\Omega_2$ will be obtained from the solution of

$$\ddot{x}_3 + \Omega_2^2 x_3 = \Omega_2^2 x_2 - x_2^2 \quad (3.28)$$

Substituting $x_2(t)$ from equation (3.27) into the right-hand side of equation (3.28), we obtain,

$$\ddot{x}_3 + \Omega_2^2 x_3 = \Omega_2^2 A(0.974868 \cos\theta + 0.0256823\ cos3\theta - 0.000665468 \cos 5\theta + 0.000140543 \cos 7\theta - 0.0000405632 \cos 9\theta + 0.0000149787 \cos 11\theta)$$
$$- A^2(0.974868 \cos\theta + 0.0256823\ c0s3\theta - 0.000665468 \cos 5\theta + 0.000140543 \cos 7\theta - 0.0000405632 \cos 9\theta + 0.0000149787 \cos 11\theta)^2$$

$$
\begin{aligned}
&= \Omega_2^2 A(0.974868 \cos\theta + 0.0256823\ cos3\theta - 0.000665468 \cos 5\theta \\
&\quad + 0.000140543 \cos 7\theta - 0.0000405632 \cos 9\theta + 0.0000149787 \cos 11\theta) \\
&\quad - A^2(0.815657\ \cos\theta + 0.193656\ cos3\theta - 0.0134172 \cos 5\theta \\
&\quad + 0.00591789\ \cos 7\theta - 0.0027829 \cos 9\theta + 0.00152773 \cos 11\theta),
\end{aligned} \tag{3.29}
$$

Secular terms will not appear in the solution for $x_3$ if the coefficient of the $\cos\theta$ term is zero, i. e,

$$0.974868 A\Omega_2^2 = 0.815657 A^2$$

$$\text{i.e. } \Omega_2 = 0.914705\sqrt{A} \tag{3.30}$$

Now we obtain from equation (3.29) without secular term as

$$
\begin{aligned}
\ddot{x}_3 + \Omega_2^2 x_3 &= \Omega_2^2 A(0.0256823\ cos3\theta - 0.000665468 \cos 5\theta \\
&\quad + 0.000140543 \cos 7\theta - 0.0000405632 \cos 9\theta \\
&\quad + 0.0000149787 \cos 11\theta) - A^2(0.193656\ cos3\theta - 0.0134172 \cos 5\theta \\
&\quad + 0.00591789 \cos 7\theta - 0.0027829 \cos 9\theta + 0.00152773 \cos 11\theta)
\end{aligned} \tag{3.31}
$$

Here the particular solution of equation (3.31) is

$$
\begin{aligned}
x_3^p &= \frac{1}{D^2 + \Omega_1^2}\{\Omega_2^2 A(0.0256823\ cos3\theta - 0.000665468 \cos 5\theta + 0.000140543 \cos 7\theta \\
&\quad - 0.0000405632 \cos 9\theta + 0.0000149787 \cos 11\theta) - A^2(0.193656\ cos3\theta \\
&\quad - 0.0134172 \cos 5\theta + 0.00591789\ \cos 7\theta - 0.0027829 \cos 9\theta \\
&\quad + 0.00152773 \cos 11\theta)\}
\end{aligned}
$$

$$
\begin{aligned}
&= \Omega_2^2 A(\frac{0.0256823}{-8\Omega_2^2}\ cos3\theta - \frac{0.000665468}{-24\Omega_2^2} \cos 5\theta + \frac{0.000140543}{-48\Omega_2^2} \cos 7\theta \\
&\quad - \frac{0.0000405632}{-80\Omega_2^2} \cos 9\theta + \frac{0.0000149787}{-120\Omega_2^2} \cos 11\theta) - A^2(\frac{0.193656}{-8\Omega_2^2}\ cos3\theta \\
&\quad - \frac{0.0134172}{-24\Omega_2^2} \cos 5\theta + \frac{0.00591789}{-48\Omega_2^2}\ \cos 7\theta - \frac{0.0027829}{-80\Omega_2^2} \cos 9\theta \\
&\quad + \frac{0.00152773}{-120\Omega_2^2} \cos 11\theta)
\end{aligned}
$$

$$
\begin{aligned}
&= A(0.0257217\ cos3\theta - 0.000640443 \cos 5\theta + 0.000144427 \cos 7\theta \\
&\quad - 0.0000410693 \cos 9\theta + 0.0000150912 \cos 11\theta)
\end{aligned} \tag{3.32}
$$

The complementary solution of equation (3.31) is

$$x_3^c = P\cos\theta, \tag{3.33}$$

where $P$ is any arbitrary constant.

The general solution of equation (3.31) is

$$\begin{aligned} x_3(t) &= x_3^c + x_3^p \\ &= P\cos\theta + A(0.0257217\ cos3\theta - 0.000640443\ \cos 5\theta + 0.000144427\ \cos 7\theta \\ &\quad - 0.0000410693\ \cos 9\theta + 0.0000150912 \cos 11\theta) \end{aligned} \tag{3.34}$$

Applying the initial condition $x_3(0) = A$ into the above equation, we have

$$P = 0.9748$$

Putting this value of $P$ into equation (3.26), we obtain,

$$\begin{aligned} x_3(t) = A(&0.9748\ \cos\theta + 0.0257217\ cos3\theta - 0.000640443\ \cos 5\theta \\ &+ 0.000144427\ \cos 7\theta - 0.0000410693\ \cos 9\theta + 0.0000150912 \cos 11\theta) \end{aligned} \tag{3.35}$$

The fourth approximation $x_4$ and the value of $\Omega_3$ will be obtained from the solution of

$$\ddot{x}_4 + \Omega_3^2 x_4 = \Omega_3^2\, x_3 - x_3^2 \tag{3.36}$$

Substituting $x_3(t)$ from equation (3.35) into the right-hand side of equation (3.36), we obtain,

$$\begin{aligned} \ddot{x}_4 + \Omega_3^2 x_4 &= \Omega_3^2 A(0.9748\ \cos\theta + 0.0257217\ cos3\theta - 0.000640443\ \cos 5\theta \\ &\quad + 0.000144427\ \cos 7\theta - 0.0000410693\ \cos 9\theta + 0.0000150912 \cos 11\theta) \\ &\quad - A^2(0.9748\ \cos\theta + 0.0257217\ cos3\theta - 0.000640443\ \cos 5\theta \\ &\quad + 0.000144427\ \cos 7\theta - 0.0000410693\ \cos 9\theta + 0.0000150912 \cos 11\theta)^2 \\ &= \Omega_3^2 A(0.9748\ \cos\theta + 0.0257217\ cos3\theta - 0.000640443\ \cos 5\theta \\ &\quad + 0.000144427\ \cos 7\theta - 0.0000410693\ \cos 9\theta \\ &\quad + 0.0000150912 \cos 11\theta) - A^2(0.815558\ \cos\theta \\ &\quad + 0.193691\ \cos 3\theta - 0.0133655\ \cos 5\theta + 0.0059299\ cos7\theta \\ &\quad - 0.00278203\ \cos 9\theta + 0.00152736\ \cos 11\theta), \end{aligned} \tag{3.37}$$

Secular terms will not appear in the solution for $x_4$ if the coefficient of the $\cos\theta$ term is zero i.e.,

$$0.9748 A\Omega_2^2 = 0.815558 A^2$$

$$\Omega_3 = 0.914681\sqrt{A} \tag{3.38}$$

Thus $\Omega_0$, $\Omega_1$, $\Omega_2$ and $\Omega_3$ can be obtained by equations (3.13), (3.22), (3.30), and (3.38) respectively, which represent different approximations of frequencies of the oscillator (3.6).

### 3.4 Results and Discussion

An Iteration method is developed based on Mickens [25] to solve 'quadratic nonlinear oscillator'. In this section, for the accuracy frequencies of the oscillator obtained by the modified technique of iteration method presented in the thesis will be compared with the existing results obtained from some different methods together with the exact frequency of the oscillator. To estimate the accuracy, we have calculated the percentage errors (denoted by Er (%)) by the definitions

$$Er = |100\{\Omega_e(A) - \Omega_i(A)\} / \Omega_e(A)\ |,\ i = 0,1,2,3,\ldots,$$

where $\Omega_i$ represents the approximate frequencies obtained by the adopted method and $\Omega_e$ represents the corresponding exact frequency of the oscillator.

Herein we have calculated the first, second, third and fourth approximate frequencies which are denoted by $\Omega_0$, $\Omega_1$, $\Omega_2$ and $\Omega_3$ respectively. A comparison among the existing results obtained by Hosen [19], Mickens and Ramadhani [68] and Belendez et al. [69] with our obtained results is shown in the following table.

**Table:** Comparison of the approximate frequencies obtained by the presented technique with other existing results and exact frequency $\Omega_e$(Belendez et al. [69]) of quadratic nonlinear oscillator.

| Exact Frequency $\Omega_e$ | $0.914681\sqrt{A}$ | | | |
|---|---|---|---|---|
| | First Approximate Frequency $\Omega_0$ Er(%) | Second Approximate Frequency $\Omega_1$ Er(%) | Third Approximate Frequency $\Omega_2$ Er(%) | Fourth Approximate Frequency $\Omega_3$ Er(%) |
| Adopted Method | $0.921318\sqrt{A}$ 0.73 | $0.915114\sqrt{A}$ 0.047 | $0.914705\sqrt{A}$ 0.0026 | $0.9146807\sqrt{A}$ 0.00003 |
| Hosen [19] | $0.921318\sqrt{A}$ 0.73 | $0.914427\sqrt{A}$ 0.028 | $0.914733\sqrt{A}$ 0.0056 | – |
| Belendez et al. [69] | $0.921318\sqrt{A}$ 0.73 | $0.914274\sqrt{A}$ 0.045 | $0.914711\sqrt{A}$ 0.0032 | – |
| Mickens and Ramadhani [68] | $0.921318\sqrt{A}$ 0.73 | $0.914044\sqrt{A}$ 0.070 | – | – |

It is noted that Mickens and Ramadhani [68] presented only second approximate frequencies by harmonic balance method. Belendez et al. [69] put forward up to third approximate frequencies by using modified He's homotopy perturbation method. Hosen [19] also presented up to third approximate frequencies by using modified harmonic balance method.

Clearly from the table, it is seen that the forth-order approximate frequency obtained by the adopted method is almost same with exact frequency. The table also shows that in most of the cases our solution gives significantly better result than other existing results. The advantages of this method include its simplicity and computational efficiency.

### 3.5 Convergence and Consistency Analysis

We know the basic idea of iteration methods is to construct a sequence of solutions $x_k$ (as well as frequencies $\Omega_k$) that have the property of convergence

$$x_e = \lim_{k \to \infty} x_k \text{ or } \Omega_e = \lim_{k \to \infty} \Omega_k$$

Here $x_e$ is the exact solution of the given nonlinear oscillator.

In the present method, it has been shown that the solution in each iterative step yields less error compared to the previous iterative step and finally, $|\Omega_3 - \Omega_e| = |0.9146807\sqrt{A} - 0.914681\sqrt{A}| < \varepsilon$, where $\varepsilon$ is a small positive number and $A$ is chosen to be unity. From this, it is clear that the adopted method is convergent.

An iterative method of the form represented by equation (3.3) with initial guesses given in equations (3.4) and (3.5) is said to be consistent if

$$\lim_{k \to \infty} |x_k - x_e| = 0 \text{ or } \lim_{k \to \infty} |\Omega_k - \Omega_e| = 0.$$

In the present analysis we see that

$$\lim_{k \to \infty} |\Omega_k - \Omega_e| = 0 \text{ as } |\Omega_3 - \Omega_e| = 0.$$

Thus the consistency of the method has been achieved.

# CHAPTER 4

## CONCLUSION

The iteration method is a powerful and effective mathematical tool in solving nonlinear differential equations of mathematical physics, applied mathematics and engineering. In this thesis, an iteration method has been employed for analytic treatment of the quadratic nonlinear differential equation. The adopted method is convergent and the obtained solution is consistent. The performance of this method is reliable, simple and gives many new solutions. The results obtained by the presented technique are not only fit to be used in the case of small nonlinearities but also fit to be used in the case of high nonlinearities.

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