A STUDY ON 0-DISTRIBUTIVE NEARLATTICE

BY

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DEDICATED TO MY PARENTS WHO HAVE PROFOUNDLY INFLUENCED MY LIFE

Declaration

This is to certify that the thesis work entitled " A study on 0-distributive nearlattice" has been carried out by Md. Zaidur Rahman in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

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Rahman

(Md. Zaidur Rahman)

Abstract

In this thesis study of the nature of the 0-distributive nearlattices is presented. By a nearlattice S we will always mean a meet semilattice together with the property that any two elements possessing a common upper bound, have a supremum. Cornish and Hickman [14] referred this property as the upper bound property and a semilattice of this nature as a semilattice with the upperbound property. Cornish and Noor [15] preferred to call these semilattices as nearlattices, as the behaviour of such a semilattice is close to that of a lattice than an ordinary semilattice. Of course a nearlattice with a largest element is a lattice. Since any semilattice satisfying the descending chain condition has the upper bound property, so all finite semilattices are nearlattices. In lattice theory, it is always very difficult to study the non-distributive and non-modular lattices. Gratzer [20] studied the non-distributive lattices by introducing the concept of distributive, standard and neutral elements in lattices. Cornish and Noor [15] extended those concepts for nearlattices to study non-distributive nearlattices. On the other hand, J.C Varlet [66] studied another class of non-distributive lattices with 0 by introducing the concept of 0-distributivity. In fact this concept also generalizes the idea of pseudocomplement in a general lattice. This thesis extend the concept of 0-distributivity in a nearlattice to study a larger class of non-distributive nearlattices. A nearlattice S with 0 is called 0-distributive if for all $x, y, z \in S$ with $x \wedge y = 0 = x \wedge z$ and $y \vee z$ exists imply $x \wedge (y \vee z) = 0$.

Chapter 1 gives a detailed description of nearlattices. Here we discuss ideals, congruences, SemiBoolean algebra and many other results on nearlattice which are basic to this thesis.

In Chapter 2 we introduce the concept of modular element in a nearlattice. Gratzer and Schmidt [23] introduced the notion of some special elements, e.g. distributive, standard and neutral elements, to study a larger class of non-distributive lattices. Then Cornish and Noor [15] used these concepts to nearlattices. Again Talukder and Noor [64] introduced the notion of modular elements in a join semilattice directed below. The

notion of modular element is also applicable for general lattices. In this chapter, we have introduced the concept of modular and strongly distributive elements for nearlattices. Here we have given several characterizations of modular and strongly distributive elements. By studying these elements and ideals we obtained many information on a class of non-distributive nearlattices.

Chapter 3 and 4 are the key chapters of this thesis. In Chapter 3 we introduce the 0-distributivity in a nearlattice with 0. We include several characterizations of distributive nearlattices. We prove that a nearlattice S with 0 is 0-distributive if and only if all maximal filter of S are prime. We also show that S is 0-distributive if and only if $I(S)$, the lattice of all ideals of S is pseudocomplemented. Then we include some prime separation properties. In this chapter we also include the notion of semi-prime ideals by extending the notion of 0-distributivity. In lattices, the notion of semi-prime ideals was given by Y. Rav [52]. By using these semi-prime ideals, we generalize the prime separation theorem of nearlattices in terms of annihilator ideals. Finally, we extend the concept of Glivenko congruence for 0-distributive nearlattices as well as for semi-prime ideals to establish a generalized version of prime separation theorem.

In chapter 4 we discuss different properties of 0-distributive nearlattices and included several characterizations of these nearlattices Annulets and α-ideals in a distributive lattice have been studied extensively by Cornish [13]. Recently Ayub Ali, Noor and Islam [4], Noor , Ayub Ali and Islam [41] extended this concept for distributive nearlattices. In this chapter we study the annulets and α-ideals in a 0-distributive nearlattice. We give several characterizations of α-ideals. We also include a prime separation theorem for α-ideals. Finally we show that a 0-distributive nearlattice is quasicomplemented if and only if $A_0(S)$ (the dual nearlattice of annulets) is a Boolean subalgebra of $A(S)$, where $A(S)$ is the set of all annihilator ideals of S. Moreover, S is sectionally quasicomplemented if and only if $A_0(S)$ is relatively complemented.

Chapter 5 brings the notions of 0-modular nearlattices. Ayub Ali, Hafizur Rahman and Noor [5], Jayaram [30], Noor , Ayub Ali and Islam [41] and Varlet [65] have studied different properties of 0-distributivity and 0-modularity in lattices and in semilattices. In this chapter we extend their work and include several characterizations of 0-modular nearlattices.

Many mathematician including Cornish [11] have studied the normal lattices and p-algebras in presence of distributivity. Recently Nag, Begum and Talukder[38] studied them in presence of 0-dirtributivity. They have generalized many results of S-algebras and D-algebras. Since the idea of pseudocomplementation is not appropriate for a nearlattice, we study the sectional pseudocomplementation for a nearlattice. In chapter 6 we extend and generalize some results of Nag, Begum and Talukder [38] on D-algebras and S-algebras. We prove that every $[0,x]$, $x \in S$ is an S-algebra if and only if it is a D-algebra where the nearlattice S is sectionally p-algebra with the condition that $[0,x]$ for each $x \in S$ is 1-distributive and S is 0-modular. We conclude the thesis by giving a characterization of sectionaly S-algebra whenever $[0,x]$ for each $x \in S$ is 1-distributive.

STATEMENT OF ORIGINALITY

This thesis does not incorporate without acknowledgment any material previously submitted for a degree or diploma in any university and to the best of my knowledge and belief, does not contain material previously published or written by another person except where due reference is made in the text.

Rahman

(Md. Zaidur Rahman)

Approval

This is to certify that the thesis work submitted by Md. Zaidur Rahaman entitled " A Study on 0-distributive nearlattice" has been approved by the board of examiners for the partial fulfillment of the requirements for the degree of Ph.D. in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh in November 2014.

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List of Publications

1. A.S.A.Noor and **Md. Zaidur Rahman**, *"Characterizations of some Special elements in a nearlattice by relative annihilators",News Bull. Cal. Math. Soc., 34, (4-6) 11-16 (2011).*
2. **Md. Zaidur Rahman** and A.S.A.Noor, *"Modular and Strongly distributive elements in a nearlattice", JMCMS, Vol-7,No-1, July (2012), 988-997.*
3. A. S. A. Noor, **Md. Zaidur Rahman** and Md. Bazlar Rahman, *"Some characterization of 0-distributive Nearlattice", Annals of Pure and Applied Mathematics, Vol. 2, No. 2, (2012), 194- 199.*
4. **Md. Zaidur Rahman**, Md. Bazlar Rahman and A. S. A. Noor, *"0-distributive Nearlattice", Annals of Pure and Applied Mathematics, Vol. 2, No. 2, (2012), 177-184.*
5. **Md. Zaidur Rahman**, Md. Bazlar Rahman and A. S. A. Noor, *"Some properties of 0-distributive Nearlattice", Annals of Pure and Applied Mathematics, Vol. 2, No. 2,(2012), 185-193.*
6. **Md. Zaidur Rahman**, Md. Bazlar Rahman and A. S. A. Noor, *On semi prime Ideal in Nearlattices, Annals of Pure and Applied Mathematics, Vol. 3, No. 1, (2012), 1- 9.*
7. **Md. Zaidur Rahman**, A.S.A.Noor and Md. Bazlar Rahman, *"0-Modular Nearlattices", Annals of Pure and Applied Mathematics, Vol. 3, No. 2,(2013), 100-107.*
8. A.S.A.Noor and **Md. Zaidur Rahman**, *"On Glivenko Congruence of a 0-Distributive Nearlattices", Annals of Pure and Applied Mathematics, Vol. 3, No. 2, (2013), 108-112.*
9. **Md. Zaidur Rahman** and A.S.A.Noor, "α*-ideals in a 0-Distributive Nearlattice", Annals of Pure and Applied Mathematics, Vol. 3, No. 2,(2013), 113-118.*
10. **Md. Zaidur Rahman** and Md. Bazlar Rahman, *"Modular ideal in a nearlattices", Journal of Engineering Science, Vol 4, No 1, June (2013), 61-65.*

IDEALS AND CONGRUNCES

1.1 Preliminaries

The intention of this section is to outline and fix the notation for some of the concepts of nearlattices which are basic to this thesis. We also formulate some results on arbitrary nearlattices for later use. For the background material in lattice theory we refer the reader to the text of Birkhoff [10] , Gratzer [19], [20] and Davey [16].

By a nearlattice S we will always mean a lower (meet) semilattice which has the property that any two elements possessing a common upper bound have a supremum. Cornish and Hickman [14], referred this property as the *upper bound property* and a semilattice of this nature as *a semilattice with the upper bound property*. The behaviour of such a semilattice is closer to that of a lattice than an ordinary semilattice.

Of course, a nearlattice with a largest element is a lattice. Since any semilattice satisfying the descending chain condition has the upper bound property, so all finite semilattices are nearlattices.

Now we give an example of a meet semilattice which is not a nearlattice.

Example: In R^2 let us consider the set, $S = \{(0,0)\} \cup \{(1,0)\} \cup \{(0,1)\} \cup \{(1,y) \mid y > 1\}$ shown in the Figure 1.1

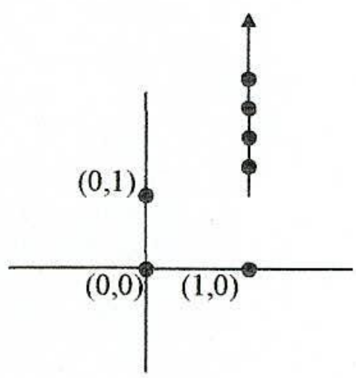


Figure 1.1

Let us define the partial ordering "$\leq$" on S by $(x,y)\leq(x_1,y_1)$ if and only if $x\leq x_1$ and $y\leq y_1$. Clearly, $(S;\leq)$ is a meet semilattice. Both (1,0) and (0,1) have common upper bounds. In fact $\{(1,y)\mid y>1\}$ are common upper bounds of them. But the supremum of (1,0) and (0,1) does not exist. Therefore $(S;\leq)$ is not a nearlattice.

The upper bound property appears in Gratzer and Lakser [21], while Rozen [54] show that it is the result of placing certain associativity conditions on the partial join operation. Moreover, Evans [18] referred nearlattices as *conditional lattices*. By a conditional lattice he means a lower semilattice S with the condition that for each $x\in S$, $\{y\in S \mid y\leq x\}$ is a lattice; and it is very easy to check that this condition is equivalent to the upper bound property of S. Also Nieminen [39] in his paper refers to nearlattices as *"partial lattices"*. Whenever a nearlattice has a least element we will denote it by 0. If $x_1,x_2,\cdots,x_n$ are elements of a nearlattice then by $x_1\vee x_2\vee\cdots\vee x_n$, we mean that the supremum of $x_1,x_2,\cdots,x_n$ exists and $x_1\vee x_2\vee\cdots\vee x_n$ symbolizing this supremum.

A non-empty subset K of a nearlattice S is called a *subnearlattice* of S if for any $a,b\in K$, both $a\wedge b$ and $a\vee b$ (whenever it exists in S) belong to K ($\wedge$ and $\vee$ are taken in S), and the $\wedge$ and $\vee$ of K are the restrictions of the $\wedge$ and $\vee$ of S to K. Moreover, a subnearlattice K of a nearlattice S is called a *sublattice* of S if $a\vee b\in K$ for all $a,b\in K$.

A nearlattice S is called *modular* if for any $a,b,c\in S$ with $c\leq a$, $a\wedge(b\vee c)=(a\wedge b)\vee c$ whenever $b\vee c$ exists.

A nearlattice S is called distributive if for any $x,x_1,x_2,\cdots,x_n$, $x\wedge(x_1\vee x_2\vee\cdots\vee x_n)=(x\wedge x_1)\vee(x\wedge x_2)\vee\cdots\vee(x\wedge x_n)$ whenever $x_1\vee x_2\vee\cdots\vee x_n$ exists. Notice that the right hand expression always exists by the upper bound property of S.

Lemma 1.1.1. *A nearlattice S is distributive (modular) if and only if* $\{y \in S \mid y \leq x\}$ *is a distributive (modular) lattice for each* $x \in S$. •

Let us consider the following two lattices: pentagonal lattice N_5 and Diamond lattice M_5. Many lattice theorists study on these two lattices and given several results.

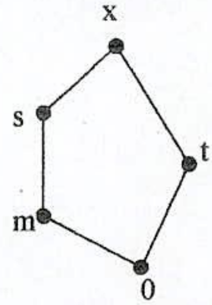


Figure-1.2

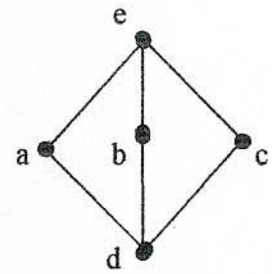


Figure-1.3

Hickman in [28] has given the following extensions of a very fundamental results of lattice theory.

Theorem 1.1.2. *A nearlattice S is distributive if and only if S does not contain a sublattice isomorphic to* N_5 *or* M_5 [in Figure 1.2 and 1.3]. •

Theorem 1.1.3. *A nearlattice S is modular if and only if S does not contain a sublattice isomorphic to* N_5. •

In this context it should be mentioned that many lattice theorists (e.g. R. Bables, J. C. Varlet, R. C. Hickman and K. P. Shum) have worked with a class of semilattice S which has the property that for each $x, a_1, a_2, \cdots, a_r \in S$, if $a_1 \vee a_2 \vee \cdots \vee a_r$ exists then $(x \wedge a_1) \vee (x \wedge a_2) \vee \cdots \vee (x \wedge a_r)$ exists and equals $x \wedge (a_1 \vee a_2 \vee \cdots \vee a_r)$. Bables [7] called them as prime semilattices while Shum [58] referred them as weakly distributive semilattices.

Hickman in [28] has defined a ternary operation j by $j(x,y,z) = (x \wedge y) \vee (y \wedge z)$, on a nearlattice S (which exists by the upper bound property of S). In fact he has shown, which can also be found in Lyndon [34] Theorem 4, that the resulting algebras of the type

$(S;j)$ form a variety, which is referred to as the variety of join algebras and following are its defining identities.

(i) $j(x,x,x)=x$

(ii) $j(x,y,x)=j(y,x,y)$

(iii) $j(j(x,y,x),z,j(x,y,x))=j(x,j(y,z,y),x)$

(iv) $j(x,y,z)=j(z,y,x)$

(v) $j(j(x,y,z),j(x,y,x),j(x,y,z))=j(x,y,x)$

(vi) $j(j(x,y,x),y,z)=j(x,y,z)$

(vii) $j(x,y,j(x,z,x))=j(x,y,x)$

(viii) $j(j(x,y,j(w,y,z)),j(x,y,z),j(x,y,j(x,y,z)))=j(x,y,z)$

We do not elaborate it further as it is beyond the scope of this thesis.

We call a nearlattice S a medial nearlattice if for all $x,y,z \in S$, $m(x,y,z)=(x\wedge y)\vee(y\wedge z)\vee(z\wedge x)$ exists. For a (lower) semilattice S, if $m(x,y,z)$ exists for all $x,y,z\in S$, then it is not hard to see that S has the upper bound property and hence is a nearlattice. Distributive medial nearlattices were first studied by Sholander [56, 57], and then by Evans [18]. Sholander preferred to call these as *medial semilattices.* He showed that every medial nearlattice S can be characterized by means of an algebra $(S;m)$ of type $\langle 3\rangle$, known as *medial algebra*, satisfying the following two identities:

(i) $m(a,a,b)=a$

(ii) $m(m(a,b,c),m(a,b,d),e)=m(m(c,d,e),a,b)$.

A nearlattice S is said to have the **three** property if for any $a,b,c\in S$, $a\vee b\vee c$ exists whenever $a\vee b$, $b\vee c$ and $c\vee a$ exists. Nearlattices with the **three** property were discussed by Evans [18], where he referred it as strong conditional lattices.

The equivalence of (i) and (iii) of the following lemma is trivial, while the proof of (i) $<=>$ (ii) is inductive.

Lemma 1.1.4. {Evans [18]}. *For a nearlattice S the following conditions are equivalent:*

(i) S *has the* ***three*** *property.*

(ii) *Every pair of a finite number* $\text{n}\ (\geq 3)$ *of elements of* S *posses a supremum ensures the existence of the supremum of all the n elements.*

(iii) S *is medial.* •

A family $\mathcal{A}$ of a subset of a set A is called a closure system on A if

(i) $A \in \mathcal{A}$ and

(ii) $\mathcal{A}$ is closed under arbitrary intersection.

Suppose $\mathcal{B}$ is a subfamily of A. $\mathcal{B}$ is called a directed system if for any $\text{X}, \text{Y} \in \mathcal{B}$ there exists Z in $\mathcal{B}$ such that $\text{X}, \text{Y} \subseteq \text{Z}$.

If $\cup \{\text{X} : \text{X} \in \mathcal{B}\} \in \mathcal{A}$ for every directed system $\mathcal{B}$ contained in the closure system $\mathcal{A}$, then $\mathcal{A}$ is called algebraic. When ordered by set inclution, an algebraic closure system forms an algebraic lattice.

1.2 Ideals of Nearlattices

A non-empty subset I of a nearlattice S is called a down set if for any $x \in S$ and $y \in I$, $x \leq y$ implies $x \in I$.

A non-empty subset I of a nearlattice S is called an ideal if it is a down set and closed under existent finite suprema. We denote the set of all ideals of S by $I(S)$, which is a lattice. If S has a smallest element 0 then $I(S)$ is an algebraic closure system on S and is consequently an algebraic lattice.

However, if S does not possess smallest element then we can only assert that $I(S) \cup \{\Phi\}$ is an algebraic closure system, where Φ is the empty subset of S.

For any subset K of a nearlattice S, $(K]$ denotes the ideal generated by K.

Infimum of two ideals of a nearlattice is their set theoretic intersection. Supremum of two ideals I and J in a lattice L is given by $I \vee J = \{x \in L \mid x \leq i \vee j \text{ for some } i \in I, j \in J\}$. Cornish and Hickman in [14] showed that in a distributive nearlattice S for two ideals I and J, $I \vee J = \{i \vee j \mid i \in I, j \in J \text{ where } i \vee j \text{ exists}\}$. But in a general nearlattice the fomula for the supremum of two ideals is not very easy. Let us consider the following lemma which gives the formula for the supremum of two ideals. It is in fact an exercise in Gratzer [19], p-54 for partial lattice.

Theorem 1.2.1. *Let I and J be ideals of a nearlattice S. Let $A_0 = I \cup J$, $A_n = \{x \in S \mid x \leq y \vee z;\ y \vee z \text{ exists and } y, z \in A_{n-1}\}$ for $n = 1, 2, \cdots$, and $K = \bigcup_{n=0}^{\infty} A_n$. Then $K = I \vee J$.*

Proof: Since $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$, K is an ideal containing I and J. Suppose H is any ideal containing I and J. Of course, $A_0 \subseteq H$. We proceed by induction.

Suppose $A_{n-1} \subseteq H$ for some $n \geq 1$ and let $x \in A_n$. Then $x \leq y \vee z$ with $y, z \in A_{n-1}$. Since $A_{n-1} \subseteq H$ and H is an ideal, $y \vee z \in H$ and so $x \in H$. That is $A_n \subseteq H$ for every n. Thus $K = I \vee J$.●

Theorem.1.2.2. *Let K be a non-empty subset of a nearlattice S. Then $(K] = \bigcup_{n=0}^{\infty}\{A_n \mid n \geq 0\}$, where $A_0 = \{t \in S \mid t = j(k_1, t, k_2)$ for some $k_1, k_2 \in K\}$ and $A_n = \{t \in S \mid t = j(a_1, t, a_2)$ for some $a_1, a_2 \in A_{n-1}\}$ for $n \geq 1$.*

Proof: For any $k \in K$ clearly $k = j(k,k,k)$ and so $K \subseteq A_0$. Similarly, for any $a \in A_{n-1}$, $a = j(a,a,a)$ implies that $A_{n-1} \subseteq A_n$. Thus $K \subseteq A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{n-1} \subseteq A_n \subseteq \cdots$.

Let $t \in \bigcup_{n=0}^{\infty} A_n ; n = 0,1,2,\cdots$, and $t_1 \in S$ such that $t_1 \leq t$. Then $t \in A_m$ for some $m \geq 0$. Clearly, $t_1 = j(t, t_1, t)$ and so $t_1 \in A_{m+1}$. Thus $\bigcup_{n=0}^{\infty} A_n$ is down set.

Now suppose, $t_1, t_2 \in \bigcup_{n=0}^{\infty} A_n$ and $t_1 \vee t_2$ exists. Let $t_1 \in A_r$ and $t_2 \in A_s$ for some $r, s \geq 0$ with $r \leq s$ (say). Then $t_1, t_2 \in A_s$ and $t_1 \vee t_2 = j(t_1, t_1 \vee t_2, t_2)$ provides $t_1 \vee t_2 \in A_{s+1}$.

Finally, suppose H is an ideal containing K. If $x \in A_0$, then $x = j(k_1, x, k_2) = (k_1 \wedge x) \vee (k_2 \wedge x)$ for some $k_1, k_2 \in K$. As $K \subseteq H$ and H is an ideal, $k_1 \wedge x, k_2 \wedge x \in H$ and so $x \in H$. Thus $A_0 \subset H$. Again we use the induction. Suppose $A_{n-1} \subseteq H$ for some $n \geq 1$. Let $x \in A_n$ so that $x = j(a_1, x, a_2)$ for some $a_1, a_2 \in A_{n-1}$. Then $x \in H$ as $a_1, a_2 \in H$ and $x = (a_1 \wedge x) \vee (a_2 \wedge x)$.●

Theorem 1.2.3. *A non empty subset K of a nearlattice S is an ideal if and only if $x \in K$ whenever $x \in S$ and $x = j(k_1, x, k_2)$ for some $k_1, k_2 \in K$.*●

We now give an alternative formula for the supremum of two ideals in an arbitrary nearlattice.

Theorem 1.2.4. *For any two ideals* K_1 *and* K_2, $K_1 \vee K_2 = \bigcup_{n=0}^{\infty} B_n$ *where* $B_0 = \{x \in S \mid x = j(k_1, x, k_2), k_i \in K_i\}$ *and* $B_n = \{x \in S \mid x = j(b_1, x, b_2), b_1, b_2 \in B_{n-1}\}$, $n = 1, 2, \cdots$.

Proof : Clearly, $K_1, K_2 \subseteq B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_{n-1} \subseteq B_n \subseteq \cdots$. Suppose $b \in \bigcup_{n=0}^{\infty} B_n$ and $b_1 \le b;\ b_1 \in S$. Then $b \in B_m$ for some $m \ge 0$. Also, $b_1 = j(b, b_1, b)$ and so $b_1 \in B_{m+1}$. Thus $\bigcup_{n=0}^{\infty} B_n$ is a down set. Now suppose $t_1, t_2 \in \bigcup_{n=0}^{\infty} B_n$ such that $t_1 \vee t_2$ exists. Then there exist $r, s \ge 0$ such that $t_1 \in B_r$ and $t_2 \in B_s$. If $r \le s$ then $t_1, t_2 \in B_s$ and $t_1 \vee t_2 = j(t_1, t_1 \vee t_2, t_2)$ implies that $t_1 \vee t_2 \in B_{s+1}$. Hence , $\bigcup_{n=0}^{\infty} B_n$ is an ideal.

Finally, suppose H is an ideal containing K_1 and K_2. If $x \in B_0$ then $x = j(k_1, x, k_2) = (k_1 \wedge x) \vee (k_2 \wedge x)$ for some $k_1 \in K_1$ and $k_2 \in K_2$. Hence H is an ideal and $K_1, K_2 \subseteq H$, clearly $x \in H$. Then using the induction on n it is very easy to see that $H \supseteq B_n$ for each n. ●

In a lattice L, it is well known that for a convex sublattice C of L. $C = (C] \cap [C)$. The following figure (Fig:1.4) shows that for a convex subnearlattice C in a general nearlattice, this may not be true.

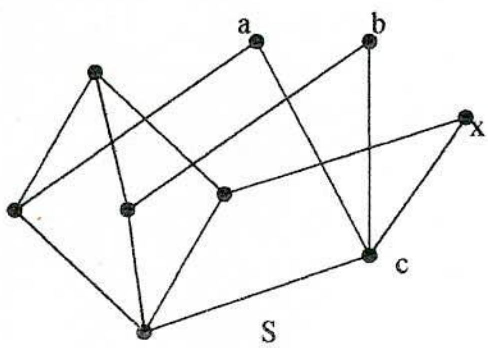


Figure 1.4

Here $C = \{a, b, c\}$ is a convex subnearlattice of S. Observe that $(C] = S$ and $[C) = \{a, b, c.x\}$, hence $(C] \cap [C) \neq C$.

Recently, Shiuly Akter [60] has proved that for a convex sublattice C of a distributive nearlattice S, $(C] = \{x \in S \mid x = (x \wedge c_1) \vee (x \wedge c_2) \vee \cdots \vee (x \wedge c_n)$ *for some* $c_1, c_2, \cdots, c_n \in C\}$. With the help of this result Rosen[54] have proved that $C = (C] \cap [C)$ when S is distributive. But in a non-distributive nearlattice of S, it is easy to show that $C = (C] \cap [C)$ when S is medial.

Theorem 1.2.5. {Cornish and Hickman [14, Theorem 1.1]}. *The following conditions on a nearlattice S are equivalent:*

(i) *S is distributive.*

(ii) *For any $H \in H(S)$, $(H] = \{h_1 \vee h_2 \vee \cdots \vee h_n \mid h_1, h_2, \cdots, h_n \in H\}$.*

(iii) *For any $I, J \in I(S)$, $I \vee J = \{a_1 \vee a_2 \vee \cdots \vee a_n \mid a_1, a_2, \cdots, a_n \in I \cup J\}$.*

(iv) *$I(S)$ is a distributive lattice.*

(v) *The map $H \to (H]$ is a lattice homomorphism of $H(S)$ onto $I(S)$ (which preserves arbitrary suprema).* ●

Observe here that by Theorem1.2.4, (iii) of above could easily be improved to (iii)$'$: For any $I, J \in I(S)$, $I \vee J = \{i \vee j \mid i \in I, j \in J\}$.

Let $I_f(S)$ denote the set of all *finitely generated ideals* of a nearlattice S. Of course $I_f(S)$ is an upper subsemilattice of $I(S)$. Also for any $x_1, x_2, \cdots, x_m \in S$, $(x_1, x_2, \cdots, x_m]$ is clearly equal to $(x_1] \vee (x_2] \vee \cdots \vee (x_m]$. When S is distributive, $(x_1, x_2, \cdots, x_m] \cap (y_1, y_2, \cdots, y_m] = ((x_1] \vee (x_2] \vee \cdots \vee (x_m]) \cap ((y_1] \vee (y_2] \vee \cdots \vee (y_m])$ $= \underset{i\,j}{\vee}(x_i \wedge y_j]$ for any $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n \in S$ and so $I_f(S)$is a distributive sublattice of $I(S)$.

A nearlattice S is said to be *finitely smooth* if the intersection of two finitely generated ideals is itself finitely generated. For example, distributive nearlattices, finite nearlattices, lattices, are finitely smooth. Hickman in [28] exhibited a nearlattice which is not finitely smooth.

From Cornish and Hickman [14], we know that a nearlattice S is distributive if and only if $I(S)$ is so. Our next result shows that the case is not the same with the modularity.

Theorem 1.2.6. *Let S be a nearlattice. If $I(S)$ is modular then S is also modular but the converse is not necessarily true.*

Proof: Suppose $I(S)$ is modular. Let $a,b,c \in S$ with $c \leq a$ and $b \vee c$ exists. Then $(c] \subseteq (a]$. Since $I(S)$ is modular, so, $(a \wedge (b \vee c)] = (a] \wedge ((b] \vee (c])$ $= ((a] \wedge (b]) \vee (c] = ((a \wedge b) \vee c]$. This implies that $a \wedge (b \vee c) = (a \wedge b) \vee c$, and so S is modular.

Nearlattice S of Figure 1.5 shows that the converse of this result is not true.

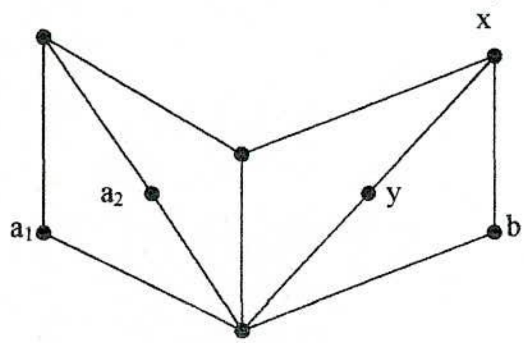


Figure 1.5

Notice that $(r]$ is modular for each $r \in S$. But in $I(S)$, clearly $\{(0], (a_1], (a_1, y], (a_2, b], S\}$ is a pentagonal sublattice. ●

The following theorem is due to Bazlar Rahman [9]

Theorem 1.2.7. {Bazlar Rahman[9]} *Let I and J be two ideals in a distributive nearlattice S. If $I \wedge J$ and $I \vee J$ are principal, then both I and J are principal.* ●

A non empty subset F of a nearlattice S is called an up set if for $x \in S$, $y \in F$ with $x \geq y$ imply $x \in F$.

A non empty subset F of a nearlattice S is called a filter if it is an up set and $f_1 \wedge f_2 \in F$ for all $f_1, f_2 \in F$.

An ideal P in a nearlattice S is called a prime ideal if $P \neq S$ and $x \wedge y \in P$ implies $x \in P$ or $y \in P$.

A filter F is called a prime filter if either $x \in F$ or $y \in F$ whenever $x \vee y$ exists and is in F.

It is not hard to see that a filter F of a nearlattice S is prime if and only if $S - F$ is a prime ideal. The set of all filters of a nearlattice is an upper (join) semilattice ; yet it is not a lattice in general, as there is no guarantee that the intersection of two filters is non empty. The join $F_1 \vee F_2$ of two filters is given by $F_1 \vee F_2 = \{s \in S \mid s \geq f_1 \wedge f_2 \text{ for some } f_1 \in F_1, f_2 \in F_2\}$. The smallest filter containing a subsemilattice H of S is $\{s \in S \mid s \geq h \text{ for some } h \in H\}$ and is denoted by $[H)$.
Moreover, the description of the join of filters shows that for all $a, b \in S$, $[a) \vee (b] = [a \wedge b)$.

Following theorem and corollary is due to Noor and Rahman [42] which is an extension of Stone's separation theorem of Gratzer [19] theorem 15, pp74.

Theorem 1.2.8. {Noor and Rahman[42]} *Let S be a nearlattice. The following conditions are equivalent:*

(i) *S is distributive.*

(ii) *For any ideal I and any filter F of S, such that $I \cap F = \Phi$, there exists a prime ideal $P \supseteq I$ and disjoint from F.* ●

Corollary 1.2.9. *A nearlattice S is distributive if and only if every ideal is the intersection of all prime ideals containing it.* ●

Lemma 1.2.10. *A subset F of a nearlattice S is a filter if and only if $S-F$ is a prime down set.*

Proof: Let $x \in S-F$ and $t \leq x$. Then $x \notin F$, and so $t \notin F$, as F is a filter. Hence $t \in S-F$, and so $S-F$ is a down set. Now let $x, y \in S$ such that $x \wedge y \in S-F$. It follows that $x \wedge y \notin F$. This implies either $x \notin F$ or $y \notin F$, as F is a filter. That is, either $x \in S-F$ or $y \in S-F$, and so $S-F$ is a prime down set.

Conversely, suppose $S-F$ is a prime down set. Let $x \in F$ and $t \geq x$. Then $x \notin S-F$ and so $t \notin S-F$ as $S-F$ is a prime down set. Thus $t \in F$ and so F is an upset. Finally let $x, y \in F$. Then $x \notin S-F$, $y \notin S-F$. Since $S-F$ is a prime, so $x \wedge y \notin S-F$. Therefore $x \wedge y \in F$, and so F is a filter. ●

Following result is an easy consequence of above lemma.

Lemma 1.2.11. A subset F *of a nearlattice* S *is a prime filter if and only if* $S-F$ is a prime ideal. ●

Now we include a generalization of theorem 1.2.8 in a general nearlattice.

Theorem 1.2.12. *Let S be a nearlattice. F be a filter and I be a down set such that $I \cap F = \Phi$. Then there exists a prime down set P containing I but disjoint to F.*

Proof: Let χ be the collection of all filter containing F *and disjoint to* I. Then χ is non-empty as $F \in \chi$. Suppose C is a chain in χ. Set $M = \cup\{X \mid X \in C\}$. Let $x \in M$ and $y \geq x$. Then $x \in X$ for some $X \in C$. Since X is a filter, so $y \in X$ and hence $y \in M$. Thus M is an upset. Now let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in C$. Since C is a chain, so either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$. This implies $x, y \in Y$, and so $x \wedge y \in Y$ as Y is a filter. It follows that $x \wedge y \in M$ and hence, M is a filter containing F. Moreover $M \cap I = \phi$. Therefore, M is the largest element of C. Thus by Zorn's lemma, M is a maximal filter containing F. Therefore by Lemma 1.2.10, $L-M$ is a minimal prime down set containing I but disjoint to F. ●

Corollary 1.2.13. *Let S be a nearlattice with 0 and F be a proper filter of S . Then there exists a prime down set P such that $F \cap P = \Phi$.*●

The following lemma is very useful in proving many results of distributive nearlattice.

Lemma 1.2.14. *If S_1 is a subnearlattice of a distributive nearlattice S and P_1 is a prime ideal in S_1, then there exists a prime ideal P in S such that $P_1 = S_1 \cap P$.*●

Following theorem is a generalization of Lemma 1.2.14, which will be needed in establishing some results in other chapters.

Theorem 1.2.15. *Let S_1 be a subnearlattice of S. and P_1 be a prime down set of S_1. Then there exists a prime down set P of S such that $P_1 = P \cap S_1$.*

Proof: Let H be a down set generated by P_1 in S. Then $H \cap (S_1 - P_1) = \Phi$. Now $S_1 - P_1$ is an upset in S_1 and $H \cap [S_1 - P_1) = \Phi$ where, $[S_1 - P_1)$ is the filter generated by $S_1 - P_1$ in S. Then by Theorem 1.2.12 , there exists a prime down set $P \supseteq H$ and disjoint to $[S_1 - P_1)$. Now $P_1 \subseteq H \cap S_1 \subseteq P \cap S_1$. Also $P \cap S_1 \subseteq P_1$. Hence, $P_1 = P \cap S_1$.●

1.3 Congruences

An equivalence relation Θ of a nearlattice S is called a congruence relation if $x_i \equiv y_i(\Theta)$ for $i = 1,2$ $(x_i, y_i \in S)$, then

(i) $x_1 \wedge x_2 \equiv y_1 \wedge y_2(\Theta)$, and

(ii) $x_1 \vee x_2 \equiv y_1 \vee y_2(\Theta)$ provided $x_1 \vee x_2$ and $y_1 \vee y_2$ exists.

It can be easily shown that for an equivalence relation Θ on S, the above conditions are equivalent to the conditions that for $x, y \in S$ if $x \equiv y(\Theta)$, then

(i$'$) $x \wedge t \equiv y \wedge t(\Theta)$ for all $t \in S$ and

(ii$'$) $x \vee t \equiv y \vee t(\Theta)$ for all $t \in S$ provided both $x \vee t$ and $y \vee t$ exists.

The set $C(S)$ of all congruences on S is an algebraic closure system on $S \times S$ and hence, when ordered by set inclusion, is an algebraic lattice.

Cornish and Hickman [14] showed that for an ideal I of a distributive nearlattice S, the relation $\Theta(I)$, defined by $x \equiv y(\Theta(I))$ if and only if $(x] \vee I = (y] \vee I$, is the smallest congruence containing I as a class. Moreover the equivalence relation $R(I)$, is defined by $x \equiv y(R(I))$ if and only if for any $s \in S$, $s \wedge x \in I$ is equivalent to $s \wedge y \in I$. In fact , this is the largest congruence of S having I as a class.

Suppose S is a distributive nearlattice and $x \in S$ we will use Θ_x as an abbreviation for $\Theta((x])$. Moreover ψ_x denote the congruence, defined by $a \equiv b\ (\psi_x)$ if and only if $a \wedge x = b \wedge x$.

Cornish and Hickman [14] also showed that for any two elements a,b of a distributive nearlattice S with $a \leq b$, the smallest congruence identifying a and b is equal to $\psi_a \cap \Theta_b$ and we denote it by $\Theta(a,b)$. Also in a distributive nearlattice S, they observed that if S has a smallest element 0, then clearly $\Theta_x = \Theta(0,x)$ for any $x \in S$.

Moreover, we see that:

(i) $\Theta_a \vee \psi_a = \tau$, the largest congruence of S.

(ii) $\Theta_a \cap \psi_a = \omega$, the smallest congruence of S and

(iii) $\Theta(a,b)' = \Theta_a \vee \psi_a$ where $a \le b$ and $(\,'\,)$ denotes the complement.

Now suppose S is an arbitrary nearlattice and $E(S)$ denote the lattice of equivalence relations. For $\Phi_1, \Phi_2 \in E(S)$ with $\Phi_1 \vee \Phi_2$ denoting their supremum $x \equiv y(\Phi_1 \vee \Phi_2)$ if and only if there exist $x = z_0, z_1, \cdots, z_n = y$ such that $z_{i-1} \equiv z_i(\Phi_1 \text{ or } \Phi_2)$ for $i = 1,2,\cdots,n$.

The following result was stated by Gratzer and Lakser in [21] without proof and a proof given below, appeared in Cornish and Hickman [14].

Theorem 1.3.1. *For any nearlattice S, $C(S)$ is a distributive (complete) sublattice of $E(S)$.*

Proof: Suppose $\Theta, \Phi \in C(S)$, Define ψ to be the supremum of Θ *and* Φ in the lattice of equivalence relations $E(S)$ on S. Let $x \equiv y(\psi)$. Then there exists $x = z_0, z_1, \cdots, z_n = y$ such that $z_{i-1} \equiv z_i(\Phi_1 \text{ or } \Phi_2)$. Thus, for any $t \in S$, $z_{i-1} \wedge t \equiv z_i \wedge t(\Phi_1 \text{ or } \Phi_2)$ as $\Theta, \Phi \in C(S)$.

Hence $x \wedge t \equiv y \wedge t(\psi)$ and consequently ψ is a semilattice congruence. Then, in particular $x \wedge y \equiv x(\psi)$ and $x \wedge y \equiv y(\psi)$. To show that ψ is a congruence, let $x \equiv y(\psi)$, with $x \le y$, and choose any $t \in S$ such that both $x \vee t$ and $y \vee t$ exists. Then there exists $z_0, z_1, z_2, \cdots, z_n$ such that $x = z_0, z_n = y$ and $z_{i-1} \equiv z_i(\Phi_1 \text{ or } \Phi_2)$. Put $w_i = z_i \wedge y$ for all $i = 0,1,\cdots,n$. Then $x = w_0, w_n = y$, $w_{i-1} \equiv w_i(\Phi_1 \text{ or } \Phi_2)$. Hence by the upper bound property, $w_i \vee t$ exists for all $i = 0,1,\cdots,n$ (as $w_i \vee t \le y \vee t$) and $w_{i-1} \vee t \equiv w_i \vee t(\Phi_1 \text{ or } \Phi_2)$ for all $i = 0,1,\cdots,n$ (as $\Theta, \Phi \in C(S)$), *i.e.* $x \vee t \equiv y \vee t(\psi)$. Then by Cornish and Noor [15] Lemma 2.3 ψ is a congruence on S. Therefore, $C(S)$ is a sublattice of the lattice $E(S)$.

To show the distributivity of $C(S)$, let $x \equiv y(\Theta \cap (\Theta_1 \vee \Theta_2))$. Then $x \wedge y \equiv y(\Theta)$ and $x \wedge y \equiv y(\Theta_1 \vee \Theta_2)$. Also $x \wedge y \equiv x(\Theta)$ and $x \wedge y \equiv x(\Theta_1 \vee \Theta_2)$.

Since $x \wedge y \equiv y(\Theta_1 \vee \Theta_2)$, there exists $t_0, t_1, \cdots, t_n$ such that (as we have seen in the proof of the first part), $x \wedge y = t_0, t_n = y$, $t_{i-1} \equiv t_i(\Theta_1 \text{ or } \Theta_2)$ and $x \wedge y = t_0 \leq t_i \leq y$ for each $i = 0,1,\cdots,n$. Hence $t_{i-1} \equiv t_i(\Theta)$ for all $i = 0,1,\cdots,n$ and so $t_{i-1} \equiv t_i(\Theta \cap \Theta_1)$ or $t_{i-1} \equiv t_i(\Theta \cap \Theta_2)$. Thus $x \wedge y \equiv y((\Theta \cap \Theta_1) \vee (\Theta \cap \Theta_2))$. By symmetry, $x \wedge y \equiv x((\Theta \cap \Theta_1) \vee (\Theta \cap \Theta_2))$ and the proof completes by transitivity of the congruences.●

In lattice theory it is well known that a lattice is distributive if and only if every ideal is a class of some congruence. Following theorem gives a generalization of this result in case of nearlattices.

This also characterizes the distributivity of a nearlattice, which is an extension of Cornish and Hickman [14] Theorem 3.1.

Thoerem 1.3.2. A nearlattice S *is distributive if and only if every ideal is a class of some congruence.*

Proof: Suppose S is distributive . Then by Cornish and Hickman [14] Theorem 3.1 for each ideal I of S $\Theta(I)$ is the smallest congruence containing I as a congruence class.

To prove the converse, let each ideal of S be a congruence class with respect to some congruence on S. Suppose S is not distributive. Then by Theorem 1.1.2, we have either N_5 (Figure 1.2) or M_5 (Figure 1.3) as a sublattice of S. In both cases consider $I = (a]$ and suppose I is a congruence class with respect to Θ. Since $d \in I$, $d \equiv a(\Theta)$. Now $b = b \wedge c = b \wedge (a \vee c) \equiv b \wedge (d \vee c) = b \wedge c = d(\Theta)$ That is, $b \equiv d(\Theta)$ and this implies $b \in I$, *i.e.* $b \leq a$ which is a contradiction. Thus S is distributive. ●

Following results are due to Noor and Rahman [41].

Theorem 1.3.3.{ Noor and Rahman [41]} *Let S be a distributive nearlattice then,*

(i) *For ideals I and J, $\Theta\left(I \cap J\right) = \Theta(I) \cap \Theta(J)$.*

(ii) *For ideals j_i $i \in A$ an indexed set, $\Theta\left(\vee J_i\right) = \vee\Theta(J_i)$.* •

Theorem 1.3.4.{ Noor and Rahman [41]} *For a distributive nearlattice S, the mapping $I \to \Theta(I)$ is an embedding from the lattice of ideals to the lattice of congruences.* •

1.4 SemiBoolean algebra

A lattice L with 0 and 1 is called a complemented lattice if for each $a \in L$ there exists $a' \in L$ such that $a \wedge a' = 0$ and $a \vee a' = 1$.

A distributive complemented lattice is called a Boolean lattice.

An algebra $(L; \wedge, \vee, ', 0, 1)$ is called a Boolean algebra if

(i) L is distributive lattice.

(ii) For each $a \in L$, $a \wedge 0 = 0 \wedge a = 0, a \vee 0 = 0 \vee a = a$.

(iii) For each $a \in L$, $a \wedge 1 = 1 \wedge a = a, a \vee 1 = 1 \vee a = 1$.

(iv) For each $a \in L$, there exist $a' \in L$ such that $a \wedge a' = 0$, $a \vee a' = 1$.

An interesting class of distributive nearlattices is provided by those semilattices in which each principal ideal is a Boolean algebra. These semilattices have been studied by Abbott [1],[2],[3] under the name of SemiBoolean algebras and mainly from the view of Abbott's implication algebras. An implication algebra is a groupoid $(I; \bullet)$ satisfying:

(i) $(ab)\, a = a$

(ii) $(ab)\, b = (ba)\, a$,

(iii) $a\,(bc) = b(ac)$

Abbott showed in [1, pp.227-236] that each implication algebra determines a SemiBoolean algebra and conversely each SemiBoolean algebra determines an implication algebra.

Following result gives a characterization of SemiBoolean algebras which is due to Cornish and Hickman [14]

Theorem 1.4.1. {Cornish and Hickman [14] Theorem2.2}. *A semilattice S is a SemiBoolean algebra if and only if the following conditions are satisfied:*

(i) *S has the upper bound property.*

(ii) *S is distributive.*

(iii) *S has a 0 and for any $x \in S$, $(x]^* = \{y \in S \mid y \wedge x = 0\}$ is an ideal and $(x] \vee (x]^* = S$.* ●

A nearlattice S is relatively complemented if each interval $[x, y]$ in S is complemented. That is, for $x \le t \le y$, there exists t' in $[x, y]$ such that $t \wedge t' = x$ and $t \vee t' = y$.

A nearlattice S with 0 is called sectionally complemented if $[0, x]$ is complemented for each $x \in S$. Of course every relatively complemented nearlattice S with 0 is sectionally complemented. It is not hard to show that S is SemiBoolean if and only if it is sectionally complemented and distributive. We denote the set of all prime ideals of S by $P(S)$.

There is a well known result in Lattice Theory due to Nachbin in 1947, [19] Theorem 22, pp-76 that a distributive lattice is Boolean if and only if its prime ideals are unordered. Following theorem is a generalization to this result which is due to Cornish and Hickman [14].

Theorem 1.4.2. {Cornish and Hickman [14]} *For a distributive nearlattice S with 0, the following conditions are equivalent:*

(i) *S is SemiBoolean.*

(ii) *$I_f(S)$ is a generalized Boolean algebra.*

(iii) *$P(S)$, the set of all prime ideals is unordered by set inclusion.* ●

Noor and Rahman [42] has proved the following theorem which is an extension of above result.

Theorem 1.4.3. {Noor and Rahman [42]} *Let S be a distributive nearlattice. S is relatively complemented if and only if $P(S)$ is unordered.* •

Theorem 1.4.4. {Noor and Rahman [42]}For a nearlattice S with 0 the following conditions are equivalent.

(i) S is SemiBoolean.

(ii) $I(S)$ is isomorphic to $C(S)$.

(iii) For all ideal I, $\Theta(I)=R(I)$. •

SOME SPECIAL ELEMENTS IN A NEARLATTICE

2.1 Introduction :

Gratzer and Schmidt [24] introduced the notion of some special elements e.g. distributive, standard and neutral elements to study a larger class of non-distributive lattices. Then Cornish and Noor [15] extended the concepts of standard and neutral elements for nearlattices. They also studied a new type of element known as strongly distributive element.

Recently Talukder and Noor [64] introduced the notion of modular elements in a join semilattice directed below. This notion is also applicable for general lattices.

In this chapter we introduce the concept of modular elements in a nearlattice. We have given several characterization of modular and strongly distributive elements. So by studying these elements and ideals, we will be able to study a larger class of non-distributive nearlattices.

In a lattice L an element $m \in L$ is called a *modular element* if for all $x, y \in L$ with $y \le x$, $x \wedge (m \vee y) = (x \wedge m) \vee y$ .Of course, in a modular lattice, every element is a modular element. Moreover, if every element of a lattice is modular, then the lattice itself is a modular lattice.

In the pentagonal lattice of Figure 1.2, observe that m is modular but t is not. Because, here $m < s$ and $s \wedge (t \vee m) = s$, But $(s \wedge t) \vee m = m$.

Let S be a nearlattice. An element $m \in S$ is called a *modular element* if for all $t, x, y \in S$ with $y \le x$, $x \wedge [(t \wedge m) \vee (t \wedge y)] = (t \wedge m \wedge x) \vee (t \wedge y)$. Of course, a nearlattice is modular if and only if its every element is modular.

In a lattice L, an element d is called a *distributive element* if for all $x, y \in L$, $d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y)$.

In order to introduce this notion for nearlattices, Cornish and Noor [15] could not give a suitable definition for distributive elements. But they discovered an element $d \in S$, such that $t \wedge d$ is a distributive element in the lattice $(t]$ for every $t \in S$. They found that these elements are also new even in case of lattices, and in fact, they are much stronger than the distributive elements. So they referred them as "*strongly distributive*" elements.

An element d of a nearlattice S is called a *strongly distributive* element if for all $t, x, y \in S$ $(t \wedge d) \vee (t \wedge x \wedge y) = [(t \wedge d) \vee (t \wedge x)] \wedge [(t \wedge d) \vee (t \wedge y)]$
In other words $t \wedge d$ is distributive in $(t]$ for each $t \in S$.

An element $s \in S$ is called a standard element if for all $t, x, y \in S$, $t \wedge [(x \wedge y) \vee (x \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$.

Due to Zaidur Rahman and Noor [67] we know that $s \in S$ is standard if and only if it is both modular and strongly distributive.

An element $s \in S$ is called neutral if (i) it is standard and (ii) for all $x, y, t \in S$, $s \wedge [(t \wedge x) \vee (t \wedge y)] = (s \wedge t \wedge x) \vee (s \wedge t \wedge y)$.

Let S be a nearlattice. For $a, b \in S$ we define $\langle a, b \rangle = \{x \in S \mid x \wedge a \leq b\}$, we call this a relative annihilator. Clearly $\langle a, b \rangle$ is a down set. Moreover, when S is distributive, this is an ideal, which is known as a relatively annihilator ideal.

In a lattice L, for $a, b \in L$ dually we define $\langle a,b\rangle_d = \{x \in L | x \vee a \geq b\}$. This is known as dual relative annihilator. This is an up set and in presence of distributive property of L this is a filter. Thus in a nearlattice S for $a, b, t \in S$ we define $\langle t \wedge a, t \wedge b\rangle_d = \{x \in (t] \| x \vee (t \wedge a) \geq t \wedge b\}$, that is a dual relative annihilator in $(t]$

In this chapter we give several characterizations of modular, strongly distributive, standard and neutral elements of a nearlattice.

2.2 Some special elements in a nearlattice

Theorem 2.2.1. *The definition of modular element in a nearlattice* S *coincides with the definition of modular element of a lattice, when* S *is a lattice.*

Proof: Suppose m is a modular element of the lattice S. Let $t,x,y\in S$ with $y\le x$, then $t\wedge y\le t\wedge x$. Since m is modular, so $(t\wedge m\wedge x)\vee(t\wedge y)=(t\wedge x)\wedge[m\vee(t\wedge y)]$ $=x\wedge[t\wedge(m\vee(t\wedge y))]=x\wedge[(t\wedge m)\vee(t\wedge y)]$, which is the definition of modularity of m in a nearlattice.

Conversely, Let m be modular according to the definition given for a nearlattice. Let $x,y\in S$ with $y\le x$.

Choose $t=m\vee y$. Then $x\wedge(m\vee y)=x\wedge((t\wedge m)\vee(t\wedge y))$

$$=(t\wedge m\wedge x)\vee(t\wedge y)$$
$$=(m\wedge x)\vee y$$

Hence m is modular according to the definition of modular element in a nearlattice. ●

Here is a characterization of modular elements in a lattice.

Theorem 2.2.2. *Let* L *be a lattice and* $m\in L$. *Then the following conditions are equivalent.*

(i) m *is modular.*

(ii) *For* $y\le x$ *with* $m\vee x=m\vee y$ *and* $m\wedge x=m\wedge y$ *implies* $x=y$.

Proof: $(i)\Rightarrow(ii)$; Suppose m is modular $y\le x$ and $m\vee x=m\vee y$, $m\wedge x=m\wedge y$.

Then $x=x\wedge(m\vee x)=x\wedge(m\vee y)=(x\wedge m)\vee y$ (by modularity of m)

$$=(y\wedge m)\vee y=y.$$

$(ii)\Rightarrow(i)$; Suppose (ii) holds.

Let $y\le x$, then $(x\wedge m)\vee y\le x\wedge(m\vee y)$ always holds.

Let $x\wedge(m\vee y)=p$ and $(x\wedge m)\vee y=q$. Then $q\le p$.

Now $p\wedge m=x\wedge m$

Also, $q \wedge m = m \wedge [(x \wedge m) \vee y] = m \wedge [(x \wedge m) \vee (x \wedge y)] = (m \wedge x) \wedge [(x \wedge m) \vee (x \wedge y)] = x \wedge m$.

Thus $p \wedge m = q \wedge m$.

Again, $q \vee m = y \vee m$

$$p \vee m = [x \wedge (m \vee y)] \vee m \leq (m \vee y) \vee m$$
$$= y \vee m = q \vee m \leq p \vee m$$

as $q \leq p$. Thus $p \vee m = q \vee m = y \vee m$.

Hence by (ii) $p = q$, that is $x \wedge (m \vee y) = (x \wedge m) \vee y$ and so m is modular. •

Now we extend the above result and give a characterization of a modular element m in a nearlattice.

Theorem 2.2.3. *Let* S *be a nearlattice and* $m \in S$. *Then the following conditions are equivalent.*

(i) m *is modular.*

(ii) *For* $t, x, y \in S$ *with* $y \leq x$, $(t \wedge m) \vee (t \wedge x) = (t \wedge m) \vee (t \wedge y)$ *and* $t \wedge m \wedge x = t \wedge m \wedge y$ *implies* $t \wedge x = t \wedge y$.

Proof: $(i) \Rightarrow (ii)$; Suppose m is modular, let $t, x, y \in S$ with $y \leq x$, $(t \wedge m) \vee (t \wedge x) = (t \wedge m) \vee (t \wedge y)$ and $t \wedge m \wedge x = t \wedge m \wedge y$.

Then $t \wedge x = (t \wedge x) \wedge [(t \wedge m) \vee (t \wedge x)] = (t \wedge x) \wedge [(t \wedge m) \vee (t \wedge y)]$

$$= (t \wedge m \wedge x) \vee (t \wedge y) \text{ (by modularity of m)}$$
$$= (t \wedge m \wedge y) \vee (t \wedge y) = t \wedge y.$$

$(ii) \Rightarrow (i)$; Suppose (ii) holds. Let $t, x, y \in S$ with $y \leq x$

Now $x \wedge [(t \wedge m) \vee (t \wedge y)] \geq (t \wedge m \wedge x) \vee (t \wedge y)$ always holds.

Let $x \wedge [(t \wedge m) \vee (t \wedge y)] = p$ and $(t \wedge m \wedge x) \vee (t \wedge y) = q$. Then $p \geq q$.

Choose $r = (t \wedge m) \vee (t \wedge y)$. Then $r \wedge p = p$ and $r \wedge q = q$.

$$r \wedge m = m \wedge [(t \wedge m) \vee (t \wedge y)] = (t \wedge m) \wedge [(t \wedge m) \vee (t \wedge y)] = t \wedge m.$$

Thus, $(r \wedge m) \vee (r \wedge q) = (t \wedge m) \vee q = (t \wedge m) \vee (t \wedge m \wedge x) \vee (t \wedge y) = (t \wedge m) \vee (t \wedge y) = r$.

Then $(r \wedge m) \vee (r \wedge p) \leq r = (r \wedge m) \vee (r \wedge q) \leq (r \wedge m) \vee (r \wedge p)$ as $q \leq p$

Hence $(r \wedge m) \vee (r \wedge p) = (r \wedge m) \vee (r \wedge q) = r$,

Also, $r \wedge m \wedge p = m \wedge p = m \wedge x \wedge [(t \wedge m) \vee (t \wedge y)] = x \wedge (t \wedge m) \wedge [(t \wedge m) \vee (t \wedge y)] = x \wedge t \wedge m$

and $r \wedge m \wedge q = m \wedge q = m \wedge [(t \wedge m \wedge x) \vee (t \wedge y)] = m \wedge t \wedge x \wedge [(t \wedge m \wedge x) \vee (t \wedge y)] = x \wedge t \wedge m$.

Thus $r \wedge m \wedge p = r \wedge m \wedge p$ and so by (ii) $r \wedge p = r \wedge q$, Hence $p = q$ and so m is modular. ●

Now we include the following result in a nearlattice which is parallel to the characterization theorem for modular elements in a lattice given in Theorem 2.2.2. But this cannot be considered as a definition of a modular element in a nearlattice.

Theorem 2.2.4. *Let* S *be a nearlattice and* $m \in S$. The *following conditions are equivalent.*

(i) *For all* $x, y \in S$ *with* $y \leq x$

$x \wedge (m \vee y) = (x \wedge m) \vee y$ *provided* $m \vee y$ *exists.*

(ii) *For all* $x, y \in S$ *with* $y \leq x$ *if* $m \vee x$, $m \vee y$ *exist and* $m \vee x = m \vee y$, $m \wedge x = m \wedge y$, *then* $x = y$.

Proof: $(i) \Leftrightarrow (ii)$ holds by the proof similar to the proof of Theorem.2.1.2 For the last part, let us consider the following nearlattice.

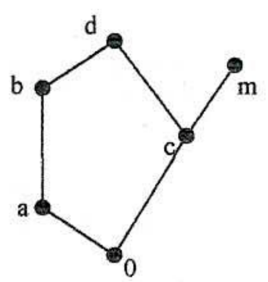


Figure-2.1

Observe that m satisfies the condition of Theorem 2.1.4

Here $a < b$ and $b \wedge [(d \wedge m) \vee (d \wedge a)] = b \wedge (c \vee a) = b \wedge d = b$.

But $(b \wedge d \wedge m) \vee (d \wedge a) = 0 \vee a = a$, so m is not modular. ●

Theorem 2.2.5. *In a Lattice, every strongly distributive element is distributive but the converse is not necessarily true.*

Proof. Let d be a strongly distributive element of a lattice L. Suppose $x, y \in L$ and $t = x \vee y \vee d$.

Then $d \vee (x \wedge y) = (t \wedge d) \vee (t \wedge x \wedge y) = [(t \wedge d) \vee (t \wedge x)] \wedge [(t \wedge d) \vee (t \wedge y)]$

$= (d \vee x) \wedge (d \vee y)$, and so d is distributive.

Now consider the lattice in figure 2.2.

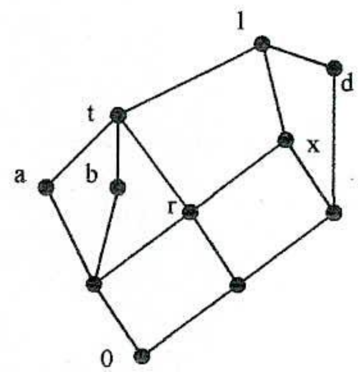


Figure 2.2

Here d is distributive but $(t \wedge d) \vee (t \wedge a \wedge b) = r < t = [(t \wedge d) \vee (t \wedge a)] \wedge [(t \wedge d) \vee (t \wedge b)]$ and so it is not strongly distributive. ●

Following characterization of strongly distributive elements in a nearlattice is due to Cornish and Noor [15].

Theorem 2.2.6. *Let* S *be a nearlattice and* $d \in S$. *Then the following conditions are equivalent.*

(i) d *is strongly distributive.*

(ii) *For all* $x, y, t \in S$, $(x \wedge [(t \wedge y) \vee (t \wedge d)]) \vee (t \wedge d) = (t \wedge x \wedge y) \vee (t \wedge d)$. ●

An element $s \in S$ is called a *standard element* if for all $t, x, y \in S$ $t \wedge [(x \wedge y) \vee (x \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$.

In a distributive nearlattice every element is standard. If every element of S is standard then S is itself a distributive nearlattice.

Theorem 2.2.7. *Every standard element in a nearlattice* S *is modular but a modular element may not be standard.*

Proof: Let $s \in S$ be standard, let $t, x, y \in S$ with $y \le x$

$$x \wedge [(t \wedge s) \vee (t \wedge y)] = x \wedge [(t \wedge y) \vee (t \wedge s)]$$

$$= (t \wedge x \wedge y) \vee (t \wedge s \wedge x)$$

$$= (t \wedge s \wedge x) \vee (t \wedge y)$$

So s is modular.

Conversely, consider the lattice of Figure 1.2

Here m is modular

But $\quad s \wedge (m \vee t) = s \wedge x = s$

$$(s \wedge m) \vee (s \wedge t) = m \vee 0 = m$$

So m is not standard. •

Theorem 2.2.8. *Every standard element is strongly distributive but the converse may not be true.*

Proof. Suppose s is standard in S. Let $t, a, b \in S$

Then, $\quad [(t \wedge s) \vee (t \wedge a)] \vee [(t \wedge s) \vee (t \wedge b)]$

$= ([(t \wedge s) \vee (t \wedge a)] \wedge (t \wedge s)) \vee ([(t \wedge s) \vee (t \wedge a)] \wedge (t \wedge b))$ (as s is standard.)

$= (s \wedge [(t \wedge a) \vee (t \wedge s)]) \vee (b \wedge [(t \wedge a) \vee (t \wedge s)])$

$= (t \wedge a \wedge s) \vee (t \wedge s) \vee (t \wedge a \wedge b) \vee (t \wedge a \wedge s)$

$= (t \wedge s) \vee (t \wedge a \wedge b)$

so s is strongly distributive.

In Figure 2.2, observes that t is strongly distributive, but it is not standard, because $d \wedge (x \vee t) > (d \wedge x) \vee (d \wedge t)$. •

Remark:

In the pentagonal lattice of Figure 1.2, m is modular and t is strongly distributive. Observe that, $m \le s$ and $s \wedge (t \vee m) = s \wedge x = s$, but $(s \wedge t) \vee m = 0 \vee m = m$. Thus t is not modular. On the other hand, $(x \wedge m) \vee (x \wedge s \wedge t) = m \vee 0 = m$, but $[(x \wedge m) \vee (x \wedge s)] \wedge [(x \wedge m) \vee (x \wedge t)] = (m \vee s) \wedge (m \vee t) = s \wedge x = s$ implies m is not strongly distributive.

We conclude the section with the following characterization of standard elements in a nearlattice.

Theorem 2.2.9. *Let* S *be a nearlattice. An element* $s \in S$ *is standard if and only if it is both modular and strongly distributive.*

Proof: If s is standard then by Theorem 2.2.7 and Theorem 2.2.8, s is both modular and strongly distributive. Conversely, suppose s is both modular and strongly distributive. Let $t, x, y \in S$.

Then, $(t \wedge x \wedge y) \vee (t \wedge x \wedge s) = (t \wedge x) \wedge [(x \wedge s) \vee (t \wedge x \wedge y)]$ (as s is modular)

$= (t \wedge x) \wedge [(x \wedge s) \vee (t \wedge x)] \wedge [(x \wedge s) \vee (x \wedge y)]$ (as s is strongly distributive)

$= t \wedge x \wedge [(x \wedge s) \vee (x \wedge y)] = t \wedge [(x \wedge s) \vee (x \wedge y)]$

so s is standard. ●

2.3 Relative annihilators

Relative annihilators have been studied by many authors. Mandelker [35] use the relative annihilators to characterize distributive and modular lattices. Noor and Islam [44] extended those results for nearlattices. In this section we use the relative annihilators to study different elements in both lattices and nearlattices.

Theorem 2.3.1. *Let L be a lattice. An element $s \in L$ is a distributive element if and only if $\langle s,b\rangle_d$ is a filter for all $b \in L$.*

Proof: Let s be a distributive element of a lattice L. Choose any $b \in L$, $x, y \in \langle s,b\rangle_d$.

Then $x \vee s \geq b$; $y \vee s \geq b$

Then $b \leq (s \vee x) \wedge (s \vee y) = s \vee (x \wedge y)$, as s is distributive, this implies $x \wedge y \in \langle s,b\rangle_d$.

If $x \in \langle s,b\rangle_d$ and $t \geq x$, then $s \vee x \geq b$ and so $s \vee t \geq b$; which implies $t \in \langle s,b\rangle_d$.

Therefore $\langle s,b\rangle_d$ is a filter.

Conversely, suppose $\langle s,b\rangle_d$ is a filter for all $b \in L$.

Let $x, y \in L$, then $s \vee x, s \vee y \geq (s \vee x) \wedge (s \vee y)$.

This implies $x \in \langle s,(s \vee x) \wedge (s \vee y)\rangle_d$ which implies $s \vee (x \wedge y) \geq (s \vee x) \wedge (s \vee y)$

Since the reverse inclusion is trivial, so $s \vee (x \wedge y) = (s \vee x) \wedge (s \vee y)$

Therefore s is distributive in L. ●

Theorem 2.3.2. *Let* S *be a nearlattice and $m \in S$.* m *is modular if and only if* for all *$a,b,t \in S$ with $b \leq a$, $t \wedge x \leq b$ and $t \wedge m \in \langle a,b\rangle$ then, $(t \wedge x) \vee (t \wedge m) \in \langle a,b\rangle$.*

Proof: Suppose m is modular, let $a,b,t \in S$, with $t \wedge x \leq b \leq a$ and $t \wedge m \in \langle a,b\rangle$.

Then $a \wedge t \wedge m \leq b$.

So, $a \wedge [(t \wedge m) \vee (t \wedge x)] = (a \wedge t \wedge m) \vee (t \wedge x) \leq b$.

Therefore, $(t \wedge m) \vee (t \wedge x) \in \langle a,b\rangle$.

Conversely, let the condition holds. Let $t, x, z \in S$ with $z \leq x$.

Then $(t \wedge z) \vee (t \wedge x \wedge m) \leq x$ and $t \wedge z \in ((t \wedge z) \vee (t \wedge x \wedge m)]$.

Also, $t \wedge x \wedge m \leq (t \wedge z) \vee (t \wedge x \wedge m)$.

This implies $t \wedge m \in \langle x, (t \wedge z) \vee (t \wedge x \wedge m) \rangle$.

Hence by the given condition $(t \wedge z) \vee (t \wedge m) \in \langle x, (t \wedge z) \vee (t \wedge x \wedge m) \rangle$.

This implies $x \wedge [(t \wedge m) \vee (t \wedge z)] \leq (t \wedge x \wedge m) \vee (t \wedge z)$.

Since the reverse inequality is trivial.

So, $x \wedge [(t \wedge m) \vee (t \wedge z)] = (t \wedge x \wedge m) \vee (t \wedge z)$.

Therefore m is modular. •

Theorem 2.3.3. *Let* S *be a nearlattice and* $s \in S$ *is strongly distributive if and only if* $\langle t \wedge s, t \wedge r \rangle_d$ *is a filter in* $(t]$, for all $t, r \in S$.

Proof: Suppose s is strongly distributive. Let $t \wedge x \in \langle t \wedge s, t \wedge r \rangle_d$.

Then $(t \wedge s) \vee (t \wedge x) \geq t \wedge r$.

If $t \wedge p \geq t \wedge x$, then $(t \wedge s) \vee (t \wedge p) \geq (t \wedge s) \vee (t \wedge x) \geq t \wedge r$.

This implies $t \wedge p \in \langle t \wedge s, t \wedge r \rangle_d$.

Now let $t \wedge x, t \wedge y \in \langle t \wedge s, t \wedge r \rangle_d$.

Then $(t \wedge s) \vee (t \wedge x) \geq t \wedge r$ and $(t \wedge s) \vee (t \wedge y) \geq t \wedge r$.

So $[(t \wedge x) \wedge (t \wedge y)] \vee (t \wedge s) = [(t \wedge s) \vee (t \wedge x)] \wedge [(t \wedge s) \vee (t \wedge y)]$ (as s is strongly distributive)

$$\geq t \wedge r$$

And so $(t \wedge x) \wedge (t \wedge y) \in \langle t \wedge s, t \wedge r \rangle_d$.

Therefore $\langle t \wedge s, t \wedge r \rangle_d$ *is a filter in* $(t]$.

Conversely, suppose $\langle t \wedge s, t \wedge r \rangle_d$ *is a filter in* $(t]$ for all $r \in S$. Let $x, y \in S$.

Suppose $r = [(t \wedge s) \vee (t \wedge x)] \wedge [(t \wedge s) \vee (t \wedge y)] = t \wedge r$.

Then $(t \wedge s) \vee (t \wedge x) \geq t \wedge r$ and $(t \wedge s) \vee (t \wedge y) \geq t \wedge r$.

This implies $t \wedge x,\ t \wedge y \in \langle t \wedge s,\ t \wedge r \rangle_d$.

Since $\langle t \wedge s,\ t \wedge r \rangle_d$ *is a filter in* $(t]$.

So, $t \wedge x \wedge y \in \langle t \wedge s,\ t \wedge r \rangle_d$.

Therefore, $(t \wedge s) \vee (t \wedge x \wedge y) \geq t \wedge r$.

That is, $(t \wedge s) \vee (t \wedge x \wedge y) \geq [(t \wedge s) \vee (t \wedge x)] \wedge [(t \wedge s) \vee (t \wedge y)]$.

But the reverse inequality is trivial.

Therefore, $(t \wedge s) \vee (t \wedge x \wedge y) = [(t \wedge s) \vee (t \wedge x)] \wedge [(t \wedge s) \vee (t \wedge y)]$

and so s is strongly distributive. ●

Here is a characterization of standard elements.

Theorem 2.3.4. *Let* S *be a nearlattice and* $s \in S$. *Then s is standard if and only if* for all $a, b, t, x \in S$ *with* $t \wedge x, t \wedge s \in \langle a,\ b \rangle$ *implies* $(t \wedge x) \vee (t \wedge s) \in \langle a,\ b \rangle$.

Proof: Suppose s is standard. Let $t \wedge x, t \wedge s \in \langle a,\ b \rangle$.

Then $a \wedge t \wedge x \leq b$, $a \wedge t \wedge s \leq b$.

Thus $a \wedge [(t \wedge x) \vee (t \wedge s)] = (a \wedge t \wedge x) \vee (a \wedge t \wedge s) \leq b$.

Therefore, $(t \wedge x) \vee (t \wedge s) \in \langle a,\ b \rangle$.

Conversely, suppose the condition holds for all $a, b \in S$, let $t, x, y \in S$.

Now $t \wedge x \wedge y \leq (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$ and $t \wedge x \wedge s \leq (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$

So, $t \wedge y,\ t \wedge s \in \langle x, (t \wedge x \wedge y) \vee (t \wedge x \wedge s) \rangle$.

Hence by the given condition, $(t \wedge y) \vee (t \wedge s) \in \langle x, (t \wedge x \wedge y) \vee (t \wedge x \wedge s) \rangle$.

This implies $x \wedge [(t \wedge y) \vee (t \wedge s)] \leq (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$.

Since the reverse inequality is trivial,

so, $x \wedge [(t \wedge y) \vee (t \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$.

Hence s is standard.●

Theorem 2.3.5. *Let* S *be a nearlattice and* $s \in S$. *s is neutral if and only if*

(i) $\langle s, b\rangle$ *is an ideal for all* $b \in S$

(ii) For all $a, b \in S$ *and* $t \wedge y, t \wedge s \in \langle a, b\rangle$ *implies* $(t \wedge y) \vee (t \wedge s) \in \langle a, b\rangle$.

Proof: Suppose s is neutral. Let $x, y \in \langle s, b\rangle$ for $b \in S$ and $x \vee y$ exists.

Then $x \wedge s \leq b$, $y \wedge s \leq b$.

So, $s \wedge (x \vee y) = (s \wedge x) \vee (s \wedge y)$ (as s is dual distributive)

$\leq b$

This implies $x \vee y \in \langle s, b\rangle$.

Hence $\langle s, b\rangle$ is an ideal.

Since s is neutral, so it is standard.

Hence (ii) follows immediately from above theorem.

Conversely, suppose the conditions holds.

By above theorem condition (ii) implies that s is standard.

Let $x, y \in S$ with $x \vee y$ exists.

Now, $s \wedge x \leq (s \wedge x) \vee (s \wedge y)$, $s \wedge y \leq (s \wedge x) \vee (s \wedge y)$.

So, $x, y \in \langle s, (s \wedge x) \vee (s \wedge y)\rangle$.

Since $\langle s, (s \wedge x) \vee (s \wedge y)\rangle$ is an ideal,

So, $x \vee y \in \langle s, (s \wedge x) \vee (s \wedge y)\rangle$.

Then $s \wedge (x \vee y) \leq (s \wedge x) \vee (s \wedge y)$.

So s is dual distributive and so it is neutral. ●

An element s is called an *Upper* element of a nearlattice S if $s \vee x$ exists for all $x \in S$.

By Cornish and Noor [15], an element $s \in S$ is called a *Central* element if

i) s is upper and neutral

and ii) s is complemented in each interval containing it.

We conclude this section with the following characterization of central elements in a nearlattice. Since for a central element s, $s \vee x$ exists for every $x \in S$, so we can talk about the dual relative annihilators $< s, y >_d$ for every $y \in S$.

Theorem 2.3.6. *Let s be an upper element of a nearlattice S. Then s is central if and only if* *i) s is neutral,*

and *ii) for all $a, b \in S$, there exists $t \in S$ with $a \wedge s \leq t \leq b \vee s$ such that $t \in < s, a > \cap < s, b >_d$.*

Proof. Suppose s is central. Then of course s is neutral. Now for $a, b \in S$, $a \wedge s \leq s \leq b \vee s$.

Then there exists $t \in S$ such that $s \wedge t = a \wedge s \leq a$ and $s \vee t = b \vee s \geq b$. This implies $t \in < s, a > \cap < s, b >_d$. Also $a \wedge s = s \wedge t \leq t \leq s \vee t = b \vee s$, and so (ii) holds.

Conversely, suppose (i) and (ii) hold. Let $a \leq s \leq b$. Then by (ii) there exists $t \in S$ with $a \wedge s \leq t \leq b \vee s$ (*then* $a \leq t \leq b$) such that $t \in < s, a > \cap < s, b >_d$. Then $s \wedge t \leq a \leq s \wedge t$ and $s \vee t \geq b \geq s \vee t$, and so $s \wedge t = a$ and $s \vee t = b$. Thus t is the relative complement of s in $[a, b]$. Also by (i) s is neutral and hence s is central. ●

2.4 Modular ideals in a nearlattice

An ideal M of a nearlattice S is called a modular ideal if it is a modular element of the ideal lattice $I(S)$. That is, M is modular if for all $I, J \in I(S)$ with $J \subseteq I$, $I \cap (M \vee J) = (I \cap M) \vee J$.

An ideal I of a nearlattice S is called a standard ideal if it is standard element of the ideal lattice $I(S)$.

Of course, every standard ideal of a nearlattice (lattice) is modular, but the converse need not be true. In this section we include several characterizations of modular ideals of a nearlattice.

Due to Cornish and Noor[15] we know that the supremum of two ideals in a nearlattice is not very easy to handle.

But due to Talukder and Noor[64], we know that for a standard ideal K of a nearlattice S and for any $J \in I(S)$, $K \vee J = \{k \vee j \mid k \in K, j \in J\}$

But in case of a modular ideal M of a nearlattice, we are unable to give a simple description of $M \vee J$. Even $x \in M \vee J$ does not imply $x \leq m \vee j$ for some $m \in M$ and $j \in J$.

For example, consider the following nearlattice S of Figure 2.3 and ideal lattice $I(S)$ of Figure 2.4.

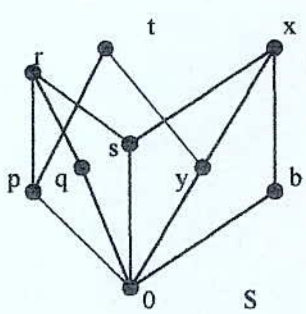


Figure 2.3

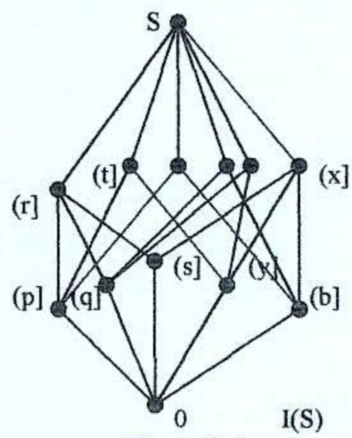


Figure 2.4

Here S is a modular nearlattice by Theorem 2.2.1. In $I(S)$, $(b]$ is modular. Now $q \in (t] \vee (b]$. But $q \nleq p \vee q$ for any $p \in (t]$ and $q \in (b]$.

Theorem 2.4.1. *Let L be a lattice and $m \in L$, m is modular if and only if $(m]$ is modular in $I(L)$.*

Proof : Suppose m is modular in L. Suppose $J \subseteq I$. Let $x \in I \cap ((m] \vee J)$.

Then $x \in I$ and $x \in (m] \vee J$.

This implies $x \leq m \vee j$ for some $j \in J$.

So $x \vee j \leq m \vee j$.

Now $j \in J \subseteq I$.

Thus $x \vee j \in I$ and $x \vee j = (x \vee j) \wedge (m \vee j) = ((x \vee j) \wedge m) \vee j$ (as m is modular)

$\in (I \cap (m]) \vee J$.

Therefore, $x \in (I \cap (m]) \vee J$.

Since the reverse inclusion is trivial, so $I \cap ((m] \vee J) = (I \cap (m]) \vee J$.

Hence $(m]$ is modular in $I(L)$.

Conversely, let $(m]$ be modular in $I(L)$.

Suppose $z \leq x$. Then $(x] \wedge ((m] \vee (z]) = ((x] \wedge (m]) \vee (z]$

That is, $(x \wedge (m \vee z)] = ((x \wedge m) \vee z]$

Therefore, $x \wedge (m \vee z) = (x \wedge m) \vee z$, and so m is modular. ●

Our next result shows that in a nearlattice S, Theorem 2.4.1 is not true.

Theorem 2.4.2. *For an element m of a nearlattice S, if $(m]$ is modular in $I(S)$, then m is modular, but the converse may not be true.*

Proof: Suppose $(m]$ is a modular ideal in S. Let $z \leq x$.

Then for all $t \in S$ $t \wedge z \leq t \wedge x \leq x$ implies $(t \wedge z] \subseteq (x]$.

Now $(t \wedge x] \wedge [(t \wedge m] \vee (t \wedge z]] \subseteq (t \wedge x] \wedge [(m] \vee (t \wedge z]]$

$$= ((t \wedge x] \wedge (m]) \vee (t \wedge z] \subseteq (t \wedge x] \wedge [(t \wedge m] \vee (t \wedge z]]$$

So $(t \wedge x] \wedge [(t \wedge m] \vee (t \wedge z]] = ((t \wedge x] \wedge (t \wedge m]) \vee (t \wedge z]$.

This implies $((t \wedge x) \wedge ((t \wedge m) \vee (t \wedge z))] = ((t \wedge x \wedge m) \vee (t \wedge z)]$.

And so, $x \wedge [(t \wedge m) \vee (t \wedge z)] = (t \wedge x) \wedge [(t \wedge m) \vee (t \wedge z)] = (x \wedge t \wedge m) \vee (t \wedge z)$

Therefore, m is modular in S.

To prove the converse, let us consider the following nearlattice and its ideal lattice.

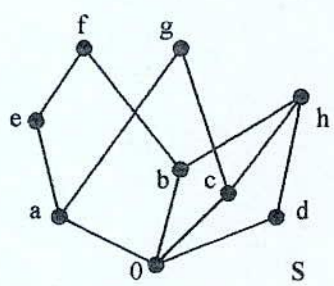


Figure 2.5

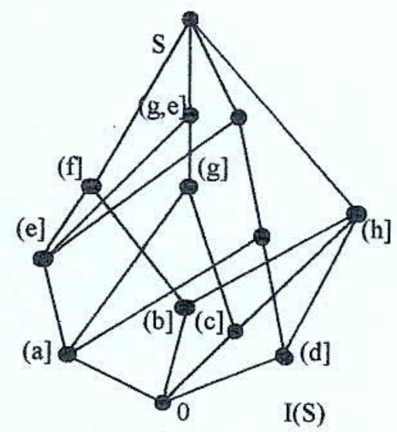


Figure 2.6

Here d is modular in S. But in $I(S)$ (Figure 2.6), $\{(0], (d], (g], (g,e], S\}$ is a pentagonal sublattice. Hence $(d]$ is not a modular ideal. ●

Theorem 2.4.3. *Let S be a nearlattice,* $I, J \in I(S)$ *and* $I, J \in (a]$ *for some* $a \in S$. *Then* $I \vee J = \{x \in S \mid x \leq i \vee j \text{ for some } i \in I, j \in J\}$

Proof: Let $x \in I \vee J$. Then by Theorem 1.2.1, $x \leq i \vee j$ for some $i, j \in A_{n-1}$, where $A_0 = I \cup J$.

Since $i, j \in A_{n-1}$, so $i \leq i_1 \vee j_1$, $j \leq i_2 \vee j_2$ for some $i_1, i_2, j_1, j_2 \in A_{n-2}$.

Then $x \leq i_1 \vee i_2 \vee j_1 \vee j_2$, the supremum exists by the upper bound property of S as $i_1, i_2, j_1, j_2 \leq a$. Thus proceding in this way $x \leq (p_1 \vee \cdots \vee p_n) \vee (q_1 \vee \cdots \vee q_n)$ for some $p_i, q_i \in A_0 = I \cup J$, and the supremum exists by the upper bound property again.

Therefore, $x \le i \vee j$ for some $i \in I$, $j \in J$. •

Theorem 2.4.4. *Let M be a modular ideal of a nearlattice S and J be an ideal. If $x \le m \vee j$ for some $m \in M, j \in J$, then $x \vee j = m_1 \vee j$ for some $m_1 \in M$.*

Proof: Let $x \le m \vee j$, then $x \vee j \le m \vee j$.

Thus, $x \vee j \in (x \vee j] \cap (M \vee (j]) = ((x \vee j] \cap M) \vee (j]$.

So by Theorem 2.3.3, $x \vee j \le p \vee q$ for some $p \in (x \vee j] \cap M$ and $q \in (j]$.

Since $p \in (x \vee j] \cap M$, so $p \in M$ and $p \le x \vee j$.

Thus $x \vee j \le p \vee q \le p \vee j \le x \vee j$ implies $x \vee j = p \vee j$, where $p \in M$. •

Here is a characterizations of modular ideals in a nearlattice.

Theorem 2.4.5. *Let M be an ideal of a nearlattice S with the condition that for all ideals J of S, and $M \vee J = \{x \in S \mid x \le m \vee j,\ m \vee j$ exists for some $m \in M, j \in J\}$. Then the following conditions are equivalent.*

(i) M is modular.

(ii) $x \in M \vee J$ implies $x \vee j = m \vee j$ for some $m \in M$, $j \in J$.

Proof: $(i) \Rightarrow (ii)$; Suppose M is modular. Let $x \in M \vee J$. Then by the given condition, $x \le m \vee j$ for some $m \in M$, $j \in J$.

Then by theorem 2.4.4,

$x \vee j = m_1 \vee j$ and so (ii) holds.

$(ii) \Rightarrow (i)$; Suppose (ii) holds.

Let $I, J \in I(S)$ with $J \subseteq I$

Suppose $x \in I \cap (M \vee J)$. Then $x \in I$ and $x \in M \vee J$.

Thus by given condition, $x \vee j = m \vee j$ for some $m \in M$, $j \in J$.

Now, $m \le x \vee j$ implies $m \in I \cap M$.

Therefore, $x \in (I \cap M) \vee J$, and so $I \cap (M \vee J) \subseteq (I \cap M) \vee J$.

Since the reverse inclusion is trivial. so $I \cap (M \vee J) = (I \cap M) \vee J$.

Hence M is modular. ●

In lattices, we know from [64] that an element m is modular if and only if for all $b \leq a$ with $a \wedge m = b \wedge m$ & $a \vee m = b \vee m$ imply $a = b$.

We conclude the chapter with the following result which is proved by above characterization of modular elements.

Theorem 2.4.6. *Let M be a modular ideal of a nearlattice S. If $I \cap M$ and $I \vee M$ are principal, then I is principal.*

Proof: Let $I \vee M = (a]$ and $I \cap M = (b]$.

Then by Theorem 2.3.3, $a \leq i \vee m$ for some $i \in I, m \in M$.

Thus, $(a] = M \vee I \supseteq M \vee (b \vee i] \supseteq M \vee (i] \supseteq (a]$.

This implies $M \vee I = M \vee (b \vee i]$.

Also, $(b] = M \cap I \supseteq M \cap (b \vee i] \supseteq (b]$.

implies $M \cap I = M \cap (b \vee i]$.

Moreover, $(b \vee i] \subseteq I$.

Therefore, $I = (b \vee i]$ as M is modular.●

0-DISTRIBUTIVE NEARLATTICE & SEMI-PRIME IDEALS IN A NEARLATTICE

3.1 Introduction:

J.C. Varlet [66] has given the definition of a 0-distributive lattice to generalize the notion of pseudocomplemented lattice. According to him a lattice L with 0 is called a 0-distributive lattice if for all $a,b,c \in L$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$. In other words, a lattice with 0 is 0-distributive if and only if for each $a \in L$, the set of elements disjoint to a is an ideal of L. Of course, every distributive lattice with 0 is 0-distributive. Also, every pseudocomplemented lattice is 0-distributive. In fact, in a pseudocomplemented lattice L, the set of all elements disjoint to $a \in L$, is a principal ideal $(a^*]$. Many authors including Balasubramani and Venkatanarasimhan [6], Jayaram [30] and Pawar and Thakare [51] studied the 0-distributive and 0-modular properties in lattices and meet semilattices. In fact, Jayaram [30] has referred the condition of 0-distributive nearlattice given in this chapter as weakly 0-distributive semilattice in a general meet semilattice.

Recently, Rav [52] has generalized the concept of 0-distributivity and gave the definition of semi-prime ideals in a lattice. An ideal I of a lattice L is called a *semi-prime ideal* if for all $x, y, z \in L$, $x \wedge y \in I$ and $x \wedge z \in I$ imply $x \wedge (y \vee z) \in I$. Thus, for lattice L with 0, L is called *0-distributive* if and only if (0] is a semi-prime ideal. In a distributive lattice L, every ideal is a semi-prime ideal. Moreover, every prime ideal is semi-prime. In a pentagonal lattice (Figure 3.1) (0] is semi-prime but not prime. Here (b] and (c] are prime, but $(a]$ is not even semi-prime. Again in Figure 3.2, $(0]$, $(a]$, $(b]$, $(c]$ are not semi-prime.

In this chapter provide a number of characterization of 0-distributive these nearlattices. We also extend the concept of 0-distributivty and give the notion of semi-prime ideals in nearlattice. Then we include a number of separation properties in a general nearlattice with respect to the annihilator ideals. Moreover, by studying a congruence related to Glivenko congruence we give a separation theorem related to separation properties in distributive nearlattices given by Noor and Bazlar Rahman [42].

Let us define a 0-distributive nearlattice as follows: A nearlattice S with 0 is called 0-distributive if for all $x, y, z \in S$ with $x \wedge y = 0 = x \wedge z$ and $y \vee z$ exists imply $x \wedge (y \vee z) = 0$.

It can be easily proved that it has the following alternative definition:

S is 0-distributive if for all $x, y, z, t \in S$ with $x \wedge y = 0 = x \wedge z$ imply $x \wedge ((t \wedge y) \vee (t \wedge z)) = 0$; $(t \wedge y) \vee (t \wedge z)$ exists by the upper bound property of S. Of course, every distributive nearlattice S with 0 is 0-distributive. Figure 3.1 is an example of a non-modular nearlattice which is 0-distributive, while Figure 3.2 gives a modular nearlattice which is not 0-distributive.

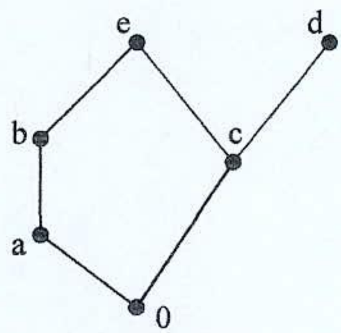


Figure 3.1

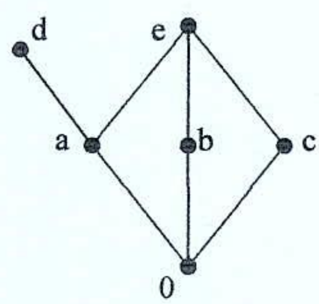


Figure 3.2

A proper filter M of a nearlattice S is called *maximal* if for any filter Q with $Q \supseteq M$ implies either $Q = M$ or $Q = S$. Dually, we define a *minimal prime ideal* (*down set*)

Let L be a lattice with 0. An element a^* is called the *pseudocomplement* of a if $a \wedge a^* = 0$ and if $a \wedge x = 0$ for some $x \in L$, then $x \leq a^*$. A lattice L with 0 and 1 is called *pseudocomplemented* if its every element has a pseudocomplement. Since a nearlattice S with 1 is a lattice, so the concept of pseudocomplementation is not possible in a general nearlattice. A nearlattice S with 0 is called sectionally pseudocomplemented if the interval $[0, x]$ for each

$x \in S$ is pseudocomplemented. For $A \subseteq S$, we denote $A^{\perp} = \{x \in S \mid x \wedge a = 0 \text{ for all } a \in A\}$. If S is distributive then clearly $A^{\perp}$ is an ideal of S.

Moreover, $A^{\perp} = \bigcap_{a \in A} \{\{a\}^{\perp}\}$. If A is an ideal, then obviously $A^{\perp}$ is the pseudocomplement of A in $I(S)$ and we denote it by A^{*}. Therefore, for a distributive nearlattice S with 0, I(S) is pseudocomplemented.

3.2 0-Distributive Nearlattice

Theorem 3.2.1. *If a nearlattice S with 0 is sectionally pseudocomplemented, then $I(S)$ is pseudocomplemented.*

Proof. Suppose S is sectionally pseudocomplemented. Let $I \in I(S)$. $I^{\perp} = \{x \in S \mid x \wedge i = 0$ for all $i \in I\}$. Suppose $x \in I^{\perp}$ and $t \leq x$. Then $x \wedge i = 0$ for all $i \in I$ and so $t \wedge i = 0$ for all $i \in I$. Hence $t \in I^{\perp}$. Now let $x, y \in I^{\perp}$ and $x \vee y$ exists. Let $r = x \vee y$. Then $0 \leq x, y, r \wedge i \leq r$ for all i, and $x \wedge (r \wedge i) = 0 = y \wedge (r \wedge i)$. Since $[0, r]$ is pseudocomplemented, $x, y \leq (r \wedge i)^{+}$ for all $i \in I$, where $(r \wedge i)^{+}$ is the relative pseudocomplement of $r \wedge i$ in $[0, r]$. Then $x \vee y \leq (r \wedge i)^{+}$, and so $r \wedge i \wedge (x \vee y) = 0$. That is $i \wedge (x \vee y) = 0$ for all $i \in I$. This implies $x \vee y \in I^{\perp}$. Therefore, $I^{\perp}$ is an ideal. Clearly $I^{\perp}$ is the pseudocomplement of I in $I(S)$. Hence $I(S)$ is pseudocomplemented. ●

Following example (Figure 3.3) shows that $I(S)$ can be pseudocomplemented but S is not sectionally pseudocomplemented.

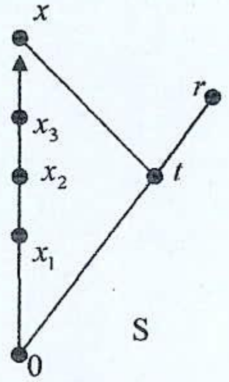


Figure 3.3

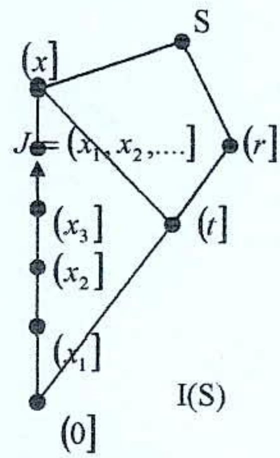


Figure 3.4

In S, observe that t has no sectionally pseudocomplement in $[0, x]$. But $I(S)$ is pseudocomplement and the ideal J is the pseudocomplement of both $(t]$ and $(r]$. Again, Figure 3.1 gives a non-distributive nearlattice S where $I(S)$ is pseudocomplemented.

Theorem 3.2.2. *If the intersection of all prime ideals of a nearlattice S with 0 is* $\{0\}$, *then S is 0-distributive.*

Proof. Let $a,b,c \in S$ such that $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists. Let P be any prime ideal of S. If $a \in P$, then $a \wedge (b \vee c) \leq a$ implies that $a \wedge (b \vee c) \in P$. If $a \notin P$, then by the primeness of P, $b,c \in P$, and so $b \vee c \in P$. This implies $a \wedge (b \vee c) \in P$. Thus $a \wedge (b \vee c)$ is in every prime ideal P of S, and hence $a \wedge (b \vee c) = 0$, proving that S is 0-distributive. ●

From Bazlar Rahman [9] we know that a nearlattice S is distributive if and only if $I(S)$ is distributive, which is also equivalent to that $D(S)$, the lattice of filters of S is distributive. Thus if S is a nearlattice with 0 such that $I(S)$ (similarly $D(S)$) is distributive, then S is 0-distributive.

Following lemma are needed for further development of the thesis.

Lemma 3.2.3. *Every proper filter of a nearlattice with 0 is contained in a maximal filter.*

Proof. Let F be a proper filter in S with 0.Let $\mathcal{F}$ be the set of all proper filters containing F. Then $\mathcal{F}$ is non-empty as $F \in \mathcal{F}$. Let C be a chain in $\mathcal{F}$ and let $M = \bigcup\{X \mid X \in C\}$. We claim that M is a filter with $F \subseteq M$. Let $x \in M$ and $y \geq x$. Then $x \in X$ for some $X \in C$. Hence $y \in X$ as X is a filter. Therefore, $y \in M$. Let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in C$. Since C is a chain, either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$. So $x, y \in Y$. Then $x \wedge y \in Y$ as Y is a filter. Hence $x \wedge y \in M$. Moreover M contains F. So M is a maximum element of C. Then by Zorn's lemma $\mathcal{F}$ has a maximal element, say Q with $F \subseteq Q$. ●

Lemma 3.2.4. *Let S be a nearlattice with 0. A proper filter M in S is maximal if and only if for any element* $a \notin M$ *there exists an element* $b \in M$ *with* $a \wedge b = 0$.

Proof. Suppose M is maximal and $a \notin M$. Let $a \wedge b \neq 0$ for all $b \in M$. Consider $M_1 = \{y \in S \mid y \geq a \wedge b, \textit{for some } b \in M\}$. Clearly M_1 is a filter and is proper as $0 \notin M$. For every $b \in M$ we have $b \geq a \wedge b$ and so $b \in M_1$. Thus $M \subseteq M_1$. Also $a \notin M$ but $a \in M_1$. So $M \subset M_1$, which contradicts the maximality of M. Hence there must exist some $b \in M$ such that $a \wedge b = 0$.

Conversely, if the proper filter M is not maximal, then as $0 \in S$, there exists a maximal filter N such that $M \subset N$. For any element $a \in N - M$ there exists an element $b \in M$ such that $a \wedge b = 0$. Hence $a, b \in N$ imply $0 = a \wedge b \in N$, which is a contradiction. Thus M must be a maximal filter. •

Following result gives several nice characterizations of 0-distributive nearlattice.

Theorem 3.2.5. *For a nearlattice S with 0, the following conditions are equivalent:*

(i) *S is 0-distributive.*

(ii) $\{a\}^{\perp}$ *is an ideal for all* $a \in S$.

(iii) $A^{\perp}$ *is an ideal for all* $A \subseteq S$.

(iv) $I(S)$ *is pseudocomplemented.*

(v) $I(S)$ *is 0-distributive.*

(vi) *Every maximal filter is prime.*

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ are trivial.

$(iii) \Rightarrow (iv)$; For any ideal I of S, $I^{\perp}$ is clearly the pseudocomplement of I in $I(S)$ if $I^{\perp} \in I(S)$, and so (iv) holds.

$(iv) \Rightarrow (v)$; Since every pseudocomplemented lattice is 0-distributive, so $(iv) \Rightarrow (v)$.

$(v) \Rightarrow (vi)$; Let $I(S)$ be 0-distributive and F be a maximal filter. Suppose $f, g \notin F$ with $f \vee g$ exists. By Lemma 3.2.4, there exist $a, b \in F$ such that $a \wedge f = 0 = b \wedge g$. Hence $(f] \wedge (a \wedge b] = (0]$ and $(g] \wedge (a \wedge b] = (0]$. Then $(f \vee g] \wedge (a \wedge b] = ((f] \vee (g]) \wedge (a \wedge b] = (0]$, by 0-distributivity of $I(S)$. Hence $(f \vee g) \wedge (a \wedge b) = 0$. Since F is maximal, $0 \notin F$. Therefore $f \vee g \notin F$, and so F is prime.

$(vi) \Rightarrow (i)$; Let (vi) holds. Suppose $a,b,c \in S$ such that $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists. If $a \wedge (b \vee c) \neq 0$, then by Lemma 3.2.3, $a \wedge (b \vee c) \in F$ for some maximal filter F of S. Then $a \in F$ and $b \vee c \in F$. As F is prime, by assumption, so either $a \in F$ and $b \in F$ or $c \in F$. That is, either $a \wedge b \in F$ or $a \wedge c \in F$. This implies $0 \in F$, which gives a contradiction and hence $a \wedge (b \vee c) = 0$. In other words, S is 0-distributive. •

Corollary 3.2.6. *In a 0-distributive nearlattice, every proper filter is contained in a prime filter.* •

Theorem 3.2.7. *Every prime down set of a nearlattice contains a minimal prime down set.*

Proof. Let P be a prime down set of L and let χ be the set of all prime down sets J such that $J \subseteq P$. Then P is non-empty since $P \in \chi$. Let C be a chain in χ and let $M = \cap\{X : X \in C\}$.

We claim that M is a prime down set. M is non-empty as $0 \in M$. Let $a \in M$ and $b \leq a$. Then $a \in X$ for all $X \in C$. Hence $b \in X$ for all $X \in C$ as X is a down set. Then $b \in M$. Now let $x \wedge y \in M$ for some $x, y \in S$. Then $x \wedge y \in X$ for all $X \in C$. As X is a prime down set, so either $x \in X$ or $y \in X$. Thus either $M = \cap\{X : x \in X\}$ or $M = \cap\{X : y \in X\}$, proving that either $x \in M$ or $y \in M$. Thus M is a prime down set. Thus by applying the dual form of Zorn's Lemma, we conclude the existence of a minimal member of P. •

Theorem 3.2.8. *In a 0-distributive nearlattice S, if $\{0\} \neq A$ is the intersection of all non-zero ideals of S, then $A^{\perp} = \left\{x \in S \,\middle|\, \{x\}^{\perp} \neq \{0\}\right\}$.*

Proof. Let $x \in A^{\perp}$. Then $x \wedge a = 0$ for all $a \in A$. Since $A \neq \{0\}$, so $\{x\}^{\perp} \neq \{0\}$. Thus $x \in \{x \in S \mid \{x\}^{\perp} \neq \{0\}\}$. That is $A^{\perp} \subseteq \{x \in S \mid \{x\}^{\perp} \neq \{0\}\}$.

Conversely, let $x \in \{x \in S \mid \{x\}^{\perp} \neq \{0\}\}$. Since S is 0-distributive, so $\{x\}^{\perp}$ is a non-zero ideal of S. Then $A \subseteq \{x\}^{\perp}$ and so $A^{\perp} \supseteq \{x\}^{\perp\perp}$. This implies $x \in A^{\perp}$,which completes the proof. ●

Theorem 3.2.9. *Let S be a nearlattice with 0. S is 0-distributive if and only if for any filter F disjoint with $\{x\}^{\perp}$; $x \in S$, there exist a prime filter containing F and disjoint with $\{x\}^{\perp}$.*

Proof. Let S be 0-distributive. Consider the set $\mathcal{F}$ of all filters of S containing F and disjoint with $\{x\}^{\perp}$. Clearly $\mathcal{F}$ is non-empty as $F \in \mathcal{F}$. Then using Zorn's lemma, there exists a maximal element Q in $\mathcal{F}$. Now we claim that $x \in Q$. If not, then $Q \vee [x) \supset Q$. So by the maximality of Q, $\{Q \vee [x)\} \cap \{x\}^{\perp} \neq \phi$. Then there exists $t \in Q \vee [x)$ and $t \in \{x\}^{\perp}$. Then $t \geq q \wedge x$ for some $q \in Q$ and $t \wedge x = 0$. Thus, $0 = t \wedge x \geq q \wedge x$, and so $q \wedge x = 0$. This implies $q \in \{x\}^{\perp}$, which contradicts the fact that $Q \cap \{x\}^{\perp} = \phi$. Therefore $x \in Q$. Finally, let $z \notin Q$. Then $\{Q \vee [z)\} \cap \{x\}^{\perp} \neq \phi$. Let $y \in \{Q \vee [z)\} \cap \{x\}^{\perp}$. Then $y \wedge x = 0$ and $y \geq q \wedge z$ for some $q \in Q$. Thus $0 = y \wedge x \geq q \wedge x \wedge z$, which implies $q \wedge x \wedge z = 0$. Now $x \in Q$ implies $q \wedge x \in Q$, and $z \wedge (q \wedge x) = 0$. Hence by Lemma 3.2.4, Q is a maximal filter of S, and so by Theorem 3.2.5, Q is prime.

Conversely, let $x \wedge y = 0 = x \wedge z$ and $y \vee z$ exists. If $x \wedge (y \vee z) \neq 0$. Then $y \vee z \notin \{x\}^{\perp}$. Thus $[y \vee z) \cap \{x\}^{\perp} = \phi$. So, there exists a prime filter Q containing $[y \vee z)$ and disjoint with $\{x\}^{\perp}$. As $y, z \in \{x\}^{\perp}$, so $y, z \notin Q$. Thus $y \vee z \notin Q$, as Q is prime. This implies $[y \vee z) \not\subset Q$, a contradiction. Hence $x \wedge (y \vee z) = 0$ and so S is 0-distributive. ●

Pawar and Thakare [51] have mentioned as a corollary to the above result that for distinct elements $a, b \in S$ for which $a \wedge b \neq 0$ are separated by a prime filter in a 0-distributive semilattice , which is not true. For example, Figure 3.1 is an example of a 0-

distributive nearlattice, where a,b are distinct and $a \wedge b \neq 0$. But there does not exist any prime filter containing b but not containing a.

Now we give few more characterizations for 0-distributive nearlattices.

Theorem 3.2.10. *Let S be a nearlattice with 0. Then the following conditions are equivalent:*

(i) *S is 0-distributive.*

(ii) *Every maximal filter of S is prime.*

(iii) *Every minimal prime down set of S is a minimal prime ideal.*

(iv) *Every proper filter of S is disjoint from a minimal prime ideal.*

(v) *For each non-zero element $a \in S$, there is a minimal prime ideal not containing a.*

(vi) *Each non-zero element $a \in S$ is contained in a prime filter.*

Proof. $(i) \Leftrightarrow (ii)$; follows from Theorem 3.2.5.

$(ii) \Leftrightarrow (iii)$; Let A be a minimal prime down set. Then S-A is a maximal filter. Then by (ii), S-A is a prime filter, and so A is an ideal. That is, A is a minimal prime ideal.

$(iii) \Rightarrow (ii)$; Let F be a maximal filter of S . Then S-F is a minimal prime down set . Thus by (iii) S-F is a minimal prime ideal and so F is a prime filter.

$(i) \Rightarrow (iv)$; Let F be a proper filter of S. Then by Corollary 3.2.6, there is a prime filter $Q \supseteq F$. Then S-Q is a minimal prime ideal disjoint from F.

$(iv) \Rightarrow (v)$; Let $a \in S$ and $a \neq 0$. Then $[a)$is a proper filter. Then by (iv) there exists a minimal prime ideal A such that $A \cap [a) = \phi$. Thus $a \notin A$.

$(v)\Rightarrow(iv)$; Let $a\in S$ and $a\neq 0$. Then by (v) there is a minimal prime ideal P such that $a\notin P$. Thus $a\in L-P$ and L-P is a prime filter.

$(vi)\Rightarrow(i)$; Let S be not 0-distributive. Then there exist $a,b,c\in S$ such that $a\wedge b=0=a\wedge c$ and $b\vee c$ exists but $a\wedge(b\vee c)\neq 0$. Then by (vi) there exists a prime filter Q such that $a\wedge(b\vee c)\in Q$. Let $F=[a\wedge(b\vee c))$. This is proper as $0\notin F$ and $F\subseteq Q$. Now, $a\wedge(b\vee c)\in Q$ implies $a\in Q$ and $b\vee c\in Q$. Since $a\wedge b=0=a\wedge c$, so $b,c\notin Q$ as $0\notin Q$, but $b\vee c\in Q$, which contradicts that Q is prime. Hence $a\wedge(b\vee c)=0$ and so S is 0-distributive. ●

Theorem 3.2.11. *Let S be a 0-distributive nearlattice and $x\in S$. Then a prime ideal P containing $\{x\}^{\perp}$ is a minimal prime ideal containing $\{x\}^{\perp}$ if and only if for $p\in P$ there is $q\in S-P$ such that $p\wedge q\in\{x\}^{\perp}$.*

Proof. Let P be a prime ideal of S containing $\{x\}^{\perp}$ such that the given condition holds. Let K be a prime ideal containing $\{x\}^{\perp}$ such that $K\subseteq P$. Let $p\in P$. Then there is $q\in S-P$ such that $p\wedge q\in\{x\}^{\perp}$. Hence $p\wedge q\in K$. Since K is prime and $q\notin K$, so $p\in K$. Thus , $P\subseteq K$ and so $K=P$. Therefore, P must be a minimal prime ideal containing $\{x\}^{\perp}$.

Conversely, let P be a minimal prime ideal containing $\{x\}^{\perp}$. Let $p\in P$. Suppose for all $q\in S-P$, $p\wedge q\notin\{x\}^{\perp}$. Set $D=(S-P)\vee[p)$. We claim that $\{x\}^{\perp}\cap D=\varphi$. If not , let $y\in\{x\}^{\perp}\cap D$. Then $y\geq r\wedge p$ for some $r\in S-P$. Thus , $p\wedge r\leq y\in\{x\}^{\perp}$, which is a contradiction to the assumption. Then by Theorem 3.2.9, there exists a maximal (prime) filter $Q\supseteq D$ and disjoint with $\{x\}^{\perp}$. By the proof of Theorem 3.1.9, $x\in Q$. Let M = S-Q. Then M is prime ideal. Since $x\in Q$, so $x\notin M$. Let $t\in\{x\}^{\perp}$. Then $t\wedge x=0\in M$ implies $t\in M$ as M is prime. Thus $\{x\}^{\perp}\subseteq M$.

Now $M \cap D = \phi$. Therefore, $M \cap (S - P) = \phi$, and hence $M \subseteq P$. Also $M \neq P$, because $p \in D$ implies $p \notin M$ but $p \in P$. Hence M is a prime ideal containing $\{x\}^{\perp}$ which is properly contained in P. This gives a contradiction to the minimal property of P.
Therefore, the given condition holds. ●

Now we refer the reader about a conjecture made by Noor and Bazlar Rahman [42] that whether the well known Stone's separation property holds in a 0-distributive nearlattice. Separation theorem for distributive nearlattices is given in [42]. Unfortunately this does not hold even in case of a 0-distributive lattice. Consider the pentagonal lattice $\{0, a, b, c, 1;\ 0 < a < b < 1,\ 0 < c < 1\}$, which is 0-distributive. Consider $I = (a]$ and $F = [b)$. Here $I \cap F = \phi$ and there does not exist any prime filter Q containing F and disjoint with I.

But in a 0-distributive nearlattice, instead of a general ideal, we can give a separation theorem for an annihilator ideal $I = J^{\perp}$ when J is a subset of S. An ideal I in a nearlattice S with 0 is called an annihilator ideal if $I = J^{\perp}$ for some $J \subseteq S$.

Recently, Zaidur Rahman, Bazlar Rahman and Noor [68] have studied the semi-prime ideals in a nearlattice. This concept was given by Rav [52] in a general lattice. An ideal I of a nearlattice S is called a *semi-prime ideal* if for all $x, y, z \in S$, $x \wedge y \in I$ and $x \wedge z \in I$ imply $x \wedge (y \vee z) \in I$ provided $y \vee z$ exists. Thus, for nearlattice S with 0, S is called *0-distributive* if and only if (0] is a semi-prime ideal in S. In a distributive nearlattice S, every ideal is a semi-prime ideal. Moreover, every prime ideal is semi-prime. From [68], it is known that for any subset A of a nearlattice S, $A^{\perp}$ is a semi-prime ideal if S is 0-distributive. Here we give a separation theorem by using the semi-prime ideals.

Theorem 3.2.12. *(The Separation Theorem) A nearlattice S is 0-distributive if and only if for a proper filter F and an annihilator $I = J^{\perp}$, where J is a non empty subset of S, with $F \cap I = \phi$, there exists a prime filter Q containing F such that $Q \cap I = \phi$.*

Proof. Suppose S is 0-distributive and $I = J^{\perp}$ for some non-empty subset J of S. Let $\mathcal{F}$ be the set of all filters containing F, and disjoint with I. Then using Zorn's lemma, there exists a maximal filter Q containing F and disjoint with I. Since by Theorem 5 of [68] I is semi-prime, so by Theorem 10 of [68], Q is prime.

Conversely, suppose the condition holds. Suppose S is not 0-distributive. Then there exist $a,b,c \in S$ such that $a \wedge b = 0$, $a \wedge c = 0$ and $a \wedge (b \vee c) \neq 0$, $b \vee c$ exists. Then $b \vee c \notin \{a\}^{\perp}$. Let $F = [b \vee c)$. Since $0 \notin F$, F is proper. Then proceeding according to the proof of converse part of Theorem 3.2.9, we find that $a \wedge (b \vee c) = 0$, and so S is 0-distributive.●

Define *a relation* R on a nearlattice with *0* by aRb if and only if $a \wedge x = 0$ is equivalent to $b \wedge x = 0$. Equivalently, aRb if and only if $(a]^{\perp} = (b]^{\perp}$.

Theorem 3.2.13. *Let S be a nearlattice with 0. then the above relation is a meet congruence. Moreover, when S is distributive, then it is a nearlattice congruence.*

Proof. Clearly R is an equivalence relation. Let $a \equiv b(R)$ and $t \in S$. Then $a \wedge x = 0$ if and only if $b \wedge x = 0$. Let $(a \wedge t) \wedge x = 0$ for some $x \in S$. Then $a \wedge (t \wedge x) = 0$ implies $b \wedge (t \wedge x) = 0$, and so $(b \wedge t) \wedge x = 0$. Similarly $(b \wedge t) \wedge x = 0$ implies $(a \wedge t) \wedge x = 0$. Therefore $a \wedge t \equiv b \wedge t(R)$ and so R is a meet congruence.

Now let $a \vee t, b \vee t$ exists and $a \equiv b(R)$. Then for any $x \in S$, $a \wedge x = 0$ if and only if $b \wedge x = 0$ .Let $(a \vee t) \wedge x = 0$ for some $x \in S$.Then $a \wedge x = 0$ and $t \wedge x = 0$ which implies $b \wedge x = 0$ and $t \wedge x = 0$, and so $(b \vee t) \wedge x = 0$ as S is distributive. Similarly $(b \vee t) \wedge x = 0$ for some $x \in S$ implies $(a \vee t) \wedge x = 0$. Therefore, R is a nearlattice congruence.●

Theorem 3.2.14. *If S is 0-distributive, then R is a nearlattice congruence.*

Proof. Suppose $(a \vee t) \wedge x = 0$ for some $x \in S$. Since $a \wedge x, t \wedge x \leq (a \vee t) \wedge x$ so $a \wedge x = t \wedge x = 0$, which implies $b \wedge x = t \wedge x = 0$ as $a \equiv b(R)$. This implies $(b \vee t) \wedge x = 0$ as

S is 0-distributive. Similarly $(b \vee t) \wedge x = 0$ implies $(a \vee t) \wedge x = 0$. Hence R is a nearlattice congruence. ●

By Theorem 3.2.5, we have the Corollary.

Corollary 3.2.15. *If I(S) is pseudocomplemented, then R is a nearlattice congruence .* ●

A nearlattice S with 0 is called Weakly complemented if for any pair of distinct elements a, b of S, there exists an element c disjoint from one of these elements but not from the other.

Theorem 3.2.16. *S is weakly complemented if and only if R is an equality relation and hence is a nearlattice congruence .*

Proof. Suppose S is weakly complemented. Let $a \equiv b(R)$. Suppose $a \neq b$. Then there exists c such that $a \wedge c = 0$ but $b \wedge c \neq 0$. This implies $a \not\equiv b(R)$, contradiction . Hence $a = b$. So, R is an equality relation. That is, R is a nearlattice congruence.

Suppose R is equality. We need to prove S is weakly complemented . Let $a, b \in S$ and $a \neq b$. Then $a \not\equiv b(R)$. This implies there exists $c \in S$ such that $a \wedge c = 0$ but $b \wedge c \neq 0$. Hence S is weakly complemented. ●

In the following nearlattice S, R is a nearlattice congruence. Here the classes are {0},{a},{b},{1},{c,d,e}. But S is neither 0-distributive nor weakly complemented.

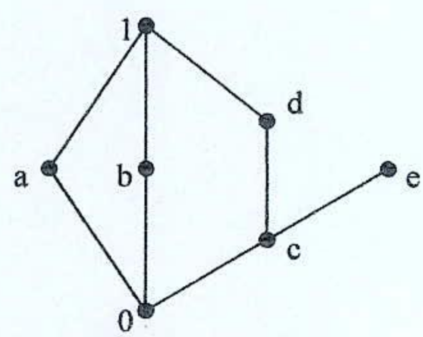


Figure 3.5

Theorem 3.2.17. *For any nearlattice S, the quotient lattice* $\frac{S}{R}$ *is weakly complemented. Furthermore, a nearlattice S with 0 is 0-distributive if and only if* $\frac{S}{R}$ *is a distributive nearlattice and R is a nearlattice congruence.*

Proof. Let A and B be two classes in $\frac{S}{R}$ such that A< B. Then there exists $a \in A$ and $b \in B$ such that a< b in S . So, by the definition of R there is an element $c \in S$, such that $a \wedge c = 0$ but $b \wedge c \neq 0$. Suppose $x \in [0]$. Then $x \equiv 0(R)$ and so $0 \wedge x = 0$ which implies $x \wedge x = x = 0$. So $[0] = \{0\}$. This implies $A \wedge C = [a] \wedge [c] = \{0\}$ but $B \wedge C \neq \{0\}$. Hence $\frac{S}{R}$ is weakly complemented.

Now let S be a nearlattice for which R is a nearlattice congruence and $\frac{S}{R}$ is distributive. Let $a, b, c \in S$ with $a \wedge b = 0 = a \wedge c$ such that $b \vee c$ exists. Then $[a] \wedge ([b] \vee [c]) = ([a] \wedge [b]) \vee ([a] \wedge [c]) = [0] \vee [0] = [0]$. This implies $[a \wedge (b \vee c)] = [0]$. Since $[0] = \{0\}$, so $a \wedge (b \vee c) = 0$. Hence S is 0-distributive .

Conversely, let S be 0-distributive. Then by Theorem 3.2.13, R is a nearlattice congruence. Let $[a],[b],[c] \in \frac{S}{R}$. We need to prove $[a] \wedge ([b] \vee [c]) = ([a] \wedge [b]) \vee ([a] \wedge [c])$ provided $[b] \vee [c]$ exists. Suppose $[b] \vee [c] = [d]$. Then $[b] = [b] \wedge [d] = [b \wedge d]$, $c = [c] \wedge [d] = [c \wedge d]$, and so $[b] \vee [c] = [(b \wedge d) \vee (c \wedge d)]$. So we need to prove that $[a \wedge ((b \wedge d) \vee (c \wedge d))] = [(a \wedge b \wedge d) \vee (a \wedge c \wedge d)]$. Let $a \wedge ((b \wedge d) \vee (c \wedge d)) \wedge x = 0$. Since $(a \wedge b \wedge d) \vee (a \wedge c \wedge d) \leq a \wedge ((b \wedge d) \vee (c \wedge d))$ so, $((a \wedge b \wedge d) \vee (a \wedge c \wedge d)) \wedge x = 0$. On the other hand, if $((a \wedge b \wedge d) \vee (a \wedge c \wedge d)) \wedge x = 0$, then $a \wedge b \wedge d \wedge x = 0 = a \wedge c \wedge d \wedge x$ and by 0-distributivity of S, $a \wedge ((b \wedge d) \vee (c \wedge d)) \wedge x = 0$.

Thus $a \wedge ((b \wedge d) \vee (c \wedge d)) \equiv (a \wedge (b \wedge d)) \vee (a \wedge (c \wedge d))(R)$ and hence $[a] \wedge ([b] \vee [c]) = ([a] \wedge [b]) \vee ([a] \wedge [c])$. •

Theorem 3.2.18. *If a 0-distributive nearlattice S is weakly complemented then S is distributive*

Proof. If S is weakly complemented. Then by Theorem 3.1.15, R is an equality relation and so by above theorem $S \cong \frac{S}{R}$ implies S is distributive. . ●

A nearlattice S with 0 is called Sectionally complemented if the intervals [0,x] are complemented for each $x \in S$. A nearlattice which is sectionally complemented and distributive is called a Semi Boolean nearlattice.

Corollary 3.2.19. *If a 0-distributive nearlattice S is sectionally complemented and weakly complemented, then S is semi Boolean.* ●

Theorem3.2.20. *Suppose S is sectionally complemented and in every interval [0,x] , every element has a unique relative complement . Then S is semi Boolean if and only if it is 0-distributive.*

Proof. Let S be 0-distributive and for every $x \in S$, the interval [0,x] is unicomplemented. Let $x, y \in S$ with $x \neq y$. If they are comparable, without loss of generality, suppose x< y. Then $0 \leq x < y$. Then there exists a unique $t \in [0, y]$ such that $t \wedge x = 0$ and $t \vee x = y$. Thus $t \wedge x = 0$ but $t \wedge y = t \neq 0$. If x, y are not comparable, then $0 \leq x \wedge y < x$ and $0 \leq x \wedge y < y$. Then there exist $s, t \in S$ such that $x \wedge y \wedge s = 0$, $(x \wedge y) \vee s = x$, $x \wedge y \wedge t = 0$ and $(x \wedge y) \vee t = y$. Now $s \wedge t \leq x \wedge y$ implies $s \wedge t \leq x \wedge y \wedge s = 0$, which implies $s \wedge t = 0$. Now $s \wedge t = 0$ and $s \wedge x \wedge y = 0$ implies $0 = s \wedge ((x \wedge y) \vee t) = s \wedge y$ as S is 0-distributive, but $s \wedge x \neq 0$. Therefore, S is weakly complemented and so by above corollary, S is semi Boolean. Since the reverse implication always holds in a Semi-Boolean nearlattice, this completes the proof. ●

There is another characterization of 0-distributive nearlattices.

Theorem 3.2.21. *Let S be a nearlattice with 0. Then S is 0-distributive if and only if $[0,x]$ is a 0-distributive lattice for every $x \in S$.*

Proof: Let S is a nearlattice with 0 then S *is* 0-distributive. Then trivially $[0,x]$ is also 0-distributive.

Conversely, suppose $[0,x]$ for all $x \in S$. Let $a,b,c \in S$ with $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists. Let $a \wedge (b \vee c) = t$ Consider the interval $[0, b \vee c]$. Then $t \in [0, b \vee c]$. Also $b,c \in [0, b \vee c]$

Now $\quad t \wedge b = a \wedge (b \vee c) \wedge b = a \wedge b = 0$

$\quad t \wedge c = a \wedge (b \vee c) \wedge c = a \wedge c = 0$

Since $[0, b \vee c]$ is 0-distributive, so, $t \wedge (b \vee c) = 0$.

So, $0 = t \wedge (b \vee c) = a \wedge (b \vee c) \wedge (b \vee c) = a \wedge (b \vee c)$

Hence, S is 0-distributive. ●

Now we give a generalization of theorem 1.4.1

Theorem 3.2.22. *Let S be a 0-distributive nearlattice and $[0,x]$ be 1-distributive for every $x \in S$, then the following conditions are equivalent.*

(i) *S is sectionally complemented.*

(ii) $(x] \vee (x]^{\perp} = (x] \vee (x]^{*} = S$ *for every* $x \in S$

(iii) *The prime ideals of $[0,x]$ are unordered for each $x \in S$.*

Proof: $(i) \Rightarrow (ii)$; Suppose S is sectionally complemented. Then for every $x \in S$, $[0,x]$ is complemented. If (ii) does not holds, then there exist elements $s,t \in S$ such that $s \notin (t] \vee (t]^{*}$. Now $0 \leq s \wedge t \leq s$. Then by (i), there exists $r \in [0,s]$ such that $r \wedge s \wedge t = r \wedge t = 0$ and $r \vee (s \wedge t) = s$. Thus $r \in (t]^{*}$ and so $s = r \vee (s \wedge t) \in (t]^{*} \vee (t]$ gives a contradiction. Therefore, (ii) must holds.

$(ii) \Rightarrow (iii)$; Suppose *(ii)* holds but *(iii)* does not. Then there exist prime ideal P, Q of some $[0,x]$, $x \in S$ such that $P \subset Q$. Thus there exists $y \in Q - P$. Since Q is a prime ideal of $[0,x]$, $x \notin Q$. By (ii) $(y] \vee (y]^* = S$ Thus $x \in (y] \vee (y]^*$. Then $x \leq p \vee q$ for some $p \in (y]$ and $q \in (y]^*$. Then $q \wedge y = 0 \in P$. Since $y \notin P$ and P is prime, so $q \in P \subset Q$. Also $p \leq y$ implies $p \in Q$. Therefore, $x \leq p \vee q$ implies $x \in Q$ gives a contradiction. Hence the prime ideals of $[0,x]$ for each $s \in S$ are unordered.

$(iii) \Rightarrow (i)$; Since here every $[0,x]$ is both a 0-distributive and 1-distributive lattice, so by [53], $[0,x]$ must be complemented.

3.3 Semi-prime ideals in a Nearlattice

An ideal I of a nearlattice S is called a *semi-prime ideal* if for all $x, y, z \in S$, $x \wedge y \in I$ and $x \wedge z \in I$ imply $x \wedge (y \vee z) \in I$ provided $y \vee z$ exists. Thus, for a nearlattice S with 0, S is called *0-distributive* if and only if (0] is a semi-prime ideal. In a distributive nearlattice S, every ideal is a semi-prime ideal. Moreover, every prime ideal is semi-prime. Of course every nearlattice S with 0 itself is semi-prime. In the nearlattice of Figure 3.1, (b] and (d] are prime, (c] is not prime but semi-prime and $(a]$ is not even semi-prime. Again in Figure 3.2, (0], $(a]$, $(b]$, $(c]$ and (d] are not semi-prime.

Lemma 3.3.1. *Non empty intersection of all prime (semi-prime) ideals of a nearlattice is a semi-prime ideal.*

Proof: Let $a, b, c \in S$ and $I = \bigcap\{P : P$ is a prime ideal $\}$ and I is nonempty. Let $a \wedge b \in I$ and $a \wedge c \in I$. Then $a \wedge b \in P$ and $a \wedge c \in P$ for all P. Since each P is prime (semi-prime), so $a \wedge (b \vee c) \in P$ for all P. Hence $a \wedge (b \vee c) \in I$, and so I is semi-prime. ●

Corollary3.3.2. *Intersection of two prime(semi-prime) ideals is a semi-prime ideal.* ●

Lemma 3.3.3. *Every filter disjoint from an ideal I is contained in a maximal filter disjoint from I.*

Proof: Let F be a filter in L disjoint from I. Let $\mathcal{F}$ be the set of all filters containing F and disjoint from I. Then $\mathcal{F}$ is nonempty as $F \in \mathcal{F}$. Let C be a chain in $\mathcal{F}$ and let $M = \bigcup(X : X \in C)$. We claim that M is a filter. Let $x \in M$ and $y \geq x$. Then $x \in X$ for some $X \in C$. Hence $y \in X$ as X is a filter. Therefore, $y \in M$. Let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in C$. Since C is a chain, either $X \subseteq Y$ or $Y \subseteq X$. Without loss of generality suppose $X \subseteq Y$. So $x, y \in Y$. Then $x \wedge y \in Y$ and so $x \wedge y \in M$. Moreover, $M \supseteq F$. So M is a maximum element of C. Then by Zorn's Lemma, $\mathcal{F}$ has a maximal element, say $Q \supseteq F$. ●

Lemma 3.3.4. *Let I be an ideal of a nearlattice S. A filter M disjoint from I is a maximal filter disjoint from I if and only if for all $a \notin M$, there exists $b \in M$ such that $a \wedge b \in I$.*

Proof: Let M be maximal and disjoint from I and $a \notin M$. Let $a \wedge b \notin I$ for $b \in M$. Consider $M_1 = \{y \in L : y \geq a \wedge b, \quad b \in M\}$. Clearly M_1 is a filter. For any $b \in M$, $b \geq a \wedge b$ implies $b \in M_1$. So $M_1 \supseteq M$. Also $M_1 \cap I = \phi$. For if not, let $x \in M_1 \cap I$. This implies $x \in I$ and $x \geq a \wedge b$ for some $b \in M$. Hence $a \wedge b \in I$, which is a contradiction. Hence $M_1 \cap I \neq \phi$. Now $M \subset M_1$ because $a \notin M$ but $a \in M_1$. This contradicts the maximality of M. Hence there exists $b \in M$ such that $a \wedge b \in I$.

Conversely, if M is not maximal disjoint from I, then there exists a filter $N \supset M$ and disjoint with I. For any $a \in N - M$, there exists $b \in M$ such that $a \wedge b \in I$. Hence, $a, b \in N$ implies $a \wedge b \in I \cap N$, which is a contradiction. Hence M must be a maximal filter disjoint with I. ●

Theorem 3.3.5. *Let S be a 0-distributive nearlattice. Then for $A \subseteq S$, $A^{\perp} = \{x \in S : x \wedge a = 0$ for all $a \in A\}$ is a semi-prime ideal.*

Proof: We have already mentioned that $A^{\perp}$ is a down set of S. Let $x, y \in A^{\perp}$ and $x \vee y$ exists. Then $x \wedge a = 0 = y \wedge a$ for all $a \in A$. Since S is 0-distributive, so $a \wedge (x \vee y) = 0$ for all $a \in A$. This implies $x \vee y \in A^{\perp}$ and so $A^{\perp}$ is an ideal.

Now let $x \wedge y \in A^{\perp}$ and $x \wedge z \in A^{\perp}$ and $y \vee z$ exists. Then $x \wedge y \wedge a = 0 = x \wedge z \wedge a$ for all $a \in A$. This implies $(x \wedge a) \wedge y = 0 = (x \wedge a) \wedge z$ and so by 0-distributivity again, $x \wedge a \wedge (y \vee z) = 0$ for all $a \in A$. Hence $x \wedge (y \vee z) \in A^{\perp}$ and so $A^{\perp}$ is a semi-prime ideal. ●

Let $A \subseteq S$ and J be an ideal of S. We define $A^{\perp_J} = \{x \in S : x \wedge a \in J$ *for all* $a \in A\}$. This is clearly a down set containing J. In presence of distributivity, this is an ideal. $A^{\perp_J}$ is called an annihilator of A relative to J. We denote $I_J(S)$, by the set of all ideals containing J. Of course, $I_J(S)$ is a bounded lattice with J and S as the smallest and the largest elements. If $A \in I_J(S)$, and $A^{\perp_J}$ is an ideal, then $A^{\perp_J}$ is called an annihilator ideal and it is the pseudocomplement of A in $I_J(S)$.

Theorem 3.3.6. *Let A be a non-empty subset of a nearlattice S and J be an ideal of S. Then* $A^{\perp_J} = \bigcap(P : P \textit{ is minimal prime down set containing J but not containing } A)$.

Proof. Suppose $X = \bigcap(P : A \not\subset P, P \textit{ is a minimal prime down set})$. Let $x \in A^{\perp_J}$. Then $x \wedge a \in J$ for all $a \in A$. Choose any P of right hand expression. Since $A \not\subset P$, there exists $z \in A$ but $z \notin P$. Then $x \wedge z \in J \subseteq P$. So $x \in P$, as P is prime. Hence $x \in X$.

Conversely, let $x \in X$. If $x \notin A^{\perp_J}$, then $x \wedge b \notin J$ for some $b \in A$. Let D = $[x \wedge b)$. Hence D is a filter disjoint from J. Then by Lemma 3.2.3, there is a maximal filter $M \supseteq D$ but disjoint from J. Then L-M is a minimal prime down set containing J. Now $x \notin S - M$ as $x \in D$ implies $x \in M$. Moreover, $A \not\subseteq S - M$ as $b \in A$, but $b \in M$ implies $b \notin S - M$, which is a contradiction to $x \in X$. Hence $x \in A^{\perp_J}$. ●

Following Theorem gives some nice characterization of semi-prime ideals.

Theorem 3.3.7. *Let S be a nearlattice and J be an ideal of S. The following conditions are equivalent.*

(i) J is semi-prime.

(ii) $\{a\}^{\perp_J} = \{x \in S : x \wedge a \in J\}$ is a semi-prime ideal containing J.

(iii) $A^{\perp_J} = \{x \in S : x \wedge a \in J \text{ for all } a \in A\}$ is a semi-prime ideal containing J.

(iv) $I_J(S)$ is pseudocomplemented

(v) $I_J(S)$ is a 0-distributive lattice.

(vi) Every maximal filter disjoint from J is prime.

Proof: $(i) \Rightarrow (ii)$; $\{a\}^{\perp_J}$ is clearly a down set containing J. Now let $x, y \in \{a\}^{\perp_J}$ and $x \vee y$ exists. Then $x \wedge a \in J, y \wedge a \in J$. Since J is semi prime, so $a \wedge (x \vee y) \in J$. This implies $x \vee y \in \{a\}^{\perp_J}$, and so it is an ideal containing J. Now let $x \wedge y \in \{a\}^{\perp_J}$ and $x \wedge z \in \{a\}^{\perp_J}$ with

$y \vee z$ exists. Then $x \wedge y \wedge a \in J$ and $x \wedge z \wedge a \in J$. Thus, $(x \wedge a) \wedge y \in J$ and $(x \wedge a) \wedge z \in J$. Then $(x \wedge a) \wedge (y \vee z) \in J$, as J is semi-prime. This implies $x \wedge (y \vee z) \in \{a\}^{\perp_J}$, and so $\{a\}^{\perp_J}$ is semi-prime.

$(ii) \Rightarrow (iii)$; This is trivial by Lemma 3.2.1, as $A^{\perp_J} = \bigcap(\{a\}^{\perp_J}; a \in A)$.

$(iii) \Rightarrow (iv)$; Since for any $A \in I_J(S)$, $A^{\perp_J}$ is an ideal, it is the pseudocomplement of A in $I_J(S)$, so $I_J(S)$ is pseudocomplemented.

$(iv) \Rightarrow (v)$; This is trivial as every pseudocomplemented lattice is 0-distributive.

$(v) \Rightarrow (vi)$; Let $I_J(S)$ is 0-distributive. Suppose F is a maximal filter disjoint from J. Suppose $f, g \notin F$ and $f \vee g$ exists. By Lemma 3.2.4, there exist $a, b \in F$ such that $a \wedge f \in J, b \wedge g \in J$. Then $f \wedge a \wedge b \in J, g \wedge a \wedge b \in J$. Hence $(f] \wedge (a \wedge b] \subseteq J$ and $(g] \wedge (a \wedge b] \subseteq J$. Then $(f \vee g] \wedge (a \wedge b] = ((f] \vee (g]) \wedge (a \wedge b] \subseteq J$, by the 0-distributive property of $I_J(S)$. Hence, $(f \vee g) \wedge a \wedge b \in J$. This implies $f \vee g \notin F$ as $F \cap J = \phi$, and so F is prime.

$(vi) \Rightarrow (i)$; Let (vi) holds. Suppose $a, b, c \in S$ with $a \wedge b \in J, a \wedge c \in J$ with $b \vee c$ exists. If $a \wedge (b \vee c) \notin J$, then $[a \wedge (b \vee c)) \cap J = \phi$. Then by Lemma 3.2.3, there exists a maximal filter $F \supseteq [a \wedge (b \vee c))$ and disjoint from J. Then $a \in F, b \vee c \in F$. By (vi) F is prime. Hence either $a \wedge b \in F$ or $a \wedge c \in F$. In any case $J \cap F \neq \phi$, which gives a contradiction. Hence $a \wedge (b \vee c) \in J$, and so J is semi-prime. ●

Corollary 3.3.8. *In a nearlattice S, every filter disjoint to a semi-prime ideal J is contained in a prime filter.* ●

Theorem 3.3.9. *If J is a semi-prime ideal of a nearlattice S and $J \neq A = \bigcap\{J_\lambda : J_\lambda$ is an ideal containing $J\}$, Then $A^{\perp_J} = \{x \in S : \{x\}^{\perp_J} \neq J\}$.*

Proof: Let $x \in A^{\perp_J}$. Then $x \wedge a \in J$ for all $a \in A$. So $a \in \{x\}^{\perp_J}$ for all $a \in A$. Then $A \subseteq \{x\}^{\perp_J}$ and so $\{x\}^{\perp_J} \neq J$ .Conversely, let $x \in S$ such that $\{x\}^{\perp_J} \neq J$. Since J is semi-prime, so $\{x\}^{\perp_J}$ is an ideal containing J. Then $A \subseteq \{x\}^{\perp_J}$, and so $A^{\perp_J} \supseteq \{x\}^{\perp_J \perp_J}$. This implies $x \in A^{\perp_J}$, which completes the proof. ●

Rav have provided a series of characterizations of 0-distributive lattices in [52]. Here we give some results on semi-prime ideals related to their results for nearlattices.

Theorem 3.3.10. *Let S be a nearlattice and J be an ideal. Then the following conditions are equivalent.*

(i) *J is semi-prime.*

(ii) *Every maximal filter of S disjoint with J is prime.*

(iii) *Every minimal prime down set containing J is a minimal prime ideal containing J.*

(iv) *Every filter disjoint with J is disjoint from a minimal prime ideal containing J.*

(v) *For each element $a \notin J$, there is a minimal prime ideal containing J but not containing a.*

(vi) *Each $a \notin J$ is contained in a prime filter disjoint to J.*

Proof. $(i) \Leftrightarrow (ii)$; follows from Theorem 3.3.7.

$(ii) \Rightarrow (iii)$; Let A be a minimal prime down set containing J. Then S-A is a maximal filter disjoint with J . Then by (ii) S-A is prime and so A is a minimal prime ideal.

$(iii) \Rightarrow (ii)$; Let F be a maximal filter disjoint with J. Then S-F is a minimal prime down set containing J. Thus by (iii), S-F is a minimal prime ideal and so F is a prime filter.

$(i) \Rightarrow (iv)$; Let F a filter of S disjoint from J. Then by Corollary 3.3.8, there is a prime filter $Q \supseteq F$ and disjoint from F.

$(iv) \Rightarrow (v)$; Let $a \in S$, $a \notin J$. Then $[a) \cap J = \varphi$. Then by (iv) there exists a minimal prime ideal A disjoint from $[a)$. Thus $a \notin A$.

$(v) \Rightarrow (vi)$; Let $a \in S$, $a \notin J$. Then by (v) there exists a minimal prime ideal P such that $a \notin P$, which implies $a \in S - P$ and S-P is a prime filter.

$(vi) \Rightarrow (i)$; Suppose J is not semi-prime . Then there exists $a, b, c \in L$ such that $a \wedge b \in J$, $a \wedge c \in J$ and $b \vee c$ exists, but $a \wedge (b \vee c) \notin J$. Then by (vi) there exists a prime filter Q disjoint from J and $a \wedge (b \vee c) \in Q$. Let $F = [a \wedge (b \vee c))$. Then $J \cap F = \varphi$ and $F \subseteq Q$. Now $a \wedge (b \vee c) \in Q$ implies $a \in Q$, $b \vee c \in Q$. Since Q is prime so either $a \wedge b \in Q$ or $a \wedge c \in Q$. This gives a contradiction to the fact that $Q \cap J = \varphi$. Therefore, $a \wedge (b \vee c) \in J$ and so J is semi-prime. ●

Now we give another characterization of semi-prime ideals with the help of Prime Separation Theorem using annihilator ideals.

Theorem 3.3.11. *Let J be an ideal in a nearlattice S. J is semi- prime if and only if for all filter F disjoint to $\{x\}^{\perp_J}$, there is a prime filter containing F disjoint to $\{x\}^{\perp_J}$.*

Proof: Using Zorn's Lemma we can easily find a maximal filter Q containing F and disjoint to $\{x\}^{\perp_J}$. We claim that $x \in Q$. If not, then $Q \vee [x) \supset Q$. By maximality of Q, $(Q \vee [x)) \cap \{x^{\perp_J}\} \neq \phi$. If $t \in (Q \vee [x)) \cap \{x\}^{\perp_J}$, then $t \geq q \wedge x$ for some $q \in Q$ and $t \wedge x \in J$. This implies $q \wedge x \in J$ and so $q \in \{x\}^{\perp_J}$ gives a contradiction. Hence $x \in Q$.

Now let $z \notin Q$. Then $(Q \vee [z)) \cap \{x\}^{\perp_J} \neq \phi$. Suppose $y \in (Q \vee [z)) \cap \{x\}^{\perp_J}$ then $y \geq q_1 \wedge z$ and $y \wedge z \in J$ for some $q_1 \in Q$. This implies $q_1 \wedge x \wedge z \in J$ and $q_1 \wedge x \in Q$. Hence by Lemma 3.3.4, Q is a maximal filter disjoint to $\{x\}^{\perp_J}$. Then by Theorem 3.3.7, Q is prime.

Conversely, let $x \wedge y \in J, x \wedge z \in J$ and $y \vee z$ exists. If $x \wedge (y \vee z) \notin J$, then $y \vee z \notin \{x\}^{\perp_J}$. Thus $[y \vee z) \cap \{x\}^{\perp_J} = \varphi$. So there exists a prime filter Q containing $[y \vee z)$ and disjoint from $\{x\}^{\perp_J}$. As $y, z \in \{x\}^{\perp_J}$, *so* $y, z \notin Q$. Thus $y \vee z \notin Q$, as Q is prime. This implies $[y \vee z) \not\subset Q$, a contradiction. Hence $x \wedge (y \vee z) \in J$, and so J is semi-prime. ●

Here is another characterization of semi-prime ideals.

Theorem 3.3.12. *Let J be a semi-prime ideal of a nearlattice S and $x \in S$. Then a prime ideal P containing $\{x\}^{\perp_J}$ is a minimal prime ideal containing $\{x\}^{\perp_J}$ if and only if for $p \in P$, there exists $q \in S - P$ such that $p \wedge q \in \{x\}^{\perp_J}$.*

Proof: Let P be a prime ideal containing $\{x\}^{\perp_J}$ such that the given condition holds. Let K be a prime ideal containing $\{x\}^{\perp_J}$ such that $K \subseteq P$. Let $p \in P$. Then there is $q \in S - P$ such that $p \wedge q \in \{x\}^{\perp_J}$. Hence $p \wedge q \in K$. Since K is prime and $q \notin K$, so $p \in K$. Thus, $P \subseteq K$ and so $K = P$. Therefore, P must be a minimal prime ideal containing $\{x\}^{\perp_J}$.

Conversely, let P be a minimal prime ideal containing $\{x\}^{\perp_J}$. Let $p \in P$. Suppose for all $q \in S - P$, $p \wedge q \notin \{x\}^{\perp_J}$. Let $D = (S - P) \vee [p)$. We claim that $\{x\}^{\perp_J} \cap D = \varphi$. If not, let $y \in \{x\}^{\perp_J} \cap D$. Then $p \wedge q \leq y \in \{x\}^{\perp_J}$, which is a contradiction to the assumption. Then by Theorem 3.3.11, there exists a maximal (prime) filter $Q \supseteq D$ and disjoint to $\{x\}^{\perp_J}$. By the proof of Theorem 3.3.11, $x \in Q$. Let $M = S - Q$. Then M is a prime ideal. Since $x \in Q$, so $t \wedge x \in J \subseteq M$ implies $t \in M$ as M is prime. Thus $\{x\}^{\perp_J} \subseteq M$. Now $M \cap D = \varphi$. This implies $M \cap (S - P) = \varphi$ and hence $M \subseteq P$. Also $M \neq P$, because $p \in D$ implies $p \notin M$ but $p \in P$. Hence M is a prime ideal containing $\{x\}^{\perp_J}$ which is properly contained in P.

This gives a contradiction to the minimal property of P. Therefore the given condition holds. ●

Observe that by Theorem 3.3.7 we can easily give a Seperation theorem in a 0-distributive neralattice for $A^{\perp}$, when A is a finite subset of S. But now we are in a position to give a proof of the theorem for any subset A.

Theorem 3.3.13. *Let F be a filter of a 0-distributive nearlattice S such that $F \cap A^{\perp} = \phi$ for any non-empty subset A* of S. *Then there exists a prime filter $Q \supseteq F$ such that $Q \cap A^{\perp} = \phi$.*

Proof: By Theorem 3.2.5, $A^{\perp}$ is a semi-prime . Thus by Lemma 3.3.3, there exists a maximal filter $Q \supseteq F$ *such that* $Q \cap A^{\perp} = \phi$. Since $A^{\perp}$ is semi-prime, so by Theorem 3.3.7, Q is prime. ●

3.4 Glivenko Congruence

We already known that the *relation* R on a nearlattice with *0, defined* by aRb if and only if $a \wedge x = 0$ is equivalent to $b \wedge x = 0$ *is a* congruence relation on S, when S is 0-distributive. We call it as Glivenko congruence. Now we extended that result.

Proposition 3.4.1. Let J be a semi-prime ideal in a nearlattice S. Define a relation R on S by $x \equiv y(R)$ if and only if $\{x\}^{\perp J} = \{y\}^{\perp J}$. Then R is a nearlattice congruence on S.

Proof: Clearly R is an equivalence relation on S. Now let $x \equiv y(R)$ and $t \in S$. Then $\{x\}^{\perp J} = \{y\}^{\perp J}$. Suppose $a \in \{x \wedge t\}^{\perp J}$. Then $a \wedge x \wedge t \in J$ which implies $a \wedge t \in \{x\}^{\perp J} = \{y\}^{\perp J}$. Thus, $a \wedge y \wedge t \in J$ and so $a \in \{y \wedge t\}^{\perp J}$. Therefore $\{x \wedge t\}^{\perp J} \subseteq \{y \wedge t\}^{\perp J}$. Similarly , $\{y \wedge t\}^{\perp J} \subseteq \{x \wedge t\}^{\perp J}$ and so $\{x \wedge t\}^{\perp J} = \{y \wedge t\}^{\perp J}$. Hence $x \wedge t \equiv y \wedge t(R)$. Now let $x \equiv y(R)$ and $x \vee t$, $y \vee t$ exist for some $t \in S$. Let $a \in \{x \vee t\}^{\perp J}$. Then $a \wedge (x \vee t) \in J$ and so $a \wedge x, a \wedge t \in J$. This implies $a \wedge y, a \wedge t \in J$ as $\{x\}^{\perp J} = \{y\}^{\perp J}$. Therefore $a \wedge (y \vee t) \in J$ as J is semi-prime. It follows that R is a nearlattice congruence on S. ●

Note: Let S be a nearlattice and Θ a congruence on S. We denote by $\frac{S}{\Theta}$ the quotient nearlattice of S modulo Θ and consider the elements of $\frac{S}{\Theta}$ as subsets of S. If $\frac{S}{\Theta}$ has a zero element $\overline{0}$, then $\overline{0}$ is called the kernel of Θ. Clearly $\overline{0}$ is then an ideal of S. Notice that we do not require S itself to have a zero element. If J is an ideal of S, we shall say that J is the kernel of a homomorphism if there exists a congruence Θ on S such that J is the kernel of Θ. Thus an ideal J is a kernel provided J is a complete congruence class for some congruence Θ on S. Since for every $x \in S$ and any $i \in J$, $x \geq x \wedge i \in J$, hence $\frac{[x]}{\Theta} \geq J$ in $\frac{S}{\Theta}$, so J is the zero element of $\frac{S}{\Theta}$.

Theorem 3.4.2. *Let S be a nearlattice and J be an ideal of S. Then the following conditions are equivalent.*

(i) J *is semi-prime.*

(ii) J *is the kernel of some homomorphism of* S *onto a distributive nearlattice with 0.*

(iii) J *is the kernel of some homomorphism of* S *onto a 0-distributive nearlattice.*

Proof: $(i)\Rightarrow(ii)$; Consider the elements $[x],[y],[z]$ in $\frac{S}{R}$ such that $y\vee z$ exists let $s\equiv x\wedge(y\vee z)(R)$. Then $\{s\}^{\perp J}=\{x\wedge(y\vee z)\}^{\perp J}$. Suppose $t\in\{s\}^{\perp J}$. Then $t\wedge(x\wedge(y\vee z))\in J$, hence $t\wedge x\in\{y\vee z\}^{\perp J}=\{y\}^{\perp J}\cap\{z\}^{\perp J}$. Therefore, $t\in\{x\wedge y\}^{\perp J}\cap\{x\wedge z\}^{\perp J}=\{(x\wedge y)\vee(x\wedge z)\}^{\perp J}$. Thus $\{s\}^{\perp J}\le\{(x\wedge y)\vee(x\wedge z)\}^{\perp J}$, equivalently, $\frac{[s]}{R}\le\frac{[(x\wedge y)\vee(x\wedge z)]}{R}$, hence $\frac{[x]}{R}\wedge\left(\frac{[y]}{R}\vee\frac{[z]}{R}\right)\le\left(\frac{[x]}{R}\wedge\frac{[y]}{R}\right)\vee\left(\frac{[x]}{R}\wedge\frac{[z]}{R}\right)$.

Since the reverse inequality is trivial, so $\frac{S}{R}$ is a distributive nearlattice.

Furthermore, for any $i,j\in J$, $i\equiv j(R)$, hence J is contained in some congruence class. But for any $i\in J$, $i\equiv a(R)$ implies $\{a\}^{\perp J}=\{i\}^{\perp J}=S$. This implies $a\in J$. Thus $J=\frac{[a]}{R}$ is a complete congruence class modulo R. That is, J is the kernel of R. Thus *(ii)* holds.

$(ii)\Rightarrow(iii)$; By (ii) $J=ker\,\Theta$ for some congruence Θ on S and $\frac{S}{\Theta}$ is a distributive nearlattice. Since every distributive nearlattice with 0 is 0-distributive, so $\frac{S}{\Theta}$ is 0-distributive and so (iii) holds.

$(iii) \Rightarrow (i)$; Let Θ be a congruence on S for which J is the zero element of the 0-distributve nearlattice $\frac{S}{\Theta}$. Let $x \wedge y \in J$ and $x \wedge z \in J$ such that $y \vee z$ exists. This implies $\frac{[x]}{\Theta} \wedge \frac{[y]}{\Theta} = \frac{[x \wedge y]}{\Theta} = J = \frac{[y \wedge z]}{\Theta} = \frac{[y]}{\Theta} \wedge \frac{[z]}{\Theta}$. Since $\frac{S}{\Theta}$ is 0-distributive it follows that $\frac{[x]}{\Theta} \wedge \frac{([y] \vee [z])}{\Theta} = J$. That is $\frac{[x \wedge (y \vee z)]}{\Theta} = J$ and so $x \wedge (y \vee z) \in J$. Therefore J is semi-prime. ●

Theorem 3.4.3. *If J is a semi-prime ideal of S , then the congruence R on S defined by $x \equiv y(R)$ if and only if $(x]^{\perp J} = (y]^{\perp J}$ is the largest congruence of S containing J as a class.*

Proof: By above theorems R is congruence of S containing J as a class. Let φ be any congruence of S containing J as a class. Suppose $x \equiv y(\varphi)$. Let $t \in (x]^{\perp J}$. Then $t \wedge x \in J$. Now $t \wedge x \equiv t \wedge y(\varphi)$. Now J as a class of φ implies $t \wedge y \in J$ and so, $t \in (y]^{\perp J}$. Similarly $t \in (y]^{\perp J}$ implies $t \in (x]^{\perp J}$. This implies $\{x\}^{\perp J} = \{y\}^{\perp J}$ and so $x \equiv y(R)$. It follows that $\varphi \subseteq R$, and so R is the largest congruence containing J as a class. ●

Now we give a separation theorem for semi-prime ideals.

Theorem 3.4.4. *Let J be a semi-prime ideal of a nearlattice S and F be a filter of S disjoint to J. Then there exists a prime ideal $P \supseteq J$ such that $P \cap F = \phi$.*

Proof: Define a relation R on S by $x \equiv y(R)$ if and only if $\{x\}^{\perp J} = \{y\}^{\perp J}$. Then by Theorem 3.4.1 and 3.4.2, R is a nearlattice congruence and the quotient nearlattice $\frac{S}{R}$ is distributive. Since $F \cap J = \phi$, so $\frac{F}{R}$ is a proper filter of $\frac{S}{R}$. It follows now from the prime separation theorem for distributive nearlattice due to Bazlar Rahman [9] Theorem 1.2.5, That there exist a prime ideal $\frac{P}{R}$ of $\frac{S}{R}$ disjoint to $\frac{F}{R}$. Then clearly, $P = h^{-1}\left(\frac{P}{R}\right)$ is a prime ideal of S containing J and disjoint from F, where h is the canonical homomorphism of S onto $\frac{S}{\Theta}$. ●

Theorem 3.4.5. *Let J be a semi-prime ideal of a nearlattice S and suppose that for some $a, b \in S$, $a \wedge b \in J$. Then there exist semi-prime ideals A and B (possibly improper) such that $a \in A$, $b \in B$ and $J = A \cap B$.*

Proof: If $a \in J$, then by choosing $A = J$ and $B = S$, the theorem trivially holds. So assume hence for that neither a nor b is in J. Now define the relation R on S by $x \equiv y(R)$ if and only if $\{x\}^{\perp_J} = \{y\}^{\perp_J}$. Since J is semi-prime, so by Theorem 3.4.1 and Theorem3.4.2, R is a nearlattice congruence and $\frac{S}{R}$ is a distributive nearlattice. Let $h : S \to \frac{S}{R}$ be the canonical homomorphism with kernel J. Put $S' = \frac{S}{R}$. Thus S' is a distributive nearlattice with $0' = J$. Hence $a' \wedge b' = 0'$, where $a' = h(a)$ and $b^* = h(b)$. By hypothesis $a \notin J$, $b \notin J$, hence $a' \neq 0' \neq b'$. Choose the ideals $A' = ((a'] \vee (b'])\cap(b']^*$ and $B' = ((a'] \vee (b'])\cap(a']^*$ in $\frac{S}{R}$. Since $a' \wedge b' = 0'$ it follows that $a' \in A'$ and $b' \in B'$. Clearly $A' \cap B' = (0']$. Since $\frac{S}{R}$ is distributive, A', B' are semi-prime Putting $A = h^{-1}A'$ and $B = h^{-1}B'$ yields the semi-prime ideals A and B. ●

Lemma 3.4.6. *Let S be a nearlattice containing a semi-prime ideal J disjoint from a filter F. Then given any $c \in S$, there exists an ideal J' and a filter F' of S such that,*

(i) $J \subseteq J'$ *and* $F \subseteq F'$

(ii) $J' \cap F' = \phi$

(iii) $c \in J' \cup F'$

(iv) J' *is semi-prime.*

Proof: If $c \in J \cup F$, then the result immediately follows. If $c \notin J \cup F$, then consider $G = F \vee [c)$. Clearly $G \cap J = \phi$. Thus applying Theorem 3.4.4, there exists a prime ideal P of S containing J and disjoint from G. Thus the lemma holds by choosing $P = J'$ and $G = F'$. ●

Theorem 3.4.7. *A nearlattice S is distributive if and only if for every ideal I and a filter F of S for which $I \cap F = \phi$ there exists a semi-prime ideal $J \supseteq I$ such that $J \cap F = \phi$.*

Proof: Suppose S is distributive. Then by prime separation theorem [9, theorem 1.2.5], there exists a prime ideal $P \supseteq I$ such that $P \cap F = \phi$. Since every prime ideal is semi-prime, so choosing $J = P$ we get the result.

Conversely, suppose the condition holds. If S is not distributive. Then there exist $x, y, z \in S$ with $y \vee z$ exists such that $x \wedge (y \vee z) > (x \wedge y) \vee (x \wedge z)$. Consider $I = ((x \wedge y) \vee (x \wedge z)]$ and $F = [x \wedge (y \vee z))$. Clearly $I \cap F = \phi$. Then by the given condition, there exists a semi-prime ideal $J \supseteq I$ such that $J \cap F = \phi$. Now $x \wedge y \in J$ and $x \wedge z \in J$. Since J is semi-prime , so $x \wedge (y \vee z) \in J$. This implies $J \cap F \neq \phi$, which gives a contradiction. Hence S must be distributive. ●

Proposition 3.4.8. *A nearlattice S is distributive if and only if every ideal of S is the kernel of some homomorphism if and only if every principal ideal of S is the kernel of some homomorphism.*

Proof: Suppose S is distributive. Since every ideal of a distributive nearlattice is semi-prime, so by Theorem 3.4.2, every ideal and every principal ideal of S is the kernel of some homomorphism.

Conversely, suppose every principal ideal of S is the kernel of some congruence. Then by a well known result in lattice theory S does not contain any sublattice isomorphic to M_5 or N_5. Therefore, S must be distributive. ●

ANNULETS AND α-IDEALS IN A 0-DISTRIBUTIVE NEARLATTICE

4.1 Introduction:

Recall that a nearlattice S with 0 is 0-distributive if for all $a,b,c \in S$ with $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists imply $a \wedge (b \vee c) = 0$. In this chapter we will study different properties of 0-distributive nearlattices. Moreover we will include several characterizations of 0-distributive lattices.

Annulets and α-ideals in a distributive lattice have been studied extensively by [13]. In a lattice L with 0, set of all ideals of the form $(x]^*$ can be made into a lattice $A_0(L)$, which Cornish [13] called the lattice of annulets of L.

By a *"dual nearlattice"* we will mean a join semilattice with the lower bound property. That is, a dual nearlattice S is a join semilattice together with the property that any two elements possessing a common lower bound, have an infimum. So the concept of a dual nearlattice is dual to the concept of a nearlattice.

Since by Theorem 3.2.5, S is 0-distributive if and only if $I(S)$ is pseudocomplemented, so 0-distributitivity of S is essential in the chapter.

An ideal J of S is called an annihilator ideal if $J = J^{**}$. The pseudocomplement of an ideal J is the annihilator ideal $J^* = \{x \in S \mid x \wedge j = 0 \ \textit{for all} \ j \in J\}$. It is well known that in the set of annihilator ideals $A(S)$, the supremum is given by $J \underline{\vee} K = (J^* \cap K^*)^*$ by Gratzer[19] Theorem 4, pp58. Moreover $A(S)$ is a Boolean lattice. Ideals of the form $(x]^*; x \in S$ are called the annulets of S. Thus for two annulets $(x]^*$ and $(y]^*$, $(x]^* \underline{\vee} (y]^* = ((x]^{**} \cap (y]^{**})^* = ((x \wedge y]^{**})^* = (x \wedge y]^*$. Hence the set of all annulets $A_0(S)$ of S

is a join subsemilattice of $A(S)$. Note that $A_0(S)$ is not necessarily a meet semilattice. But for any $x, y \in S$ if $x \vee y$ exists then $(x]^* \cap (y]^* = (x \vee y]^*$.

Recently, Ayub Ali, Noor and Islam[4] has studied the annulets in a distributive nearlattice S with 0. On the other hand α-ideals have been studied in distributive nearlattice due to Noor, Ayub Ali and Islam [41].

An ideal I in a distributive nearlattice S with 0 is called an α-ideal if $x \in I$ ($x \in S$) implies $(x]^{**} \subseteq I$.

In section 2 of this chapter, we have discussed different properties of 0-distributive nearlattice and included several characterizations of these nearlattice.

Section 3 discusses annulets in 0-distributive nearlattices and characterizes the quasi-complemented nearlattice.

Finally in section 4, we study the α-ideals in presence of 0-distributivity.

4.2. Some Characterizations of 0-distributive Nearlattice

We start this section with the following lemma which is very trivial.

Lemma 4.2.1. *In a nearlattice S, F is a proper filter if and only if $S-F$ is a prime down set.*

Proof: Let F be a proper filter. Let $x \in S-F$ and $t \leq x$. Then $x \notin F$ and so $t \notin F$ as F is a filter. Hence $t \in S-F$ and so $S-F$ is a down set.

Now let $a \wedge b \in S-F$ for some $a,b \in S$. Then $a \wedge b \notin F$. This implies either $a \notin F$ or $b \notin F$ and so either $a \in S-F$ or $b \in S-F$. Therefore $S-F$ is prime.

Conversely, suppose $S-F$ is a prime down set. Let $a \in F$ and $t \geq a$ $(t \in S)$. Then $a \notin S-F$ and so $t \notin S-F$ as it is a down set. Thus $t \in F$ and so F is an up set. Now let $a,b \in F$, then $a \notin S-F$ and $b \notin S-F$. Since $S-F$ is prime, so $a \wedge b \notin S-F$. This implies $a \wedge b \in F$, and so F is a filter. •

Now we give some characterization of 0-distributive nearlattice.

Theorem 4.2.2. *Let S be a nearlattice with 0. Then the following conditions are equivalent.*

(i) *S is 0-distributive.*

(ii) *If A is a non-empty subset of S and B is a proper filter intersecting A, then there is a minimal prime ideal containing $A^{\perp}$ and disjoint from B.*

(iii) *For each non-zero element $a \in S$ and each proper filter B containing a, there is a prime ideal containing $\{a\}^{\perp}$ and disjoint from B.*

(iv) *For each non-zero element $a \in S$ and each proper filter B containing a, there is a prime filter containing B and disjoint from $\{a\}^{\perp}$.*

(v) *For each non-zero element $a \in S$ and each prime down set B not containing a, there is a prime filter containing $S-B$ and disjoint from $\{a\}^{\perp}$.*

Proof: $(i) \Rightarrow (ii)$; Suppose (i) holds. Let A be a non-empty subset of S and B is a proper filter such that $B \cap A \neq \phi$. By Lemma 4.2.1, $S-B$ is a prime down set and so by Theorem

3.2.7, $S-B$ contains a minimal prime down set N. Clearly $N \cap B = \phi$. Also $S - B \not\supseteq A$ and so $N \not\supseteq A$. Then there exists $p \in A$ such that $p \notin N$. Now suppose $x \in A^{\perp}$. Then $x \wedge a = 0$ for all $a \in A$. Thus $x \wedge p = 0 \in N$. Since N is prime and $p \notin N$, so $x \in N$ and so $A^{\perp} \subset N$. Since S is 0-distributive but we know by Jayaram[31] theorem 9, so N is a minimal ideal.

$(ii) \Rightarrow (iii)$;. Suppose (ii) holds. Now $\{a\} \subset S$ and suppose B is a proper filter containing a. Then $B \cap \{a\} \neq \phi$. Thus by (ii) there is a prime ideal containing $\{a\}^{\perp}$ and disjoint from B.

$(iii) \Rightarrow (iv)$; This is trivial as P is a prime ideal of a nearlattice S if and only if $S-P$ is a prime filter.

$(iv) \Rightarrow (v)$; This is trivial by Lemma 4.2.1.

$(v) \Rightarrow (i)$; Suppose (v) holds and a be a non-zero element of S. Then by Lemma 4.2.1, $B = S - [a)$ is a prime down set and $a \notin B$. Then by (v) there is a prime filter Q containing $S-B$ and disjoint from $\{a\}^{\perp}$. Then $a \in Q$ and as by Varlet [66] theorem 9, so S is 0-distributive. ●

For a subset A of a nearlattice S, we define $A^0 = \{x \in S \mid x \wedge a = 0 \ for \ some \ \ a \in A\}$. It is easy to see that A^0 is a down set. Moreover, $\{a\}^{\perp} = \{a\}^0 = [a)^0$.

Theorem 4.2.3. *Let S be a nearlattice with 0. Then the following conditions are equivalent.*

(i) *S is 0-distributive.*

(ii) A^0 *is an ideal for every filter A of S.*

(iii) $\{a\}^0$ *is an ideal.*

Proof: $(i) \Rightarrow (ii)$; Let S be a 0-distributive. We already know that A^0 is a down set. Now let $x, y \in A^0$ and $x \vee y$ exists. Then $x \wedge a = 0 = y \wedge b$ for some $a, b \in A$. Hence

$x \wedge a \wedge b = 0 = y \wedge a \wedge b$, $a \wedge b \in A$ as it is a filter. Thus $a \wedge b \wedge (x \vee y) = 0$ as S is 0-distributive. Therefore $x \vee y \in A^0$ and so A^0 is an ideal.

(ii)$\Rightarrow$(iii); This is trivial as $\{a\}^0 = \{a\}^\perp$.

(iii)$\Rightarrow$(i); Suppose (iii) holds. Let $a,b,c \in S$ with $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists. Consider $[a)$. Then $b,c \in [a)^0$. Since $[a)^0$ is an ideal, so $b \vee c \in [a)^0$. Thus $a \wedge (b \vee c) = 0$, and hence S is 0-distributive. ●

Lemma 4.2.4. *Let A and B be filters of a nearlattice S with 0 such that* $A \cap B^0 = \phi$. *Then there is a minimal prime down set N containing* B^0 *and* $N \cap A = \phi$.

Proof: Since $A \cap B^0 = \phi$, so $0 \notin A \vee B$. So $A \vee B$ is a proper filter of S. Then by Lemma 3.2.3, $A \vee B \subseteq M$ for some maximal filter M. Now $B \subseteq M$ and consequently $M \cap B^0 = \phi$. By Lemma 4.2.1, $S - M = N$ is a minimal prime down set. Clearly $B^0 \subseteq N$ and $N \cap A = \phi$. ●

Theorem 4.2.5. Let S be a nearlattice with 0. Then the following conditions are equivalent.

(i) S is 0-distributive.

(ii) If A and B are filter of S such that A and B^0 are disjoint, there is a minimal prime ideal containing B^0 and disjoint from A.

(iii) If A is a filter of S and B is a prime down set containing A^0, there is a minimal prime ideal containing A^0 and contained in B.

(iv) If A is a filter of S and B is a prime down set containing A^0, there is a prime filter containing $S - B$ and disjoint from A^0.

(v) For each non-zero element $a \in S$ and each prime down set B containing $\{a\}^0$, there is a prime filter containing $S - B$ and disjoint from $\{a\}^0$.

Proof: $(i) \Rightarrow (ii)$; Suppose (i) holds. Let A and B be filter of S such that $A \cap B^0 = \phi$. Then by Lemma 4.2.4, there is a minimal prime down set N such that $N \supseteq B^0$ and $N \cap A = \phi$. Since S is 0-distributive it follows from Theorem 3.2.10 that N is a minimal prime ideal.

$(ii) \Rightarrow (iii)$; Suppose (ii) holds. Let A be a filter of S and B is a prime down set containing A^0. Then by Lemma 4.2.1, $S - B$ is a filter such that $(S - B) \cap A^0 = \phi$. Then by (ii), there is a minimal prime ideal containing A^0 and disjoint from $S - B$, that is contained in B.

$(iii) \Rightarrow (iv)$; This is trivial by Lemma 4.2.1.

$(iv) \Rightarrow (v)$; Let a be a non-zero element of S and B be a prime down set containing $\{a\}^0$. Let $A = [a)$. Then $B \supset \{a\}^0 = [a)^0 = A^0$. Then by (iv), there is a prime filter containing $S - B$ and disjoint from $\{a\}^0$.

$(v) \Rightarrow (i)$; Suppose (v) holds and let a be any non-zero element of S. By Lemma 4.2.1, $S - [a)$ is a prime down set not containing a. since $(a] \cap \{a\}^0 = (0] \subset S - [a)$, it follows that $S - [a)$ contains $\{a\}^0$ as $S - [a)$ is prime. Then by (v), there is a prime filter B containing $[a) = S - (S - [a))$ and disjoint from $\{a\}^0$. Clearly $a \in B$. Hence by Theorem 3.2.10, S is 0-distributive. •

Lemma 4.2.6. *Let S be a nearlattice with 0. Suppose* $A, B \in I(S)$ *and* $a, b \in S$, *then we have the following:*

(i) *If* $A \cap B = (0]$, *then* $B \subseteq A^{\perp}$.

(ii) $A \cap A^{\perp} = (0]$.

(iii) $a \le b$ *implies* $\{b\}^{\perp} \subseteq \{a\}^{\perp}$ *and* $\{a\}^{\perp\perp} \subseteq \{b\}^{\perp\perp}$.

(iv) $\{a\}^{\perp} \cap \{a\}^{\perp\perp} = (0]$

(v) $\{a \wedge b\}^{\perp\perp} = \{a\}^{\perp\perp} \cap \{b\}^{\perp\perp}$.

(vi) $\{a\} \subseteq \{a\}^{\perp\perp}$.

(vii) $\{a\}^{\perp} = \{a\}^{\perp\perp\perp}$.

(viii) *For all* $a,b,c \in S$, $\{(a \wedge b) \vee (a \wedge c)\}^{\perp} = \{a \wedge b\}^{\perp} \cap \{a \wedge c\}^{\perp}$.

Proof: (i) Let $b \in B$. Then $a \wedge b = 0$ for all $a \in A$ as $A \cap B = (0]$. Therefore, $b \in A^{\perp}$ and so $B \subseteq A^{\perp}$.

(ii) Let $x \in A \cap A^{\perp}$. Then $x \in A$ and $x \wedge a = 0$ for all $a \in A$. Thus in particular, $x = x \wedge x = 0$. Hence $A \cap A^{\perp} = (0]$.

(iii) Let $a \leq b$. Suppose $x \in \{b\}^{\perp}$. Then $x \wedge b = 0$ and $x \wedge a \leq x \wedge b = 0$ implies $x \wedge a = 0$. Thus $x \in \{a\}^{\perp}$. Therefore, $\{b\}^{\perp} \subseteq \{a\}^{\perp}$. Now let $x \in \{a\}^{\perp\perp}$. Then $x \wedge p = 0$ for all $p \in \{a\}^{\perp}$. Let $q \in \{b\}^{\perp}$. Since $\{b\}^{\perp} \subseteq \{a\}^{\perp}$, so $q \in \{a\}^{\perp}$. Hence $x \wedge q = 0$, which implies $x \in \{b\}^{\perp\perp}$. Therefore $\{a\}^{\perp\perp} \subseteq \{b\}^{\perp\perp}$.

(iv) Let $x \in \{a\}^{\perp} \cap \{a\}^{\perp\perp}$. Then $x \in \{a\}^{\perp}$ and $x \in \{a\}^{\perp\perp}$, and so $x \wedge p = 0$ for all $p \in \{a\}^{\perp}$. Thus in particular $x = x \wedge x = 0$, implies $\{a\}^{\perp} \cap \{a\}^{\perp\perp} = (0]$.

(v) Let $x \in \{a\}^{\perp\perp} \cap \{b\}^{\perp\perp}$ and $y \in \{a \wedge b\}^{\perp}$. Then we have $(y \wedge a) \wedge b = 0$, which implies $y \wedge a \in \{b\}^{\perp}$. Since $x \in \{b\}^{\perp\perp}$, we get $(x \wedge y) \wedge a = x \wedge (y \wedge a) = 0$. This implies $x \wedge y \in \{a\}^{\perp}$. Since $x \in \{a\}^{\perp\perp}$, we have $x \wedge y \in \{a\}^{\perp\perp}$ as $\{a\}^{\perp\perp}$ is a down set. Thus $x \wedge y \in \{a\}^{\perp} \cap \{a\}^{\perp\perp} = (0]$. Hence $x \wedge y = 0$ for all $y \in \{a \wedge b\}^{\perp}$, which implies $x \in \{a \wedge b\}^{\perp\perp}$. Therefore, $\{a\}^{\perp\perp} \cap \{b\}^{\perp\perp} \subseteq \{a \wedge b\}^{\perp\perp}$.

Conversely, since $a \wedge b \leq a,b$, so by (iii) $\{a \wedge b\}^{\perp\perp} \subseteq \{a\}^{\perp\perp}$ and $\{a \wedge b\}^{\perp\perp} \subseteq \{b\}^{\perp\perp}$. Hence $\{a \wedge b\}^{\perp\perp} = \{a\}^{\perp\perp} \cap \{b\}^{\perp\perp}$.

(vi) $x \in \{a\}^{\perp\perp}$ implies $x \wedge p = 0$ for all $p \in \{a\}^{\perp}$. Now $p \in \{a\}^{\perp}$ implies $p \wedge a = 0$. Thus we have $a \wedge p = 0$ for all $p \in \{a\}^{\perp}$, which implies $a \in \{a\}^{\perp\perp}$.

(vii) Since $\{a\}^{\perp} \cap \{a\}^{\perp\perp} = (0]$ so by (i) $\{a\}^{\perp} \subseteq \{a\}^{\perp\perp\perp}$.

Conversely, let $x \in \{a\}^{\perp\perp\perp}$. Then $x \wedge p = 0$ for all $p \in \{a\}^{\perp\perp}$. But by (vi) we have $a \in \{a\}^{\perp\perp}$. Therefore $x \wedge a = 0$ and so $x \in \{a\}^{\perp}$. Thus $\{a\}^{\perp\perp\perp} \subseteq \{a\}^{\perp}$, and so $\{a\}^{\perp} = \{a\}^{\perp\perp\perp}$.

(viii) $a \wedge b \leq (a \wedge b) \vee (a \wedge c)$ implies $(a \wedge b)^{\perp} \supseteq \{(a \wedge b) \vee (a \wedge c)\}^{\perp}$.
Similarly, $(a \wedge c)^{\perp} \supseteq \{(a \wedge b) \vee (a \wedge c)\}^{\perp}$ implies $(a \wedge b)^{\perp} \cap (a \wedge c)^{\perp} \supseteq \{(a \wedge b) \vee (a \wedge c)\}^{\perp}$.

Conversely, let $x \in (a \wedge b)^{\perp} \cap (a \wedge c)^{\perp}$ implies $x \wedge a \wedge b = 0 = x \wedge a \wedge c$ implies $x \wedge ((a \wedge b) \vee (a \wedge c)) = 0$ as S is 0-distributive. i e. $x \in ((a \wedge b) \vee (a \wedge c))^{\perp}$ so $(a \wedge b)^{\perp} \cap (a \wedge c)^{\perp} \subseteq \{(a \wedge b) \vee (a \wedge c)\}^{\perp}$. Hence $\{(a \wedge b) \vee (a \wedge c)\}^{\perp} = (a \wedge b)^{\perp} \cap (a \wedge c)^{\perp}$ ●

Theorem 4.2.7. *Let S be a nearlattice with 0. Then the following conditions are equivalent.*

(i) *S is 0-distributive.*

(ii) *For any non-empty subset A of S, $A^{\perp}$ is the intersection of all the minimal prime ideals not containing A.*

(iii) *For any ideal A of S and any family of ideals $\{A_i \mid i \in I\}$ of S,*

$$\left(A \cap \left(\bigvee_{i \in I} A_i\right)\right)^{\perp} = \bigcap_{i \in I} (A \cap A_i)^{\perp}$$

(iv) *For any three ideals A, B, C of S, $(A \cap (B \vee C))^{\perp} = (A \cap B)^{\perp} \cap (A \cap C)^{\perp}$.*

(v) *For all $a, b, c \in S$, $(a \wedge (b \vee c))^{\perp} = (a \wedge b)^{\perp} \cap (a \wedge c)^{\perp}$ provided $b \vee c$ exists.*

Proof: $(i) \Rightarrow (ii)$; Let N be a minimal prime down set not containing A. Then there exists $t \in A$ such that $t \notin N$. Suppose $x \in A^{\perp}$. Then $x \wedge a = 0$ for all $a \in A$. Thus $x \wedge t = 0 \in N$. Since N is prime, so $x \in N$. Hence $A^{\perp} \subseteq N$.

Thus, $A^{\perp} \subseteq \cap\{$ All minimal prime down set containing $A\} = X\,(say)$

Suppose $A^{\perp} \subset X$. Then there exists $x \in X$ such that $x \notin A^{\perp}$. Then for some $b \in A$, $x \wedge b \neq 0$. Thus $F = [x \wedge b)$ is a proper filter. Hence by Lemma 3.2.3. there is a maximal filter $M \supseteq F$. Then $S - M$ is a minimal prime down set such that $(S - M) \cap F = \phi$. Since $b \in F \subseteq M$, so $b \notin S - M$ and so $S - M \not\supseteq A$. Also $x \in F \subseteq M$ implies $x \notin S - M$.

Thus $x \notin X$, which is a contradiction. Therefore, $A^{\perp} = X = \cap\{$ All minimal prime down set containing $A\}$. Since S is 0-distributive, so by Theorem 3.2.10, the result follows.

$(ii) \Rightarrow (iii)$; Suppose (ii) holds. Let $A \subseteq I(S)$ and $\{A_i \mid i \in I\} \subseteq I(S)$. If Q is any minimal prime ideal of S such that $Q \not\supseteq A \cap \left(\underset{i \in I}{\vee} A_i \right)$, then $Q \not\supseteq A \cap A_j$ for some $j \in I$. By (ii) it follows that $\left\{ A \cap \left(\underset{i \in I}{\vee} A_i \right) \right\}^{\perp} \supseteq (A \cap A_i)^{\perp}$. On the other hand, $A \cap A_i \subseteq A \cap \left(\underset{i \in I}{\vee} A_i \right)$. Then by Lemma 4.2.6 $\{A \cap A_i\}^{\perp} \supseteq \left\{ A \cap \left(\underset{i \in I}{\vee} A_i \right) \right\}^{\perp}$. Therefore (iii) holds.

$(iii) \Rightarrow (iv)$; is obvious.

$(iv) \Rightarrow (v)$; Let $A = (a]$, $B = (b]$, $C = (c]$. Then by (iv), $\{(a] \cap ((b] \vee (c])\}^{\perp} = ((a] \cap (b])^{\perp} \cap ((a] \cap (c])^{\perp}$. Thus $(a \wedge (b \vee c)]^{\perp} = (a \wedge b]^{\perp} \cap (a \wedge c]^{\perp}$, and so $\{a \wedge (b \vee c)\}^{\perp} = (a \wedge b)^{\perp} \cap (a \wedge c)^{\perp}$.

$(v) \Rightarrow (i)$; Suppose (v) holds. Let $a, b, c \in S$ with $a \wedge b = 0 = a \wedge c$ such that $b \vee c$ exists. Then $(a \wedge b)^{\perp} = S = (a \wedge c)^{\perp}$. So by (v), $\{a \wedge (b \vee c)\}^{\perp} = S$. Thus, $\{a \wedge (b \vee c)\}^{\perp\perp} = S^{\perp} = (0]$. Hence by Lemma 4.2.6, $a \wedge (b \vee c) = 0$, and so S is 0-distributive. ●

Theorem 4.2.8. *Let A be a meet sub semi-lattice of a 0-distributive nearlattice S. Then* A^0 *is a semi-prime ideal.*

Proof: A^0 is obviously a down set. Now let $x, y \in A^0$ and suppose $x \vee y$ exists. Then $x \wedge a = 0 = y \wedge b$ for some $a, b \in A$. Then $x \wedge a \wedge b = 0 = y \wedge a \wedge b$ and $a \wedge b \in A$. Since S is 0-distributive so $a \wedge b \wedge (x \vee y) = 0$. Thus $x \vee y \in A^0$ and so A^0 is an ideal. Now suppose $p \wedge q \in A^0$ and $p \wedge r \in A^0$ and $q \vee r$ exists. Then $p \wedge q \wedge c = 0 = p \wedge r \wedge d$ for some $c, d \in A$. Thus $p \wedge q \wedge c \wedge d = 0 = p \wedge r \wedge c \wedge d$. Since S is 0-distributive $p \wedge c \wedge d \wedge (q \vee r) = 0$. That is $p \wedge (q \vee r) \wedge (c \wedge d) = 0$. Therefore $p \wedge (q \vee r) \in A^0$ as $c \wedge d \in A$. Hence A^0 is semi-prime. ●

Theorem 4.2.9. *Let S be a nearlattice with 0. Then the following conditions are equivalent.*

(i) *S is 0-distributive.*

(ii) *For a proper filter A, there exists a minimal prime ideal disjoint to A but containing* A^0.

(iii) *For a non-zero element* $a \in S$, *there is a minimal prime ideal containing* $\{a\}^0$ *but not containing* a.

Proof: $(i) \Rightarrow (ii)$; Since A is a proper filter, so A^0 is an ideal by Theorem 4.2.8. Now $A \cap A^0 = \phi$. For if $x \in A \cap A^0$ then $x \in A$ and $x \wedge a = 0$ for some $a \in A$. This implies $0 \in A$, which is a contradiction as A is a proper filter. Then by Lemma 3.3.3, there exists a maximal filter $M \supseteq A$ such that $M \cap A^0 = \phi$. Since S is 0-distributive, so by Theorem 3.2.5, M is a prime filter. This implies $S - M$ is a minimal prime ideal containing A^0 and disjoint to A.

$(ii) \Rightarrow (iii)$; is trivial by considering the filter $[a)$.

$(iii) \Rightarrow (i)$; Suppose (ii) holds but S is not 0-distributive. Then there exists $a, b, c \in S$ with $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists but $a \wedge (b \vee c) \neq 0$. Then by (ii) , there exists a minimal prime ideal P such that $a\wedge(b\vee c)\notin P$ but $\{a\wedge(b\vee c)\}^0\subseteq P$. Now $b\wedge[a\wedge(b\vee c)]=a\wedge b=0$ and $c\wedge[a\wedge(b\vee c)]=a\wedge c=0$ imply $b,c\in\{a\wedge(b\vee c)\}^0\subseteq P$. Since P is an ideal, so $b\vee c\in P$ and hence $a\wedge(b\vee c)\in P$ which gives a contradiction. Therefore S must be 0-distributive.●

Theorem 4.2.10. *Let S be a nearlattice with 0. Then the following conditions are equivalent.*

(i) *S is 0-distributive.*

(ii) *If A is a an ideal and $\{A_i \mid i\in I\}$ is a family of ideals of S such that $A\cap A_i=(0]$ for all I, then $A\cap\left(\underset{i\in I}{\vee}A_i\right)=(0]$.*

(iii) *If $a_1,a_2,\cdots,a_n\in S$ such that $a\wedge a_1=\cdots=a\wedge a_n=0$, then $(a]\wedge((a_1]\vee(a_2]\vee\cdots\vee(a_n])=(0]$.*

Proof: $(i)\Rightarrow(ii)$; By Lemma 1.3.1, we know that $\underset{i\in I}{\vee}A_i=\overset{\infty}{\underset{n=0}{\cup}}B_n$, where $B_0=\underset{i\in I}{\cup}A_i$ and $B_n=\{x\in S \mid x\le p\vee q \text{ for some } i,j\in B_{n-1}, i\vee j \text{ exists}\}$. Here clearly $B_0\subseteq B_1\subseteq B_2\subseteq\cdots\cdots$ and each B_n are down sets. Now $A\cap\left(\underset{i\in I}{\vee}A_i\right)=A\cap\left(\overset{\infty}{\underset{n=0}{\cup}}B_n\right)=\overset{\infty}{\underset{n=0}{\cup}}(A\cap B_n)$. Since $A\cap A_i=(0]$ for each i, so $A\cap B_0=A\cap\left(\underset{i\in I}{\cup}A_i\right)=\underset{i\in I}{\cup}(A\cap A_i)=(0]$. Now we use the method of induction. Suppose $A\cap B_{k-1}=(0]$. Then let $x\in A\cap B_k$. This implies $x\in A$ and $x\le r\vee s$ for some $r,s\in B_k$ and $r\vee s$ exists. Since $A\cap B_{k-1}=(0]$, so $x\wedge r=0=x\wedge s$. Then by the 0-distributively of S, $x\wedge(r\vee s)=0$. That is, $x=0$. Hence $A\cap B_k=(0]$. Therefore $A\cap B_n=(0]$ for all positive integer n, and so $A\cap\left(\underset{i\in I}{\vee}A_i\right)=\overset{\infty}{\underset{n=0}{\cup}}(A\cap B_n)=(0]$.

$(ii)\Rightarrow(iii)$; is trivial by considering $A=(a]$, $A_1=(a_1],\cdots\cdots,A_n=(a_n]$.

(iii)$\Rightarrow$(i); Let $a,b,c \in S$ with $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists. Then by (iii) $(a] \wedge ((b] \vee (c]) = (0]$. This implies $(a \wedge (b \vee c)] = (0]$, and so $a \wedge (b \vee c) = 0$. Hence S is 0-distributive. ●

Theorem 4.2.11. *Let S be a nearlattice with 0. Then the following conditions are equivalent.*

(i) *S is 0-distributive.*

(ii) *For any filter A of S, A^0 is the intersection of all the minimal prime ideals disjoint from A .*

(iii) *For all $a,b,c \in S$, $\{a \wedge (b \vee c)\}^0 = (a \wedge b)^0 \cap (a \wedge c)^0$ provided $b \vee c$ exists.*

Proof: $(i) \Rightarrow (ii)$ Let N be a minimal prime down set disjoint from A. If $x \in A^0$, then $x \wedge a = 0$ for some $a \in A$. Thus $x \wedge a \in N$. But $N \cap A = \phi$ implies $a \notin N$. So $x \in N$ as N is prime. Therefore, $A^0 \subseteq N$.

Now let $y \in S - A^0$. Then $a \wedge y \neq 0$ for all $a \in A$. Hence $A \vee [y) \neq S$. Then by Lemma 3.2.3, there exists a maximal filter $M \supseteq A \vee [y)$. Thus, $S - M$ is a minimal prime down set such that $(S - M) \cap A = \phi$ and $y \notin S - M$. Therefore, A^0 is the intersection of all minimal prime down sets disjoint from A. Since S is 0-distributive, so by Theorem 3.2.10, all minimal prime down set are minimal prime ideals and this proves (ii) .

(ii)$\Rightarrow$(iii); Let $A = [a) \vee ([b) \cap [c))$. Suppose Q is a minimal prime ideal disjoint from A. Then $Q \cap [a) = \phi$ and $Q \cap [b) \cap [c) = \phi$. Then either $Q \cap ([a) \vee [b)) = \phi$ or $Q \cap ([a) \vee [c)) = \phi$. If not suppose $x \in Q \cap ([a) \vee [b))$ and $y \in Q \cap ([a) \vee [c))$. Then $x, y \in Q$ and $x \geq a \wedge b$, $y \geq a \wedge c$. This implies $a \wedge b, a \wedge c \in Q$. Since $Q \cap A = \phi$, so $a \notin Q$. Thus $b, c \in Q$ as Q is a prime ideal. Hence $b \vee c \in Q$. Also $b \vee c \in [b) \cap [c)$, which contradicts the fact $Q \cap [b) \cap [c) = \phi$. Therefore either $Q \cap ([a) \vee [b)) = \phi$ or $Q \cap ([a) \vee [c)) = \phi$. Hence by (ii), $([a) \vee ([b) \cap [c)))^0 \supseteq ([a) \vee [b))^0 \cap ([a) \vee [c))^0$. Since the reverse inclusion is obvious, so $([a) \vee ([b) \cap [c)))^0 = ([a) \vee [b))^0 \cap ([a) \vee [c))^0$. This implies $[a \wedge (b \vee c))^0 = [a \wedge b)^0 \cap [a \wedge c)^0$ and so, $\{a \wedge (b \vee c)\}^0 = (a \wedge b)^0 \cap (a \wedge c)^0$.

(iii)$\Rightarrow$(i) ; Suppose (iii) holds. Let $a,b,c \in S$ with $a \wedge b = 0 = a \wedge c$ and $b \vee c$ exists. Then $(a \wedge b)^0 = S = (a \wedge c)^0$. Hence by (iii) $(a \wedge (b \vee c))^0 = S$. It follows that $a \wedge (b \vee c) = 0$. Thus S is 0-distributive. ●

4.3 Annulets in a 0-distributive Nearlattice

Recall that in a 0-distributive nearlattice S, the ideal lattice $I(S)$ is psuedocomplemented. The ideals of the form $(x]^* ; x \in S$, are called the annulets of S. The set of all annihilator ideals of S in a Boolean lattice, denoted by $A(S)$; while the set of annulets is denoted by $A_0(S)$.

Proposition 4.3.1. *Let S be a 0-distributive nearlattice. Then $A_0(S)$ is a dual nearlattice and it is a dual subnearlattice of $A(S)$. Moreover $A_0(S)$ has the same largest element $S = (0]^*$ as $A(S)$.*

Proof: We have already shown that $A_0(S)$ is a join subsemilattice of $A(S)$. Now suppose $(x]^* \supseteq (t]^*$ and $(y]^* \supseteq (t]^*$ for some $x, y, t \in S$. Then $(x]^* \cap (y]^* = ((x]^* \underline{\vee} (t]^*) \cap ((y]^* \underline{\vee} (t]^*)$ $= (x \wedge t]^* \cap (y \wedge t]^* = ((x \wedge t) \vee (y \wedge t)]^*$ as $((x \wedge t) \vee (y \wedge t))$ exist by the upper bound property of S. This shows that $A_0(S)$ has the lower bound property. Hence $A_0(S)$ is a dual nearlattice and so a dual subnearlattice of $A(S)$. ●

Proposition 4.3.2. *Let S be a 0-distributive nearlattice. $A_0(S)$ has a smallest element (then of course, it is a lattice) if and only if S possesses an element d such that $(d]^* = (0]$.*

Proof: *If there is an element $d \in S$ with $(d]^* = (0]$ then clearly $(0]$ is the smallest element in $A_0(S)$.*

Conversely, if $A_0(S)$ has a smallest element $(d]^*$, then for any $x \in S$, $(x]^* = (x]^* \underline{\vee} (d]^* = (x \wedge d]^*$. Thus $x \wedge d = 0$ implies $(x]^* = (0]^* = S$, so that $x = 0$, and hence $(d]^* = (0]$. ●

A nearlattice S with 0 is called quasi-complemented if for each $x \in S$, there exists $y \in S$ such that $x \wedge y = 0$ and $((x] \vee (y])^* = (x]^* \cap (y]^* = (0]$.

A 0-distributive nearlattice S is called quasi-complemented if for each $x \in S$, there exist $x' \in S$ such that $x \wedge x' = 0$ and $((x] \vee (x'])^* = (0]$.

A nearlattice S with 0 is called sectionally quasi-complemented if each interval $[0, x]$, $x \in S$ is quasi-complemented.

Theorem 4.3.3. *A 0-distributive nearlattice S is quasi-complemented if and only if for each $x \in S$ there exist $y \in S$ such that $(x]^{**} = (y]^*$.*

Proof: Let S be quasi-complemented. Suppose $x \in S$ then there exist $y \in S$ such that $x \wedge y = 0$ and $(x]^* \cap (y]^* = (0]$. This implies $(y]^* \subseteq (x]^{**}$.

Again $x \wedge y = 0$ implies $(x] \cap (y] = (0]$, so $(x] \subseteq (y]^*$. Therefore $(x]^{**} \subseteq (y]^{***} = (y]^*$ and hence $(x]^{**} = (y]^*$.

Conversely, let $x \in S$ implies $(x]^{**} = (y]^*$ for some $y \in S$. Then $x \in (x]^{**} = (y]^*$ implies $x \wedge y = 0$. Also $(x]^{**} = (y]^*$ implies $(x]^* \cap (y]^* = (x]^* \cap (x]^{**} = (0]$ and so S is quasi-complemented ●

Theorem 4.3.4. *Let S be a 0-distributive nearlattice. Then S is quasi-complemented if and only if it is sectionally quasi-complemented and possesses an element d such that $(d]^* = (0]$.*

Proof: Suppose S is quasi-complemented. Then there exists an element d such that $0 \wedge d = 0$ and $(d]^* = ((0] \vee (d])^* = (0]$. We now show that an arbitrary interval $[0, x]$ is quasi-complemented. Let $y \in [0, x]$. Then there exists $y' \in S$ such that $y \wedge y' = 0$ and $((y] \vee (y'])^* = (0]$. Put $z = x \wedge y'$. Then $z \wedge y = (x \wedge y') \wedge y = x \wedge (y \wedge y') = 0$ and $z \in [0, x]$. If $w \in [0, x]$ and $(w] \wedge ((y] \vee (z]) = (0]$, then $(w \wedge y] = (0] = (w \wedge z] = (w \wedge x \wedge y'] = (w \wedge y']$.

Thus $(w]\wedge((y]\vee(y'])=(0]$ as by Theorem3.2.5, $I(S)$ is 0-distributive. Hence $w=0$, and so $[0,x]$ is quasi-complemented.

Conversely, suppose S is sectionally quasi-complemented and there exists an element $d\in S$ with $(d]^*=(0]$. Let $x\in S$ and consider the interval $[0,d]$. Then $x\wedge d\in[0,d]$. Since S is sectionally quasi-complemented, so there exists an element $x'\in[0,d]$ with $x\wedge d\wedge x'=0$ and $\{y\in[0,d] \mid y\wedge((x\wedge d)\vee x')=0\}=(0]$. Now let $z\in((x]\vee(x'])^*$. Then $z\wedge r=0$ for all $r\in(x]\vee(x']$. Since $(x\wedge d)\vee x'\in(x]\vee(x']$, so $z\wedge((x\wedge d)\vee x')=0$. Thus $z\wedge d\wedge((x\wedge d)\vee x')=0$ and $z\wedge d\in[0,d]$; so $z\wedge d=0$. This implies $z\in(d]^*=(0]$. Hence $z=0$ and $x\wedge d\wedge x'=0$ implies $x\wedge x'=0$. Therefore S is quasi-complemented •

Theorem 4.3.5. A 0-distributive nearlattice is quasi-complemented if and only if $A_0(S)$ is a Boolean subalgebra of $A(S)$.

Proof: Suppose S is quasi-complemented. Then by Theorem 4.3.4, S has an element d such that $(d]^*=(0]$. Then by Proposition 4.3.2, $A_0(S)$ has a smallest element and so it is a sublattice of $A(S)$. Moreover for each $x\in S$ there exists $x'\in S$ such that $x\wedge x'=0$ and $(x]^*\cap(x']^*=(0]$. Then $(x]^*\underline{\vee}(x']^*=(x\wedge x']^*=(0]^*=S$. Therefore $A_0(S)$ is a Boolean subalgebra of $A(S)$.

Conversely, if $A_0(S)$ is a Boolean subalgebra of $A(S)$, then for any $x\in S$ there exists $y\in S$ such that $(x]^*\cap(y]^*=(0]$ and $(x]^*\underline{\vee}(y]^*=S$. But $(x]^*\underline{\vee}(y]^*=(x\wedge y]^*$ and $x\wedge y=0$. Therefore, S is quasi-complemented. •

Let us introduce the following lemma, whose proof is trivial.

Lemma 4.3.6. Let $I=[0,x]$, $0<x$ be an interval in a 0-distributive nearlattice. For $a\in I$ $(a]^+=\{y\in I \mid y\wedge a=0\}$ is the annihilator of $(a]$ with respect to I. Then

(i) if $a,b\in I$ and $(a]^+\subseteq(b]^+$ then $(a]^*\subseteq(b]^*$

(ii) if $w \in S$, $(w]^{*} \cap I = (w \wedge x]^{+}$. •

The above lemma is useful to prove the generalization of the proposition 2.5 in [13] by Cornish Let $I = [0, x]$, $0 < x$ be an interval in a distributive lattice with 0. For $a \in I$ $(a]^{+} = \{y \in I \mid y \wedge a = 0\}$ is the annihilator of $(a]$ with respect to I. Then

(i) if $a, b \in I$ and $(a]^{+} \subseteq (b]^{+}$ then $(a]^{*} \subseteq (b]^{*}$

(ii) if $w \in L$, $(w]^{*} \cap I = (w \wedge x]^{+}$.

Theorem 4.3.7. For a 0-distributive nearlattice S, $A_0(S)$ is relatively complemented if and only if S is sectionally quasi-complemented.

Proof: Suppose $A_0(S)$ is relatively complemented. Consider the interval $I = [0, x]$ and let $a \in I$; then $(x]^{*} \subseteq (a]^{*} \subseteq (0]^{*} = S$. Since the interval $[(x]^{*}, S]$ is complemented in $A_0(S)$, there exists $w \in S$ such that $(a]^{*} \cap (w]^{*} = (x]^{*}$ and $(a]^{*} \veebar (w]^{*} = S$. Then $(a]^{*} \veebar (w]^{*} = (a \wedge w]^{*}$ gives $a \wedge w = 0$. Then $a \wedge w \wedge x = 0$ and $w \wedge x \in I$. Moreover, intersecting $(a]^{*} \cap (w]^{*} = (x]^{*}$ with $(x]$ and using the Lemma 4.3.6, we have $(a]^{+} \cap (w \wedge x]^{+} = (0]$. This shows that I is quasi-complemented.

Conversely, suppose S is sectionally quasi-complemented. Since $A_0(S)$ is 0-distributive, it suffices to prove that the interval $[(a]^{*}, S]$ is complemented for each $a \in S$. Let $(b]^{*} \in [(a]^{*}, S]$. Then $(a]^{*} \subseteq (b]^{*} \subseteq S$, so $(b]^{*} = (a]^{*} \veebar (b]^{*} = (a \wedge b]^{*}$. Now consider the interval $I = [0, a]$ in S. Then $a \wedge b \in I$. Since I is quasi-complemented, there exists $w \in I$ such that $w \wedge a \wedge b = 0$ and $(w]^{+} \cap (a \wedge b]^{+} = (0] = (a]^{+}$. This implies $(w \vee (a \wedge b)]^{+} = (a]^{+}$, as $w \vee (a \wedge b)$ exists in S. Then by Lemma 4.2.6 $(a]^{*} = (w \vee (a \wedge b)]^{*} = (w]^{*} \cap (a \wedge b]^{*} = (w]^{*} \cap (b]^{*}$. Also $w \wedge a \wedge b = 0$ we have $w \wedge b = 0$, hence $(w]^{*} \veebar (b]^{*} = S$. Therefore $A_0(S)$ is relatively complemented. •

4.4 α-Ideals in a 0-distributive Nearlattice.

α-ideals have been studied by Jayaram [31] in case of distributive lattices. Recently Noor, Ayub Ali and Islam[41] have generalized those results for distributive nearlattices. In recent years many authors e. g. Ayub Ali, Hafizur Rahman and Noor[5] and Jayaram [31] have studied the α-ideals in a general lattice.

Recall that an ideal I in a 0-distributive nearlattice S is called an α-ideal if for each $x \in S$, $x \in I$ implies $(x]^{**} \subseteq I$.

According to Cornish[13] , for an ideal J in S we define $\alpha(J) = \{(x]^* \mid x \in J\}$. Also for a filter F in $A_0(S)$, $\bar{\alpha}(F) = \{x \in S \mid (x]^* \in F\}$. Clearly $\alpha(J)$ is a filter in $A_0(S)$and $\bar{\alpha}(F)$is a ideal in S. An ideal J in S is called an α-ideal if $\bar{\alpha}(\alpha(J)) = J$. It is easy to cheek that the two definitions of α-ideals are equivalent.

In this section we would like to study the α-ideals in a 0-distributive nearlattice.

By Theorem 3.2.5, we know that for a non-empty subset A of S, $A^{\perp}$ is an ideal if S is 0-distributive and if A is an ideal, then $A^{\perp} = A^*$ is the annihilator ideal.

Theorem 4.4.1. *For any ideal I in a 0-distributive nearlattice S the set $I^e = \{x \in S \mid (a]^* \subseteq (x]^* \text{ for some } a \in I\}$ is the smallest α-ideal containing I and ideal I in S is an α-ideal if and only if $I = I^e$.*

Proof: Let $x \in I^e$. Then $(a]^* \subseteq (x]^*$ for some $a \in I$ and so $(x]^{**} \subseteq (a]^{**}$. Suppose $y \in (a]^{**}$. Thus $(y] \subseteq (a]^{**}$ and so $(a]^* \subseteq (y]^*$. This implies $y \in I^e$. Therefore, $(a]^{**} \subseteq I^e$ and so $(x]^{**} \subseteq I^e$. It follows that I^e is an α-ideal. Now suppose $x \in I$, Then by definition, $x \in I^e$, and so $I \subseteq I^e$. Suppose K is an α-ideal containing I. Let $x \in I^e$. Then

$(a]^* \subseteq (x]^*$ for some $a \in I \subseteq K$. This implies $(x]^{**} \subseteq (a]^{**} \subseteq K$ as K is an α-ideal. Thus $(x] \subseteq K$ and so $x \in K$. Hence $I^e \subseteq K$. That is I^e is the smallest α-ideal containing I. ●

Theorem 4.4.2. *Every annihilatior ideal in a 0-distributive nearlattice S is an α-ideal.*

Proof : Let $I = A^*$ be the annihilator ideal of S. Suppose $y \in I = A^*$. Then $y \wedge a = 0$ for all $a \in A$. Then $(y] \wedge (a] = (0]$ and so $(y] \subseteq (a]^*$. Thus $(y]^{**} \subseteq (a]^{***} = (a]^*$ for all $a \in A$. Hence, $(y]^{**} \subseteq \bigcap_{a \in A} (a]^* = A^* = I$ and so I is an α-ideal. ●

Theorem 4.4.3. *For any ideal I in a 0-distributive nearlattice S the following are equivalent.*

(i) I *is an* α*-ideal.*

(ii) $I = \bigcup_{x \in I} (x]^{**}$

(iii) *For any $x, y \in S$, if $x \in I$ and $(x]^* = (y]^*$ then $y \in I$.*

Proof: *(i)*$\Rightarrow$*(ii)*; Let $x \in I$. Then $(x]^{**} \subseteq I$ as I is an α- ideal. So, $\bigcup_{x \in I} (x]^{**} \subseteq I$. On the other hand, for any $t \in I$, $t \in (t]^{**}$ implies $t \in \bigcup_{x \in I} (x]^{**}$.

Thus $I \subseteq \bigcup_{x \in I} (x]^{**}$, and so *(ii)* holds.

(ii)$\Rightarrow$*(iii)*; Let $x \in I$ and $(x]^* = (y]^*$. Then by *(ii)* $(y]^{**} = (x]^{**} \subseteq I$, and so $y \in (x]^{**} \subseteq I$.

(iii)$\Rightarrow$*(i)*; Let $x \in I$ and $t \in (x]^{**}$. Then $(t] \subseteq (x]^{**}$ implies $(x]^* \subseteq (t]^*$. Now choose any $r \in S$. Then $(r \wedge t] \subseteq (x]^{**}$. Again $(r \wedge t] \subseteq (t]^{**}$. Hence $(r \wedge t] \subseteq (x]^{**} \cap (t]^{**} = (x \wedge t]^{**}$. This implies $(x \wedge t]^* \subseteq (r \wedge t]^*$. Thus $(x \wedge t]^* = (x \wedge t]^* \cap (r \wedge t]^* = ((x \wedge t) \vee (r \wedge t)]^*$.

Now $x \wedge t \in I$. So by *(iii)*, $(x \wedge t) \vee (r \wedge t) \in I$. Then $r \wedge t \in I$ for all $r \in S$. In particular, choose $r = t$. This implies $t \in I$. Hence $(x]^{**} \subseteq I$ and so I is an α-ideal.●

Theorem 4.4.4. *Let S be a 0-distributive nearlattice. A be a meet subsemilattice of S. Then A^0 is an α-ideal, where $A^0 = \{x \in S \mid x \wedge a = 0$ for some $a \in A\}$.*

Proof: By Theorem 4.2.8 A^0 is an ideal. Now let $x \in A^0$ and $y \in (x]^{**}$. Clearly $x \in A^0$ implies $x \wedge a = 0$ for some $a \in A$. But then $a \in (x]^*$ and hence $y \wedge a = 0$. This shows that $y \in A^0$, consequently $(x]^{**} \subseteq A^0$. Hence A^0 is an α-ideal of S.●

Theorem 4.4.5. *If a prime ideal P of a 0-distributive nearlattice S is non-dense then P is an α-ideal.*

Proof: By assumption $P^* \neq (0]$. Hence there exists $x \in P^*$ such that $x \neq 0$. But then $(x]^* \supseteq P^{**}$ gives $(x]^* \supseteq P$ as $P \subseteq P^{**}$. Furthermore if $t \in (x]^*$, then $x \wedge t = 0 \in P$. But as P is a prime ideal, so $t \in P$ (since $P \cap P^* = (0] \Rightarrow x \notin P$). This implies $(x]^* \subseteq P$. Combining both the inclusions, we get $P = (x]^*$. Hence P is an annihilator ideal and so by Theorem 4.4.2, P is an α-ideal.●

Corollary 4.4.6. *Every non-dense prime ideal of a 0-distributive nearlattice is an annulet.* ●

Lemma 4.4.7. *For an α-ideal I of a 0-distributive nearlattice S, $I = \{y \in S \mid (y] \subseteq (x]^{**}$ for some $x \in I\}$.*

Proof: Let $a \in I$. Then $(a] \subseteq (a]^{**}$ implies that $a \in \{y \in S \mid (y] \subseteq (x]^{**}$ *for some* $x \in I\}$. Conversely, let $a \in \{y \in S \mid (y] \subseteq (x]^{**}$ *for some* $x \in I\}$. Then $(a] \subseteq (x]^{**}$ for some $x \in I$. Since I is an α-ideal, so $(x]^{**} \subseteq I$ and so $(a] \subseteq I$. Hence $a \in I$.●

Now we include a prime Separation Theorem for α-ideals in a 0-distributive nearlattice. This result is also a generalization of the Theorem11 presented in [31] by Jayaram.

Theorem 4.4.8. *Let F be a filter and I be an α-ideal in a 0-distributive nearlattice S such that $I \cap F = \phi$. Then there exists a prime α-ideal $P \supseteq I$ such that $P \cap F = \phi$.*

Proof: Let χ be the collection of all filters containing F and disjoint from I. χ is non-empty as $F \in \chi$ Then by Lemma 3.3.3, there exists a maximal filter Q containing F and disjoint from I. Suppose Q is not prime . Then there exist $f, g \notin Q$ such that $f \vee g$ exists and $f \vee g \in Q$. Then by Lemma 3.3.4, there exist $a \in Q$, $b \in Q$ such that $a \wedge f \in I$ and $b \wedge g \in I$. Thus we have $a \wedge b \wedge f \in I$ and $a \wedge b \wedge g \in I$. Then by Lemma 4.3.8, $(a \wedge b \wedge f] \subseteq (x]^{**}$ and $(a \wedge b \wedge g] \subseteq (y]^{**}$ for some $x, y \in I$. Choose any $t \in Q$. Then $(a \wedge b \wedge f] \wedge (t] \subseteq (t]^{**} \wedge (x]^{**}$. That is $(a \wedge b \wedge t \wedge f] \subseteq (t \wedge x]^{**}$. Similarly, $(a \wedge b \wedge t \wedge g] \subseteq (t \wedge y]^{**}$.

Thus we have $(a \wedge b \wedge t \wedge f] \wedge (t \wedge x]^{*} = (0] = (a \wedge b \wedge t \wedge g] \wedge (t \wedge y]^{*}$. That is $(a \wedge b \wedge t] \wedge (t \wedge x]^{*} \wedge (t \wedge y]^{*} \wedge (f] = (0] = (a \wedge b \wedge t] \wedge (t \wedge x]^{*} \wedge (t \wedge y]^{*} \wedge (g]$. Since $I(S)$ is 0-distributive, it follows that $(a \wedge b \wedge t] \wedge (t \wedge x]^{*} \wedge (t \wedge y]^{*} \wedge ((f] \vee (g]) = (0]$. That is, $(a \wedge b \wedge t] \wedge ((t \wedge x) \vee (t \wedge y)]^{*} \wedge (f \vee g] = (0]$, $(t \wedge x) \vee (t \wedge y)$ exists by the upper bound property of S and $(t \wedge x) \vee (t \wedge y) \in I$ as $x, y \in I$. Therefore, $(a \wedge b \wedge t] \wedge (f \vee g] \subseteq ((t \wedge x) \vee (t \wedge y)]^{**}$, which implies by Lemma 4.3.8 that $a \wedge b \wedge t \wedge (f \vee g) \in I$. But $a \in Q, b \in Q, t \in Q$, $f \vee g \in Q$ imply $a \wedge b \wedge t \wedge (f \vee g) \in Q$ which is a contradiction to $Q \cap I = \phi$. Therefore, Q must be prime. Thus $P = S - Q$ is a prime ideal containing I such that $P \cap Q = \phi$.

Let $x \in P$. If $x \in I$, then $(x]^{**} \subseteq I \subseteq P$. Again if $x \in P - I$, then by maximality of Q, there exists $a \in Q$ such that $a \wedge x \in I$. Thus, $(a]^{**} \wedge (x]^{**} \subseteq I \subseteq P$. Since $(a]^{**} \not\subseteq P$, so $(x]^{**} \subseteq P$ as P is prime. Therefore P is an α-ideal. ●

Proposition 4.4.9. *Let S be a 0-distributive nearlattice, then the following statements hold:*

(i) *For any ideal I in S, $\alpha(I)=\{(x]^{*} \mid x\in I\}$ is a filter in $A_0(S)$.*

(ii) *For a filter F in $A_0(S)$, $\bar{\alpha}(F)=\{x\in S \mid (x]^{*}\in F\}$ is an ideal in S.*

(iii) *If I_1, I_2 are ideals in S then $I_1\subseteq I_2$ implies that $\alpha(I_1)\subseteq\alpha(I_2)$; and if F_1, F_2 are filters in $A_0(S)$ then $F_1\subseteq F_2$ implies that $\bar{\alpha}(F_1)\subseteq\bar{\alpha}(F_2)$.*

(iv) *The map $I\to\bar{\alpha}\alpha(I)$ $\{=\bar{\alpha}(\alpha(I))\}$ is a closure operation on the lattice of ideals, i.e.*

(a) $\bar{\alpha}\alpha(\bar{\alpha}\alpha(I))=\bar{\alpha}\alpha(I)$,

(b) $I\subseteq\bar{\alpha}\alpha(I)$,

(c) $I\subseteq J$ *implies that* $\bar{\alpha}\alpha(I)\subseteq\bar{\alpha}\alpha(J)$ *for any ideal* $I,J\in S$.

Proof: (i) By Proposition 4.3.1, $A_0(S)$ is a join semilattice with the lower bound property. Let $(x]^{*},(y]^{*}\in\alpha(I)$, and $(t]^{*}\in A_0(S)$, where $x,y\in I, t\in S$. Then $((t]^{*}\underline{\vee}(x]^{*})\wedge((t]^{*}\underline{\vee}(y]^{*})=(t\wedge x]^{*}\wedge(t\wedge y]^{*}=((t\wedge x)\vee(t\wedge y)]^{*}\in\alpha(I)$, as $(t\wedge x)\vee(t\wedge y)\in I$. Also, if $(x]^{*}\in\alpha(I)$ and $(t]^{*}\in A_0(S)$ with $(x]^{*}\subseteq(t]^{*}$, then $(t]^{*}=(t]^{*}\underline{\vee}(x]^{*}=(t\wedge x]^{*}\in\alpha(I)$. So, $\alpha(I)$ is a filter in $A_0(S)$.

(ii) Let $x,y\in\bar{\alpha}(F)$ and $t\in S$, then $(x]^{*},(y]^{*}\in F$, and $(t]^{*}\in A_0(S)$. Since F is a filter of $A_0(S)$, so $((t]^{*}\underline{\vee}(x]^{*})\wedge((t]^{*}\underline{\vee}(y]^{*})\in F$ implies that $((t\wedge x)\vee(t\wedge y)]^{*}\in F$ implies that $(t\wedge x)\vee(t\wedge y)\in\bar{\alpha}(F)$. Also, if $x\in\bar{\alpha}(F)$ and $t\in S$, with $t\leq x$, then $(t]^{*}\supseteq(x]^{*}$ and $(x]^{*}\in F$. So, $t\in\bar{\alpha}(F)$. Hence $\bar{\alpha}(F)$ is an ideal in S.

(iii) Let $(x]^{*}\in\alpha(I_1)$, then $x\in I_1\subseteq I_2$ implies that $(x]^{*}\in\alpha(I_2)$ implies that $\alpha(I_1)\subseteq\alpha(I_2)$. Let $x\in\bar{\alpha}(F_1)$, then $(x]^{*}\in F_1\subseteq F_2$ implies that $x\in\bar{\alpha}(F_2)$ implies that $\alpha(F_1)\subseteq\alpha(F_2)$.

(iv) is trivial. ●

Proposition 4.4.10. *The α - ideals of a nearlattice S with 0 form a complete distributive lattice isomorphic to the lattice of filters, ordered by set inclusion of $A_0(S)$.*

Proof: Let $\{I_i\}$ be any class of α - ideals of S. Then $\bar{\alpha}\alpha(I_i)=I_i$ for all i. By Proposition 4.4.9 (iv), $\cap I_i \subseteq \bar{\alpha}\alpha(\cap I_i)$. Again $\bar{\alpha}\alpha(\cap I_i) \subseteq \bar{\alpha}\alpha(I_i)=I_i$ for all i implies that $\bar{\alpha}\alpha(\cap I_i) \subseteq \cap I_i$, and so $\bar{\alpha}\alpha(\cap I_i)=\cap I_i$. Thus $\cap I_i$ is an α - ideal. Trivially lattice of α - ideals is distributive. Hence α - ideals form a complete distributive lattice. •

For an α - ideal I, $\bar{\alpha}\alpha(I)=I$. Also, it is clear that for any filter F of $A_0(S)$, $\bar{\alpha}\alpha(F)=F$. Moreover, by Proposition 4.4.9 (iii), both α and $\bar{\alpha}$ are isotone. Hence the lattice of α - ideals of S is isomorphic to the lattice of filters.

Corollary 4.4.11. *Let S be a 0-distribtive lattice. Then the set of prime α - ideals of S are isomorphic to the set of prime filters of $A_0(S)$.* •

A 0-distributive nearlattice S is called disjunctive if for $0 \le a < b$ $(a,b \in S)$ there is an element $x \in S$ such that $a \wedge x = 0$ where $0 < x \le b$. It is easy to cheek that S is disjunctive if and only if $(a]^* = (b]^*$ implies $a = b$ for any $a,b \in S$.

Proposition 4.4.12. *In a 0-distirbutive nearlattice S the following conditions are equivalent:*

- *(i)* *each ideal is an α - ideal.*
- *(ii)* *each prime ideal is an α - ideal.*
- *(iii)* *S is disjunctive.*

Proof: $(i) \Rightarrow (ii)$; Suppose P is any prime ideal of S then by (i) P is an α - ideal, that is $\bar{\alpha}\alpha(P)=P$. Let I be any ideal of S then we have $I=\cap(P \mid P \supseteq I)$ implies $\bar{\alpha}\alpha(I)=\bar{\alpha}\alpha(\cap(P \mid P \supseteq I))=\cap(\bar{\alpha}\alpha(P) \mid P \supseteq I)=\cap(P \mid P \supseteq I)=I$ implies that $\bar{\alpha}\alpha(I)=I$. So I is an α -ideal.

$(ii) \Rightarrow (i)$; is trivial.

$(i) \Rightarrow (iii)$; For any $x, y \in S$, $(x]^* = (y]^*$. Since $(x]$ is an α - ideal, so by definition of α - ideal, $y \in (x]$. Therefore, $y \leq x$. Similarly $x \leq y$, and so $x = y$. Hence S is disjunctive.

$(iii) \Rightarrow (i)$; Suppose I is any ideal of S. By proposition 4.4.9, $(x]^* \subseteq \bar{\alpha}\alpha(I)$. For the reverse inclusion, let $x \in \bar{\alpha}\alpha(I)$. Then by definition $(x]^* \in \alpha(I)$, and so $(x]^* = (y]^*$ for some $y \in (x]^*$. This implies $x = y$, as S is disjunctive. So $x \in I$, and hence $\bar{\alpha}\alpha(I) = I$. Therefore I is an α - ideal of S. ●

Lemma 4.4.13. *A 0-distribtuive nearlattice S is relatively complemented if and only if every prime filter is an ultra filter (Proper and maximal.)*

Proof: By Theorem 2.11 in [41,] we have S is relatively complemented if and only if its prime ideals are unordered. Thus the result follows. ●

We conclude the chapter with the following result.

Theorem 4.4.14. *Let S be a 0-distributive nearlattice. Then the following conditions are equivalent:*

(i) *S is sectionally quasi-complemented.*

(ii) *each prime α - ideal is a minimal prime ideal.*

(iii) *each α - ideal is an intersection of minimal prime ideals.*

Moreover, the above conditions are equivalent to S being quasi complemented if and only if there is an element $d \in S$ such that $(d]^* = (0]$.

Proof: $(i) \Rightarrow (ii)$; Suppose S is sectionally quasi-complemented. Then by Theorem 4.3.7, $A_0(S)$ is relatively complemented. Hence its every prime filter is an ultra filter. Then by Corollary 4.4.11, each prime α - ideal is a minimal prime ideal.

$(ii) \Rightarrow (iii)$; It is not hard to show that each ideal of S is an intersection of prime α - ideals. This shows $(ii) \Rightarrow (iii)$.

$(iii) \Rightarrow (ii)$; This is obvious by the minimality property of prime α - ideals.

$(ii) \Rightarrow (i)$; Suppose (ii) holds. Then by Corollary 4.4.11, each prime filter of $A_0(S)$ is maximal. Then by Lemma 4.4.13, $A_0(S)$ is relatively complemented, and so by Proposition2.7 in [13,] by Cornish S is sectionally quasi-complemented.●

0-MODULAR NEARLATTICE

5.1 Introduction:

J.C Varlet [66] introduced the concept of 0-distributive and 0-modular lattices to study a larger class of non-distributive lattices. Recall that a lattice L with 0 is called 0-distributive if for all $a,b,c \in L$ with $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$. A lattice L with 0 is called 0-modular if for all $a,b,c \in L$ with $c \leq a$ and $a \wedge b = 0$ imply $a \wedge (b \vee c) = c$. Of course, every distributive lattice is both 0-distributive and 0-modular. Every pseudocomplemented lattice is 0-distributive but not necessarily 0-modular. Ayub Ali, Hafizur Rahman, and Noor [5], Jayaram [30], Pawar and Thakare, [51] and Varlet [66] have studied different properties of 0-distributivity and 0-modularity in lattices and in semilattices. Recently Zaidur Rahman, Bazlar Rahman and Noor [69] have studied 0-distributive nearlattices. In this chapter, we study some properties of 0-modular nearlattices.

A nearlattice S with 0 is called a 0-modular nearlattice if for all $a,b,c \in S$ with $c \leq a$, $a \wedge b = 0$ imply $a \wedge (b \vee c) = c$ provided $b \vee c$ exists. Clearly this definition is equivalent to " for all $t,a,b,c \in S$ with $c \leq a$ $a \wedge b = 0$ imply $a \wedge [(t \wedge b) \vee (t \wedge c)] = t \wedge c$". Moreover it is easy to show that the definition of 0-modular nearlattice coincides with the definition of 0-modular lattice when S is a lattice . Of course every modular nearlattice with 0 is 0-modular. Due to Varlet [66] we know that S with 0 is 0-modular if it contains no non-modular five element pentagonal sublattice including 0. Also S with 0 is 0-distributive if it contains no five element modular but not distributive sublattice including 0. Now we include some examples:

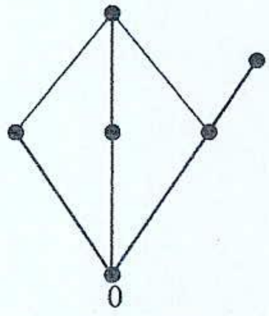


Figure 5.1

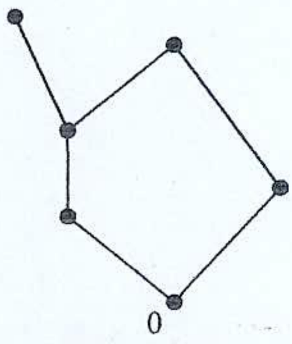


Figure 5.2

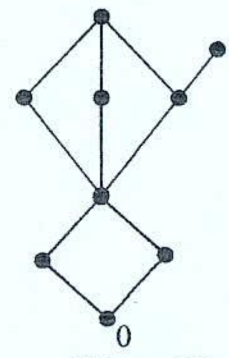
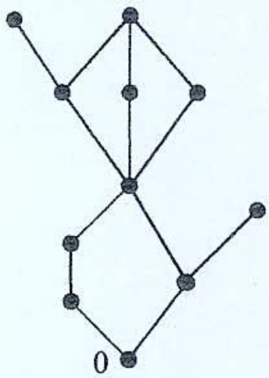
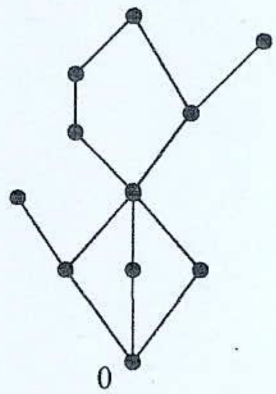


Figure 5.3



0

Figure5.4



0

Figure 5.5

0

Figure 5.6

Figure 5.1 is 0-modular but not 0-distributive, Figure 5.2 is 0-distributive but not 0-modular, Figure 5.3 is both 0-modular and 0-distributive, figure 5.4 is 0-distributive but not 0-modular , Figure 5.5 is 0-modular but not 0-distributive, Figure 5.6 is both 0-modular and 0-distributive.

A lattice L with 1 is called 1-distributive if for all $a,b,c \in L$ with $a \vee b = a \vee c = 1$ imply $a \vee (b \wedge c) = 1$. A lattice L with 1 is called 1-modular if for all $a,b,c \in L$ with $c \geq a$ and $a \vee b = 1$ imply $a \vee (b \wedge c) = c$.

A lattice L with 0 is semi-complemented if for any $a \in L$, $(a \neq 1)$ there exists $b \in L,\ b \neq 0$ such that $a \wedge b = 0$. Dually a lattice L with 1 is called dually semi-complemented if for any $a \in L$, $(a \neq 0)$ there exists $b \in L,\ b \neq 1$, such that $a \vee b = 1$.

A lattice L with 0 and 1 is called complemented if for any $a \in L$ there exist $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$.

A nearlattice S with 0 is called weakly complemented if for any distinct elements $a,b \in S$, there exists $c \in S$ such that $a \wedge c = 0$ but $b \wedge c \neq 0$.

An element a of a nearlattice S is called meet prime if $b \wedge c \leq a$ implies either $b \leq a$ or $c \leq a$. A non-zero element x of a nearlattice S with 0 is an atom if for any $y \in S$, with $0 \leq y \leq x$ implies either $0 = y$ or $y = x$. Dually in a lattice L with 1, an element x is called a dual atom if for any $y \in L$, $x \leq y \leq 1$ implies $x = y$ or $y = 1$.

A non-empty subset F of a nearlattice S is called a filter if for $x, y \in S$, $x \wedge y \in F$ if and only if $x \in F$ and $y \in F$.

The set of all filters of a nearlattice is just a join semi-lattice. But in case of a lattice, the set of filters is again a lattice.

5.2 0-Modular Nearlattice

Theorem 5.2.1. *A nearlattice S with 0 is 0-modular if for all* $a,b,c \in S$ *with* $c \leq a$, $a \wedge b = 0$, $a \vee b = c \vee b$ *imply* $a = c$, *provided* $a \vee b$ *exist.*

Proof: Suppose S is 0-modular and $a,b,c \in S$ with $c \leq a$, $a \wedge b = 0$ and $a \vee b = c \vee b$. If $a \vee b$ exists then $c \vee b$ exists by the upper bound property. Then $a = a \wedge (a \vee b) = a \wedge (b \vee c) = c$.

Conversely, let the stated conditions are satisfied in S. Let $a,b,c \in S$ with $c \leq a$, $a \wedge b = 0$ and $b \vee c$ exists. Here $c \leq a \wedge (b \vee c)$ and $b \wedge [a \wedge (b \vee c)] = b \wedge a = 0$. Now $a \wedge (b \vee c) \leq b \vee c$, so $b \vee [a \wedge (b \vee c)] \leq b \vee c$. Also $c \leq a \wedge (b \vee c)$ implies $b \vee [a \wedge (b \vee c)] \geq b \vee c$ and so $b \vee c = b \vee [a \wedge (b \vee c)]$, so by the given conditions $c = a \wedge (b \vee c)$, which implies S is 0-modular. ●

Theorem 5.2.2. *A nearlattice S with 0 is 0-modular if and only if the interval* $[0,x]$ *for each* $x \in S$ *is 0-modular.*

Proof: If S is 0-modular then trivially $[0,x]$ is 0-modular for each $x \in S$.

Conversely, let $[0,x]$ is 0-modular for each $x \in S$. Let $a,b,c \in S$ with $a \wedge b = 0$, $c \leq a$ and $b \vee c$ exist. Choose $t = b \vee c$.

Then $a \wedge (b \vee c) = a \wedge [(t \wedge b) \vee (t \wedge c)] = (t \wedge a) \wedge [(t \wedge b) \vee (t \wedge c)] = t \wedge c = c$ as the interval $[0,t]$ is 0-modular. ●

In a similar way we can easily prove the following result.

Theorem 5.2.3. *A nearlattice S with 0 is 0-distributive if and only if the interval* $[0,x]$ *for each* $x \in S$ *is 0-distributive.* ●

Theorem 5.2.4. *For a nearlattice S with 0, if* $I(S)$ *is 0-modular, then S is 0-modular, but the converse need not be true.*

Proof: Suppose $I(S)$ is 0-modular. Let $a,b,c \in S$ with $a \wedge b = 0$, $c \leq a$ and $b \vee c$ exist. Then $(a] \wedge ((b] \vee (c]) = (c]$ as $I(S)$ is 0-modular. Thus $(a \wedge (b \vee c)] = (c]$ and so $a \wedge (b \vee c) = c$, which implies S is 0-modular.

For the converse, we consider the nearlattice S given below which is due to Abbott [2].

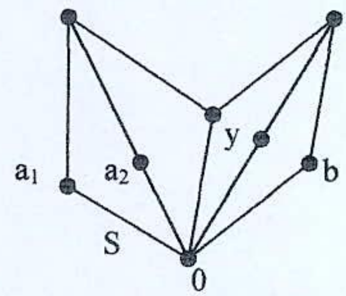


Figure 5.7

Here S is 0-modular. But in $I(S)$, $\{(0], (a_1], (a_1, y], (a_2, b], S\}$ is a pentagonal sublattice including 0. So $I(S)$ is not 0-modular. •

Theorem 5.2.5. *A nearlattice S with 0 is 0-modular if and only if the lattice of filter of the interval $[0,x]$ for each $x \in S$ is 1-modular.*

Proof: Let S be 0-modular. Choose any $x \in S$. Then $[0,x]$ is also 0-modular. Let F, G, H be filters of the lattice $[0,x]$ such that $H \supseteq F$, $F \vee G = [0)$.

Then $F \vee (G \cap H) \subseteq H$ is obvious. Let $h \in H$. Now $F \vee G = [0)$ implies $0 = f \wedge g$ for some $f \in F$ and $g \in G$. Thus $h \wedge f \leq f$ and $f \wedge g = 0$ implies $f \wedge [g \vee (h \wedge f)] = h \wedge f$ as S is 0-modular. So $h \wedge f \in F \vee (G \cap H)$ and hence $h \in F \vee (G \cap H)$. Therefore, $F \vee (G \cap H) = H$ and so the lattice of filters of $[0,x]$ is 1-modular.

Conversely, suppose the lattice of filters of $[0,x]$ is 1-modular. Let $a,b,c \in [0,x]$, $(x \in S)$ such that $c \leq a$, $a \wedge b = 0$. Then $[a) \subseteq [c)$ and $[a) \vee [b) = [0)$. So by 1-modular property, $[a) \vee ([b) \wedge [c)) = [c)$. Thus $[a \wedge (b \vee c)) = [c)$ and hence $a \wedge (b \vee c) = c$. This implies $[0,x]$ is 0-modular. Therefore by Theorem5.2.2, S is 0-modular. •

Theorem 5.2.6. *If a nearlattice S is 0-distributive and the interval $[0,x]$ for each $x \in S$ is semi-complemented, then the interval $[0,x]$ is 1-distributive for all $x \in S$.*

Proof: Let $a,b,c \in [0,x]$ with $a \vee b = x = a \vee c$. Suppose $a \vee (b \wedge c) \neq x$. Then there exists $p \neq 0$ in $[0,x]$ such that $p \wedge (a \vee (b \wedge c)) = 0$. Then $a \wedge p = 0 = (b \wedge c) \wedge p$. Thus $p \wedge b \wedge a = 0 = (p \wedge b) \wedge c$ which implies $(p \wedge b) \wedge (a \vee c) = 0$ as S is 0-distributive. This implies $0 = p \wedge b \wedge x = p \wedge b$. Then using the 0-distributivity of S again, $p \wedge (a \vee b) = 0$. That is, $0 = p \wedge x = p$, which gives a contradiction. Therefore , $a \vee (b \wedge c) = x$ and so $[0,x]$ is 1-distributive. ●

Theorem 5.2.7. *If a dual nearlattice S with 1 is 1-distributive and $[x,1]$ is dual semi-complemented for each $x \in S$, then the interval $[x,1]$ is 0-distributive for each $x \in S$.*

Proof: This is trivial by a dual proof of Theorem 5.2.6. ●

A nearlattice S with 0 is called a semi Boolean lattice if it is distributive and the interval $[0,x]$ for each $x \in S$ is complemented.

Theorem 5.2.8. *If a sectionally complemented nearlattice S is 0-distributive, then it is semi Boolean.*

Proof: Let $a < b$ for some $a,b \in S$. Then $0 \leq a < b$. Since $[0,b]$ is complemented, so there exists $c \in [0,b]$ such that $c \wedge a = 0,\ \ c \vee a = b$. Now if $b \wedge c = 0$, then by the 0-modularity of S, $b = b \wedge (c \vee a) = a$, which is a contradiction. Therefore, $b \wedge c \neq 0$. This implies S is weakly complemented. Since S is also 0-distributive. Therefore, by Theorem5.2.3 and Varlet[66] corollary2.2 $[0,x]$ is Boolean for each $x \in S$ and so S is semi Boolean. ●

Theorem 5.2.9. *Let S be a 0-modular nearlattice and F, G are two filters such that $F \vee G = [0)$ and $F \cap G = [x)$ for some $x \in S$. Then both F and G are principal filters.*

Proof: Suppose $F \vee G = [0)$ and $F \cap G = [x)$. Then $0 \geq f \wedge g$ for some $f \in F$ and $g \in G$. That is, $f \wedge g = 0$. Let $b = x \wedge f$ and $c = x \wedge g$. Then $b \in F$ and $c \in G$. We claim that

$F = [b)$ and $G = [c)$. Indeed if for instance $G \neq [c)$, then there exists $a \in G$ such that $a < c$. Then $\{0, a, c, b, x\}$ is a pentagonal sublattice of S. This implies S is not 0-modular and this gives a contradiction.

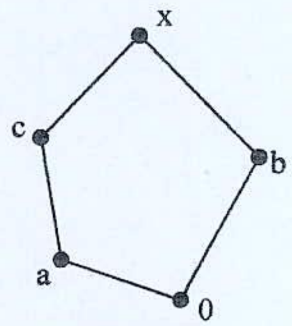


Figure 5.8

Therefore, $G = [c)$. Similarly $F = [b)$ and so both F and G are principal. ●

Lemma 5.2.10. *In a bounded semi complemented lattice L, every meet prime element is a dual atom.*

Proof: Suppose x is a meet prime element. Let $x \leq y < 1$. Then $0 \leq y < 1$. Since L is semi complemented, so there exists $t \neq 0 \in L$ such that $t \wedge y = 0$. Since $x \leq y$, so $t \wedge x = 0$. Since x is meet prime so this implies either $t \leq x$ or $y \leq x$. Now $t \leq x$ implies $t = t \wedge x = 0$, which is a contradiction. Thus $y \leq x$ and so $x = y$. Therefore x is a dual atom. ●

Lemma 5.2.11. *Let L be a bounded semi complemented lattice. If 0 is the meet of a finite number of meet prime elements of L, then L is dual semi complemented and 0-distributive.*

Proof: Let x be a non-zero element of L. Then by hypothesis, there is a meet prime element p in L such that $x \not\leq p$. Since L is semi complemented, so by lemma 5.2.10 p is a dual atom and $x \vee p = 1$. Therefore, L is dual semi complemented. Now suppose $a \wedge b = 0 = a \wedge c$ for some $a, b, c \in L$. Let us assume that $0 = \overset{n}{\underset{i=1}{\wedge}} p_i$ where p_i are meet prime elements in L. Observe that for each i, $p_i \geq a \wedge b$ and $p_i \geq a \wedge c$. Then for each i, $p_i \in [a)$ or $p_i \in [b) \cap [c)$. Therefore for each i $p_i \in [a) \vee ([b) \cap [c))$. This implies $[a) \vee ([b) \cap [c)) = [0)$, consequently, $a \wedge (b \vee c) = 0$, and so L is 0-distributive. ●

Lemma 5.2.12. *Let L be a bounded 0-modular lattice. If $b \in L$ is a dual atom and $a \wedge b = 0$ for some $a \neq 0$, $(a \in L)$, then a is an atom.*

Proof: Suppose $0 < c \leq a$ for some $c \in L$. As $c \leq a$ and $a \wedge b = 0$, so by 0-modularity, $a \wedge (b \vee c) = c$. Since $0 < c$, it follows that $b < b \vee c$ and so $b \vee c = 1$ as b is a dual atom. Consequently, $a = a \wedge 1 = a \wedge (b \vee c) = c$ by 0-modular. Therefore, a is an atom. ●

Lemma 5.2.13. *Let S be a 0-modular nearlattice and $[0, x]$ is semi complemented for each $x \in S$. If for each $x \in S$, 0 is the meet of a finite number of meet prime elements in $[0, x]$. Then x is the join of finite number of atoms in $[0, x]$.*

Proof: Let $0 = \overset{n}{\underset{i=1}{\wedge}} p_i$, where p_i's are meet prime elements in $[0, x]$. Observe that by lemma 5.2.10, each p_i is a dual atom in $[0, x]$. Since each $p_i \neq x$, and $[0, x]$ is semi complemented, so there exists $q_i \in [0, x]$ such that $p_i \wedge q_i = 0$, $i = 1, 2, ----, n$. Also by lemma 5.2.12, each q_i is an atom in $[0, x]$. Now let $c = \overset{n}{\underset{i=1}{\vee}} q_i$. Then $c \vee p_i = x$ as p_i is a dual atom for each i. As $[0, x]$ is bounded, semi-complemented and 0 is the meet of finite number of meet primes, by Lemma 5.2.11, $[0, x]$ is 0-distributive and so by Theorem5.2.5, $[0, x]$ is 1-distributive. Therefore, $c \vee \left(\overset{n}{\underset{i=1}{\wedge}} p_i \right) = x$. That is, $c = c \vee 0 = x$. Hence $\overset{n}{\underset{i=1}{\vee}} q_i = x$. ●

Theorem 5.2.14. *A nearlattice S with 0 is a semi Boolean lattice if and only if the following conditions are satisfied*

(i) $[0, x]$ *for each* $x \in S$ *is 1-distributive.*

(ii) *S is 0-distributive.*

(iii) $F([0, x])$ *is semi complemented for each* $x \in S$.

Proof: From Jayaram [30], Theorem3, every $[0, x]$, $x \in S$ is a finite Boolean algebra. Therefore, S is semi Boolean. ●

We conclude this section with the following Theorem which also trivially follows from Jayaram [30], Theorem4.

Theorem 5.2.15. *For a nearlattice S with 0, S is semi-Boolean if and only if the following conditions are satisfied.*

(i) $[0,x]$ *is semi complemented for each* $x \in S$.

(ii) *S is 0-modular.*

(iii) *0 is the meet of a finite number of meet primes.* •

SECTIONALLY PSEUDOCOMPLEMENTED NEARLATTICE

6.1 Introduction :

Pseudocomplemented lattices have been studied many authors such as Davey [17], Gratzer and Lakser [22], Gratzer and Schmidt [25], Katrinak [32], Katrinak [33], but they have studied these lattices in presence of distributivity and modularity. Since the concept of pseudocomplementedness is not appropriate for a nearlattice, many authors including Noor, Rahman and Azad [44], Noor and Islam [45], Shuily Akhter and Noor [60], Shuily Akhter and Noor [61] have studied the relative pseudocomplement in a distributive nearlattices. On the other hand normal lattices and nearlattices have been studied by Cornish [11], Cornish [12], Noor, Rahman and Azad [43], Noor and Latif [47] in presence of distributivity.

A lattice L with 0 is called normal if every prime ideal of L contains a unique minimal prime ideal.

Similarly a nearlattice S with 0 is called a normal nearlattice if its every prime ideal contains a unique minimal prime ideal.

Also we discuss p-algebra, S-algebra and D-algebra in this chapter.

In section 2 of this chapter we studied the normal nearlattices in presence of 0-distributivity. Here we included some characterizations of normal nearlattices.

In section 3 we have included a nice characterization of sectionally S-algebras when $[0,x]$ is 1-distributive for each $x \in S$. We also showed that S is sectionally S-algebra if and only if S is sectionally D-algebra when $[0,x]$ is 1-distributive for each $x \in S$ and S is 0-modular.

6.2 Normal Nearlattice

Let S be a nearlattice with 0 and P be a prime down set of S. We define $0(P)=\{x\in S: x\wedge y=0 \text{ for some } y\in S-P\}$. Since P is a prime down set, so by Lemma 1.2.10, S-P is a filter. Clearly $0(P)$ is a down set and $0(P)\subseteq P$.

Lemma 6.2.1. *If S is 0-distributive, then for a prime down set P, $0(P)$ is a semi-prime ideal.*

Proof: Let $a,b\in 0(P)$. Suppose $a\vee b$ exists. Then $a\wedge v=0=b\wedge s$ for some $v,s\in S-P$. Thus $a\wedge v\wedge s=0=b\wedge v\wedge s$. Since S is 0-distributive, so $v\wedge s\wedge(a\vee b)=0$ and $v\wedge s\in S-P$ as it is a filter. Hence $a\vee b\in 0(P)$. Hence $0(P)$ is an ideal as it is a down set. Now suppose $x\wedge y, x\wedge z\in 0(P)$. Then $x\wedge y\wedge v=0=x\wedge z\wedge s$ for some $v,s\in S-P$. Then by 0-distributivity of S, $[(x\wedge y)\vee(x\wedge z)]\wedge v\wedge s=0$ where $v\wedge s\in S-P$. This implies $(x\wedge y)\vee(x\wedge z)\in 0(P)$, we have $0(P)$ is a semi-prime ideal.●

Lemma 6.2.2. *Let S be a 0- distributive nearlattice and P be a prime down set. If Q is a minimal prime down set containing $0(P)$ such that $Q\subseteq P$,then for any $y\in Q-P$, there exists $z\notin Q$ such that $y\wedge z\in 0(P)$.*

Proof: If this is not true , then suppose for all $z\notin Q$, $y\wedge z\notin 0(P)$. Set $D=(S-Q)\vee[y)$. We claim that $0(P)\cap D=\phi$. If not , let $t\in 0(P)\cap D$. Then $t\in 0(P)$ and $t\geq a\wedge y$ for some $a\in S-Q$. Now $a\wedge y\leq t$ implies $a\wedge y\in 0(P)$, which is a contradiction to the assumption . Thus, $0(P)\cap D=\phi$. Then using Zorn's lemma as in Lemma 3.2.3, there exists a maximal filter $R\supseteq D$ such that, $R\cap 0(P)=\phi$.

Since $0(P)$ is semi-prime , so by Theorem 3.3.7, R is a prime filter. Therefore $S-R$ is a minimal prime ideal containing $0(P)$. Moreover $S-R\subseteq Q$ and $S-R\neq Q$ as $y\in Q$ but $y\notin S-R$. This contradicts the minimality of Q.Therefore, there must exist $z\notin Q$ such that $y\wedge z\in 0(P)$. ●

Proof of the following result is similar to the above proof.

Corollary 6.2.3. *Let S be a 0-distributive nearlattice and P,Q be distinct minimal prime down sets. Then for any $y \in Q-P$, there exists $z \in P-Q$ such that $y \wedge z = 0$.*•

Lemma 6.2.4. *Let P be a prime down set of a 0-ditributive nearlattice S. Then each minimal prime down set containing $0(P)$ is contained in P.*

Proof: Let Q be a minimal prime down set containing $0(P)$. If $Q \not\subseteq P$, then choose $y \in Q-P$. Then by Lemma 6.2.2, $y \wedge z \in 0(P)$ for some $z \notin Q$. Here $y \wedge z \wedge x = 0$ for some $x \notin P$. As P is prime, $y \wedge x \notin P$. This implies $z \in 0(P) \subseteq Q$, which is a contradiction. Hence $Q \subseteq P$.•

Theorem 6.2.5. *If P is a prime ideal in a 0-distributive nearlattice S, then the ideal $0(P)$ is the intersection of all the minimal prime ideals contained in P.*

Proof: Let Q be a prime ideal such that $Q \subseteq P$. Suppose $x \in 0(P)$ .Then $x \wedge y = 0$ for some $y \in S-P$. Since $y \notin P$, so $y \notin Q$. Then $x \wedge y = 0 \in Q$ implies $x \in Q$. Thus $0(P) \subseteq Q$. Hence $0(P)$ is contained in the intersection of all minimal prime ideals contained in P. Thus $0(P) \subseteq \bigcap\{Q$, the prime ideals contained in $P\} \subseteq \bigcap\{Q$, the minimal prime ideals contained in $P\} = X$ (say).

Now, $0(P) \subseteq X$. If $0(P) \neq X$, then there exists $x \in X$ such that $x \notin 0(P)$. Then $[x) \cap 0(P) = \phi$. So by Zorn's lemma as in Lemma 3.2.3, there exists a maximal filter $F \supseteq [x)$ and disjoint to $0(P)$.

Then by Theorem 3.3.7, F is a prime filter as $0(P)$ is semi-prime. Therefore $S-F$ is a minimal prime ideal containing $0(P)$. But $x \notin S-F$ implies $x \notin X$ gives a contradiction. Hence $0(P) = X = \bigcap\{Q$, the prime down sets contained in $0(P)\}$.•

A nearlattice S with 0 is called a *normal nearlattice* if its every prime ideal contains a unique minimal prime ideal. Cornish [11] has given nice characterizations of normal lattices in presence of distributivity. Now we generalize a part of his result in case of a 0-distributive nearlattice.

Theoren 6.2.6. *For a 0-distributive nearlattice S, the following conditions are equivalent.*

(i) *Every prime ideal contains a unique minimal prime ideal, i,e, S is normal.*

(ii) *$0(P)$ is a prime ideal for every prime ideal P.*

Proof. $(i)\Rightarrow(ii)$; is a direct consequence of Theorem 6.2.5.

$(ii)\Rightarrow(i)$; Suppose (ii) holds. Let P be a prime ideal. Then by Lemma 6.2.4 and Theorem 6.2.5, $0(P)$ is the intersection of all minimal prime ideals contained in P. Since by (ii) $0(P)$ is prime, so $0(P)$ is the only minimal prime ideal contained in P. Thus (i) holds. In other words, S is normal. •

Two ideal P and Q of a nearlattice S are called comaximal if $P\vee Q=S$. A nearlattice S with 0 is said to be a comaximal nearlattice if any two minimal prime ideals of S are comaximal.

Theorem 6.2.7. *Every comaximal nearlattice S is normal but the converse need not be true.*

Proof: Let P be a prime ideal. By Islam [62] Lemma 2.1.1, P conains a minimal prime ideal. Suppose P contains two minimal prime ideals Q and R. Since S is comaximal, so $Q\vee R=S\subseteq P$ which is a contradiction. Therefore, P must contain exactly one minimal prime ideal, and so S is normal.

For the converse, consider the nearlattice S_1 below

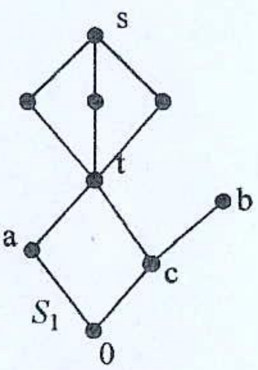


Figure 6.1

Here the ideals $(a]$, $(s]$ and $(b]$ are the only prime ideals. But only $(a]$ and $(b]$ are minimal. This shows that every prime ideal contains a unique minimal prime ideal. Thus S_1 is normal. But S_1 is not co-maximal as $(a]\vee(b]\neq S_1$. •

6.3 P-algebra

An algebra $\mathcal{L}=(L;\wedge,\vee,*,0,1)$ of type $(2,2,1,0,0)$ is called p-algebra if (i) $\mathcal{L}=(L;\wedge,\vee,*,0,1)$ is a bounded lattice, and (ii) for all $a\in L$, there exists an a^* such that $x\leq a^*$ if and only if $x\wedge a=0$. The element a^* is called the pseudo complemented of a.We have already mentioned that every p-algebra is 0-distributive. Figure 3.3 is an example of a 0-distributive lattice which is not a p-algebra . Of course, every finite 0-distributive lattice is a p-algebra.

A p-algebra $\mathcal{L}$. is called an S-algebra if it satisfies the following Stone identity; for all $a\in L$, $a^*\vee a^{**}=1$.

The De-Morgan identity: for any $a,b\in L$, $(a\wedge b)^*=a^*\vee b^*$ may not hold in a general p-algebra. Observe that the following lattice is a p-algebra , but $(a\wedge b)^*=1\neq c=a^*\vee b^*$.

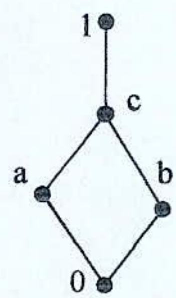


Figure 6.2

A p-algebra $\mathcal{L}$ is said to be a D-algebra if for any $a,b\in L$, $(a\wedge b)^*=a^*\vee b^*$.

A nearlattice S with 0 is called sectionally pseudocomplemented if $[0,x]$ is pseudocomplemented for each $x\in S$.

A nearlattice S with 0 is called sectionally S-algebra if $[0,x]$ is an S-algebra for each $x \in S$. Thus for each $t \in [0,x]$ $t^+ \vee t^{++} = x$, where t^+ is the relative pseudocomplement of t in $[0,x]$.

A nearlattice S with 0 is called sectionally D-algebra if $[0,x]$ is a D-algebra for each $x \in S$.

Theorem 6.3.1. *Every sectionally D-algebra is sectionally S-algebra.*

Proof: Let S be a sectionally D-algebra. Choose $x \in S$. Then for any $t \in [0,x]$ $t^+ \vee t^{++} = (t \wedge t^+)^+ = 0^+ = x$, where t^+ is the relatively pseudocomplemented of t in $[0,x]$. So S is sectionally S-algebra. •

Using the following example Nag, Begum and Talukder shows that that not every S-algebra is a D-algebra in [38].

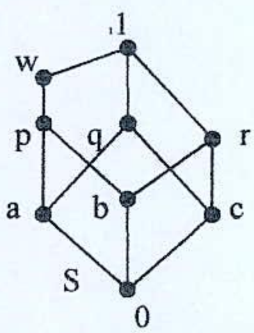


Figure 6.3

Here S is clearly an S-algebra. But $(q \wedge r)^* = c^* = w \neq p = b \vee a = q^* \vee r^*$ implies that it is not a D-algebra.•

Theorem 6.3.2. *A sectionally p-algebra is a sectionally D-algebra if and only if* $(x \vee y)^{++} = x^{++} \vee y^{++}$ *for each* $t \in S$ *and* $x, y \in [0,t]$.

Proof: Let S be a sectionally D-algebra . Then for each $x, y \in [0,t]$, $t \in S$ $(x \vee y)^{++} = (x^{+} \wedge y^{+})^{+} = x^{++} \vee y^{++}$.

Conversely, let the given identity holds. Then for $x, y \in [0,t]$ $(x \wedge y)^{+} = (x \wedge y)^{+++} = (x^{++} \wedge y^{++})^{+} = (x^{+} \vee y^{+})^{++} = x^{+++} \vee y^{+++} = x^{+} \vee y^{+}$.

Hence $(t]$ is a D-algebra. •

For any p-algebra, $\mathcal{L}$ define $D(L) = \{a \in L \mid a^{*} = 0\}$. It is well known that $\partial(L)$ is a filter in L.

Following lemma is needed for further development of the chapter.

Lemma 6.3.3. *Let S be a 0-distributive lattice and P be a prime ideal of S. If P is minimal then the following conditions hold:*

(i) $x \in P$ *implies* $(x]^{*} \not\subseteq P$.

(ii) $x \in P$ *implies* $(x]^{**} \subseteq P$.

(iii) $P \cap D(S) = \phi$.

Proof : (i) Let P be minimal and let (i) fail, that is, $(a]^{*} \subseteq P$ for some $a \in P$. Let $D = (S-P) \vee [a)$. We claim that $0 \notin D$. Indeed, if $0 \in D$, then $0 = q \wedge a$ for some $q \in S - P$, which implies that $q \in (a]^{*} \subseteq P$, a contradiction. Then $(a]^{*} \not\subseteq D$, for otherwise $(0] = (a] \wedge (a]^{*} \subseteq D$. Hence $D \cap (a]^{*} = \phi$. Then by Corollary 3.3.8, there exists a prime filter $F \supseteq D$ and disjoint to (a]*. Hence $Q = S - F$ is a prime ideal disjoint to D. Then $Q \subseteq P$ since $Q \cap (S - P) = \phi$ and $Q \neq P$, so $a \notin Q$, cotradicting the minimality of P.

(ii) $(x]^{*} \wedge (x]^{**} = (0] \subseteq P$ for any $x \in L$; thus if $x \in P$, then by (i), $(x]^{*} \not\subseteq P$, implying that $(x]^{**} \subseteq P$, as P is prime.

(iii) If $a \in P \cap D(S)$ for some $a \in L$, then $(a]^{**} = S \not\subseteq P$, a contradiction to (ii). Thus $P \cap D(S) = \phi$. ●

Nag, Begum and Talukder [38] have proved that every S-algebra in which the underlying lattice is both 0-modular and 1-distributive is also a D-algebra. Now in the following Theorem we generalize the result for a nearlattice.

To prove this we need the following lemma which is due to Noor and Razia Sultana [40]

Lemma 6.3.4. {Noor and Razia Sultana [40]} *A lattice L with 1 is 1-distributive if and only if for any $a \neq 1$ in L there is a prime ideal of L containing a.* ●

Theorem 6.3.5. *Let S be a sectionally p-algebra. Suppose for each $x \in S$, $[0,x]$ is 0-modular and 1-distributive. Then the following conditions are equivalent.*

(i) S *is sectionally S-algebra.*

(ii) S *is sectionally D-algebra.*

(iii) *For $x, y \in [0,t]$ with $x \wedge y = 0$ implies $x^+ \vee y^+ = t$.*

(iv) *For two minimal prime ideal P and Q of $(t]$ $(t \in S)$, $P \vee Q = (t]$.*

(v) *For any $t \in S$ every prime ideal of $(t]$ contains a unique minimal prime ideal of $(t]$.*

Proof: $(i) \Rightarrow (ii)$; Suppose S is a sectionally S-algebra. Let $a, b \in [0,x]$. By (i) $a^+ \vee a^{++} = x = b^+ \vee b^{++}$. Thus $(a^+ \vee b^+) \vee b^{++} = x = (a^+ \vee b^+) \vee a^{++}$. Since $[0,x]$ is 1-distributive, so $a^+ \vee b^+ \vee (a^{++} \wedge b^{++}) = x$. Now $a \wedge b \wedge a^+ = 0 = a \wedge b \wedge b^+$ imply $a^+, b^+ \leq (a \wedge b)^+$ and so $a^+ \vee b^+ \leq (a \wedge b)^+$. Also $(a \wedge b)^+ \wedge (a^{++} \wedge b^{++}) = (a \wedge b)^+ \wedge (a \wedge b)^{++} = 0$. Thus by 0-modularity of S, $(a \wedge b)^+ = (a \wedge b)^+ \wedge x = (a \wedge b)^+ \wedge [(a^{++} \wedge b^{++}) \vee (a^+ \vee b^+)] = a^+ \vee b^+$. And so (ii) holds.

$(ii) \Rightarrow (iii)$; is trivial.

$(iii) \Rightarrow (iv)$; Suppose (iii) holds. Let P and Q be two distinct minimal prime ideals of $(t]$. Let $a \in P - Q$. Since $a \wedge a^+ = 0$, so $a^+ \vee a^{++} = t$. Now by Lemma 6.3.3 $a \in P$ implies $a^+ \notin P$ and $a^{++} \in P$ as P is minimal and prime. Also $a \wedge a^+ = 0 \in Q$ and $a \notin Q$ imply $a^+ \in Q$ as Q is prime. Moreover $a^{++} \notin Q$. Thus $a^+ \in Q - P$ and $a^{++} \in P - Q$. This implies $t = a^{++} \vee a^+ \in P \vee Q$ and hence $P \vee Q = (t]$.

$(iv) \Rightarrow (v)$; is trivial.

$(v) \Rightarrow (i)$; Suppose (v) holds. If S is not a sectionally S-algebra. Then there exists $x \in S$ such that $[0, x]$ is not S-algebra. So there exists $a \in [0, x]$ such that $a^+ \vee a^{++} \neq x$. Since $[0, x]$ is 1-distributive, so by Lemma 6.3.4, there exists a prime ideal R of $(x]$ containing $a^+ \vee a^{++}$. Then $((x] - R) \vee [a^+) \neq (x]$. For if $((x] - R) \vee [a^+) = (x]$ then $0 = q \wedge a^+$ for some $q \in (x] - R$. Then $q \leq a^{++}$ implies $a^{++} \in (x] - R$ which is a contradiction. Thus $((x] - R) \vee [a^+) \neq (x]$. Then by Lemma3.2.3, there exists a maximal filter F containing $((x] - R) \vee [a^+)$, and it is also prime by Theorem 3.2.5.

Similarly, let G be a (maximal) prime filter containing $((x] - R) \vee [a^{++})$. We set $P = (x] - F$ and $Q = (x] - G$. Then P, Q are minimal prime ideals of $(x]$. Moreover $P \neq Q$; because $a^+ \in F$ implies $a^+ \notin P$ and so $a^{++} \in P$; but $a^{++} \notin Q$. Finally $P, Q \subseteq R$ contradicts the condition (v). Therefore, $a^+ \vee a^{++} = t$ and so $[0, t]$ is a sectionally S-algebra. •

We conclude the thesis with the following characterization of sectionally S-algebra.

Theorem 6.3.6. *Let S be a sectionally p-algebra. Suppose for each $x \in S$, $[0, x]$ is 1-distributive. Then the following condition are equivalent.*

(i) S *is sectionally S-algebra.*

(ii) *Any two distinct minimal prime ideals of $(t]$ are comaximal for each $t \in S$.*

(iii) *Every prime ideal in $(t]$ contains a unique minimal prime ideals of $(t]$; $t \in S$.*

(iv) *For each prime ideal P in $(t]$, $0(P)$ is a prime ideal for each $t \in S$.*

(v) *For any $x, y \in (t]$, $x \wedge y = 0$ implies $x^+ \vee y^+ = t$ for each $t \in S$.*

Proof: $(i) \Rightarrow (ii)$; Suppose S is sectionally S-algebra. Let P and Q be two distinct minimal prime ideals. Choose $x \in P - Q$. Then by Lemma 6.3.3, $x^+ \notin P$ but $x^{++} \in P$. Now $x \wedge x^+ = 0 \in Q$ implies $x^+ \in Q$, as Q is prime. Therefore $x = x^{++} \vee x^+ \in P \vee Q$. Hence $P \vee Q = (x]$. That is P and Q are comaximal.

$(ii) \Rightarrow (iii)$; is trivial.

$(iii) \Rightarrow (iv)$; follows from Theorem 6.2.6.

$(iv) \Rightarrow (v)$; Suppose (iv) holds and yet (v) does not. Then there exists $a, b \in (x]$ with $a \wedge b = 0$ but $a^+ \vee b^+ \neq x$. Then by Lemma 6.3.4, there is a prime ideal P containing $a^+ \vee b^+$. If $a \in 0(P)$, then $a \wedge r = 0$ for some $r \in (x] - P$. This implies $r \leq a^+ \in P$ gives a contradiction. Hence $a \notin 0(P)$. Similarly $b \notin 0(P)$. But by (iv) $0(P)$ is prime, and so $a \wedge b = 0 \in 0(P)$ is contradictory. Thus $(iv) \Rightarrow (v)$

$(v) \Rightarrow (i)$; Since $a \wedge a^+ = 0$, so by (v) $a^{++} \vee a^+ = x$ and $(x]$ is a sectionally S-algebra. •

It should be mentioned that in presence of distributivity, Abbott has proved that a lattice L is normal if and only if for all $x, y \in L$ with $x \wedge y = 0$, $(x]^* \vee (y]^* = L$ [2]. But in a 0-distributive nerlattice this need not be true. For example, consider the nearlattice of Figure 6.1. Here S_1 is 0-distributive and only prime ideals are $(a]$, $(b]$ and they are in fact minimal prime ideals . Thus S_1 is normal. Here $a \wedge b = 0$, but $(a]^* \vee (b]^* = (b] \vee (a] = \{0, a, b, c, t\} \neq S_1$.

1. Abbott, J. C., "*Sets, lattices and Boolean algebras*", *Allyn and Bacon, Boston, (1969).*
2. Abbott, J. C., "*Semiboolean algebra*", *Mat. Vesnik 4(19)(1967),177-198.*
3. Abbott, J. C., "*Implication algebra*", *Bull. Math. de la Soc.Sci. de la R.S. de Romonic 11(59)(1967),1-21.*
4. Ayub Ali,M, Noor, A. S. A and Islam,A.K.M.S, "*Annulets in a Distributive Nearlattice*", *Annals of Pure and Applied Mathematics, vol. 1.(2012), 91-96.*
5. Ayub Ali, M, Hafizur Rahman, R. M and Noor, A. S. A, "*Prime separation theorem for* α*-ideals in a 0-distributive lattice*", *IJPAMST vol-12(1), (2012), 16-20 India.*
6. Balasubramani, P. and Venkatanarasimhan, P.V., "*Characterizations of the 0-Distributive Lattice*", *Indian J.pure appl.Math. 32(3), (2001) 315-324*
7. Balbes, R., "*A representation theory for prime and implicative semilattices*", *Trans. Amer. Math. Soc. 126(1969), 261-267.*
8. Banaschewski, B. , "*on lattice –ordered groups*" *Fund. Math. 55(1964), 113-122.*
9. Bazlar Rahman, M., "*A study on distributive nearlattice*", *Ph.D Thesis, Rajshahi University. Bangladesh. (1994).*
10. Birkhoff, G., "*Lattice Theory*", Amer. *Math. Soc. Colloq. Pub. 25, 3rd Edition,(1967).*
11. Cornish, W.H., "*Normal lattices*", *J. Austral. Math. Soc.14 (1972), 200-215.*
12. Cornish, W. H., "*n-Normal lattice*", *Proc. Amer. Math. Soc. 1 (45) (1974),48-54.*
13. Cornish W. H, "*Annulets and* α*-ideals in a distributive lattice*" *J. Aust. Math. Soc. 15 (1) (1975), 70-77.*
14. Cornish W. H. and. Hickman R. C, "*Weakly distributive semilattices*", *Acta. Math. Acad. Sci. Hungar 32(1978),5-16.*
15. Cornish W. H. and Noor A. S.A.,"*Standard elements in a nearlattice*", *Bull. Austral. Math. Soc. 26(2),(1982),185-213.*
16. Davey, B. A. and Prestley, H. A., "*Introduction to lattices and order*", *University press, Cambridge (1992).*
17. Davey, B. A., Some annihilator conditions on distributive lattices, Alg. Univ. 3(4) (1974), 316-322.
18. Evans, E. , "*Median lattices and convex sublagebras*", *Manuscript.*
19. Gratzer, G., "*Lattice theory First concept and distributive lattices*", *Freeman, San Francisco,(1971).*

20. Gratzer, G., *"General Lattice theory", Birkhauser. Verlag. Basel und Stuttgart, (1978).*

21. Gratzer G. and Lakser H., *"Extension theorems on congruences of partial lattices", I. Notices Amer. Math. Soc. 15(1968),785-786.*

22. Gratzer G. and Lakser H., *"The structure of pseudocomplemented distributive lattice III. Injuctives and absolute subretracts, "Trans. Amer. Math. Soc. 169(1972) 475-487.*

23. Gratzer,G. and Schmidt, E.T., *"Standrad ideals in lattices", Acta. Math. Acad. Sci. Hung. 12(1961); 17-86.*

24. Gratzer, G. and Schmidt, E. T., *"characterizations of relatively complemented distributive lattice", Publ. Math. Debrecen 5(1958) 275-287.*

25. Gratzer, G. and Schmidt, E. T.,"On problem of M.H. Stone" Acta Math. Acad. Sci. Hunger. 8(1957), 455-460.

26. Gillman, L and Jerison, M. , *"Rings of continuous functions", Van Nostrand (1960).*

27. Hafizur Rahman, R. M., Ayub Ali, M. and Noor, A. S. A., *"On Semi prime Ideals of a lattice"*, J. Mech. Cont. & Math. Sci. Vol-7(2),(2013),1094-1102 .

28. Hickman R. C., *"Join algebra", Manuscripts.*

29. Hickman, R. C., *"Distributivity in semilattices", Ph.D Thesis, Flinders University of South Australia, (1978).*

30. Jayaram, C., *"0-modular semilattices", Studia Sci. Math. Hung. 22(1987), 189-195.*

31. Jayaram, C. , *"Prime α-ideals in a 0-distributive lattice", Indian J. Pure Appl. Math. 17 3 (1986), 331-337.*

32. Katrinak, T, *"Congruence lattices of distributive P-algebras Algebra Universalis, 7(1977), 265-271.*

33. Katrinak, T, *"Remarks on Stone lattice", I' (Russian) Math. –fyz. casopis 16(1966), 128-142.*

34. Lyndon, R.C.,*"Identities in two-valued calculi", Trans. Amer. Math. Soc. 71(1951),457-465.*

35. Mandelker, M. , *"Relative annihilators in lattice", Duke. Math. J. 40(1970), 377-386.*

36. Medarly, P. ,Sur L, *"arithmetique des filters premiers", C. R. Acad. Sci. Paris. 225 (1947), 846-848.*

37. Nachbin, L.,*"Une propriete characteristique des algebras booleiennes", Portugaliac math. 6 (1947), 115-118.*

38. Nag C., Begum S.N. and Talukder M.R, Some Characterization of Subclasses of P-algebra, submitted.

39. Nieminen, *"About generalized ideals in a distributive lattice", Manuscripte math., Springer verlag 7(1972),13-21.*

40. Noor, A.S.A and Razia Sultana., *"0-modularity in p-algebra", Submitted.*

41. *Noor, A. S. A. Ayub Ali, M and Islam, A. K. M. S," α-ideals in a distributive nearlattice", Journal of Physical Science, vol-16,(2012), 1-7, Bidyasagar University, India.*

42. Noor, A.S.A and Bazlar Rahman, M., *"Separation properties in nearlattices", The Rajshahi University Studies (Part B) 22(1994), 181-188.*

43. Noor, A.S.A, Rahman, M.B. and Azad M.A.K., *"Relatively normal nearlattices", The Rajshahi University Studies (Part B), 32(2004), 77-82.*

44. Noor, A. S. A and Islam, A. K. M. S., *"Relative annihilators in nearlattices", The Rajshahi University Studies (part-B) 25(1997); 117-120.*

45. Noor, A.S.A and Islam, M. A. K. S. , *"Relatively Stone nearlattices", The Rajshahi University studies(Part B) 25(1997)183-186.*

46. Noor, A.S.A. and Islam, A. K. M. S. , *"Generalized Stone Nearlattices", Journal of Science, Jahangirnagar University, Vol. 33,No.1 (2010), pp. 105-111.*

47. Noor, A. S. A and Latif , M. A., " normal nearlattices " , *The Rajshahi University Studies 9(1982), 69-74.*

48. Noor, A. S. A., Zaidur Rahman, M. and Bazlar Rahman, M. *"Some characterization of 0-distributive Nearlattice", Annals of Pure and Applied Mathematics, Vol. 2, No. 2, (2012), 194- 199.*

49. Pawar Y.S., Khopade S. S., *"α-ideals and Annihilator Ideals in 0-distributive Lattice" Acta. Univ. Palacki. Olomue., Fac. rer. nat. Mathematica 49, 1 (2010) 63-74*

50. Pawer , Y. S. and Mane, D. N., *"α-ideals in a 0-distributive semilattices and 0-distributive lattices", Indian J. Pure Appl. Math. 24, 7-8 (1993), 435-443.*

51. Pawar, Y. S and Thakare, N. K., *"0-Distributive semilattices", Canad . Math. Bull. Vol . 21(4) (1978) , 469-475.*

52. Rav, Y., *"Semi prime ideals in general lattices", Journal of pure and Applied Algebra, 56(1989) 105- 118.*

53. Razia Sultana, Ayub Ali, M. and Noor A. S. A., *"Some Properties of 0-distributive and 1-distributive Lattice", Annals of Pure and Applied Mathematics, Vol-1,No 2, (2012), 168-175.*

54. Rosen, V. V., *"Partial operations in ordered sets (Russian)", Publishing house of Saratov University, Saratov (1973), 17-20.*

55. Rutherford, D. E., *"Introduction to lattice theory", Oliver and Boyd, (1965).*

56. Sholander, M., *"Medians and betweenness", Proc. Amer. Math. Soc. 5(1954) , 801-807.*

57. Sholander, M., *"Medians lattices and trees", Proc. Amer. Math. Soc. 5(1954) , 808-812.*

58. Shum K. P., Chan M. W., Lai C. K. and So K. Y., *"Characterizations for prime semilattices", Can. J. Math. Vol. 37. No. 6(1985), 1059-1073.*

59. Shiuly Akhter, *"A study on principal n-ideals of a nearlattice", Ph.D Thesis, Rajshahi University. Bangladesh. (2003).*

60. Shuily Akhter and Noor, A. S. A.,*"Characterizations of those $P_n(S)$ which are relatively stone nearlattice"*, Accepted in the Journal of Science and Research (2012).

61. Shuily Akhter and Noor, A. S. A.,*"Principal n-ideals which form generalized stone nearlattices"*, Accepted in the Journal of Science and Research (2012).

62. Islam A. K. M. S, *" Sectionally Pseudocomplemented Distributive nearlattice", Ph.D Thesis, Rajshahi University Bangladesh. (1998).*

63. Subramanian, H, *"l-prime ideals in a f-rings" Bull. Soc. Math. France 95 (1967), 193-204.*

64. Talukder, M. R. and Noor, A. S. A., *"Modular ideals of a join semilattice Directed below", SEA Bull. Math 22(1998); 215-218.*

65. Varlet J. C., *"On separation properties in semilattices Semigroup", Forum 10(1975),200-228.*

66. Varlet, J. C., *"A generalization of the notion of pseudo-complementedness", Bull.Soc. Sci. Liege, 37(1968), 149-158.*

67. Zaidur Rahman ,M. and Noor, A. S. A., *"Modular and Strongly Distributive elements in a nearlattice", JMCMS,* Vol-7,No-1, July(2012), 988-997.

68. Zaidur Rahman, M., Bazlar Rahman, M. and Noor, A.S.A., *"On Semi prime Ideals in Nearlattices", Annals of Pure and Applied Mathematics, Vol. 3, No. 1,(2012), 1- 9.*

69. Zaidur Rahman, M., Bazlar Rahman, M. and Noor, A. S. A., *"0-distributive Nearlattice", Annals of Pure and Applied Mathematics, Vol. 2, No. 2,(2012), 177-184.*

70. Zaidur Rahman,M., Bazlar Rahman, M. and Noor, A. S. A., *"Some properties of 0-distributive Nearlattice," Annals of Pure and Applied Mathematics, Vol. 2, No. 2, (2012), 185-193.*