Study of Principal n-ideals of a Lattice

A Thesis Submitted for the Degree of
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in **Mathematics**

by

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to

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This thesis does not incorporate, without acknowledgement, any material previously submitted for a degree or diploma in any University, and to the best of my knowledge and belief, does not contain material previously published or written by another person except where due reference is made in the text.

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Md. Abul Kalam Azad

Md. Abul Kalam Azad

DEDICATED TO
MY PARENTS,
Who have Profoundly
influenced my life.

ABSTRACT

This thesis studies extensively the Principal n-ideals of a lattice. The idea of n-ideals in a lattice was first introduced by Cornish and Noor in studying the kernels around a particular element n, of a skeletal congruence on a distributive lattice. Then Latif and Ayub Ali in their thesis studied thoroughly on the n-ideals and established many valuable results. For a fixed element n of a lattice L, a convex sublattice of L containing n is called an n-ideal. If L has a "0", then replacing n by 0, an n-ideal becomes an ideal and if L has a "1" then it becomes a filter by replacing n by 1. Thus, the idea of n-ideals is a kind of generalization of both ideals and filters of lattices. The n-ideal generated by a finite number of elements of a lattice is called a finitely generated n-ideal, while the n-ideal generated by a single element is known as a principal n-ideal. Latif in his thesis has given a neat description on finitely generated n-ideals of a lattice and has provided a number of important results on them. For a lattice L, the lattice of all n-ideals of L and the lattice of all finitely generated n-ideals of L are denoted by I_n (L) and F_n (L) respectively, while P_n (L) represents the set of principal n-ideals of L. In this thesis, we devote ourselves in studying several properties on P_n (L) and F_n (L) which will certainly enrich many branches of lattice theory. Our results in this thesis generalize many results on normal, relatively normal, m-normal and relatively m-normal lattices. We also introduce the concept of n-annulets and α -n-ideal in studying $P_n(L)$.

In this connection it should be mentioned that if L has a 0, then putting n = 0 we find that $F_n(L)$ is the set of all principal ideals of L which is isomorphic to L. Thus, for every result on $F_n(L)$ in this thesis, we can obtain a result for the lattice L with 0 by substituting n = 0. Hence the result in each chapter of the thesis regarding $F_n(L)$ are generalizations of the corresponding results in lattice theory.

In chapter 2, we discuss some fundamental properties of n-ideals, which are basic to this thesis. Here we give an explicit description of $F_n(L)$ and $P_n(L)$ which are essential for the development of the thesis. Though $F_n(L)$ is always a lattice, $P_n(L)$ is not even a semilattice. But when n is a neutral element, $P_n(L)$ becomes a meet semilattice. Moreover, we show that $P_n(L)$ is a lattice if and only if n is a central element, and then in fact, $P_n(L) = F_n(L)$. We also show that, for a neutral element n, the lattice L is complemented if and only if $P_n(L)$ is so. In this chapter we also discuss on prime n-ideals. We give several properties and characterizations of prime n-ideals. We include a proof of the generalization of Stone's separation theorem. We also include a new proof of the result that for a distributive lattice L, $F_n(L)$ is generalized Boolean if and only if prime n-ideals are unorderd.

Chapter 3 discusses on minimal prime n-ideals of a lattice. We give some characterizations on minimal prime n-ideals which are essential for the further development of this chapter. Here we provide a number of results which are generalizations of the results on normal lattices.

We prove that for a distributive lattice L, $F_n(L)$ is normal if and only if each prime n-ideal of L contains a unique minimal prime n-ideal. We also show that if n is central in L, then $P_n(L)$ is a normal lattice if and only if any two minimal prime n-ideal are comaximal which is also equivalent to $<x>_n \cap <y>_n = \{n\}$ implies $<x>_n^* \vee <y>_n^* = L$.

In chapter 4 we introduce the notion of relative n-annihilators $<a,b>^n$. We characterize distributive and modular lattices in terms of relative n-annihilators. Then we generalize several results of Mandelker on annihiltors. We use these to characterize those $F_n(L)$ which are relatively normal lattices. Among many results we have shown that for a central element n, $P_n(L)$ is a relatively normal lattice, if and only if any two incomparable prime n-ideal are comaximal . What is more , this is also equivalent to the condition $<<a>_n,<b>_n> \vee <<b>_n,<a>_n> = L$ for all a,b $\in$ L .

Pseudocomplemented distributive lattices satisfying Lee's identities form equational subclasses denoted by B_m , $-1 \le m < \omega$. Cornish have studied distributive lattices analogues to B_m-lattices and relatively B_m-lattices .He referred then as m-normal lattices.Moreover, Beazer and Deavy have each independently obtained several characterizations of (sectionally) B_m–lattices and relatively B_m–lattices.

In chapter 5 we generalize their results by studying finitely generated n-ideals which form a m-normal and a relatively m-normal lattice .We show that for a central element $n \in L$, $P_n(L)$ is m-normal if and only if for any m+1 distinct minimal prime n-ideals $P_0, \ldots\ldots\ldots\ldots, P_n$ of L, $P_0 \vee \ldots\ldots\ldots\ldots\ldots \vee P_m = L$. In this chapter we also show that for a central element $n \in L$, $P_n(L)$ is relatively m-normal if and only if any m+1 pairwise incomparable prime n-ideals are comaximal.

Chapter 6 introduces the concept of n-annulets and α-n-ideals of a lattice. Here we include several result on the set of n-annulets $A_n(L)$ when n is a central element of L. We proved $A_n(L)$ is relatively complemented if and only if $P_n(L)$ is sectionally quasi-complemented.

In section 2 we studied α-n-ideals. We have shown that n-ideal $n(P)$ where P is a prime n-ideal is an α-n-ideals. Moreover, all the minimal prime n-ideals are α-n-ideals. Then we generalize all the results of Cornish [11] in terms of α-n-ideals. We also show that for a central element n, P_n (L) is disjunctive if and only if each n-ideal is an α-n-ideals. We conclude the thesis by characterizing $P_n(L)$ to be generalized Stone in terms of α-n-ideals.

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Chapter-1

INTRODUCTION

In this thesis we have studied the Principal n-ideals of a lattice. For a fixed element n of a lattice L, a convex sublattice of L containing n is called an n-ideal. If L has a '0', then replacing n by 0, an n-ideal becomes an ideal and if L has a '1' then it becomes a filter by replacing n by 1. Thus, the idea of n-ideals is a kind of generalization of both ideals and filters of lattices. The n-ideal generated by a finite number of elements of a lattice is called a finitely generated n-ideal, while the n-ideal generated by a single element is known as a principal n-ideal. Latif [30] in his thesis has given a neat description on finitely generated n-ideals of a lattice and has provided a number of important results on them. Balbes and Horn [1], Chen and Gratzer [7] and many others have studied the minimal prime ideals in a distributive lattice. The n-ideals of a lattice have been studied extensively by Noor and Latif [32], [35], [49], [53] etc. For a lattice L, the lattice of all n-ideals of L and the lattice of all finitely generated n-ideals of L are denoted by I_n (L) and F_n (L) respectively, while P_n (L) represents the set of principal n-ideals of L.

Many authors including Mandelker [38] and Varlet [60] have studied relative annihilators in lattices and semilattices. Also Noor and Ali [45] have introduced the notion of relative annihilators around a fixed element.

Cornish [9] have studied distributive lattice analogues to B_m-lattices and relatively B_m-lattices. He referred them as m-normal lattices. Moreover, Beazer [3] and Deavy [13] have each independently obtained several characterizations of (sectionally) B_m–lattices and relatively B_m–lattices.

Normal lattice have been studied by several authors including Cornish [9] and Monteiro [39]. Cignoli [8] and Cornish [9] introduced the notion of k-normal and k-completely normal lattice.

Subramanian [57] studied h-ideals with respect to the space of maximal l-ideals in f-ring. Bigard [4] has studied α-ideals in the context of lattice ordered groups. Noor and Islam [54] has studied the annulets and α-ideals in a distributive near lattice.

Here we devote ourselves in studying several properties on $P_n(L)$ and $F_n(L)$ which will certainly enrich many branches of lattice theory. The results in this thesis generalize many results on normal, relatively normal, m-normal and relatively m-normal lattices. We also introduce the concept of n-annulets and α-n-ideals in studying $P_n(L)$.

In this connection it should be mentioned that if L has a 0, then putting n = 0, we find that $P_n(L)$ is the set of all principal ideals of L which is isomorphic to L. Thus, from every result on $F_n(L)$ in this thesis, we can obtain a result for the lattice L with 0 by substituting n = 0. Hence the results in this thesis regarding $F_n(L)$ are generalizations of the corresponding results in lattice theory.

Here we also have given explicit descriptions of $F_n(L)$ and $P_n(L)$ which are essential for the development of the thesis. Since $F_n(L)$ is always a lattice and $P_n(L)$ is not even a semilattice, but when n is a neutral element, $P_n(L)$ becomes a meet semilattice. We have shown that $P_n(L)$ is a lattice if and only if when n is a central element, and then in fact, $P_n(L) = F_n(L)$. We also have shown that, for a neutral element n, the lattice L is complemented if and only if $P_n(L)$ is also complemented. We also have discussed on prime n-ideals and given several properties and characterizations of prime n-ideals. We included a proof of the generalization of Stone's separation theorem and included new results that can be established from the generalization, especially we have shown that for a distributive lattice L, $F_n(L)$ is generalized Boolean if and only if prime n-ideals are unordered.

We have proved here that for a distributive lattice L, $F_n(L)$ is normal if and only if each prime n-ideal of L contains a unique minimal prime n-ideal. If n is central in L, then $P_n(L)$ is a normal lattice if and only if any two minimal prime n-ideal are comaximal which is also equivalent to $<x>_n \cap <y>_n = \{n\}$ implies $<x>_n^* \vee <y>_n^* = L$.

We introduce the notion of relative n-annihilators $<a,b>^n$ and characterize distributive and modular lattices in terms of relative n-annihilators. Many results have been introduced that shown - for a central element n, $P_n(L)$ is a relatively normal lattice.

For a central element $n \in L$, $P_n(L)$ is m-normal if and only if for any m+1 distinct minimal prime n-ideals $P_0, \ldots\ldots\ldots, P_n$ of L, $P_0 \vee \ldots\ldots\ldots\ldots \vee P_m = L$.

We introduces the concept of n-annulets and α-n-ideals of a lattice. Here we include several result on the set of n-annulets $A_n(L)$ when n is a central element of L. We proved $A_n(L)$ is relatively complemented if and only if $P_n(L)$ is sectionally quasi-complemented.

We have shown that n-ideal $n(P)$, where P is a prime n-ideal, is an α-n-ideals. Moreover, all the minimal prime n-ideals are α-n-ideals. Then we have generalized all the results of Cornish [11] in terms of α-n-ideals. We conclude by characterizing $P_n(L)$ to be generalized Stone in terms of α-n-ideals.

CHAPTER -2

Chapter-2

n-ideals of a lattice.

Introduction:

The intention of this chapter is to outline and fix the notation for some of the concepts of n-ideals of a lattice which are basic to this thesis. The idea of n-ideals was first introduced by **Cornish** and **Noor** in several papers including [12] . The n-ideals have also been used in proving some results in [42].

A non empty subset I of a lattice L is called an *ideal* of L if

(i) for $x, y \in I$, $x \vee y \in I$

(ii) $x \in I$ and $y \leq x, (y \in L)$ imply $y \in I$.

An ideal P of L is called a *prime ideal* if for $x, y \in L$, $x \wedge y \in P$ implies either $x \in p$ or $y \in P$.

Similarly a non-empty subset F of L is called a *dual ideal* (filter) if (i) for $x, y \in F$, $x \wedge y \in F$ and (ii) $x \in F$ and $y \geq x, y \in L$, imply $y \in F$ .A filter F is called a *prime filter* if for $x, y \in L$, $x \vee y \in F$ implies either $x \in F$ or $y \in F$.

The n-ideals of a lattice have been studied extensively by Noor and Latif in [30], [32], [33], [34], [31], [49], [50], [51], [52] and [53]. For a fixed element n of a lattice L, a convex sublattice containing n is called an n-*ideal* . If L has 0 then replacing n by 0 an n-ideal becomes an ideal. Moreover if L has 1, and n-ideal becomes a filter by replacing n by 1. Thus the idea of n-ideals is a kind of generalization of both ideals and filters of lattices. So any result involving n-ideals of a lattice L will give a generalization of the results on ideals if $0 \in L$ and filters if $1 \in L$.

The set of all n-ideals of a lattice L is denoted by $I_n(L)$, which is an algebraic lattice under set inclusion. Moreover, $\{n\}$ and L are respectively the smallest and the largest elements of $I_n(L)$, while the set theoretic intersection is the infimum.

For any two n-ideals 1 and J of a lattice L, it is easy to check that $I \cap J = \{ x : x = m\,(i,\ n, j)$ for some $i \in I\ j \in J$ where $m\,(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and $I \vee J = \{ x : i_1 \wedge j_1 \le x \le i_2 \vee j_2$, for some $i_1, i_2 \in I$ and $j_1, j_2 \in J \}$.

The n-ideal generated by $a_1,\ a_2, \ldots\ldots\ldots\ \ldots\ldots\ldots., a_m$ is denoted by $< a_1,\ a_2, \ldots\ldots\ldots\ \ldots\ldots\ldots\ \ldots\ldots\ldots\ldots, a_m >_n$.
Clearly $< a_1,\ a_2, \ldots\ldots\ldots\ \ldots\ldots\ldots.., a_m >_n$
$= < a_1 >_n \vee < a_2 >_n \vee \ldots\ldots\ldots\ \ldots. \vee < a_m >_n$.

The n-ideal generated by a finite number of elements is called a *finitely generated n-ideal*. The set of all finitely generated n-ideals is denoted by $F_n\,(L)$. Of course, $F_n\,(L)$ is a lattice. The n-ideal generated by a single element is called a *principal n-ideal* .The set of all principal n-ideals of a lattice L is denoted by $P_n\,(L)$. We have
$< a >_n = \{x \in L : a \wedge n \le x \le a \vee n\}$.

The median operation $m(x, \quad y, \quad z)=(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ is very well known in lattice theory . This has been used by several authors including Birkhoff and Kiss [6] for bounded distributive lattices, Jakubik and Kalibiar [22] for distributive lattices and Sholander [56] for median algebras.

An n-ideal P of a lattice L is called a prime n-ideal if $m\ (x, n, y) \in P\ (x, y \ \in L)$ implies $x \ \in P$ or $y \in \ P$.

Standard and neutral elements in a lattice were studied extensively in [16] and [20] . An element s of a lattice L is called *standard* if for all x, $y \in L$,

$$x \wedge (y \vee s) = (x \wedge y) \vee (x \wedge s).$$

An element $n \in L$ is called *neutral* if it is standard and for all $x, y \in L$,

$n \wedge (x \vee y) = (n \wedge x) \vee (n \wedge y)$. By [17], we know that $n \in L$ is neutral if and only if for all $x, y \in L$,

$m(x, \quad n, \quad y) = (x \wedge y) \vee (x \wedge n) \vee (y \wedge n) = .$

$= (x \vee y) \wedge (x \vee n) \wedge (y \vee n)$. Of course 0 and 1 of a lattice are always neutral. In a distributive lattice clearly every element is standard and neutral.

Let L be a lattice with 0 and 1. For an element $a \in L$, a' is called the *complement* of a if $a \wedge a' = 0$ and $a \vee a' = 1$. A bounded lattice in which every element has a complement is called a *complemented* lattice. In a distributive lattice it is easy to see that every element has at most one complement.

An element $n \in L$ is called *central* if it is neutral and complemented in each interval containing it.

A lattice L with 0 is called *sectionally complemented* if [0, x] is complemented for all $x \in L$. A complemented distributive lattice is called a Boolean lattice, while a distributive lattice with 0, which is sectionally complemented is called a *generalized Boolean lattice*. For the background material on lattices we refer the reader to the texts of G. Gratzer [15], Birkhoff [5], Rutherford [55], Khanna [27] and Maeda and Maeda [37].

In this thesis we have studied the lattice $F_n(L)$ in different situations. If L has a 0, then putting n = 0, we find that $<a_1,\dots\dots a_m>_n = (a_1 \vee \dots\dots \vee a_m]$.
Hence for n = 0, $F_n(L)$ is the set of all principal ideals of L which is isomorphic to L. Thus for every result on $F_n(L)$ in this thesis, we can obtain a result for the lattice L by substituting n = 0. Hence the result in each chapter of the thesis regarding $F_n(L)$ are generalizations of several results on Boolean, generalized Boolean, normal, and relatively normal lattices. Chapter 4 gives generalizations of several results on those lattices which are in m-normal and relative m-normal respectively.

In section I we have given an explicit description of $F_n(L)$ and $P_n(L)$ which will be needed for the development of the thesis. We have shown that $P_n(L) = F_n(L)$ if and only if n is central. We have proved that a lattice L is (modular) distributive if and only if $F_n(L)$ is so. We have also shown that for a neutral element n, lattice L is complemented if and only if $P_n(L)$ is complemented. Moreover, if a' is the complement of a in L, then $<a'>_n$ is the complement of $<a>_n$ in $P_n(L)$.

In section 2 we have discussed on prime n-ideals. We have given several properties of prime n-ideals. We have included a proof of generalization of Stone's seperation theorem. Finally we include a new proof of the result that for a distributive lattice L, $F_n(L)$ is generalized Boolean if and only if prime n-ideals of L are unordered.

1. Finitely generated n-ideals.

We start this section with the following proposition which is due to [30], also see [33] and [49]. This gives some simpler description of $F_n(L)$.

Proposition 2.1.1. *Let* $F_n(L)$ *be a lattice and* $n \in L$.

For $a_1, \quad a_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, a_m \in L,$

(i) $<a_1, a_2, \ldots\ldots\ldots\ldots, a_m>_n \subseteq \{ y \in L : (a_1] \cap \ldots\ldots \cap (a_m] \cap (n]$
$\subseteq (y] \subseteq (a_1] \vee \ldots\ldots \vee (a_m] \vee (n] \}$;

(ii) $< a_1, \quad a_2, .., a_m >_n = \{ y \in L : a_1 \wedge a_2 \wedge \ldots\ldots \wedge a_m \wedge n$
$\leq y \leq a_1 \vee a_2 \vee \ldots\ldots \vee a_m \vee n \}$:

(iii) $<a_1, \quad a_2, \ldots\ldots\ldots\ldots, a_m >_n = \{ y \in L : a_1 \wedge a_2 \wedge \ldots\ldots \wedge a_m \wedge n \leq y$
$= (y \wedge a_1) \vee \ldots\ldots \vee (y \wedge a_m) \vee (y \wedge n) \}$,

where L *is distributive;*

(iv) *For any* $a \in L$,

$<a>_n$
$= \{ y \in L : a \wedge n \leq y = (y \wedge a) \vee (y \wedge n) \} =$
$\{ y \in L : y = (y \wedge a) \vee (y \wedge n) \vee (a \wedge n) \}$,

where n *is standard;*

(v) *Each finitely generated* n*-ideal is two generated.*

Indeed

$< a_1, \quad a_2, \ldots\ldots, a_m >_n = < a_1 \wedge a_2 \wedge \ldots\ldots \wedge a_m \wedge n,$

$a_1 \vee a_2 \vee \ldots\ldots\ldots \ldots \vee a_m \vee n >_n$;

(vi) $F_n(L)$ *is a lattice and its members are simply the intervals* [a, b] *such that* $a \le n \le b$ *and for each intervals* [a, b] *and* $[a_1, \ b_1]$,

$$[a, \ b] \vee [a_1, \ b_1] = [a \wedge a_1, \ b \vee b_1] \text{ and}$$

$$[a, \ b] \cap [a_1, \ b_1] = [a \vee a_1, \ b \wedge b_1] . \qquad \blacksquare$$

For $n \in L$ *suppose* $(n]^d$ *denotes the dual of the lattice* $(n]$. *Then for any* $x, \ y \in (n], \ x \vee^d y = x \wedge y$ *and* $x \wedge^d y = x \vee y$.

Theorem 2.1.2. *Let* L *be a lattice and* $n \in L$. *The maps* $\Phi : F_n(L) \to (n]^d \times [n)$ *and* $\Psi : (n]^d \times [n) \to F_n(L)$ *is given by* $\Phi([a, \ b]) = (a, b)$ *and* $\Psi((x, \ y)) = [x, y]$ *where* $[a, \ b] \in F_n(L)$ *and* $(x, \ y) \in (n]^d \times n$, *are mutually inverse lattice isomorphism's. In other words,* $F_n(L) \cong (n]^d \times [n)$.

Proof : Let $[a, \ b] \subseteq [a_1, \ b_1]$ .Then $a_1 \le a \le n \le b \le b_1$, and so $a \le^d a_1$ in $(n]^d$ and $b \le b_1$ in $[n)$. Thus, $(a, b) \le (a_1, b_1)$ in $(n]^d \times [n)$. Hence Φ is order preserving. If $(a, b) \le (a_1, b_1)$ in $(n]^d \times [n)$, then $a \le^d a_1$ in $(n]^d$ and $b \le b_1$ in $[n)$. Thus, $a_1 \le a \le n \le b \le b_1$ in L and

so $[a, \ b] \subseteq [a_1, \ b_1]$.

That is Ψ is also order preserving. But Φ and Ψ are mutually inverse and so the theorem is established. ■

When n is a neutral element of a lattice L, then it is very easy to check that $P_n(L)$ is a meet semilattice. In fact, for any
a, $b \in L$, $<a>_n \cap <b>_n = m(a, n, b)>_n$.
But $P_n(L)$ is not necessary a lattice. The case is different when n is a central element. The following theorem also gives characterization of central elements of a lattice L.

Theorem 2.1.3. *Let n be neutral element of a lattice* L. *Then* $P_n(L)$ *is a lattice if and only if* n *is central. Then of course*
$P_n(L) = F_n(L) \approx (n]^d \times [n)$.

Moreover, for a central element $n \in L$, L *is bounded if and only if* $P_n(L)$ *is bounded.*
Also if L is bounded and n is a central element of L, *then for any* x, $y \in L$ $<x>_n \vee <y>_n = <m(x, n', y)>_n$ *where* n' *is the complement of* n *in* L .

Proof : Suppose n is central. Since for all a, $b \in L$,
$<a>_n \cap <b>_n = <m(a, n, b)>_n$, we need only to check that $<a>_n \vee <b>_n \in P_n(L)$. Now,
$<a>_n \vee <b>_n = [a \wedge b \wedge n, a \vee b \vee n]$. Since n is central, there exists $c \in L$ such that $c \wedge n = a \wedge b \wedge n$ and

$c \vee n = a \vee b \vee n$ which implies that

$< a >_n \vee < b >_n = < c >_n$ and so $P_n(L)$ is a lattice.

Conversely, suppose that $P_n(L)$ is a lattice and $a \leq n \leq b$. Then $[a, b] = < a >_n \vee < b >_n$. Since $P_n(L)$ is a lattice, $< a >_n \vee < b >_n = < c >_n$ for some $c \in L$. This implies that $c \wedge n = a$ and $c \vee n = b$. This implies c is the relative complement of n in $[a, b]$. Therefore n is central.

For the second part, if L = [0, 1], then {n} and $< n' >_n$ are the smallest and the largest elements of $P_n(L)$, where n' is the complement of n in L.

Also if $P_n(L)$ is bounded, then there exists $n' \in L$ such that $< n' >_n$ is the largest element of $P_n(L)$. Therefore for any $x \in L$,

$< x >_n \subseteq < n' >_n$. That is

$n \wedge n' \leq x \wedge n \leq x \leq x \vee n \leq n \vee n'$. This implies

$n \wedge n'$ and $n \vee n'$ are the smallest and the largest elements of L and so L is bounded. Last part is easily verifiable. ■

Thus the following results are obvious from the Theorem 2.1.2.

Theorem 2.1.4. *Let* L *be a lattice. Then* $F_n(L)$ *is sectionally complemented if and only if for each* a, $b \in L$ *with* $a \leq n \leq b$, *the interval*

[a, b] *and* [n, b] *are complemented.*

Corollary 2.1.5. *For a distributive lattice* L, $F_n(L)$ *is generalized Boolean if and only if the interval* [a, n] *and* [n, b] *are complemented for each* a, $b \in L$ *with* $a \le n \le b$.

Corollary 2.1.6. *For a distributive lattice* L, $F_n(L)$ *is generalized Boolean if and only if both* $(n]^d$ *and* $[n)$ *are generalized Boolean .* ■

It is clear from the Corollary 2.1.4. that if L is relatively complemented, then $F_n(L)$ is sectionally complemented and in fact $F_n(L) = P_n(L)$. If L has 0 and 1, the largest element L of $I_n(L)$ is finitely generated . Then in fact, L = [0, 1] .

A lattice L with 0 is said to be sectionlly-semi-complemented lattice (disjunctive) if $0 \le a < b$ $(a, b \in L)$ implies that there is an element $x \in L$ such that $x \wedge a = 0$ and $0 < x \le b$, while a lattice satisfying the definition which is dual to that of a sectionally semi-complemented lattice is called a dual sectionally semi-complemented lattice (dual disjunctive).

A lattice L is called *implicative (relatively pseudocomplemented*) if for any given elements a and b, the set of all $x \in L$ such that $a \wedge x \le b$ contains a largest element which is denoted by $a \rightarrow b$. A dual implicative lattice is defined dually.

The following corollary holds because of theorem 2.1.2.

Corollary 2.1.7. *Let* L *be a lattice and* $x \in L$ *. Then*

(i) $F_n(L)$ *is sectionally-semi-complemented if and only if* $(n]$ *is dual sectionally-semi-complemented and* $[n)$ *is sectionally-semi-complemented*

(ii) $F_n(L)$ *is implicative if and only if* $(n]$ *is dual implicative and* $[n)$ *is implicative.* ■

Theorem 2.1.8. *Let* n *be a neutral element of a bounded lattice* L. *Then* L *is complemented if and only if* $P_n(L)$ *is a complemented lattice.*

Moreover, a′ *is the complemented of a in* L *if and only if* $< a' >_n$ *is the complemented of* $< a >_n$ *in* $P_n(L)$.

Proof : Suppose L is complemented. Then by Theorem 2.1.3, $P_n(L)$ is a lattice with {n} and $< n' >_n$ as the smallest and the largest elements. Moreover, $P_n(L) = F_n(L)$. Now let $< a >_n \in P_n(L)$. Suppose a' is the complement of a in L. Then

$< a >_n \cap < a' >_n = [a \wedge n, a \vee n] \cap [a' \wedge n, a' \vee n]$

$= [(a \vee a') \wedge n, (a \wedge a') \vee n] = [(a \vee a') \wedge n, (a \wedge a') \vee n]$

$= [1 \wedge n, 0 \vee n] = \{n\}$. Also $< a >_n \vee < a' >_n$

$= [a \wedge a' \wedge n, a \vee a' \vee n] = [0, \; 1] = < n' >_n$. This implies $P_n(L)$ is complemented, and $< a' >_n$ is the complement of $< a >_n$ for each $a \in L$

Conversely, suppose $P_n(L)$ is complemented . Let $a \in L$ and let $< b >_n$ be the complement of $< a >_n$ in $P_n(L)$.

Then $< a >_n \cap < b >_n = \{ n \}$ and

$< a >_n \vee < b >_n = [0, 1]$.

Thus, $[(a \vee b) \wedge n, \quad (a \wedge b) \vee n] = \{n\}$ and

$[(a \wedge b) \wedge n, \quad a \vee b \vee n] = [0, 1]$. Now

$[(a \vee b) \wedge n, \quad (a \wedge b) \vee n] = \{n$ implies $a \wedge b \leq n \leq a \vee b$.

Hence $[0, 1] = [a \wedge b \wedge n, a \vee b \vee n] = [a \wedge b, \ a \vee b]$ and so $a \wedge b = 0$ and $a \vee b = 1$. This implies b is the complement of a in L. Therefore L is complemented. ■

Thus we have the following corollary :

Corollary 2.1.9. *For a bounded distributive lattice* L *with* $n \in L$, L *is Boolean if and only if* $P_n(L)$ *is a Boolean lattice.* ■

In lattice theory, it is well known that a lattice L is modular (distributive) if and only if the lattice of ideals I (L) is modular (distributive). Our following theorems are nice generalizations of those results in terms of n-ideals when n is a neutral element which is due to [30] . Also see [49] .

Theorem 2.1.10. *For a neutral element* n *of a lattice* L, *the following conditions are equivalent* :

(i) L *is modular*

(ii) $I_n(L)$ *is modular*

(iii) $F_n(L)$ *is modular*

Following result is also due to [30]. ■

Theorem 2.1.11. *Let* L *be a lattice with a neutral element* n . *Then the following conditions are equivalent* :

(i) L *is distributive* ;

(ii) $I_n(L)$ *is distributive* ;

(iii) $F_n(L)$ *is distributive* ; ■

For any two n-ideals I and J of a lattice we have already defined $I \vee J$ in the introduction . Now we include the following result , which will be used to prove several theorems in different chapters of the thesis.

Theorem 2.1.12. *Let* I *and* J *be two* n-*ideals of a distributive lattice . Then for any* $x \in I \vee J$, $x \vee n = i_1 \vee j_1$ *and* $x \wedge n = i_2 \wedge j_2$ *for some* $i_1, i_2 \in I$, $j_1, j_2 \in J$ *with* $i_1, j_1 \geq n$ *and* $i_2, j_2 \leq n$.

Proof : Let $x \in I \vee J$. Then $i \wedge j \leq x \leq i' \vee j'$ for some $i, j' \in I$, $j, j' \in J$.

Now $x \leq i' \vee j'$ implies $x \vee n \leq i' \vee j' \vee n$. Thus

$x \vee n = (x \vee n) \wedge (i' \vee j' \vee n)$.

$= [(x \vee n) \wedge (i' \vee n)] \vee [(x \vee n) \wedge (j' \vee n)]$

But $n \leq (x \vee n) \wedge (i' \vee n) \leq i' \vee n$ implies by convexity that $(x \vee n) \wedge (i' \vee n) = i_1 (say) \in I$ .Similarly, $(x \vee n) \wedge (j' \vee n) = j_1 (say) \in J$. Thus,

$x \vee n = i_1 \vee j_1$; $i_1 \in I$, $j_1 \in J$ and $i_i \geq n$, $j_1 \geq n$. Similarly we can show that $x \wedge n = i_2 \wedge j_2$ for some $i_2 \in I$, $j_2 \in J$ with $i_2, j_2 \leq n$

We conclude this section with the following useful result which is due to [31]. This result will also be used in proving several results in different chapters of the thesis.

Theorem 2.1.13. *For a neutral element* n *of a lattice* L, *any finitely generated* n-*ideal of l which is contained in a principal* n-*ideal is a principal*

n-*ideal* . ■

2. Prime n-ideals.

Recall that an n-ideal P of a lattice L is *prime* if $m\,(x, n, y) \in P$, $x, y \in L$ implies either $x \in P$ or $y \in P$.

Since for any two n-ideals I and J of L, $I \cap J = \{ m\,(i, n, j) : i \in I,\ j \in J \}$, so it is very easy to see that for any prime n-ideal P. $I \cap J \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$.

Theorem 2.2.1. *If* P *is a prime* n-*ideal of a lattice, then for any* $x \in L$ *, at least one of* $x \wedge n$ *and* $x \vee n$ *is a member of* P.

Proof : Observe that $m\,(x \wedge n,\ n,\ x \vee n) = n \in P$. Thus either $x \wedge n \in P$ or $x \vee n \in P$. ■

Theorem 2.2.2. *If* P *is a prime* n-*ideal of a lattice, then* P *contains either* $(n]$ or $[n)$, *but not both* .

Proof : Suppose P is prime and. $P \not\supseteq (n]$ Then there exists $r < n$ such that $r \notin P$. Now let $s \in [n)$. Then $m\,(r, n, s) = (r \wedge n) \vee (n \wedge s) \vee (s \wedge r) = r \vee n \vee r = n \in P$ implies that $s \in P$. That is , $P \supseteq [n)$. Similarly, if $P \not\supseteq [n)$, then we can show $P \supseteq (n]$.

Finally suppose that P contains both $(n]$ and $[n)$ Let $t \in L$. Then $t \wedge n \in P$ and $t \vee n \in P$. Then by convexity of n-ideals $t \in P$. This implies P = L , which is contradiction to the primness of P . ■

Thus we have the following corollary :

Corollary 2.2.3. *If* P *is a prime n-ideal of a lattice* L, *then there exists at least one* $x \in L$ *such that both* $x \wedge n$ *and* $x \vee n$ *does not belong to* P. ■

Theorem 2.2.4. *Let* n *be a neutral element of a lattice* L. *Then an*
n-*ideal* P *is prime if and only if it is a prime ideal or a prime dual ideal (filter)* .

Proof : Suppose the n-ideal P is prime . Then by Theorem 2.2.2, either $P \supseteq (n]$ or $P \supseteq [n)$. Suppose $P \supseteq (n]$. Let $x \in P$ and $t \leq x$, $t \in L$. Then $t \wedge n \in (n] \subseteq P$. Thus, by convexity of P, $t \wedge n \leq t \leq x$ implies that $t \in P$. This implies that P is an ideal. Also let $a \wedge b \in P$, $a, b \in L$. Then $(a \wedge b) \vee n \in P$ and $m(a, n, b) = (a \wedge n) \vee (b \wedge n) \vee (a \wedge b) \leq (a \wedge b) \vee n$ implies that $m(a, n, b) \in P$. Thus, either $a \in P$ or $b \in P$, and so P is a prime ideal.

On the other hand if $P \supseteq [n)$, we can similarly prove that P is a prime dual ideal. ■

Following lemma is due to [30, Lemma 2.2.8] .

Lemma 2.2.5. *In a distributive lattice* L, *a prime ideal containing* n *is also a prime* n-*ideal* . ■

Dually we have the following result .

Lemma 2.2.6. *In a distributive lattice* L, *a prime dual ideal (filter) containing* n *is also a prime* n-*ideal* . ■

The set of all prime n-ideals of L is denoted by P (L) . The following separation property for distributive lattices was given by M. H. Stone [15, Theorem-15, Page-741], which is known as Stone' separation theorem .

Theorem 2.2.7. *Let* L *be a distributive lattice, let* I *be an ideal, let* D *be a dual ideal of* L, *and let* $I \cap D = \emptyset$, *then there exists a prime ideal* P *of* L *such that* $P \supseteq I$ *and* $P \cap D = \emptyset$. ■

Following result is an improvement of above theorem which is due to [31, Theorem 2.2.3]

Theorem 2.2.8. *Let* L *be a distributive lattice* , *let* I *be an ideal, let* D *be a convex sublattice of* L *and let* $I \cap D = \Phi$, *then there exists a prime ideal* P *of* L *such that* $P \supseteq I$ *and* $P \cap D = \Phi$. ■

Now we give a separation property for distributive lattices in terms of prime n-ideals which is of course an extension of Stone's separation theorem . It should be mentioned that this result has also been obtained by Latif and Noor in [53] . Here we include a separate proof of it is much more simpler than that of [53] .

Theorem 2.2.9. *In a distributive lattice* L, *suppose* I *is an* n-*ideal and* D *is a convex sublattice of* L *with* $I \cap D = \Phi$. *Then there exists a prime ideal* P *of* L *such that* $P \supseteq I$ *and* $P \cap D = \Phi$.

Proof : Since $I \cap D = \Phi$, so either $(I] \cap D = \Phi$ or $[I) \cap D = \Phi$. If $(I] \cap D = \Phi$, then by Theorem 2.2.8 , there exists a prime ideal $P \supseteq I$ such that $P \cap D = \Phi$. Similarly if $[I) \cap D = \Phi$, then there exists a prime filter $Q \supseteq [I)$ such that $Q \cap D = \Phi$. But by Lemma 2.2.5 and Lemma 2.2.6 both P and Q are prime n-ideals . ■

Corollary 2.2.10. *Every* n-*ideal* I *of a distributive lattice* L *is the intersection of all prime n-ideals containing it .*

Proof : Let $I_1 = \cap\{P \;\; : P \supseteq I, P$ is a prime n-ideal of L }. If $I \neq I_1$, then there is an element $a \in I_1 - I$. Then by above corollary , there is a prime n-ideal P with $P \supseteq I$, $a \notin P$. But $a \notin P \supseteq I$, gives a contradiction . ■

For an n-ideal I of a distributive lattice L, the congruence $\Theta(I)$ has been studied in [61] and [30]. By [61], $x \equiv y\,\Theta(I)$ if and only if $x \wedge i_1 = y \wedge i_1$ and $x \vee i_2 = y \vee i_2$ for some $i_1, \; i_2 \in I$. Moreover $\Theta(I)$ is the smallest congruence of L containing I as a class . In chapter 2 of [30], Latif has proved the following result :

Theorem 2.2.11. : *Let* L *be a distributive lattice. Then for any two* n-*ideals* I *and* J *of* L

(i) $\Theta(I \cap J) = \Theta(I) \cap \Theta(J)$;

(ii) $\Theta(I \vee J) = \Theta(I) \vee \Theta(J)$

Moreover, the correspondence $I \to \Theta(I)$ *is an embedding from* $I_n(L)$ to C (L).

Theorem 2.2.12. *For a neutral element* n *of a lattice* L, $I_n(L) \cong C(L)$ *if and only if* $F_n(L)$ *is generalized Boolean.* ■

For an n-ideal I of a distributive lattice L, Latif has also studied the congruence R (I) in [61] . By [61], the relation R (I) defined by "$x \equiv y\ R(I)$ if and only if for any $t \in L$, $m(x, n, t) \in I$ is equivalent to $m(y, n, t) \in I$" is the largest congruence of L containing I as a class . With the help of this congruence we will provide the following characterization of prime n-ideals of a distributive lattice . This result is due to [Ayub's Thesis]. We prefer to include its proof for the convenience of the readers .

Theorem 2.2.13. *Let* L *be a distributive lattice and* $n \in L$ *. An* n-*ideal* P *is prime of and only if the quotient lattice* L / R (P) *is a two element chain .*

Proof : Suppose P is prime . Let $x, y \in L - P$. Then for any $t \in L$, $m(x, n, t) \in P$ implies $t \in P$. Since $t \wedge n \le m(y, n, t) \le t \vee n$, so by convexity of P, $m(y, n, t) \in P$. Therefore $x \equiv y\ R(P)$. Moreover, let $r \equiv x\ R(P)$ for some $x \in L - P$. Then $m(r, n, x) \notin P$ as $m(x, n, x) = x \notin P$. This implies $r \notin P$. For otherwise, $r \wedge n \le m(r, n\ x) \le r \vee n$ would imply that $m(r, n, x) \in P$ by convexity of P and that is a contradiction . Thus L / R (P) is a two element chain {P, L-P} .

Conversely, suppose L / R (P) is a two-element chain. Then L-P is a congruence class of the congruence R (P) . If P is not prime, then there exists $x, y \in L - P$ such that $m(x, n, y) \in P$. Since L-P is a congruence class, so $x \equiv y\ R(P)$. Thus $m(x, n, y) \in P$ implies $m(y, n, y) = y \in P$ which is a contradiction . Therefore P must be prime . ■

For any n-ideal J of a distributive lattice L, we define

$J^{+} = \{x \in L : m(x, n, j) = n$ for all $j \in J\}$. Obviously, J^{+} is an n-ideal and $J \cap J^{+} = \{n\}$. We call J^{+} as the *annihilator* n-*ideal of* J .

It is well known from [15, Theorem 22, Page 76] that a distributive lattice with 0 is generalized Boolean if and only if the set of prime ideals is unordered. We conclude the chapter with a nice generalization of that result which is due to [30, Theorem 2.2.9]; also see [49] . Hence we prefer to include a new proof of (i) $\Rightarrow$ (iii), as it is much easier than that of [30] .

Theorem 2.2.14. *Let* L *be a distributive lattice and* $n \in L$. *Then the following conditions are equivalent :*

(i) $F_n(L)$ *is generalized Boolean ;*

(ii) *For each principal* n-*ideal*

$< x >_n, < x >_n \vee < x >_n^{+} = L, where$

$< x >_n^{+} = \{y \in L : m(x, n, y) = n$;

(iii) The set of prime n-*ideals* P (L) *is unordered by set inclusion.*

Proof : (i) $\Leftrightarrow$(ii) and (iii) $\Leftrightarrow$(i) follows from [30, Theorem 2.2.9] .

(i) $\Leftrightarrow$(iii) . Suppose (i) holds. Then by Theorem 2.1.5, the intervals [x, n] and [n, y] are complemented for each $x, \quad y \in L$ with $x \leq n \leq y$. Let P and Q be any two prime n-ideals of L . Then by Theorem 2.2.4 , P and Q are either prime ideals or prime filters of L . If one of them is a prime ideal and the other is a prime filter, then of course they are unordered. If both P and Q are prime ideals, then $P \cap [n, y]$ and $Q \cap [n, y]$ are prime ideals of [n, y] .

Since [n, y] is a complemented lattice, so by [15, Theorem 22, Page 76], $P \cap [n, y]$ and $Q \cap [n, y]$ are unordered. Therefore P and Q are unordered . If P, Q are filters , then using the same argument we find that $P \cap [n, y]$ and $Q \cap [n, y]$ are unordered . Thus P and Q are unordered and this establishes (iii) . ■

CHAPTER -3

Chapter-3

Principal n-ideals which form normal Lattices

Introduction

Normal lattice have been studied by several author including **Cornish** [9] and **Monteiro** [39] ,[31],while n-normal lattices have been studied by Cornish [11] and Devey [13]. On the other hand Cignoli in [8] and [9] introduced the notion of k-normal and k-completely normal lattice. Again Chan and Gratzer in [7] and [8] studied the constructions and the structures of Stone lattices.

A distributive lattice L with 0 is called *normal* if each prime ideal of L contains a unique minimal prime ideal. Equivalently, L is called *normal* if each prime filter of L is contained in a unique ultrafilter (maximal and proper) of L.

Minimal prime ideals in distributive lattice have been studied extensively by many authors including [1], [7], [8], [9], [21], [28], [58], and [59].

Recently, [6] , [8] introduced the concept of minimal prime n-ideal in lattices and generalized several results of minimal prime ideals .

In this chapter we study the minimal prime n-ideals and generalized some of the results on minimal prime ideals. Then we used these results to generalized several important results on normal lattices in terms of n-ideals.

A Prime n-ideal P is said to be a *minimal Prime n-ideal* belonging to n-ideal I if,

(i) $I \subseteq P$, and

(ii) There exists no prime n-ideal Q such that $Q \neq P$ and $I \subseteq Q \subseteq P$.

A prime n-ideal P of L is called a minimal prime n-ideal if there exists no prime n-ideal Q such that $Q \neq P$ and $Q \subseteq P$. Then a minimal prime n-ideal is a minimal prime n-ideal belonging to {n}.

For any n-ideal J of L, we have already defined in chapter 1 that

$$J^* = \{x \in I, : m(x, n, j) = n \quad \textit{for all} \quad j \in J\}$$

Observe that J^* is an n-ideal and $J \cap J^* = \{n\}$.

In fact, this is the largest n-ideal which annihilates J. Latif in [30] called this an annihilator n-ideal of J. We prefer to call this as the pseudocomplement of J in $F_n(L)$. Moreover, for a distributive lattice L, $F_n(L)$. is a distributive algebraic lattice and so it is pseudocomplemented. Observe that $I_n(L)$. has always the smallest element viz. {n}.

We shall call two prime n-ideals P and Q of L comaximal if $P \vee Q = L$.

In section 1. we have studied minimal prime n-ideals of L. There we have given some characterizations of minimal prime n-ideals. Also see [43]. These results give nice generalizations of several results on minimal prime ideals which will be used to prove some important results in section 2.

In section 2, we have given several characterizations of those $P_n(L)$ which are normal lattices in terms of n-ideals. Then we have proved that $P_n(L)$ is normal if and only if each prime n-ideal contains. a unique minimal prime n-ideal .

1. Minimal prime n-ideals

Recall that a prime n-ideal P is a minimal prime n-ideal belonging to an n-ideal

I if

(i) $I \subseteq P$ and

(ii) There exists no prime n-ideal Q such that $Q \neq P$ and $I \subseteq Q \subseteq P$.

Following theorem is a generalization of [15, Lemma 4, Page 169].

Lemma 3.1.1 *Let* L *be a lattice with an element* n. *Then every prime* n-*ideal contains a minimal prime* n-*ideal* .

Proof : Let P be a prime n-ideal of L and let χ denotes the set of all prime n-ideals Q contained in P. Then χ is not void, since $P \in \chi$. If C is a chain in χ and $Q - \cap (X : X \in C)$, then Q is nonvoid because $n \in Q$ and Q is an n-ideal, in fact, Q is prime. Indeed, if $m(a, n, b) \in X$ for some $a, b \in L$.
then $m(a, n, b) \in X$ for all $X \in C$. Since X is prime, either $a \in X$ or $b \in X$ Thus, either $Q = \cap (X : a \in X)$ or $Q = \cap (X : b \in X)$, proving that $a \in Q$ or $b \in Q$. Therefore, we can apply to χ the dual form of Zorn's lemma to conclude the existence of a minimal member of χ. ■

Now we give a characterization of minimal prime n-ideals of a distributive lattice L, when $P_n(L)$ is sectionally pseudocomplemented. In order to do this, we need the following lemmas:

Lemma 3.1.2 *Let* L *be a distributive lattice and* $n \in L$. *Then for any* $[a, b] \in F_n(L)$ *and for any* n-*ideal* I.

$(I \cap [a, b])^* \cap [a, b] = I^* \cap [a, b]$.

Proof : Since $[a, b] \cap I \subseteq I$ so R. H. S $\subseteq$ L. H. S. To prove the reverse inclusion , let $x \in$ L. H. S. Then $a \le x \le b$ and $m(x, n, I) = n$ for all $I \in [a, b] \cap I$. Since $x \in [a, b]$, so $m(x, n, i) \in [a, b] \cap I$ for all $i \in I$. Thus $m(x, n, m(x, n, i)) = n$. But it can be easily seen that $m(x, n, m(x, n, i)) = m(x, n, i)$.

This implies $m(x, n, i) = n$ for all $i \in I$. Hence. $X \in$ R.H.S. ■

Lemma 3.1.3 *Suppose* L *is a distributive lattice, and* $[c, d] \subseteq [a, b]$ in $F_n(L)$

then, (i) $[c, d]^+ = [c, d]^* \cap [a, b]$ and

(iii) $[c, d]^{++} = [c, d]^{**} \cap [a, b]$

Proof : (i) is trivial . For (ii) using (i) we have

$[c, d]^{++} = ([c, d]^+)^* \cap [a, b] = ([c, d]^* \cap [a, b])^* \cap [a, b]$

Thus, by Lemma 3.1.2, $[c,\ d]^{++} = [c,\ d]^{**} \cap [a,\ b]$. ■

Now we give the following characterization of minimal prime n-ideals. (Also see [43]).

Theorem 3.1.4 *Let* P_n (L) be a *sectionally pseudocomplemented distributive lattice, and* P *be a prime* n-*ideal of* L *Then the following condition are equivalent :*

(i) P *is minimal*

(ii) $x \in P$ *implies* $< x >_n^* \not\subseteq P$;

(iii) $x \in P$ *implies* $< x >_n^* \subseteq P$;

(iv) $P \cap D\ (<t>_n = \ \phi)$ *for all* $t\ \ L - P$;

Where $D\ (< t >_n) = \{x \in < t >_n : < x >_n^+ = \{n\}$.

Which is due to [43].

Proof : (i) $\Rightarrow$ (ii) . Suppose P is minimal . If (ii) fails, then there exists $x \in P$ such that $< x >_n^* \subseteq P$. Since P is a prime n-ideal . So by theorem 3.2.4,

P is a prime ideal or a prime dual ideal. Suppose P is a prime ideal . Let $D = (L - P) \vee [x)$. We claim that $n \notin D$. If $n \in D$, then $n = q \wedge x$ for some $q \in L - P$.

Then $< q >_n \cap < x >_n = < (q \wedge x) \vee (q \wedge n) \vee (x \wedge n) >_n = \{ n \}$

implies $< q >_n \subseteq < x >_n^* \subseteq P$. Thus $q \in P$, which is a contradiction.

Hence $n \notin D$. Then by Stones separation theorem for n-ideals

[53, Lemma 1.3], there exists a prime n-ideal Q with $Q \cap D = \Phi$. Then

$Q \subseteq P$ as $Q \cap (L - P) = \Phi$ and $Q \neq P$ since $x \notin Q$. But

this contradicts the minimality of P. Hence, $< x >_n^* \subseteq P$

Similarly, we can prove that $< x >_n^* \subseteq P$ if P is prime dual ideal

(ii) $\Rightarrow$ (iii). Suppose (ii) holds and $x \in P$. Then $< x >_n^* \not\subseteq P$.

Since $< x >_n^* \cap < x >_n^{**} = \{ n \} \subseteq P$, P is prime, so

$< x >_n^{**} \subseteq P$.

(iii) $\Rightarrow$ (iv) . Suppose (iii) holds and $t \in L - P$. Let

$x \in P \cap D (< t >_n)$. Then $x \in P$, $x \in D (< t >_n)$. Thus

$< x >_n^+ = \{ n \}$ and so $< x >_n^{++} = < t >_n$ By (iii) ,

$x \in P$ implies . $\langle x \rangle_n^{**} \subseteq P$. Also by

Lemma 3.1.3 . $\langle x \rangle_n^{++} = \langle x \rangle_n^{**} \cap \langle t \rangle_n$

Hence $< x >_n^{**} \cap < t >_n = < t >_n$, and so $< t >_n \subseteq < x >_n^{**} \subseteq P$. That is , $t \in P$, which is a contradiction. Therefore, $P \cap D(< t >_n) = \Phi$ for all $t \in L - P$.

(iv) $\Rightarrow$ (i) , Suppose P is not minimal . Then there exists a prime

Since $< x >_n \cap < x >_n^* = \{n\} \subseteq Q$, so

$< x >_n^* \subseteq Q \subset P$, Thus, $< x >_n \vee < x >_n^* \subseteq P$.

Choose any $t \in L - P$. Then

$< t >_n \cap (< x >_n \vee < x >_n^*) \subseteq P$. Now

$$<t>_n \cap \left(<x>_n \vee <x>_n^{*}\right) = \left(<t>_n \cap <x>_n\right) \vee \left(<t>_n \cap <x>_n^{*}\right)$$

$$= < m(t, n, x)>_n \vee ((< t >_n \cap < x >_n)^* \cap < t >_n$$

(by Lemma 3.1.2)

$$= < m(t, n, x)>_n \vee (< m(t, n, x)_n^* \cap < t >_n)$$

$$= < m(t, n, x)>_n \vee < m(t, n, x)>_n^+$$

[by Lemma 3.1.3] where $< m(t, n, x)>_n^+$ is the relative pseudocomplement of $< m(t, n, x)>_n$ in $\langle t \rangle_n$.

Since $P_n(L)$ is sectionally pseudocomplemented, so

$< m(t, n, x)>_n^+$ is a principal n-ideal and so by [33, Lemma 3.4], $< m(t, n, x)>_n \vee < m(t, n, x)>_n^+$ is a principal n-ideal contained $< t >_n$. Therefore,

$$< m\ (t,\ n,\ x) >_n \vee < m\ (t,\ n,\ x) >_n^+ = < r >_n$$

for some $r \in < t >_n$ Moreover,

$$< r >_n^+ = < m\ (t,\ n,\ x) >_n^+ \cap < m\ (t,\ n,\ x) >_n^{++} = \{ n \ .$$

Thus $r \in P \cap D\ (< t >_n)$, which is a contradiction .

Therefore P must be minimal .

2. Lattices whose principal n-ideals form normal Lattices

Recall that a distributive lattice L with 0 is called a normal lattice if its every prime ideal contains a unique minimal prime ideal. Following result is due to [9, *Theorem, 2.4,]* which gives a characterization of normal lattices.

Theorem 3.2.1. *For a distributive lattice* L *with* 0, *the following conditions are equivalent.*

i) *Any two distinct minimal prime ideals are comaximal.*

ii) L *is normal.*

iii) *For any* $x, y \in L$, $(x \wedge y]^* = (x]^* \vee (y]^*$.

iv) *For any* $x, y \in L$ *with* $x \wedge y = 0$ *implies* $(x]^* \vee (y]^* = L$.

Moreover, when L *has a largest element* 1, *then each of the above conditions is equivalent to for any* $x, y \in L$, $x \wedge y = 0$ *implies* $x_1, y_1 \in L$ *such that* $x \wedge x_1 = 0 = y \wedge y_1$ *and* $x_1 \vee y_1 = 1$ ■

By theorem 2.1.2, We know that $F_n^{(L)} \cong (n]^d \wedge [n)$, so we have the following result :

Theorem 3. 2. 2 For a distributive lattice L with $n \in L$, $F_n(L)$ is normal if and only if $(n]^d$ and $[n)$ are normal. ■

A distributive lattice L with 0 is called a generalized Stone lattice if for each $x \in L$, $(x]^* \vee (x]^{**} = L$.

By katrinak [16, Lemma 8, P-134,] we know that L is generalized Stone if and only if [0, x] is a Stone sub lattice for each $x \in L$.

Moreover by [9, Theorem 5.6] we know that a distributive lattice L with 0 is generalized Stone if and only if it is normal and. pseudocomplemented The following result is trival by 2.1.2

Corollary 3.2.3. *Suppose $F_n(L)$ is a sectionally seudocomplemented distributive lattice, then $F_n(L)$ is generalized stone if and only if $(n]$ is dual generalized stone and $[n)$ is generalized stone.* ■

Following results are needed to prove the main results of this section. These are due to [9, Theorem 2.4,] .

Lemma 3.2.4. *If* L_1 *is a sublattice of a lattice* L *and* P_1 *is a prime ideal in* L_1, *then there exist a prime ideal* P *in* L *such that* $P_1 = L_1 \cap P$. ■

Lemma 3.2.5 *Let* L_1 *be a sublattice of a lattice* L . *For every (minimal) Prime ideal* P_1 *of* L, *there exist a (minimal) prime ideal* P *of* L *such that* $P_1 = P \cap L$ *and conversely.* ■

Lemma 3.2.6. *Suppose* L *is a distributive lattice and* $n \in L$. *Let* $x, y \in L$ *with* $\langle x \rangle_n \cap \langle y \rangle_n = \{n\}$. *Then the following conditions are equivalent*

i) $\langle x \rangle_n^* \vee \langle y \rangle_n^* = L$

ii) *For any* $t \in L$, $\langle m(x, n, t) \rangle_n^+ \vee \langle m(y, n, t) \rangle_n^+ = \langle t \rangle_n$,

where $\langle m(x,n,t) \rangle_n^+$ *denotes the relatively psuedocomplement* $\langle m(x,n,t) \rangle_n$ in $[\{n\}, \langle t \rangle_n]$.

Proof:- (i) $\Rightarrow$ (ii)

Suppose (i) holds. Then for any $t \in L,$ using Lemma 3.1.3,

$$\langle m(x,n,t)\rangle_n^+ \vee \langle m(y,n,t)\rangle_n^+$$

$$= \left(\langle x\rangle_n \cap \langle t\rangle_n\right)^+ \vee \left(\langle y\rangle_n \cap \langle t\rangle_n\right)^+$$

$$= \left(\left(\langle x\rangle_n \cap \langle t\rangle_n\right)^* \cap \langle t\rangle_n\right) \vee \left(\left(\langle y\rangle_n \cap \langle t\rangle_n\right)^* \cap \langle t\rangle_n\right)$$

$$= \left(\left(\langle x\rangle_n^* \cap \langle t\rangle_n\right) \vee \left(\langle y\rangle_n^* \cap \langle t\rangle_n\right)\right) \text{ by (Lemma 3.1.2)}$$

$$= \left(\langle x\rangle_n^* \vee \langle y\rangle_n^*\right) \cap \langle t\rangle_n$$

$$= L \cap \langle t\rangle_n$$

$$= \langle t\rangle_n$$

(ii) = (i), Suppose (ii) holds and $t \in L$. By (ii),

$\langle m(x,n,t)\rangle_n^+ \vee \langle m(y,n,t)\rangle_n^+ = \langle t\rangle_n$, then by

Calculation of (i) $\Rightarrow$ (ii), we have

$$\left(\langle x\rangle_n^* \vee \langle y\rangle_n^*\right) \cap \langle t\rangle_n = \langle t\rangle_n$$

This implies $\langle t\rangle_n \subseteq \langle x\rangle_n^* \vee \langle y\rangle_n^*$ and so

$$t \in \langle x\rangle_n^* \vee \langle y\rangle_n^*.$$

Therefore, $\langle x\rangle_n^* \vee \langle y\rangle_n^* = L.$

Theorem 3.2.7 *Let* L *be a distributive lattice and* $n \in L$. *The following conditions are equivalent.*

i) $F_n(L)$ *is normal.*

ii) *Every prime n-ideal of* L *contains a unique minimal prime* n-*ideal.*

iii) *For any two minimal prime* n-*ideals* P and Q of L, $P \vee Q = L$.

Proof : (i) $\Rightarrow$ (ii)

Let $F_n(L)$ be normal since $F_n(L) \cong (n]^d \wedge [n)$ so both $(n]^d$ and $[n)$ are normal.

Suppose P is any prime n-ideal of L. Then by theorem 2.2.2, either $P \supseteq (n]$ or, $P \supseteq [n)$. without loss of generality suppose $P \supseteq (n]$. Then by Theorem 2.2.4,

P is Prime ideal of L. Hence by Lemma 3.2.4 $P_1 = P \cap [n)$ is a prime ideal of $[n)$.

Since $[n)$ is normal, so by [Theorem 3.2.1] P_1 contains a unique minimal Prime ideal R_1 of $[n)$. Therefore P contains a unique minimal prime ideal R of L where $R_1 = R \cap [n)$. Since $n \in R_1$ so $n \in R$ and hence R is a minimal prime n-ideal of L.

Thus (ii) holds

(ii) $\Rightarrow$ (i) Suppose (ii) holds

Let P_1 be a prime ideal in $[n)$. Then by [Lemma3.2.4] $P_1 = P \cap [n)$ for some prime ideal P of L. Since $n \in P_1 \subseteq P$, so P is prime n-ideal.

Therefore P contains a unique minimal prime n-ideal R of L. Thus by

[Lemma 3.2.4] P_1 contains the unique minimal prime ideal $R_1 = R \cap [n)$ of $[n)$. Hence by [2] $[n)$ is normal. Similarly, we can prove that $(n]^d$ is also normal. Since $F_n(L) \cong (n]^d \times [n)$, so $F_n(L)$ is normal

(ii) $\Rightarrow$ (iii) is trival. ■

By 2.1.3, we have the following interesting result when n is a central element of L.

Theorem 3.2.8 *Let* L *be a distributive Lattice and* $n \in L$ *be central in it .Then the following conditions are equivalent.*

(i) $P_n(L)$ *is a normal lattice*

(ii) $I_n(L)$ *is a normal lattice*

(iii) $F_n(L)$ *is a normal lattice* ■

Thus we have the following result.

Theorem 3.2.9 *Let* L *be a distributive lattice and* $n \in L$ *be central in it. The following condition are equivalent.*

(i) $P_n(L)$ *is a normal lattice*

(ii) *Every prime* n-*ideals of* L *contains a minimal prime* n-*ideal.*

(iii) *For any two minimal prime* n-*ideals* P *and* Q *of* L

$P \vee Q = L$. ■

For a prime ideal P of a distributive lattice L with 0, Cornish in [7] has defined $0(P) = \{x \in L : x \wedge y = 0 \text{ for some } y \in L - P\}$. Clearly $0(P)$ is an ideal and $0(P) \subseteq P$. Cornish in [7] has shown that $0(P)$ is the intersection of all the minimal prime ideals of L which are contained in P.

For a prime n-ideal P of a distributive lattice L, we write $n(P)=\{y\in L \;:\; m(y,n,x)=n$ for some $x\in L-P\}$. Clearly, $n(P)$ is an n-ideal and $n(P)\subseteq P$.

Lemma 3. 2. 10. *Let* P *be a prime* n-*ideal in a distributive lattice* L.
Then each minimal prime n-ideal belonging to $n(P)$ *is contained in* P.

Proof: Let Q be a minimal prime n-ideal belonging to $n(P)$.
If $Q\not\subseteq P$, then choose $y\in Q-P$. By Theorem 2.2.4, we know that Q is either an ideal or a filter. Without less of generality suppose Q is an ideal. Now let $S=\{s\in L : m(y,n,s)\in n(P)\}$. We shall show that $s\not\subseteq Q$.

If not, let $D=(L-Q)\vee[y)$. Then $n(P)\cap D=\Phi$. For otherwise, $y\wedge r\in n(P)$ for some $r\in L-Q$. Then by convexity, $y\wedge r\leq m(y,n,r)\leq(y\wedge r)\vee n$ implies $m(y,n,r)\in n(P)$. Hence $r\in S\subseteq Q$, which is a contradiction. Thus, by Stone's separation theorem for n-ideals, there exists a prime n-ideal R Containing $n(P)$ disjoint to D. Then $R\subseteq Q$.

Moreover, $R\neq Q$ as $y\notin R$, this shows that Q is not a minimal prime n-ideal belonging to $n(P)$ which is a contradiction.
Therefore, $S\not\subseteq Q$. Hence there exists $z\notin Q$ such that $m(y,n,z)\in n(P)$.
Thus $m(m(y,n,z),n,x)=n$ for some $x\in L-P$. It is easy to see that

$m(m(y,n,z),n,x) = m(m(y,n,x),n,z)$.

Hence, $m(m(y,n,x),n,z) = n$. Since P is Prime and $y, x \notin P$, so $m(y,n,x) \notin P$. Therefore, $z \in n(P) \subseteq Q$, which is a contradiction.

Hence $Q \subseteq P$. ■

Proposition 3.2.11. *If* P *is a prime* n-*ideal in a distributive lattice* L, *then* $n(P)$ *is the intersection of all-minimal prime* n-*ideals contained in* P.

Proof: Clearly $n(P)$ is contained in any prime n-ideal which is contained in P. Hence $n(P)$ is contained in the intersection of all minimal prime n-ideal contained in P.

Since L is distributive so by Corollary 2.2.10, $n(P)$ is the intersection of all minimal prime n-ideals belonging to it.

By Lemma 3.1.1, as each prime n-ideal contained a minimal prime n-ideal, above remarks and Lemma 3.2.10 establish the proposition. ■

Theorem 3.2.12, *Let* L *be a distributive lattice and* $n \in L$. *Then the following condition are equivalent,*

i) $F_n(L)$ *is normal.*

ii) *Every prime* n-*ideal contains a unique minimal prime* n-*ideal*

iii) *For each prime* n-*ideal* P, $n(P)$ *is a prime* n-*ideal*

iv) *For all* $x, y \in L$, $\langle x \rangle_n \cap \langle y \rangle_n = \{n\}$ *implies that*

$\langle x\rangle_n^* \vee \langle y\rangle_n^* = L.$

v) *For all* $x, y \in L,\quad (\langle x\rangle_n \cap \langle y\rangle_n)^* = \langle x\rangle_n^* \vee \langle y\rangle_n^*.$

Proof:- (i) $\Leftrightarrow$ (ii) holds by Theorem 3.2.7.

(ii) $\Rightarrow$ (iii) is a direct consequence of proposition 3.2.10.

(iii) $\Rightarrow$ (iv). Suppose (iii) holds. Consider $x, y \in L.$ with

$\langle x\rangle_n \cap \langle y\rangle_n = \{n\}.$ If $\langle x\rangle_n^* \vee \langle y\rangle_n^* \neq L$,

Then by theorem 2.2.9, there exists a prime n-ideal P such that

$\langle x\rangle_n^* \vee \langle y\rangle_n^* \subseteq P$, then $\langle x\rangle_n^* \subseteq P$, and $\langle y\rangle_n^* \subseteq P$, imply

$x \notin n(P)$ and $y \notin n(P)$.

But $n(P)$ is prime and so $m(x, n, y) = n \in n(P)$ in contradictory.

Therefore, $\langle x\rangle_n^* \vee \langle y\rangle_n^* = L$.

(iv)$\Rightarrow$ (v) obviously $\langle x\rangle_n^* \vee \langle y\rangle_n^* \subseteq (\langle x\rangle_n \cap \langle y\rangle_n)^*$

Conversely, let $w \in (\langle x\rangle_n \cap \langle y\rangle_n)^*$

Then $\quad \langle w\rangle_n \cap \langle x\rangle_n \cap \langle y\rangle_n = \{n\}$.

That is , $\quad \langle m(w, n, x)\rangle_n \cap \langle y\rangle_n = \{n\}$.

Thus (iv), $\quad \langle m(w, n, x)\rangle_n^* \vee \langle y\rangle_n^* = L$.

So, $w \in \quad \langle m(w, n, x)\rangle_n^* \vee \langle y\rangle_n^*$.

Therefore, $w \vee n \in \langle m(w, n, x)\rangle_n^* \vee \langle y\rangle_n^*$.

Then by theorem 2.1.12

$w \vee n = r \vee s$ for some $r \in \langle m(w, n, x)\rangle_n^*$

and $s \in \langle y \rangle_n^*$, with $r, s \geq n$.

Now $r \in \langle m(w, n, x) \rangle_n^*$ implies

$r \wedge [(w \wedge n) \vee (w \wedge x) \vee (x \wedge n)] \vee (r \wedge n) \vee$

$[(w \wedge n) \vee (x \wedge n) \vee (w \wedge x)] \wedge n = n$.

That is, $(r \wedge w \wedge n) \vee (r \wedge w \wedge x) \vee (r \wedge x \wedge n) \vee (r \wedge n) \vee (w \wedge n) \vee (x \wedge n) = n$.

or, $(w \wedge n) \vee (r \wedge w \wedge x) \vee (x \wedge n) \vee n \vee (w \wedge n) \vee (x \wedge n) = n$.

or, $(r \wedge w \wedge x) \vee n = n$.

or, $(r \vee n) \wedge (w \vee n) \wedge (x \vee n) = n$.

or, $(r \vee n) \wedge (x \vee n) = n$. as $r \vee n \leq w \vee n$.

or, $(r \wedge x) \vee n = n$.

or, $(r \wedge x) \vee (x \wedge n) \vee (r \wedge n) = n$.

or, $m(r, n, x) = n$.

Which implies $r \in \langle x \rangle_n^*$

Therefore $w \vee n \in \langle x \rangle_n^* \vee \langle y \rangle_n^*$

A dual proof of above, shows that $w \wedge n \in \langle x \rangle_n^* \vee \langle y \rangle_n^*$

So by Convexity, $w \in \langle x \rangle_n^* \vee \langle y \rangle_n^*$.

Therefore, $(\langle x \rangle_n \cap \langle y \rangle_n)^* \subseteq \langle x \rangle_n^* \vee \langle y \rangle_n^*$, and so

$\left(\langle x\rangle_n \cap \langle y\rangle_n\right)^* = \langle x\rangle_n^* \vee \langle y\rangle_n^*$, which is (v).

(v) $\Rightarrow$(iv) Let $\langle x\rangle_n \cap \langle y\rangle_n = \{n\}$, for some $x, y \in L$.

by (v), $L = \{n\}^* = \left(\langle x\rangle_n \cap \langle y\rangle_n\right)^* = \langle x\rangle_n^* \vee \langle y\rangle_n^*$.

Thus (iv) holds. (iv) $\Rightarrow$(i) Consider $[n)$. Let $x, y \in [n)$ with $x \wedge y = n$.

Then $\langle x\rangle_n \cap \langle y\rangle_n = \{n\}$. Thus, by (iv), $\langle x\rangle_n^* \vee \langle y\rangle_n^* = L$.

This implies $[\, n\,) = \left(< x >_n^* \vee < y >_n^*\right) \cap [\, n\,)$

$= \left(< x >_n^* \cap [\, n\,)\right) \vee < y >_n^* \cap [\, n\,) = < x >_n^+ \vee < y >_n^+$

Notice that both $\langle x\rangle_n$ and $\langle y\rangle_n$ are ideal in $[n)$ and $\langle x\rangle_n^+$, $\langle y\rangle_n^+$ are annihilator ideals of $\langle x\rangle_n$ and $\langle y\rangle_n$ respectively in $[n)$. This implies by [Cornish, Theorem 2.4] that $[n)$ is a normal lattice. A dual proof of above shows that $[n)^d$ is also a normal lattice. Therefore $F_n(L)$ is also normal as $F_n(L) \cong (n]^d \wedge [n)$. ■

We conclude this chapter with the following result, when n is a central element which follows immediately from the above result and theorem 2.1,3

Theorem 3.2.13. *Let* n *be a central element of a distributive lattice* L.

Then following conditions are equivalent

(i) $P_n(L)$ *is a normal*

(ii) *For all* $x, y \in L$, $\langle x\rangle_n \cap \langle y\rangle_n = \{\, n\}$ *implies that*

$$\langle x\rangle_n^* \vee \langle y\rangle_n^* = L.$$

(iii) *For all* $x, y \in L$, $\left(\langle x\rangle_n \cap \langle y\rangle_n\right)^* = \langle x\rangle_n^* \vee \langle y\rangle_n^*$. ■

CHAPTER -4

Chapter-4

Finitely generated n-ideals, which form relatively normal lattices.

Introduction

Relative annihilators in lattices and semilattices have been studied by many authors including Mandelker [38] and Varlet [60]. Cornish in [9] has used the annihilators in studying relative normal lattices. Recently Noor & Ayub in [45] have introduced the notion of relative annihilators around a fixed element $n \in L$ known as relative n-annihilators. In this chapter we intend to generalize several results on relatively normal lattices by using the relations n-annihilators .

For a, $b \in L$, $< a, \quad b > = \{x \in L \quad : x \wedge a \leq b\}$ is known as *annihilator of* a *relative* to b, or simply a *relative annihilator*. It is very easy to see that in presence of distributivity, $< a, \quad b >$ is an ideal of L.

Again for a, $b \in L$ we define $< a, \quad b >_d = \{x \quad : \quad x \vee a \geq b\}$, which we call a *dual annihilator of* a *relative to* b, or simply a *relative dual annihilator*. In presence of distributivity of L, $< a, \quad b >_d$ is a dual ideal (filter).

For a, $b \in L$ and a fixed element $n \in L$, we define

$< a, b >^n = \{x \in L : m(a, n, x) \in < b >_n\} = \{x \in L : b \wedge n \leq m(a, n, x) \leq b \vee n\}$. We call $< a, \quad b >^n$ *the annihilator of* a *relative* to b around the element n or simply a *relative* n- *annihilator*. It is easy to see that for all $a, b \in L, \quad < a, \quad b >^n$ is always a convex subset containing n . In presence of distributivity, it can be easily seen that $< a, \quad b >^n$ is an n-ideal. For two n-ideals A and B of a lattice L, <A, B> denotes $\{x \in L \quad : m(a, n, x) \in B\}$ for all $a \in A\}$. In presence of distributivity, clearly <A, B> is an n-ideal. Moreover, we can easily show that

$<a, \ b>^n = << a>_n, \ <b>_n>.$

$<a, \ b>^n = << a>_n, \ <b>_n>.$

Recall that a distributive lattice with 0 is a *normal lattice* if its every prime ideal contains a unique minimal prime ideal. A distributive lattice L is called a *relatively normal lattice* if its every interval [a, b] is normal.

In section 1 of this chapter we have included several characterizations of $< a, \quad b >^n$. If $0 \in L$, then putting n = 0 the n-ideals become ideals and $< a, \quad b >^n = < a, b >$. So the results of this section generalize several results on annihilators in [38].

In section 2 we characterize those $F_n(L)$ which are relatively normal in terms of n-ideals and relative n- annihilators. These results are certainly generalizations of several results on relatively normal lattices. At the end we show that for a central element n, $P_n(L)$ is relatively normal if and only if any two incomparable prime n-ideals of L are comaximal.

1. Relative annihilators around a neutral element of a lattice.

We start the following result due to [45], which gives a characterization of $< a,\ b >^n$

Theorem 4.1.1. *Let* L *be a lattice with a neutral element* n *in it. Then for all* a, $b \in L$, *the following conditions are equivalent :*

(i) $< a,\ b >^n$ *is an n-ideal ;*

(ii) $< a \wedge n,\ b \wedge n >_d$ *is filter and*

(iii) $< a \wedge n,\ b \vee n >$ *is an ideal .*

The following result is also due to [45] .

Theorem 4.1.2 *Let* L *be a lattice with a neutral element* n . *For all* a, $b \in L$ *the following hold :*

(i) $< a \wedge n,\ b \vee n >_d$ *is an ideal if and only if* $[n)$ *is a distributive sub lattice of* L *:*

(ii) $< a \wedge n,\ b \wedge n >_d$ *is a filter if and only if* $(n]$ *is a distributive sub lattice of* L . ■

By theorem 2.1.3, we know that for a central element $n \in L$, $P_n(L) \cong (n]^d \times [n)$, where $(n]^d$ denotes the dual of the lattice $(n]$. Thus by Theorem 4.1.1, and above result we have the following result.

Theorem 4.1.3. *Let be a lattice and $n \in L$ be neutral. Then for all $a, b \in L$, $< a, \ b >^n$ is an* n-*ideal if and only if $P_n(L)$ is distributive.* ■

Now by [30], we know that L is distributive if and only if $P_n(L)$ is distributive. Therefore, we have the following corollary which is a generalization of [38, Theorem 1].

Corollary 4.1.4. *For all* a, $b \in L$ *and for a central element* $n \in L$, $<a, \ b>^n$ *is an* n-*ideal if and only if* L *is distributive.* ■

Following result also generalizes [38, Theorem 1] which is due to [45]

Theorem 4.1.5. *Let* n *be a neutral element a lattice* L. *Then the following conditions are equivalent :*

(i) L *is distributive* ;

(ii) $< a \vee n, \ b \vee n >$ *is an ideal and* $< a \wedge n, \ b \wedge n >_d$ *is a filter whenever* $< a >_n \supseteq < b >_n$.

Theorem 4.1.6. *Let* n *be a central element of a lattice* L. *Then the following conditions are equivalent :*

(i) $P_n(L)$ *is modular ;*

(iii) *For* a, $b \in L$ *with* $< b >_n \subseteq < a >_n$, $x \in < b >_n$ *and* $y \in < a, \ b >^n$ *imply* $x \wedge y, \ x \vee y \ \in < a, \ b >^n$.

Proof : (i) $\Rightarrow$ (ii). Suppose $P_n(L)$ is modular. Then by theorem 2.1.3, $(n]$ and $[n)$ are modular. Here $< b >_n \subseteq < a >_n$.

So $a \wedge n \leq b \wedge n \leq n \leq b \vee n \leq a \vee n$. Since $x \in < b >_n$, So, $b \wedge n \leq x \leq b \vee n$. Hence,

$a \wedge n \leq b \wedge n \leq x \wedge n \leq x \vee n \leq b \vee n \leq a \vee n$. Now,

$y \in <a, \ b>^n$ implies $m(y, \ n, \ a) \in <b>_n$. Then by the

neutrality of n, $(y \vee a) \wedge (y \vee n) \wedge (a \vee n) \leq b \vee n$, and so

$((y \vee a) \wedge (y \vee n) \wedge (a \vee n)) \vee n = (y \vee n) \wedge (a \vee n) \leq b \vee n$.
Thus, using the modularity of $[n)$,

$m(x \vee y \vee n, \ n, a) = (x \vee y \vee n) \wedge (a \vee n)$

$= [(a \vee n) \wedge (y \vee n)] \vee (x \vee n)$, as $x \vee n \leq b \vee n \leq a \vee n$

This implies $m(x \vee y \vee n, n, a) \leq b \vee n$, and so $x \vee y \vee n \in <a, b>^n$

. Since n is neutral, so $a \wedge n \leq b \wedge n \leq x \wedge n$ implies that

$b \wedge n \leq (x \wedge n) \vee (y \wedge n) \vee (a \wedge n) = ((x \vee y) \wedge n) \vee (a \wedge n)$

$= m((x \vee y) \wedge n, n, a) \leq b \vee n$.

Therefore, $(x \vee y) \wedge n \in <a, \ b>^n$. Hence by the convexity of

$<a, \ b>^n, \ x \vee y \in <a, \ b>^n$. Again using the modularity of $(n]$,

a dual proof of above shows that $x \wedge y \in <a, b>^n$.

Conversely, suppose (ii) holds. Let $x, y, z \in [n)$ with $x \leq z$.

Then $x \vee (y \wedge z) \leq z$. This implies $<x \vee (y \wedge z)>_n \subseteq <z>_n$

Now $x \leq x \vee (y \wedge z)$ implies $x \in <x \vee (y \wedge z)>_n$.

Again $y \wedge z \leq x \vee (y \wedge z)$ implies

$m(y, n, z) = y \wedge z \in < x \vee (y \wedge z) >_n$ Hence

$y \in < z, \quad x \vee (y \wedge z) >^n$. Thus by (ii), $x \vee y \in < z, x \vee (y \wedge z) >^n$

That is, $(x \vee y) \wedge z \leq x \vee (y \wedge z)$ and so

$(x \vee y) \wedge z = x \vee (y \wedge z)$.

Therefore, $[n)$ is modular .

Similarly, using the condition (ii) we can easily show that $(n]$ is also modular. Hence by theorem 2.1.3 , $P_n(L)$ is modular. ■

By [49, Theorem 3.2] , we know that a lattice L is modular if and only if the lattice of all n-ideals $I_n(L)$ is modular. Following their proof if can be easily seen that L is modular if and only if $P_n(L)$ is modular. Hence we have the following result which generalizes [38, Theorem 2].

Corollary 4.1.7 *Let* n *be a central element of a lattice* L. *Then the following conditions are equivalent :*

(i) L *is modular;*

(ii) *For* a, $b \in L$ *with* $< b >_n \subseteq < a >_n$, $x \in < b >_n$ *and*

$y \in < a, b >^n$ *implies* $x \wedge y, \quad x \vee y \in < a, b >^n$ ■ .

We conclude the section with the following characterization of minimal prime n-ideals belonging to an n-ideal. Since the proof of this is almost similar to Theorem 3.1.4, we omit the proof.

Theorem 4.1.8. *Let* L *be a distributive lattice and* P *be a prime* n-*ideal of* L. *belonging to an* n-*ideal* J. *Then the following conditions are equivalent :*

(i) P *is minimal belonging to* J;

(ii) $x \in P$ *implies* $<<x>_n, \ J \ > \not\subseteq P$ ■

2. Some characterizations of those $F_n(L)$ which are relatively normal lattices.

We start this section with the following result which is a generalization of [9, lemma 3.6]. This plays an important role in proving our main result in this section.

Theorem 4.2.1. *Let* L *be a distributive lattice. Then the following hold*

(i) $\langle \langle x \rangle_n \vee \langle y \rangle_n, \langle x \rangle_n \rangle = \langle \langle y \rangle_n, \langle x \rangle_n \rangle$;

(ii) $\langle \langle x \rangle_n, J \rangle = \bigvee_{y \in J} \langle \langle x \rangle_n, \langle y \rangle_n \rangle$, *the supremum of* n-*ideals* $\langle \langle x \rangle_n, \langle y \rangle_n \rangle$ *in the lattice of n-ideals of* L, *for any* $x \in L$ *and any* n-*ideals* J.

Proof: (i) L. H. S. $\subseteq$ R. H. S. is obvious. Let $t \in$ R. H. S, then $t \in \langle \langle y \rangle_n, \langle x \rangle_n \rangle$. This implies $m\ (y, n, t) \subseteq \langle x \rangle_n$. That is $\langle m\ (y, n, t) \rangle_n \subseteq \langle x \rangle_n$ and so $(\langle y \rangle_n \cap \langle t \rangle_n) \vee (\langle x \rangle_n \cap \langle t \rangle_n) \subseteq \langle x \rangle_n$ That is, $\langle t \rangle_n \cap [\langle x \rangle_n \vee \langle y \rangle_n] \subseteq \langle x \rangle_n$ which implies $t \in \langle \langle x \rangle_n \vee \langle y \rangle_n, \langle x \rangle_n \rangle$. Thus $t \in$ R. H. S. and so (i) holds. (ii) R. H. S. $\subseteq$ L. H. S. is obvious. Let $t \in$ L. H. S, then $m\ (x, n, t) \in J$ that is $m\ (x, n, t) = j$ for some $j \in J$ This implies $t \in \langle\langle x\rangle_n, \langle j\rangle_n\rangle$. Thus $t \in$ R. H. S and so (ii) holds $x \in P$

implies $<<x>_n,\ J\ > \not\subseteq P$ ■

Following lemma will be needed for further development of this chapter. This is in fact, the dual of [9, Lemma 3.6] and is very easy to prove. So we prefer to omit the proof.

Lemma 4.2.2. *Let* L *be a distributive lattice. Then the following hold.*

(i) $\langle x \wedge y, x\rangle_d = \langle y, x\rangle_d$;

(ii) $\langle\ [x),\ F\rangle_d = \bigvee_{y \in F} \langle x, y\rangle_d$, *where* F *is a filter of* L .

(iii) $\{\langle x, a\rangle_d \vee \langle y, a\rangle_d\} \cap [a, b]$

$= \{\langle x, a\rangle_d \cap [a, b]\} \vee \{\ \langle y, a\rangle_d \cap [a, b]$.

Lemma 4.2.3 and Lemma 4.2.4 are essential for the proof of our main result of this section. There lemmas are due to [45]. We include only the proofs of Lemma 4.2.3 for the convenience of the reader.

Lemma 4.2.3, *Let* L *be a distributive lattice with* $n \in l$, *Suppose* a, b, c $\in$L.

i) *If* a, b, c $\geq n$, *then* $\langle\langle m\ (a, n, b)\rangle_n, \langle c\rangle_n\rangle$

$= \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$ *is equivalent to*

$\langle a \wedge b,\ c\rangle = \langle a, c\rangle \vee \langle b,\ c\rangle$;

ii) *If* $a,\ b,\ c \leq n$ *then*

$$\langle\langle m\ (a,n,b)\rangle_n, \langle c\rangle_n\rangle = \langle\langle a\rangle_n, \langle c\rangle_n > \vee \langle\langle < b\rangle_n,\rangle \langle c\rangle_n\rangle$$

is equivalent to $\langle a \vee b, c >_d = \langle a,c\rangle_d \vee \langle b,c\rangle\rangle_d$.

Proof: (i) Suppose a, b, c $\geq n$ and

$$\langle\langle a\rangle_n \cap \langle b\rangle_n, \langle c\rangle_n\rangle = \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle.$$

Let $x \in \langle a \wedge b, c\rangle$. Then $x \wedge a \wedge b \leq c$,

$$\langle x\rangle_n \cap \langle a \wedge b\rangle_n = \langle x\rangle_n \cap [n, \quad a \wedge b] = [n, (x \vee n) \wedge (a \wedge b)]$$
$$= [n, (x \wedge a \wedge b) \vee n] \subseteq [n, c] .$$

Hence $x \in \langle\langle a \wedge b\rangle_n, (c)_n\rangle = \langle\langle m\ (a,n,b)\rangle_n, \langle c\rangle_n\rangle$

$= \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$. Thus $x \leq p \vee q$,

where $p \in \langle\langle a\rangle_n, \langle c\rangle_n\rangle, \quad q \in \langle\langle b\rangle_n, \langle c\rangle_n\rangle$.

Then $\langle p\rangle_n \cap \langle a\rangle_n \subseteq \langle c\rangle_n$. That is

$[p \wedge n, p \vee n] \cap [n, a] \subseteq [n, c]$, Thus,

$[n, (p \vee n) \wedge a] \subseteq [n, c]$ which implies

$p \wedge a \leq c$ and so $p \in \langle a, c\rangle$ Similarly, $q \in \langle b,c\rangle$ and so

$x \in \langle a, c\rangle \vee \langle b, c\rangle$. Hence $\langle a \wedge b, c\rangle \subseteq \langle a, c\rangle \vee \langle b, c\rangle$.

But $\langle a, c\rangle \vee \langle b, c\rangle \subseteq \langle a \wedge b, c\rangle$ is obvious. Therefore,

$\langle a \wedge b , c\rangle = \langle a , c\rangle \vee \langle b , c\rangle$. Conversely, suppose

$\langle a \wedge b, c\rangle = \langle a, c\rangle \vee \langle b, c\rangle$.

Let $x \in \langle\langle m(a, n, b)\rangle_n, \langle c\rangle_n\rangle$.

Then $\langle x\rangle_n \cap \langle m(a,n,b)\rangle_n = [x \wedge n, \quad x \vee n] \cap [n, a \wedge b] \subseteq [n, c]$.

That is $[n, (x \vee n) \wedge (a \wedge b)] \subseteq [n, c]$.

Thus $[n, (x \wedge a \wedge b) \vee n] \subseteq [n, c]$ which implies

$x \wedge a \wedge b \leq c$, and so $x \in \langle a \wedge b, c\rangle = \langle a, c\rangle \vee \langle b, c\rangle$.

This implies $x = r \vee s$, where $r \in \langle a, c\rangle$ and $s \in \langle b, c\rangle$.

Then $r \wedge a \leq c$ and $s \wedge b \leq c$.

Now $\langle r\rangle_n \cap \langle a\rangle_n = [r \wedge n, r \vee n] \cap [n, a] = [n, (r \vee n) \wedge a]$

$= [n, (r \wedge a) \vee n] \subseteq [n, c] = \langle c\rangle_n$.

Hence $r \in < < a >_n , < c >_n >$ .Similarly,

$S \in \langle\langle b\rangle_n, \langle c\rangle_n\rangle$. Thus $x \in \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$ and so

$\langle\langle m(a, n, b)\rangle_n, \langle c\rangle_n\rangle \subseteq \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$.

Since $\langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle \subseteq \langle\langle m(a, n, b)\rangle_n, \langle c\rangle_n\rangle$ is obvious, so $\langle\langle m(a,n,b)\rangle_n, \langle c\rangle_n\rangle = \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$.

Dual calculation of above proof proves (ii) ■

$x \in P$ *implies* $<<x>_n, \ J \ > \not\subseteq P$

Lemma 4.2.4. *Let* L *be a distributive lattice with* $n \in L$, *Suppose*

a, b, c, $\in L$.

(i) *For* $a, b, c \geq n$,

$$<<c>_n, <a>_n \vee <b>_n> = <<c>_n, <a>_n> \vee <<c>_n, <b>_n>$$

is equivalent to $\langle c, a \vee b \rangle = \langle c, a \rangle \vee \langle c, b \rangle$;

(ii) *For* $a, b, c \leq n$, $\langle \langle c \rangle_n, \langle a \rangle_n \vee \langle b \rangle_n \rangle$

$= \langle \langle c \rangle_n, \langle a \rangle_n \rangle \vee \langle \langle c \rangle_n, \langle b \rangle_n \rangle$ *is equivalent to*

$\langle c, a \wedge b \rangle_d = \langle c, a \rangle_d \vee \langle c, b \rangle_d$. ■

Following result on Stone lattice is well known due to [15, Theorem 3, Page-161] and [9, Theorem 2.4]

A distributive lattice L with 1 is called a dual normal lattice of L^d is a normal lattice. In otherwords a distributive lattice L with 1 is called *dual normal* if every prime filter of L is contained in a unique ultrafilter (maximal and proper) of L.

In fact, this condition in a lattice is self-dual. Thus for a bounded distributive lattice, the concept of normality and dual normality coincides.

Following the technique of the proof of [9, Theorem 2.4], we can similarly prove the following result, which gives some characterization of dual normal lattices. These results are in fact the dual result of Theorem 3.2.1.

Theorem 4.2.5. *Let* L *be a distributive lattice with* 1. *Then the following conditions are equivalent.*

(i) L *is normal ;*

(ii) *Each prime filter of* L *is contained in a unique ultrafilter (maximal and proper) ;*

(iii) *For each* $x, y \in L, [x \vee y)^{*d} = [x)^{*d} \vee [y)^{*d}$;

If $x \vee y = 1, \quad x, y \in L,$ *Then* $[x)^{*d} \vee [y)^{*d} = L$ ■

Corollary 4.2.6. L *be a bounded distributive lattice. Then the following conditions are equivalent.*

(i) L *is normal*

(ii) *For each* $x, y \in L, (x \wedge y]^{*} = (x]^{*} \vee (y]^{*}$

(iii) *If* $x \wedge y = 0$, *Then* $(x]^{*} \vee (y]^{*} = L$

(iv) *For each* $x, y \in L,$ $[x \vee y)^{*d} = [x)^{*d} \vee [y)^{*d}$

(v) *If* $x \vee y = 1$, *then* $[x)^{*d} \vee [y)^{*d} = L.$

Recall that a distributive lattice L is *relatively normal* if each interval [x, y] with x < y $(x, y \in L)$ is a normal lattice.

Since for a bounded distributive lattice the concept of normality and dual normality coincides, so the concept of relative normality is self-dual in any distributive lattice.

Now we prove the following result whose technique of proof is dual to

[9, Theorem 3.7] . This will be used to prove our main result of this chapter.

Theorem 4.2.7. *Let* L *be a distributive lattice. Let* $a, b, c \in L$ *be arbitrary elements and A, B arbitrary filters. Then the following are equivalent:*

(i) L *is relatively normal*

(ii) $\langle a, b \rangle_d \vee \langle b, a \rangle_d = L$;

(iii) $\langle c, a \wedge b \rangle_d = \langle c, a \rangle_d \vee \langle c, b \rangle_d$;

(iv) $<[c), A \vee B>_d = <[c), A>_d \vee <[c), B>_d$;

(v) $\langle a \vee b, c \rangle_d = \langle a, c \rangle_d \vee \langle b, c \rangle_d$.

Proof: (i) $\Rightarrow$(ii). Let $z \in L$ be arbitrary. Consider the interval $1 = [z, a \vee b \vee z]$. Then $a \vee b \vee z$ is the largest element of I. Since by (i), I is a distributive lattice, so by Theorem 4.2.6 (v), there exist $r, s \in I$ such that $a \vee s = a \vee b \vee z = b \vee z \vee r$ and $z = s \wedge r$. Now, $a \vee s \geq b$ implies $s \in \langle a, b \rangle_d$ and $b \vee r = b \vee z \vee r = a \vee b \vee z \geq a$ implies $r \in \langle b, a \rangle_d$. Hence (ii) holds. (ii) $\Rightarrow$(iii). In (iii), R. H. S $\subseteq$ L. H. S. is obvious; Let $z \in \langle c, a \wedge b \rangle_d$, then $z \vee c \geq a \wedge b$. Since (ii) holds, so $z = x \wedge y$, where $x \in \langle a, b \rangle_d$ and $y \in \langle b, a \rangle_d$. Then $x \vee a \geq b$ and $y \vee b \geq a$. Thus, $x \vee c = x \vee z \vee c \geq x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b) \geq b$, which implies $x \in \langle c, b \rangle_d$.

Similarly, $y \in \langle c, a \rangle_d$. Hence $z = x \wedge y \in \langle c, a \rangle_d \vee \langle c, b \rangle_d$,

and so $\langle c, a \wedge b \rangle_d \subseteq \langle c, a \rangle_d \vee \langle c, b \rangle_d$.

Since the reverse inclusion is obvious, so (iii) holds.

(iii) $\Rightarrow$ (iv) follows from Lemma 4.2.2 (ii),

(iv) $\Rightarrow$ (iii) is trivial .

(iii) $\Rightarrow$ (ii) follows from Lemma 4.2.2 (i) by putting $c = a \wedge b$.

(ii) $\Rightarrow$ (v) Let $z \in \langle a \vee b, c \rangle_d$. Then by (ii) $z = x \wedge y$,

where $x \vee a \geq b$ and $y \vee b \geq a$. Also

$x \vee a = x \vee a \vee b \geq z \vee a \vee b \geq c$. This implies $x \in \langle a, c \rangle_d$.

Similarly, $y \in \langle b, c \rangle_d$. It follows that

$\langle a \vee b, c \rangle_d \supseteq \langle a, c \rangle_d \vee \langle b, c \rangle_d$. Since the reverse inequality is

obvious, so (v) holds.

(v) $\Rightarrow$(i) . Let $x \in [a, b]$, $a < b$. Let $[x^{+d}) = \{ t \in [a, b], t \vee x = b$,

then largest element of $[a, b]$ }. It is easy to see that

$[x)^{+d} = \langle x, b \rangle_d \cap [a, b]$

Now, suppose $x, y \in [a, b]$ with $x \vee y = b$, then by (v),

$$[x)^{+d} \vee [y)^{+d} = (\langle x, b \rangle_d \cap [a, b]) \vee (\langle y, b \rangle_d \cap [a, b])$$

$$= (\langle x, b \rangle_d \vee \langle y, b \rangle_d) \cap [a, b] \text{ (by Lemma 4.2.2 (iii))}$$

$$= \langle x \vee y, b \rangle_d \cap [a, b]$$

$= \langle b, b \rangle_d \cap [a, b] = L \cap [a, b] = [a, b]$.

Hence by Theorem 4.2.6, $[a,b]$ is normal and so L is a relatively normal lattice. ■

Now we prove our main results of this chapter, which are generalizations, of [9, Theorem 3.7] and [38, Theorem 5] . These give characterizations of those $F_n(L)$ and $P_n(L)$ which are relatively normal in terms of n-ideals.

Theorem 4.2.8. *Let $F_n(L)$ be distributive lattice and* A *and* B *be two n-ideals of* L, *Then for all $a, b, c \in L$, the following conditions are equivalent. $F_n(L)$ is relatively normal.*

(i) $\langle \langle a \rangle_n, \langle b \rangle_n \rangle \vee \langle \langle b \rangle_n, \langle a \rangle_n \rangle = L$;

(ii) $\langle \langle c \rangle_n, \langle a \rangle_n \vee \langle b \rangle_n \rangle = \langle \langle c \rangle_n, \langle a \rangle_n \rangle \vee \langle \langle c \rangle_n, \langle b \rangle_n \rangle$;

(iii) $\langle \langle c \rangle_n, A \vee B \rangle = \langle \langle c \rangle_n, A \rangle \vee \langle \langle c \rangle_n, B \rangle$;

(iv) $\langle \langle m(a, n, b) \rangle_n, \langle c \rangle_n \rangle = \langle \langle a \rangle_n, \langle c \rangle_n \rangle \vee \langle \langle b \rangle_n, \langle c \rangle_n \rangle$;

Proof: (i) $\Rightarrow$ (ii). Let $z \in L$, consider the interval

$I = [\langle a \rangle_n \cap \langle b \rangle_n \cap \langle z \rangle_n, \langle z \rangle_n]$ in $F_n(L)$. Then $\langle a \rangle_n \cap \langle b \rangle_n \cap \langle z \rangle_n$ is the smallest element of the interval I.

By (i), I is normal, then by Theorem 3.2.1, there exists finitely generated n-ideals [p, q] , [r, s] $\in I$ such that. $\langle a \rangle_n \cap \langle z \rangle_n \cap [p, q]$

$= \langle a \rangle_n \cap \langle b \rangle_n \cap \langle z \rangle_n$

$= \langle b \rangle_n \cap \langle z \rangle_n \cap [r, s]$ and $< z >_n = [\, p,\ q\,] \vee [\, r, s\,]$

Now, $< a >_n \cap [\, p, q] = < a >_n \cap [\, p, q\,] \cap < z >_n$

$= < a >_n \cap < b >_n \cap < z >_n \subseteq < b >_n$ implies

$[p, q] \subseteq << a >_n , < b >_n >$. Also

$< b >_n \cap [r, s] = < b >_n \cap < z >_n \cap [r, s]$

$= < a >_n \cap < b >_n \cap < z >_n \subseteq < a >_n$ implies

$[r, s] \subseteq << b >_n , < a >_n >$.Thus $\langle z \rangle_n \subseteq \langle \langle a \rangle_n , \langle b \rangle_n \rangle \vee \langle \langle b \rangle_n , \langle a \rangle_n \rangle$,

and so $z \in \langle \langle a \rangle_n , \langle b \rangle_n \rangle \vee \langle \langle b \rangle_n , \langle a \rangle_n \rangle$

Hence $\langle \langle a \rangle_n , \langle b \rangle_n \rangle \vee \langle \langle b \rangle_n , \langle a \rangle_n \rangle = L$.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. For (iii), R. H. S. $\subseteq$ L. H. S. is obvious.

Now, let $z \in \langle \langle c \rangle_n , \langle a \rangle_n \vee \langle b \rangle_n \rangle$

Then $z \vee n \in \langle \langle c \rangle_n , \langle a \rangle_n \vee \langle b \rangle \rangle$, and so

$m\,(z \vee n,\ n, c\,) \in [a \wedge b \wedge n, a \vee b \vee n]$

That is, $(z \vee n) \wedge (c \vee n) \leq a \vee b \vee n$.

Now by (ii), $z \vee n \in \langle \langle a \rangle_n , \langle b \rangle_n \rangle \vee \langle \langle b \rangle_n , \langle a \rangle_n \rangle$.

So $z \vee n \leq (p \vee n) \vee (q \vee n)$ for some

$p \vee n \in \langle \langle a \rangle_n , \langle b \rangle_n \rangle$ and $q \vee n \in << b >_n , < a >_n >$

Hence, $z \vee n = ((z \vee n) \wedge (p \vee n)) \vee ((z \vee n) \wedge (q \vee n)) = r \vee s$ (say)

Now, $m(p \vee n, n, a) = (p \vee n) \wedge (a \vee n) \leq b \vee n$. So

$(b \wedge n) \leq r \wedge (a \vee n) \leq b \vee n$.

Hence, $r \wedge (c \vee n) = r \wedge (z \vee n) \wedge (c \vee n) \leq r \wedge (a \vee b \vee n)$

$= (r \wedge (a \vee n)) \vee (r \wedge (b \vee n)) \leq b \vee n$.

This implies $r \in \langle \langle c \rangle_n, \langle b \rangle_n \rangle$, similarly,

$s \in \langle \langle c \rangle_n, \langle a \rangle_n \rangle$.

Hence $z \vee n \in \langle \langle c \rangle_n, \langle a \rangle_n \rangle \vee \langle \langle c \rangle_n, \langle b \rangle_n \rangle$.

Again $z \in \langle \langle c \rangle_n, \langle a \rangle_n \vee \langle b \rangle_n \rangle$ implies

$z \wedge n \in \langle \langle c \rangle_n, \langle a \rangle_n \vee \langle b \rangle_n \rangle$ Then a dual calculation

of above shows that $z \wedge n \in \langle \langle c \rangle_n, \langle a \rangle_n \rangle \vee \langle \langle c \rangle_n, \langle b \rangle_n \rangle$.

Thus by convexity, $z \in \langle \langle c \rangle_n, \langle a \rangle_n \rangle \vee \langle \langle c \rangle_n, \langle b \rangle_n \rangle$ and so (iii) holds.

(iii) $\Rightarrow$ (iv). Suppose (iii) holds. In (iv), R. H. S. $\subseteq$ L. H. S is obvious.

Now let $x \in \langle \langle c \rangle_n, A \vee B \rangle$. Then $x \vee n \in \langle \langle c \rangle_n, A \vee B \rangle$.

Thus $m(x \vee n, n, c) \in A \vee B$.

Now $m(x \vee n, n, c) = (x \vee n) \wedge (n \vee c) \geq n$ implies

$m(x \vee n, n, c) \in (A \vee B) \cap [n)$. Hence by

Theorem 4.2.1 (ii), $x \vee n \in \langle\langle c\rangle_n, (A \cap [n)) \vee (B \cap [n))\rangle$

$= \bigvee_{r \in (A \cap [n)) \vee (B \cap [n)))} \langle\langle c\rangle_n, \langle r\rangle_n\rangle.$

But by Theorem 2.1.12, $r \in (A \cap [n)) \vee (B \cap [n))$ *implies*

$r = s \vee t$ *for some* $s \in A$, $t \in B$ *and* $s, t \geq n$.

They by (iii), $\langle\langle c\rangle_n, \langle r\rangle_n\rangle = \langle\langle c\rangle_n, \langle s \vee t\rangle_n\rangle$

$$= \langle\langle c\rangle_n, \langle s\rangle_n \vee \langle t\rangle_n\rangle$$

$$= \langle\langle c\rangle_n, \langle s\rangle_n\rangle \vee \langle\langle c\rangle_n, \langle t\rangle_n\rangle$$

$$\subseteq \langle\langle c\rangle_n, A\rangle \vee \langle\langle c\rangle_n, B\rangle.$$

Hence $x \vee n \in \langle\langle c\rangle_n, A\rangle \vee \langle\langle c\rangle_n, B\rangle$.

Also $x \in \langle\langle c\rangle_n, A \vee B\rangle$ implies $x \wedge n \in \langle\langle c\rangle_n, A \vee B\rangle$.

Since $m(x \wedge n, \ n, c) = (x \wedge n) \vee (n \wedge c) \leq n$,

So $x \wedge n \in << c >_n, (A \vee B) \cap (n]>$

Then by Theorem 4.2.1 (ii),

$x \wedge n \in << c >_n, (A \cap (n]) \vee (B \cap (n])>$

$= \bigvee_{t \in (A \cap (n]) \vee (B \cap (n])} << c >_n, < t >_n > .$

Using Theorem 2.1.12 again, we se that $t = p \wedge q$ where

$p \in A, \quad q \in B, \quad p, q \leq n$. Then by (iii),

$$\langle \langle c \rangle_n , \langle t \rangle_n \rangle = \langle \langle c \rangle_n , \langle p \wedge q \rangle_n \rangle$$

$$= \langle \langle c \rangle_n , \langle p \rangle_n \vee \langle q \rangle_n \rangle$$

$$= \langle \langle c \rangle_n , \langle p \rangle_n \rangle \vee \langle \langle c \rangle_n , \langle q \rangle_n \rangle$$

$$\subseteq \langle \langle c \rangle_n , A \rangle \vee \langle \langle c \rangle_n , B \rangle$$

Hence $x \wedge n \in \langle \langle c \rangle_n , A \rangle \vee \langle \langle c \rangle_n , B \rangle$. Therefore by

Convexity, $x \in \langle \langle c \rangle_n , A \rangle \vee \langle \langle c \rangle_n , B \rangle$. and so (iv) holds.

(iv) $\Rightarrow$ (iii) trivial. (ii) $\Rightarrow$ (v). In (v) R. H. S. $\subseteq$ L. H. S. is obvious. Let $Z \in$ L. H. S. Then $z \in \langle \langle m(a, n, b) \rangle_n , \langle c \rangle_n \rangle$, which implies

$z \vee n \in \langle \langle m(a, n, b) \rangle_n , \langle c \rangle_n \rangle$. By (ii),

$z \vee n \in \langle \langle a \rangle_n , \langle b \rangle_n \rangle \vee \langle \langle b \rangle_n , \langle a \rangle_n \rangle$.

Then by Theorem 2.1.12, $z \vee n = x \vee y$ for some

$x \in \langle \langle a \rangle_n , \langle b \rangle_n \rangle$ and $y \in \langle \langle b \rangle_n , \langle a \rangle_n \rangle$ and $x, y \geq n$.

Thus, $\langle x \rangle_n \cap \langle a \rangle_n \subseteq \langle b \rangle_n$, and so

$\langle x \rangle_n \cap \langle a \rangle_n = \langle x \rangle_n \cap \langle a \rangle_n \cap \langle b \rangle_n \subseteq \langle z \vee n \rangle_n \cap \langle a \rangle_n \cap \langle b \rangle_n$
$= \langle z \vee n \rangle_n \cap \langle m(a, n, b) \rangle_n \subseteq \langle c \rangle_n$. This implies

$x \in \langle\langle a\rangle_n, \langle c\rangle_n\rangle$ Similarly $y \in \langle\langle b\rangle_n, \langle c\rangle_n\rangle$,

and so $z \vee n \in \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$.

Similarly, a dual calculation of above shows that

$z \wedge n \in \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$.

Thus by convexity, $z \in \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$

and so (v) holds. (v) $\Rightarrow$ (i).

Suppose (v) holds, Let a, b, c $\geq n$. By (v),

$\langle\langle m\,(a, n, b)\rangle_n, \langle c\rangle_n\rangle$

$= \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$.

But by Lemma 4.2.3 (i), this is equivalent to

$\langle a \wedge b, c\rangle = \langle a, c\rangle \vee \langle b, c\rangle$

Then by [9, Theorem 3.7], this shows that $[n)$ is a relatively normal lattice.

Similarly, for a, b, c $\leq n$, using the Lemma 4.2.3 (ii)
and Theorem 4.2.7, we find that $(n]$ is relatively normal.
Therefore $F_n(L)$ is relatively normal by Theorem 2.1.2.
Finally we need to prove (iii) $\Rightarrow$ (i). Suppose (iii) holds. Let a, b, c $\in (n]$.

By (iii), $\langle\langle c\rangle_n, \langle a\rangle_n \vee \langle b\rangle_n\rangle = \langle\langle c\rangle_n, \langle a\rangle_n\rangle \vee \langle\langle c\rangle_n, \langle b\rangle_n\rangle$.

But by Lemma 4.2.4(i), this is equivalent to $\langle c, a \vee b\rangle = \langle c, a\rangle \vee \langle c, b\rangle$

which says by [9, Theorem 3.7] $[n)$ is relatively normal.

Similarly for a, b, c $\leq n$, using the Lemma 4.2.**4** (ii) and Theorem 4.2.7, we find that $(n]$ is relatively normal. Hence by 2.1.2, $F_n(L)$ is relatively normal.

Following result is due to [9, Lemma 3.4].

Theorem 4.2.9 A lattice is relatively normal if and only if any two incomparable prime ideals are comaximal. ■

Now we generalized the above result.

Theorem 4.2.10 *Let $F_n(L)$ be a distributive lattice. Then the following conditions are equivalent:*

(i) $F_n(L)$ *is relatively normal.*

(ii) *Any two incomparable prime* n-*ideals* P *and* Q *are comaximal, that is* $P \vee Q = L$.

Proof :- Suppose (i) holds. Let P, Q be two incomparable prime n-ideals of L. Then there exist a, b $\in L$ such that $a \in P - Q$ and $b \in Q - P$. Then $\langle a\rangle_n \subseteq P - Q$, $\langle b\rangle_n \subseteq Q - P$.

Since $F_n(L)$ is relatively normal, so by Theorem 4.2.8.

$\langle\langle a\rangle_n, \langle b\rangle_n\rangle \vee \langle b\rangle_n, \langle a\rangle_n > = L$. But as P, Q are prime, so it is

easy to see that, $\langle\langle a\rangle_n, \langle b\rangle_n\rangle \subseteq Q$ and $\langle\langle b\rangle_n, \langle a\rangle_n\rangle \subseteq P$,

Therefore $L \subseteq P \vee Q$ and so $P \vee Q = L$. That is, (ii) holds.

Conversely, suppose (ii) holds. Let P_1 and Q_1 be two incomparable prime ideals of $[n)$. Then by Lemma 3.2.4 there exist incomparable prime ideals P and Q of L such that $P_1 = P \cap [n)$ and $Q_1 = Q \cap [n)$. Since $n \in P_1$ and $n \in Q_1$, so by Lemma 2.2.5, P, Q are in fact two incomparable prime n-ideals of L. Then by (ii), $P \vee Q = L$. Therefore, $P_1 \vee Q_1 = (P \vee Q) \cap [n) = [n)$. Thus by [9, Theorem 3.5], $[n)$ is relatively normal.

Similarly, considering two prime filters of $(n]$ and proceeding as above and using the dual result of [9, Theorem 3.5] we find that $(n]$ is relatively normal. Therefore by Theorem 2.1.2, $F_n(L)$ is relatively normal. ■

We already mentioned that $P_n(L) = F_n(L)$ when n is a central element of L. So we conclude the chapter with the following nice and interesting result.

Corollary 4.2.11 *Let* n *be a central element of a distributive lattice* L. *Then the following conditions are equivalent.*

(i) $P_n(L)$ *is a relatively normal lattice*

(ii) *For all* $a, b, c \in L$

$$<<a>_n, <b>_n> \vee <<b>_n, <a>_n> = L$$

(iii) For all $a, b, c \in L$

$$<<c>_n, <a>_n \vee <b>_n> = <<c>_n, <a>_n> \vee <<c>_n, <b>_n>.$$

(iv) Any two incomparable prime n-ideals P and Q are comaximal; that is $P \vee Q = L$

CHAPTER -5

Chapter-5

Characterization of finitely generated n-ideals which form sectionally and relatively m-normal lattice.

Introduction

Lee in [36] also see Lakser [29] has determined the lattice of all equational subclasses of the class of all pseudocomplemented distributive lattices. They are given by $B_{-1} \subset B_0 \subset --- \subset B_m \subset --- \subset B_\omega$, where all the inclusions are proper and B_ω is the class of all pseudocomplemented distributive lattices, B_{-1} consists of all one element algebra, B_0 is the variety of Boolean algebras while B_m, for $-1 \leq m < \omega$ consists of all algebras satisfying the equation

$$(x_1 \wedge x_2 \wedge ---- \wedge x_m)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge x_2 \wedge --- \wedge x_{i-1} \wedge x_i^* \wedge x_{i+1} \wedge ------ \wedge x_m)^* = 1$$

where x^* denotes the pseudocomplemented of x. Thus B_1 consists of all Stone algebras.

He also generalized Gratzer and Schmidt's theorem by proving that for $-1 \leq m < \omega$ the mth variety consists of all lattices such that each prime ideal contains at most m minimal prime ideals.

Beazer [3] and Davey [13] have each independently obtained several characterizations of (sectionally) B_m and relatively B_m lattices. Moreover, Gratzer and Lakser in [18] and [19] have obtained some results on this topic.

On the other hand Cornish [9] have studied the lattices analogues to B_m and relatively B_m lattices known as m-normal and relatively m-normal lattices.

A distributive lattice L with 0 is called m-normal, if each prime ideal of L contains at most m-minimal prime ideals A distributive lattice L is called *relatively* m-normal if each interval $[0, \quad x]$ is m-normal.

Recall that a family of ideals of a lattice L is comaximal if their join is L. Similarly a family of n-ideals of a lattice L is comaximal if their join is L.

In section 1 we will study finitely generated n-ideals which form a (sectionally) m-normal lattice. We will include several characterizations which generalize several results of [10], [13], [3] and [18]. We shall show that $F_n(L)$ is m-normal if and only if for any $x_1, x_2, ----, x_m \in L$, with

$m(x_i, n, x_j) = n$ implies $< x_0 >_n^+ \vee ---- \vee < x_m >_n^+ = L,$

which is also equivalent to the condition that for any m + 1 distinct minimal prime n-ideals $P_0, ------, P_m$

of L, $P_0 \vee --------- \vee P_m = L$.

In section 2 we will study those $F_n(L)$ which are relatively m-normal . Here we will include a number of characterizations of those $F_n(L)$ which give generalizations of results on relatively m-normal lattices given in [10], and [13], We show that $F_n(L)$ is relatively m-normal if and only if any m + 1 pairwise incomparable prime n-ideals are comaximal.

1. Lattices whose $P_n(L)$ form m-normal lattices

The following result is due to [13, Lemma 2.2]. This follows from the corresponding result for commutative semigroups due to Kist [28].

Lemma 5.1.1. *Let* M *be a prime ideal containing an ideal* J. *Then* M *is a minimal prime ideal belonging to* J *if and only if for all* $x \in M$, *there exists* $x' \notin M$ *such that* $x \wedge x' \in J$. ■

Now we generalize this result for n-ideals.

Lemma 5.1.2. *Let* M *be a prime n-ideal containing an n-ideal* J. *Then* M *is a minimal prime n-ideal belonging to* J *if and only if for all* $x \in M$ there exists $x' \notin M$ such that $m(x, n, x') \in J$.

Proof: Let M be a minimal prime n-ideal belonging to J and $x \in M$. Then by Theorem 4.1.8, $<<a>_n, J> \underline{\not\subset} M$. So there exists x' with $m(x, n, x') \in J$. such that $x' \notin M$.

Conversely, suppose $x \in M$, then there exists $x' \notin M$ such that $m(x, n, x') \in J$. This implies $x' \notin M$, but $x' \in <<x>_n, J>$, that is $<<x>_n, j> \underline{\not\subset}. M$. Hence by Theorem 4.1.8. M is a minimal prime n-ideals belonging to J. ■

Davey in [13, Corollary 2.3] used the following result in proving several equivalent conditions on B_m lattices. On the other hand, Cornish in [10] has used this result in studying n-normal lattice.

Proposition 5.1.3. *Let* M_0 , ------------------------------, M_n *be* n +1 *distinct minimal prime ideals. Then there exist* $a_0, -----, a_n \in L$ *such that* $a_i \wedge a_j \in J \quad (i \neq j)$ *and* $a_j \notin M_j \quad j = 0, -----n$. ■

The following result is a generalization of above result in terms of n-ideals.

Proposition 5.1.4. *Let* M_0 , ------------------------------, M_n *be* m +1 *distinct minimal prime n-ideals. Then there exist* $a_0, ----, a_n \in L$ *such that* $m(a_i, n, a_j) \in J(i \neq j)$ and $a_j \notin M_j (j = 0, -----, n)$.

Proof: For n = 1. Let $x_0 \in M_1 - M_0$ and $x_1 \in M_0 - M_1$. Then by Lemma 5.1.1, there exists $x_1' \notin M_0$ such that $m(x_1, n, x_1') \in J$. Hence $a_1 = x_1$, $a_0 = m(x_0, n, x_1')$ are the required elements.

Observe that $m(a_0, n, a_1) = m(m(x_0, n, x_1'), n, x_1)$

$$= (x_0 \wedge x_1 \wedge x_1') \vee (x_0 \wedge n) \vee (x_1 \wedge n) \vee (x_1' \wedge n)$$

$$= (x_0 \wedge m(x_1, n, x_1')) \vee (x_0 \wedge n) \vee (m(x_1, n, x_1') \wedge n)$$

$$= m(x_0, n, m(x_1, n, x_1')) .$$

Now $m(x_1, n, x_1') \wedge n \leq m(x_0, n, m(x_1, n. x_1'))$

$\leq m(x_1, n, x_1') \vee n$ *and* $m(x_1, n, x_1') \in J$, so by convexity

(ii) *For any ideals* $J_1, ---, J_n$ *in L such that* $J_i \cap J_J \subseteq J$, *for any* $i \neq j$, *there exists k such that* $J_k \subseteq J$;

(iii) J *is the intersection of at most* n – 1 *distinct Prime ideals.* ■

Our next result is a generalization of above result. This result will be needed in proving the next theorem which is the main result of this section. In fact, the following lemma is very useful in s studying those $P_n(L)$ which are m-normal .

Lemma 5.1.6. *Let* J *be an n-ideal in a lattice* L . *For a given positive integer* $m \geq 2$, *the following conditions are equivalent :*

(i) *For any* $x_1, x_2, ---, x_m \in L$ *with* $m(x_i, n, x_j) \in J$ *(that is, they are pairwise in* J) *for any* $i \neq j$, *there exists k such that* $x_k \in J$;

(ii) *For any n-ideals* $J_1, -------, J_m$ *in* L *such that* $J_i \cap J_j \subseteq J$ *for any* $i \neq j$, *there exists* k *such that* $J_k \subseteq J$;

(iii) J *is the intersection of at most* m – 1 *distinct prime n-ideals.*

Proof: (i) and (ii) are easily seen to be equivalent. (iii) ⇒ (i) . Suppose $P_1, P_2, ---, P_k$ *are* k $(1 \leq k \leq m-1)$ distinct prime –n-ideals such that $J = P_1 \cap ---- \cap P_k$. Let $x_1, x_2, -----, x_m \in L$ be such that $m(x_i, n, x_j) \in J$ *for all* $i \neq j$. Suppose no element x_i

is a member of J. Then for each r $(1 \le r \le k)$ there is at most one i $(1 \le i \le m)$ such that $x_i \in P_r$. Since k < m, there is some i such that $x_i \in P_1 \cap P_2 \cap ---- \cap P_k$.

(i) $\Rightarrow$ (iii). Suppose (i) holds for n = 2, then it implies that J is a prime n-ideal. Then (iii) is trivially true. Thus we may assume that there is a largest integer $t < m$ such that the condition (i) does not hold for J (consequently condition (i) holds for t + 1, t + 2, ------- m). For some t < m, we may suppose that there exist elements

$a_1, a_2, ----, a_t \in L$ such that $m(a_i, n, a_j) \in J$ for

$i \neq j$, $i = 1, 2, ----, t$ $j = 1, 2, ---, t$ yet $a_1, a_2, ----, a_t \notin J$.

As L is a distributive lattice, $<< a_i >_n, J >$ is an n-ideal for any $i \in \{1, 2, ------, t\}$. Each $<< a_i >_n, J >$ is in fact a prime n-ideal. Firstly $<< a_i >_n, J > \neq L$, since $a_i \notin J$. Secondly, suppose that b and c are in L and $m(b, n, c) \in << a_i >_n, J >$. Consider the set of $t+1$ elements

$\{a_1, a_2, ---, a_{i-1}, m(b, n, a_i), m(c, n, a_i), a_{i+1}, ---, a_t\}$.

This set is pairwise in J and so, either $m(b, n, a_i) \in J$ or $m(c, n, a_i) \in J$ since condition (i) holds for t + 1.

That is, $b \in \ll a_i >_n, \ J >$ or $c \in \ll a_i >_n, \ J >$ and so $\ll a_i >_n, \ J >$ is prime .

Clearly, $J \subseteq \bigcap_{l \le i \le t} \ll a_i >_n, \ J >$. If $w \in \bigcap_{l \le i \le t} \ll a_i >_n, \ J >$.

Then $w, a_1, \ a_2, \ldots\ldots, a_t$ are pairwise in J and so $w \in J$. Hence

$J = \bigcap_{l \le i \le t} \ll a_i >_n, \ J >$ is the intersection of t < m prime n-ideals . ∎

An ideal $J \neq L$ satisfying the equivalent conditions of Lemma 5.1.5 is called an m-prime ideal .

Similarly, an n-ideal $J \neq L$ satisfying the equivalent conditions of Lemma 5.1.6 is called an m-prime n-ideal .

Now we generalize a result of Davey in [13, Proposition 3.1] .

Theorem 5.1.7. *Let* J *be an n-ideal of a distributive lattice* L . *Then the following conditions are equivalent :*

(i) *For an* m + 1 *distinct prime n-ideals* $P_0, \ P_1, \ldots\ldots, P_m$ *belonging to* J, $P_0 \vee P_1 \vee \ldots\ldots \vee P_m = L$;

(ii) *Every prime n-ideal containing* J *contains at most m distinct minimal prime n-ideals belonging to* J ;

(iii) *If* $a_0, \ a_1, \ldots\ldots, a_m \in L$ *with* $m(a_i, \ n, \ a_j) \in J \ (i \neq j)$ *then* $\bigvee_j \ll a_j >_n, \ J > = L$.

Proof: (i) ⇒ (ii) is obvious. (ii) ⇒ (iii) .

Assume $a_0, \ a_1, \ldots\ldots, a_m = L$ with $m(a_i, \ n, \ a_j) \in J$

and $\underset{j}{\vee} \ll a_j >_n, \ J > \neq \ L$. It follows that $a_j \notin \ J$, for all j . Then by theorem 2.2.9 there exists a prime n-ideal P such that $\underset{j}{\vee} \ll a_j >_n, \ J > \subseteq P$. But by theorem 2.2.4, we know that P is either a prime ideal or a prime filter . Suppose P is a prime ideal .

For each j, let $F_j = \{x \wedge y \ : \ x \geq a_j, \ x, \ y \geq n, \ y \notin P$

Let $x_1 \wedge y_1, \ x_2 \wedge y_2 \ \in \ F_j$

$$(x_1 \wedge y_1) \wedge (x_2 \wedge y_2) = (x_1 \wedge x_2) \wedge (y_1 \wedge y_2).$$

Now $x_1 \wedge x_2 \geq a_j$ and $y_1 \wedge y_2 = m(y_1, \ n, \ y_2)$

So $t \geq x \wedge y$ implies $t = (t \vee x) \wedge (t \vee y)$.

Since $y \notin P$, so $t \vee y \notin P$. Hence $t \in F_j$, and so F_j is a dual ideal. We now show that $F_j \cap J = \Theta$, for all j = 0 , 1, ...m . If not, let $b \in F_j \cap J$, then $b = x \wedge y, \ \ x \geq a_j, \ \ x, \ \ y \geq n, \ \ y \notin P$.

Hence $m(a_j, n, y) = (a_j \wedge n) \vee n \vee (a_j, \wedge y) = (a_j \wedge y) \vee n = (a_j \vee n) \wedge (y \vee n)$.

But $(a_j \vee n) \wedge (y \vee n) \in F_j$ and $n \leq (a_j \wedge y) \vee n \leq b$ implies $m(a_j, n, y) \in J$.

Therefore, $m(a_j, n, y) \in F_j \cap J$. Again $m(a_j, n, y) \in J$ with $y \ \notin P$ implies $\ll a_j >_n, \ \ J > \not\subseteq P$, which is a contradiction.

Hence $F_j \cap J = \Theta$ for all j. For each j, let P_j be a minimal prime n-ideals belonging to J and $F_j \cap P_j = \Theta$. Let $y \in P_j$. If $y \notin P_j$, then $y \vee n \notin P$. Then $m(a_j, n, y \vee n) = (a_j \vee n) \wedge (y \vee n) \in F_j$.

But $m(a_j, n, y \vee n) \in \langle y \vee n \rangle_n \subseteq \langle y \rangle_n \subseteq P_j$, which is a contradiction. So $y \in P$.

Therefore $P_j \subseteq P$, and $a_j \notin P_j$. For if $a_j \in P_j$, then $a_j \vee n \in P_j$.

Now, $a_j \vee n = (a_j \vee n) \wedge (a_j \vee n \vee y) \in F_j$ for any $y \notin P$. This implies $P_j \cap F_j \neq \Theta$, which is a contradiction. So $a_j \notin P_j$.

But $m(a_i, n, a_j) \in J \subseteq P_j \quad (i \neq j)$ which implies $a_i \in P_j \quad (i \neq j)$ as P_j is prime. It follows that P_j form a set of m + 1 distinct minimal prime n-ideals belonging to J and contained in P.

This contradicts (ii) . Therefore $\underset{j}{\vee} \langle\langle a_j \rangle_n, \ J\rangle = L$.

Similarly, if P is filter, then a dual proof of above also shows that $\underset{j}{\vee} \langle\langle a_j \rangle_n, \ J\rangle = L$, and hence (iii) holds.

(iii) $\Rightarrow$ (i) . Let $P_0, \ P_1 .... P_m$ be m + 1 distinct minimal prime n-ideals belonging to J. Then by proposition 5.1.4. there exist

$$\left(i \neq j\right),\ i = 0,\quad 1, \ldots\ldots\ldots, m,\quad j = 0,\quad 1, \ldots\ldots\ldots\ldots\ldots, m$$

$$\left(x_0\right]^* \vee \left(x_1\right]^* \vee \ldots\ldots \vee \left(x_m\right]^* = L\,;$$

(iv) *For each prime ideal* P. 0 (P) is m + 1 *prime;*

(v) *L is m-normal lattice :* ■

Recall that for a prime n-ideal P of a distributive lattice L, we write $n\left(P\right) = \left\{\, y \in L \mid m\left(y, n, x\right) = n \right.$ for some $\left. x \in L - P \right\}$. Clearly $n\left(P\right)$ is an n-ideal and $n\left(P\right) \subseteq P$.

Our next result is a nice extension of above result in terms of n-ideals.

Theorem 5.1.9. *Let* L *be a distributive lattice. Then the following Conditions are equivalent :*

(i) *For any* m + 1 *distinct minimal prime n-ideals*

$P_0 \quad P_1 \ldots\ldots\ldots P_m;\ P_0 \vee P_1 \vee \ldots\ldots \vee P_m = L\,;$

(ii) *Every prime n-ideal contains at most m-minimal prime n-ideals ;*

(iii) *For any* $a_0 \quad a_1 \ldots\ldots\ldots a_m \in L$ *with* $m\left(a_i \quad n, a_j\right) = n, \left(i \neq j\right)$

$i = 0, \ldots\ldots\ldots m\ .\ j = 0, \ldots\ldots\ldots m$

$< a_0 >^+ \vee < a_1 >_n{}^+ \vee \ldots\ldots\ldots\ldots\ldots \vee < a_m >_n{}^* = L\ ;$

(iv) *For each prime n-ideal* P, n (P) *is an* m + 1 *prime n-ideal .*

Proof: (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i), easily hold by theorem 5.1.7 replacing J by {n}. To complete the proof we need to show that (iv) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (iii). Suppose (iv) holds and $x_0 \ x_1 \ldots\ldots x_m$ are m + 1 elements of L such that $m\left(x_i\ n, x_j\right)= n \ for \ (i \neq j)$.

Suppose that $< x_0 >_n^+ \vee < x_1 >_n^+ \vee \ldots\ldots\ldots \vee < x_m >_n^+ \neq L$. Then by Theorem 2.2.9 there is a prime n-ideal P such that

$< x_0 >_n^+ \vee < x_1 >_n^+ \vee \ldots\ldots\ldots \vee < x_m >_n^+ \subseteq P$.

Hence $x_0 \quad x_1 \ldots\ldots\ldots x_m \in L - n(P)$. This contradicts (iv) by Lemma 5.1.6, since $m\left(x_i, n, x_j\right)= n \ \in n\ (P)$ for all $i \neq j$.

Thus (iii) holds. (ii) $\Rightarrow$ (iv).

This follows immediately from Proposition 3.2.10 and Lemma 5.1.6 above. ■ Following result is due to [8].

Proposition 5.1.10. *Let* L *be a distributive lattice with* 0. *If the equivalent conditions of Theorem* 5.1.8 *hold, then for any* m + 1 *elements*

$$x_0, x_1, \ldots, x_m, (x_0 \wedge x_1 \wedge \ldots x_m]^* = \underset{0 \le i \le n}{\vee} (x_0 \wedge x_1 \wedge \ldots . x_{i-1} \wedge x_{i+1} \wedge \ldots . \wedge x_m]^*$$

■

Proposition 5.1.11. *Let* L *be a distributive lattice and* $n \in L$. *If the equivalent conditions of Theorem* 5.1.9 *hold then for any* m + 1 *elements*

$$x_0, x_1, \ldots\ldots\ldots\ldots, x_m ; \quad (< x_0 >_n \cap < x_1 >_n \cap \ldots . \cap < x_m >_n)^* = \underset{0 \le i \le n}{\vee}$$

$(<x_0>_n \cap .... \cap <x_{i-1}>_n \cap <x_{i+1}>_n \cap .... \cap <x_m>_n)^*$.

Proof: Let

$<b_i>_n = <x_0>_n \cap ...... \cap <x_{i-1}>_n \cap <x_{i+1}>_n \cap ..... \cap <x_m>_n$ for each

$0 \le i \le m$.

Suppose $x \in (<x_0>_n \cap ...... \cap <x_m>_n)^*$. Then

$<x>_n \cap <x_0>_n \cap ...... \cap <x_m>_n = \{n\}$. For all $i \ne j$;

$(<x>_n \cap <b_i>_n) \cap (<x>_n \cap <b_j>_n = \{n\}$.

So $(<x>_n \cap <b_0>_n)^+ \vee .... \vee <x>_n \cap <b_m>_n)^+ = L$.

Thus $x \in (<x>_n \cap <b_0>_n)^+ \vee .... \vee <x>_n \cap <b_m>_n)^+$.

Hence by theorem 2.1.2, $x \vee n = a_0 \vee .... \vee a_m$ where

$a_i \in (<x>_n \cap <b_i>_n)^+$ and $a_i \ge n$, for $i = 0, 1, ..., m$.

Then $x \vee n = (a_0 \wedge (x \vee n)) \vee .... \vee (a_m \wedge (x \vee n))$.

Now $a_i \in (<x>_n \cap <b_i>_n)^+$ implies

$<a_i>_n \cap <x>_n \cap <b_i>_n = \{n\}$.

Then by a routine calculation we find that $(a_i \wedge x \wedge b_i) \vee n = n$.

Thus, $<a_i \wedge (x \vee n)>_n \cap <b_i>_n = [n, (a_i \wedge x \wedge b_i) \vee n] = \{n\}$ implies

that $a_i \wedge (x \vee n) \in <b_i>_n^+$ and so $x \vee n \in <b_0>_n^+ \vee ...... \vee <b_m>_n^+$.

By a dual proof of above, we can easily show that

$x \wedge n \in < b_0 >_n^+ \vee .... \vee < b_m >_n^+$. Thus by convexity,

$x \in << b_0 >_n^+ \vee .... \vee < b_m >_n^+$. This proves that

L. H. S. $\subseteq$ R. H. S.

The reverse inclusion is trivial . ■

Theorem 5.1.12. For a distributive lattice L, if $F_n(L)$ *is distributive then the following conditions are equivalent.*

(i) $F_n\ (L)$ *is m-normal lattice.*

(ii) Every prime n-ideal contains at most m minimal prime n-ideals .

(iii) For any m+1 distinct minimal prime n-ideals P_0 , P_1 ,… …, P_m ;

$P_0 \vee P_1 \vee .... \vee P_m = L$.

Proof: (i) $\Rightarrow$ (ii). Let $F_n(L)$ be m-normal .

Since $F_n(L) \cong (n]^d \times [n)$, so both $(d]^d$ and $[n)$ are m-normal .

Suppose P is any prime n-ideal of L .Then by Theorem 2.2.3, either

$P \supseteq (n]$ or $P \supseteq [n)$ .Without loss of generality suppose $P \supseteq (n]$.

Then by Theorem 2.2.4 , P is a prime ideal of L .Hence by Lemma 3.2.4 ,

$P_1 = P \cap [n)$ is a prime ideal of $[n)$. Since $[n)$ is m-normal , so by Theorem 3.2 .1, P_1 contains at most m minimal prime ideals R_1, …….,R_m of $[n)$.

Therefore P contains m minimal prime ideals Q_1 ,…………,Q_m of L where $R_i = Q_i \cap [n)$ .Since $n \in R_i$, so $n \in Q_i$ and hence Q_1,… …..,Q_m are minimal prime n-ideals of L .Thus (ii) holds . (ii) $\Rightarrow$(i) Suppose (ii) holds . Let P_1 be a prime ideal in $[n)$. Then by Lemma 3.2.4 , $P_1 = P \cap [n)$ for some prime ideal P of L. Since $n \in P_1 \subseteq P$, so P is a prime n-ideal. Therefore by (ii) P contains at most m minimal prime ideals $Q_1, \ldots\ldots, Q_m$. Thus by Lemma 3.2.4, P_1 contains at most m minimal prime ideals $R_1, \ldots\ldots, R_m$ of $[n)$ such that . $R_1 = Q_i \cap [n)$. Hence by theorem 5.1.8 $[n)$ is m-normal .

Similarly we can prove that $(n]^d$ is also m-normal. Since $F_n(L) \cong (n]^d \times [n)$, so $F_n(L)$ is m-normal.

(ii) $\Leftrightarrow$ (iii) has already been prime in Theorem 5.1.9 ■

We already know that when n is a central element in L, then $P_n(L) = F_n(L)$ .Thus we have the following interesting characterization of them $P_n(L)$ which are m-normal.

Theorem 5.1.13 : Let n be a central element of distributive lattice L. Then the following conditions have equivalent.

(i) $P_n(L)$ is m-normal

(ii) For any m + 1 distinct minimal prime n-ideals
$P_0, \ldots\ldots, P_m$; $P_0 \vee P_1 \vee \ldots\ldots \vee P_m = L$;

(iii) Every prime n-ideal contains at most m minimal prime n-ideals ;

(iv) For any a_0 , a_1 , ..., a_m $\in L$ with

$m\left(a_{i,} n, a_j\right) = n, \left(i \neq j\right)$ $i = 0, 1, ..., m$

$j = 0, 1, ..., m,$

$< a_0 >^+{}_n \vee < a_1 >_n^+ \vee ...... \vee < a_m >_n^+ = L;$

(v) For each prime n-ideal P, $n\left(P\right)$ is an m +1 prime n-ideal.

2. Generalizations of some results on relatively m-normal lattice

Several characterization on relative B_m lattices have been given by Davey in [13]. Also Cornish have studied these lattices in [10] under the name of relatively n-normal lattices.

Recall that a lattice L is relatively m-normal lattice if its every interval [a, b] $(a, \quad b \in L \quad a < b)$ is a m-normal lattice.

Following result gives some characterizations of $F_n(L)$, which are relatively m-normal lattices which is a generalization of [13, Theorem 3.4].

Theorem 5.2.1. *Let L be a distributive lattice with $n \in L$. Then the following conditions are equivalent :*

(i) $F_n(L)$ *is relatively m-normal* .

(ii) *For all* $x_0, \quad x_1, \ldots\ldots, x_m \in L$

$$<<x_1>_n \cap <x_2>_n \cap \ldots\ldots \cap <x_m>_n, \quad <x_0>_n>$$

$$\vee <<x_0>_n \cap <x_2>_n \cap \ldots\ldots \cap <x_m>_n, \quad <x_1>_n>$$

$$\vee \ldots\ldots \vee <<x_0>_n \cap <x_1>_n \cap \ldots\ldots \cap <x_{m-1}>_n, \quad <x_m>_n> = L;$$

(iii) *For all* $x_0, \quad x_1, \ldots\ldots, x_m \; z \in L$,

$$<<x_0>_n \cap <x_1>_n \cap \ldots\ldots \cap <x_m>_n, \quad <z>_n>$$

$$= <<x_1>_n \cap \ldots\ldots \cap <x_m>_n, \quad <z>_n>$$

$\vee \ll x_0 >_n \; < x_2 >_n \cap \ldots\ldots\ldots\ldots \cap < x_m >_n, \;\; < z >_n >$

$\vee \ldots\ldots\vee \ll x_0 >_n \cap \; < x_1 >_n \cap \ldots\ldots\ldots \cap < x_{m-1} >_n, \;\; < z >_n >$

.

(iv) *For any* m + 1 *pairwise incomparable prime n-ideals*

$P_0, \;\; P_1 \ldots\ldots\ldots\ldots, P_m, \;\; P_0 \vee \ldots\ldots\ldots\ldots \vee P_m = L$

(v) Any prime n-ideal contains at most m mutually incomparable prime n-ideals .

Proof : $(i) \Rightarrow (ii)$. Let $z \in L$, consider the interval

$I = [< x_0 >_n \cap < x_1 >_n \cap \ldots\ldots\ldots\ldots\ldots\ldots \cap < x_m >_n \cap < z >_n , < z >_n]$

in $F_n(L)$. Then $< x_0 >_n \cap < x_1 >_n \cap \ldots\ldots\ldots\ldots \cap < x_m >_n \cap < z >_n$

is the smallest element of the interval I .For $0 \le i \prec m$, the set of

element $< t_i >_n = < x_0 >_n \cap < x_1 >_n \cap \ldots\ldots\ldots\ldots \cap < x_{i-1} >_n \cap < x_{i+1} >_n \cap$

$\ldots\ldots\ldots \cap < x_m >_n \cap < z >_n$ are obviously pairwise disjoint in the

interval I . Since I is m-normal . Then by Theorem 5.1.8,

$< t_0 >_n^0 \vee \ldots\ldots\ldots\ldots\ldots \vee < t_m >_n^0 = < z >_n$ .So by Theorem 2.1.12 ,

$z \vee n = p_0 \vee \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \vee p_m$ where $p_i \ge n$. Thus,

$< p_0 >_n \cap < t_0 >_n = < p_0 >_n \cap < t_1 >_n = \ldots\ldots\ldots = < p_m >_n \cap < t_m >_n$

= The smallest element of I .

$= < x_0 >_n \cap < x_1 >_n \cap \ldots\ldots\ldots\ldots\ldots\ldots \cap < x_m >_n \cap < z >_n$.

Now ,

$< p_0 >_n \cap < t_0 >_n = < x_0 >_n \cap < x_1 >_n \cap \text{..........} \cap < x_m >_n \cap < z >_n$

which implies $< p_0 >_n \cap < t_0 >_n \subseteq < x_0 >$. Again

$, < p_0 >_n \cap < t_0 >_n = < p_0 >_n \cap < x_1 >_n \cap \text{............} \cap < x_m >_n \cap < z >_n$

$= < p_0 >_n \cap < x_1 >_n \cap \text{..................} < x_m >_n$, as

$< p_0 >_n \subseteq < z >_n$. This implies

$, < p_0 >_n \cap < x_1 >_n \cap \text{..............} \cap < x_m >_n \subseteq < x_0 >_n$ and so

$< p_0 >_n \in << x_1 > \cap < x_2 >_n \cap \text{............} \cap < x_m >_n , < x_0 >_n >$

$< p_1 >_n \in << x_0 > \cap < x_2 >_n \cap \text{............} \cap < x_m >_n , < x_1 >_n >$

....

....

$< p_m >_n \in << x_0 >_n \cap < x_1 >_n \cap \text{.............} \cap < x_{m-1} >_n , < x_m >_n >$.

Therefor, $z \vee n \subseteq << x_1 >_n \cap < x_2 >_n \cap \text{..........} \cap < x_m >_n , < x_0 >_n >$

$\vee << x_0 >_n \cap < x_2 >_n \cap \text{..............} \cap < x_m >_n , < x_1 >_n >$

$\vee \text{.............} \vee << x_0 >_n \cap < x_1 >_n \cap \text{.........} \cap < x_{m-1} >_n , < x_m >_n >$.

By a dual proof of above we can easily show that

$z \vee n \subseteq << x_1 >_n \cap < x_2 >_n \cap \text{..............} \cap < x_m >_n , < x_0 >_n >$

$\vee << x_0 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n , < x_1 >_n >$

$\vee .... \vee << x_0 >_n \cap < x_1 >_n \cap .... \cap < x_{m-1} >_n , < x_m >_n >$.

Hence by convexity ,

$z \in << x_1 >_n \cap < x_2 > \cap .... \cap < x_m >_n , < x_0 >_n >$

$\vee << x_0 >_n \cap < x_2 > \cap .... \cap < x_m >_n , < x_1 >_n >$

$\vee .... \vee << x_0 >_n \cap < x_1 >_n \cap .... \cap < x_{m-1} >_n , < x_m >_n >$.

This implies (ii) holds .

(ii) $\Rightarrow$ (iii). Suppose

$b \in << x_0 >_n \cap < x_1 >_n \cap .... \cap < x_m >_n, \quad < z >_n >$.

Then by

(ii) and Theorem 2.1.12, $b \vee n = s_0 \vee s_1 \vee .... \vee s_m$, for some

$s_0 << x_1 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n, \quad < x_0 >_n >$

$s_1 \in << x_0 >_n \quad < x_2 >_n \cap .... \cap < x_m >_n, \quad < x_1 >_n >$

....

....

$s_m \in < x_0 >_n \cap < x_1 >_n \cap .... \cap < x_{m-1} >_n' < x_m >_n >$.

and $\quad s_i \geq n, i = 0, 1, ..., m$.

Thus $< x_1 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n \cap < s_0 >_n \subseteq < x_0 >_n$

$< x_0 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n \cap < s_1 >_n \subseteq < x_1 >_n$

....

....

$< x_0 >_n \cap < x_1 >_n \cap .... \cap < x_{m-1} >_n \cap < s_m >_n \subseteq < x_m >_n$

This implies $< x_1 >_n \cap < x_2 > \cap .... \cap < x_m >_n \cap < s_0 >_n$

$= < x_0 >_n \cap < x_1 >_n \cap .... \cap < x_m >_n \cap < s_0 >_n$

$\subseteq < x_0 >_n \cap < x_1 >_n \cap .... \cap < x_m >_n \cap < b \vee n >_n \subseteq < z >_n$

Hence $s_0 \in < < x_1 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n, < z >_n >$.

Similarly $t_m = x_0 \vee x_1 \vee .... \vee x_{m-1}$

Therefore, $b \vee n \in < x_1 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n, < z >_n >$

$\vee << x_0 >_n < x_2 >_n \cap .... \cap < x_m >_n, < z >_n >$

$\vee .... \vee << x_0 >_n \cap < x_1 >_n \cap .... \cap < x_{m-1} >_n, < Z >_n >$.

The dual proof of above gives

$b \vee n \in < x_1 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n, < z >_n >$

$\vee << x_0 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n, < z >_n > \vee ......$

$....... \vee << x_0 >_n \cap < x_1 >_n \cap .... \cap < x_{m-1} >_n, < Z >_n >$.

Thus by convexity,

$b \in << x_1 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n, < z >_n >$

$\vee << x_0 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n, < z >_n >$

$\vee .... \vee << x_0 >_n \cap < x_1 >_n \cap .... \cap < x_{m-1} >_n, < Z >_n >$.

Therefore, $<< x_0 >_n \cap < x_1 >_n \cap .... \cap < x_m >_n \cap < z >_n >$

$\subseteq < x_1 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n , < z >_n >$

$\vee << x_0 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n , \; < z >_n >$

$\vee .... \vee << x_0 >_n \cap < x_1 >_n \cap .... \cap < x_{m-1} >_n , \; < z >_n >$.

Since the reverse inequality always holds, so (iii) holds.

(iii) $\Rightarrow$ (i). Suppose , $n \le b \le d$.

Let $x_0 , \; x_1 , ..., x_m \in [b, \; d]$ such that $x_i \wedge x_j = b$, for all

$i \neq j$.

Let $t_0 = x_1 \vee x_2 \vee .... \vee x_m$

$t_1 = x_0 \vee x_2 \vee .... \vee x_m$

....

....

$t_m = x_0 \vee x_1 \vee .... \vee x_{m-1}$

Clearly , $n \le b \le t_i \le d$ and

$x_0 = t_1 \wedge t_2 \wedge .... \wedge t_m$

$x_1 = t_0 \wedge t_2 \wedge .... \wedge t_m$

....

....

$x_m = t_0 \wedge t_1 \wedge .... \wedge t_{m-1}$.

Then $[b,\ d] \cap \{<< x_0 >_n, < b >_n > \vee .... \vee < x_m >_n, \ < b >_n >\}$

$$= [b,\ d] \cap \{<< t_1 >_n \cap < t_2 >_n > \cap .... \cap << t_m >_n, \ < b >_n >$$

$$\vee << t_0 >_n, < t_2 >_n \cap .... \cap < t_m >_n >$$

$$\vee .... \vee << t_0 >_n, < t_1 >_n \vee .... \vee < t_{m-1} >_n . < b >_n >$$

$$= [b,\ d] \cap \{<< t_0 >_n \cap < t_1 >_n \cap .... \cap << t_m >_n, \ < b >_n >\}$$

$$= [b,\ d] \cap << b >_n, < b >_n > = [b,\ d] \cap L = [b,\ d]$$, so by

theorem 5.1.8, [b, d] is m-normal. Hence, $[n)$ is relatively m-normal .

A dual proof of above shows that $(n]$ is also relatively m-normal. Since $F_n(L) \cong (n]^d \times [n)$ so, $F_n(L)$ is relatively m-normal .

(ii) $\Rightarrow$ (iv) . Suppose (ii) holds. Let $P_0,\ P_1, ... P_m$ be m + 1 pairwise incomparable prime n-ideals. Then, there exist

$x_0,\ x_1, ... x_m$ L such that $x_i \in P_j - \bigcup_{i=1}^{n} P_i$. Then by (ii),

$$<< x_1 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n, \ < x_0 >_n >$$

$$\vee \ << x_0 > \cap < x_2 >_n \cap .... \cap < x_m >_n, \ < x_1 >_n >$$

$$\vee .... \vee \ << x_0 >_n \cap < x_1 >_n \cap .... \cap < x_{m-1} >_n, \ < x_m >_n > = L \ .$$

Let $t_0 \in << x_1 > \cap < x_2 >_n \cap .... \cap < x_m >_n, < x_0 >_n >$,

then $< t_0 >_n \cap < x_1 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n \subseteq < x_0 >_n \subseteq P_0$.

Now, $x_i \notin P_0$, for $i = 1, 2, ..., m$ implies that

$< x_i >_n \not\subseteq P_0$ for $i = 1, 2, \ldots\ldots\ldots\ldots\ldots, m$. Thus

$< x_1 >_n \cap < x_2 >_n \cap \ldots\ldots\ldots \cap < x_m >_n, \not\subseteq P_0$ as P_0 is prime .

This implies $< t_0 >_n \subseteq P_0$, and so $t_0 \in P_0$.

Therefore, $<< x_1 > \cap < x_2 >_n \cap \ldots\ldots\ldots \cap < x_m >_n, < x_0 >_n > \subseteq P_0$.

Similarly, $<< x_0 > \cap < x_2 >_n \cap \ldots\ldots\ldots \cap < x_m >_n, < x_1 >_n > \subseteq P_1$.

$<< x_0 > \cap < x_1 >_n \cap \ldots\ldots\ldots \cap < x_m >_n, < x_2 >_n > \subseteq P_2$.

....

....

$<< x_0 > \cap < x_1 >_n \cap \ldots\ldots\ldots \cap < x_{m-1} >_n, < x_m >_n > \subseteq P_m$.

Hence $P_0, \vee P_1 \vee \ldots\ldots\ldots\ldots \vee P_m = L$.

(iv) $\Leftrightarrow$ (v) is trivial by Stone's separation theorem.

(iv) $\Leftrightarrow$ (i). Let any m + 1 pairwise incomparable prime n-ideals of L are comaximal. Cosider the interval [b, d] in L with b, $d \geq n$, let $P_0', \; P_0', \ldots\ldots P_m'$ be m + 1 distinct minimal prime ideals of [b, d]. Then by Lemma 4.2.9 there exist prime ideals $P_0, \ldots\ldots P_m$ of L such that $P_0' = P_0 \cap [b, d] \ldots\ldots P_m' = P_m \cap [b, d]$.

Since each P_1 is an ideal, so $b \in P_i$, Moreover, $n \leq b$ implies that $n \in P_i$ Therefore each P_i is a prime n-ideal by Lemma 2.2.5. $i = 0, \; 1, \ldots\ldots, m$.

Since $P_0', \ldots\ldots P_m'$ are incomparable, so $P_0, \ldots\ldots P_m$ are also incomparable, Now by (iv), $P_0 \vee \ldots\ldots \vee P_m = L$ Hence

$P_0' \vee \ldots \vee P_m' = (P_0 \vee \ldots \vee P_m) \cap [b, d] = L \cap [b, \; d] = [b, \; d]$. Therefore by theorem 5.1.8 [b, d] is in m –normal . Hence $[n)$ is relatively in m-normal .

A dual proof of above shows that $(n]$ is relatively in dual m-normal . Since $F_n(L) \cong (n]^d \times [n)$. So $F_n(L)$ is relatively in m-normal .

Following result is also a generalization of [13, Theorem 3.4]

Theorem 5.2.2. *Let L be a distributive lattice with* $n \in L$. *Suppose* $F_n(L)$ *is relatively m-normal*

Then the following conditions are equivalent :

(i) $F_n(L)$ *is relatively m-normal .*

(ii) If $a_0,\ a_1, \ldots\ldots\ldots\ldots\ldots\ldots a_m \in L$ with

$m(a_i, n, a_j) \in <b>_n\ (i \neq j)$ then

$<<a_0>_n, <b>_n> \vee \ldots\ldots \vee <a_m>_n, <b>_n> = L.$

Proof : (i) $\Rightarrow$ (ii).

By theorem 5.2.1. (v), any prime n-ideal containing b contains at most m minimal prime n-ideals belonging to $<b>_n$. Hence by theorem 5.1.7 with $J = <b>_n$, we have $<<a_0>_n, <b>_n> \vee \ldots\ldots \vee <a_m>_n, <b>_n> = L$. Thus (ii) holds. (ii) $\Rightarrow$ (i). Consider b, $c \in [n)$ with $b \leq c$

Let $a_0, \ldots\ldots\ldots, a_m \in [b,\ c]$ with $a_i \wedge a_j\quad (i \neq j)$ then by

$m(a_i,\ n,\ a_j) = b \in <b>_n$. Then by (ii),

$<<a_0>_n, <b>_n> \vee \ldots\ldots \vee <a_m>_n, <b>_n> = L.$

So, $[b,\ c] = (<<a_0>_n, <b>_n \cap [b,\ c]) \vee \ldots\ldots \vee (<a_m>_n, <b>_n> \cap [b,\ c])$

$$= < a_0, \quad b > [b, \quad c] \vee .... \vee < a_m, \quad b > [b, \quad c]$$

Hence by theorem 5.1.8 [b, c] is m-normal Therefore $[n)$ is relatively m-normal. A dual proof of above shows that $(n]$ is relatively in dual m-normal . Therefore by Theorem 2.1.2 $F_n\,(L)$ is relatively m-normal .

We conclude this chapter with the following result.

Let n be a central element of a distributive lattice L.

(i) $P_n\,(L)$ is relatively m-normal.

(ii) *For all* $x_0, \quad x_1, ..., x_m \in L$

$$<< x_1 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n, \quad < x_0 >_n >$$

$$\vee << x_0 >_n \cap < x_2 >_n \cap .... \cap < x_m >_n, \quad < x_1 >_n >$$

$$\vee .... \vee << x_0 >_n \cap < x_1 >_n \cap .... \cap < x_{m-1} >_n, \quad < x_m >_n > = L;$$

(iii) *For all* $x_0, \quad x_1, ..., x_m \; z \in L$,

$$<< x_0 >_n \cap < x_1 >_n \cap .... \cap < x_m >_n, \quad < z >_n >$$

$$= << x_1 >_n \cap .... \cap < x_m >_n, \quad < z >_n >$$

$$\vee << x_0 >_n \quad < x_2 >_n \cap .... \cap < x_m >_n, \quad < z >_n >$$

$$\vee ...... \vee << x_0 >_n \cap < x_1 >_n \cap .... \cap < x_{m-1} >_n, \quad < z >_n > .$$

(iv) *For any* m + 1 *pairwise incomparable prime n-ideals* $P_0, \quad P_1 ...., P_m, \quad P_0 \vee .... \vee P_m = L$

(v) Any prime n-ideal contains at most m mutually incomparable

(v) *prime n-ideals .*

CHAPTER -6

Chapter-6

Annulets and α-n-ideals of a distributive lattice

Introduction

Annulets and α- ideals in a distributive lattice with 0 have been studied by W. H. Cornish in [11] . In a distributive lattice L with 0, the set of ideals of the form $(x]^*$ can be made into a lattice $A_0(L)$, called the lattice of annulets of L. $A_0(L)$ is a sublattice of the Boolean algebra of all annihilator ideals in L, while the lattice of annulets is no more than the dual of the so - called lattice of filets (carriers) as studied in l-groups and abstractly for distributive lattices in

[2]. From the basic theorem of [9] it follows that $A_0(L)$ is a sublattice of the lattice of all ideals of L if and only if each prime ideal in L contains a unique minimal prime ideal.

Subramanian [57] studied h-ideals with respect to the space of maximal l-ideals in an f-ring. Of course Cornish's α - ideals and his h-ideals were both suggested by the z-ideals of Gillman and Jerison [7]. On the other hand Bigard [4] has studied α-ideals in the context of lattice ordered groups.

Recently [54] has studied the annulets and α-ideals in a distributive near lattice .

By [11] for an ideal J in L we define $\alpha(J) = \{(x]^* \mid x \in J$. Also for a filter F in $A_0(L)$, $a^{\leftarrow}(F) = \{x \in L \mid (x]^* \in F$. It is easy to see that

$\alpha(J)$ is a filter in $A_0(L)$ and $\alpha^{\leftarrow}(F)$ is an ideal in L. An ideal J in L is called an α-*ideal* if $\alpha^{\leftarrow}\alpha(J)=J$.

In this chapter we have generalized these concepts around a central element n of L. We have introduced the notion of n-annulets and α-n-ideals in L. As mentioned earlier, for a distributive lattice with $n \in L$, the lattice of n-ideals $I_n(L)$ is a distributive algebraic lattice, and so it is pseudocomplemented. We denote the set of annihilator n-ideals (the n-ideals J such that $J = J^{**}$) by $S_n(L)$.

By [15] $(S_n(L); \cap, \veebar, *, \{n\}, L)$ is a Boolean algebra which is not necessarily a sublattice of $I_n(L)$.

We denote the set of all n-ideals of the form $<x>_n^*$ by $A_n(L)$. This is a join subsemilattice of $S_n(L)$, but it becomes a sublattice if n is a central element of L. We call $A_n(L)$ by lattice of n-annulets.

In section 1 We have studied n-annulets when n is central and generalized several results of [11]. We have proved that $A_n(L)$ is a sublattice of $I_n(L)$ If and only if $P_n(L)$ is normal. We have also shown that $A_n(L)$ is relatively complemented if and only if $P_n(L)$ is sectionally quasi-complemented. Finally we have given a characterization for $P_n(L)$ to be generalized Stone in terms of $A_n(L)$.

In section 2 we have introduced the notion of α-n-ideals. We have shown that the n-ideal n (P) where P is a prime n-ideal is an α-n-ideal. Moreover, all the minimal prime n-ideal are α-n-ideals. Then we have generalized all the results of Cornish in [11] in terms of α -n-ideals. We have shown that $P_n(L)$ is disjunctive if and only if each n-ideal is an α -n-ideal. Also $P_n(L)$ is sectionally quasi-complemented if and only if each prime α -n-ideal is a minimal prime n-ideal. We conclude the thesis by characterizing $P_n(L)$ to be generalized Stone in terms of α -n-ideals.

1. n-Annulets of a distributive lattice

For a distributive lattice L with 0, $I(L)$, the lattice of ideals of L is pseudocomplemented . Recall that an ideal J of L is an *annihilator ideal* if $J = J^{**}$. The pseudocomplement of an ideal J is the annihilator ideal $J^{*} = \{x \in L \mid x \wedge j = 0 \text{ for all } j \in J\}$. It is well known by [8] that the set of annihilator ideals $A(L)$ is a Boolean algebra, where the supremum of J and K in $A(L)$ is given by $J \underline{\vee} K = (J^{*} \cap K^{*})^{*}$.

Ideals of the form $(x]^{*}$ $(x \in L)$ are called the *annuletes* of L .Then for two annuletes $(x]^{*}$ and $(y]^{*}$,

$(x]^{*} \underline{\vee} (y]^{*} = ((x]^{**} \wedge (y]^{**})^{*} = ((x \wedge y]^{**})^{*} = (x \wedge y]^{*}$.

Moreover, $(x]^{*} \cap (y]^{*} = (x \vee y]^{*}$ Hence the set of all annulets of L denoted by $A_0(L)$ is a sublattice of $A(L)$. In general, $A(L)$ and so $A_0(L)$ are not sublattice of $I(L)$.

For a distributive lattice L with $n \in L$, the lattice of n-ideals $I_n(L)$ is a distributive algebraic Lattice with $\{n\}$ and L as the smallest and largest elements respectively. Thus $I_n(L)$ is pseudo complemented. For an n-ideal J of L, the pseudo complement of J is the *annihilator* n-*ideal* . $J^{*} = \{x \in L \mid m(x, n\ j) = n \text{ for all } j \in J\}$. We denote the set of annihilator n-ideals by $S_n(L)$, where the supremum of J and K in $S_n(L)$ is given by $J \underline{\vee} K = (J^{*} \cap K^{*})^{*}$. Recall that the n-ideals of the form $< x >_n^{*}$ $(x \in L)$ are the n-*annulets* of L. We denote the set of n-*annulets* of L by $A_n(L)$.

Thus for two annulets $<x>_n^*$ and $<y>_n^*$,

$<x>_n^* \veebar <y>_n^* = (<x>_n^{**} \cap <y>_n^{**})^* = (<x>_n \cap <y>_n)^{***}$

$= < m(x, n, y) >_n^*$

Moreover, $<x>_n^* \cap <y>_n^* = (<x>_n \vee <y>_n)^*$, which is not necessarily a member of $A_n(L)$. Thus $A_n(L)$ is only a join subsemilattice of $S_n(L)$. $S_n(L)$ is a Boolean algebra with $\{n\}^* = L$ as the largest element and $L^* = \{n\}$ as the smallest element. Of course, $S_n(L)$ is not necessarily a sublattice of $I_n(L)$. We start this section with the following result :

Proposition 6.1.1 *Let L be a distributive lattice with* n *as a central element. Then the set of n-annulets* $A_n(L)$ *of L is a lattice* $(A_n(L), \cap, \veebar)$ *and a sublattice of the Boolean algebra* $(S_n(L); \cap, \veebar, *, \{n\}, L)$ *of annihilator n -ideals of L.* $A_n(L)$, *has the same largest element* $L = \{n\}^*$ *as* $S_n(L)$ *while* $A_n(L)$ *has a smallest element if and only if L possesses an element* d *such that* $<d>_n^* = \{n\}$.

Proof : We already know that $A_n(L)$ is a join subsemilattice of $S_n(L)$. Now for $<x>_n^*, <y>_n^* \in A_n(L)$,

$<x>_n^* \cap <y>_n^* = (<x>_n \vee <y>_n)^*$

$= ([x \wedge n, x \vee n] \vee [y \wedge n, y \vee n])^*$

$= [x \wedge y \wedge n, \quad x \vee y \vee n]^*$. Since n is central so

$x \wedge y \wedge n \quad \leq n \leq x \vee y \vee n$ implies there exists $t \in L$ such

that $t \wedge n = x \wedge y \wedge n$ and $t \vee n = x \vee y \vee n$.

Therefore, $<x>_n^* \cap <y>_n^* = <t>_n^* \in A_n(L)$, and so $A_n(L)$ is a sublattice of $S_n(L)$. Since $L =< n >_n^* \in A_n(L)$ so it has the same largest element as $S_n(L)$. For the last part if there exists $d \in L$ with $< d >_n^* = \{n\}$, then $\{n\}$ is the smallest element in $A_n(L)$. Finally suppose there in an element $d \in L$ such that $< d >_n^*$ is the smallest element in $A_n(L)$. Then for any $x \in L$,
$< x >_n^* = < x >_n^* \veebar < d >_n^* = < m(x, n, d) >_n^*$. Thus $m(x, n, d) = n$ implies $< x >_n^* = \{n\}^* = L$ so that $x = n$, and hence $< d >_n^* = \{n\}$. ■
Now we generalize [11, Proposition 2.2].

Proposition 6.1.2 . *Let* L *be a distributive lattice with a central element* n . *Then* $P_n(L)$ *is normal if and only if* $A_n(L)$ *is sublattice of* $I_n(L)$.

Proof : Let $< x >_n^*, < y >_n^* \in A_n(L)$. By 3.2.13, $P_n(L)$ is normal if and only if $< x >_n^* \vee < y >_n^* = (< x >_n \cap < y >_n)^*$

$= (<x>_n \cap <y>_n)^{***} = ((<x>_n \cap <y>_n)^{**})^* = (< x >_n^{**} \cap < y >_n^{**})^*$

$= < x >_n^* \veebar < y >_n^*$. That is $\veebar$ in $A_n(L)$ is same as $\vee$ in $I_n(L)$. This proves the proposition. ■

A distributive lattice L with 0 is called *disjunctive* if for $0 \le a < b \quad (a,b \in L)$ there is an element $x \in L$ such that $a \wedge x = 0$ where $0 < x \le b$. This is also known as *sectionally semi-complemented distributive lattice.* It is easy to check that L is disjunctive if and only if $(a]^* = (b]^*$ implies a = b for any a, b $\in L$.

Similarly a distributive lattice L with 1 is called *dual disjunctive* if for $c < d \le 1 \quad (c, d \in L)$ there is an element $y \in L$ such that $d \vee y = 1$ where $c \le y < 1$. Since $F_n(L) \cong (n]^d \times [n)$, so $F_n(L)$ is disjunctive if and only if $(n]$ is dual disjunctive and $[n)$ is disjunctive . By [10] we also know that $F_n(L)$ is disjunctive if and only if $< a >_n = < a >_n^{**}$ for each $a \in L$.

Following result is a generalization of [11 , Proposition 3.3]

Proposition 6. 1. 3 *For a distributive lattice* L *with a central element* n, *if* $P_n(L)$ *is disjunctive and normal then* $P_n(L)$ *is dual isomorphic to* $A_n(L)$. *Hence* $0, 1 \in L$ *if and only if there is an element* $d \in L$ *such that* $< d >_n^* = \{n\}$.

Proof : Since n is central, so $F_n(L) = P_n(L)$. Define

$\varphi : P_n(L) \to A_n(L)$ by $\varphi(< a >_n) = < a >_n^*$. Then for $< a >_n , < b >_n \in P_n(L)$,

$$\varphi(< a >_n \vee < b >_n) = \varphi([a \wedge b \wedge n,\ a \vee b \vee n]) = \varphi(< t >_n) = < t >_n^*,$$

where t is the relative complement of n in $[a \wedge b \wedge n,\ a \vee b \vee n]$.

On the other hand

$$\varphi(< a >_n) \cap \varphi(< b >_n) = < a >_n^* \cap < b >_n^* = (< a >_n \vee < b >_n)^*$$

$= [a \wedge b \wedge n, \quad a \vee b \vee n]^* = <t>_n{}^*$.

Therefore, $\varphi(<a>_n \vee <b>_n) = \varphi(<a>_n) \cap \varphi(<b>_n)$. Again as $P_n(L)$ is normal, so by Theorem 3.2.13

$\varphi\ (<a>_n \cap <b>_n) = \varphi(<m(a,n,b)>_n) = <m(a,n,b)>_n{}^*$

$= (<a>_n \cap <b>_n)^* = <a>_n{}^* \vee <b>_n{}^* = \varphi(<a>_n) \vee \varphi(<b>_n)$ Therefore, φ is a dual homomorphism from $P_n(L)$ onto $A_n(L)$. Now let $\varphi(<a>_n) = \varphi(<b>_n)$. Then $<a>_n{}^* = <b>_n{}^*$ and so $<a>_n{}^{**} = <b>_n{}^{**}$. Thus by [10], $<a>_n = <b>_n$ as $P_n(L)$ is disjunctive . Hence $P_n(L)$ is dual isomorphic to $A_n(L)$.

Finally if $0, 1 \in L$, Then $[0,1]$ is the largest element of $P_n(L)$, and so from the dual isomorphism $A_n(L)$ has a smallest element. Then by proposition 6.1.1, there is an element $d \in L$ such that $<d>_n{}^* = \{n\}$. Conversely if for some $d \in L$, $<d>_n{}^* = \{n\}$, Then $A_n(L)$ has a smallest element and so $P_n(L)$ has a largest element, which implies $0, 1 \in L$. ■

Following result is due to [11, Proposition 2.5]

Proposition 6. 1.4 *The lattice of annulets of a generalized Stone lattice* L *is a relatively complemented sublattice of the lattice of ideals of* L. ■

We generalize the above result for n- annulets.

Theorem 6. 1.5. *Let* n *be a central elements of a distributive lattice* L . *If* $P_n(L)$ *is generalized Stone, then* $A_n(L)$ *is a relatively complemented sublattice of* $I_n(L)$

Proof : Since every generalized Stone lattice is normal, so by Proposition 6. 1.2, $A_n(L)$ is a sublattice of $I_n(L)$. We therefore write $\vee$ for $\underline{\vee}$. Since $A_n(L)$ is a distributive lattice with largest element L, so $A_n(L)$ will be relatively complemented if and only if each interval of the form $[I, L]$, $I \in A_n(L)$ is complemented. Thus let $J = \left[<x>_n^*, L \right]$ be an interval in $A_n(L)$ and let $<y>_n^* \in J$. As $P_n(L)$ is generalized Stone, so by $<y>_n^* \vee <y>_n^{**} = L$ and $<y>_n^* \cap <y>_n^{**} = \{n\}$ always holds. Hence $(<x>_n \cap <y>_n^*) \vee \left(<x>_n \cap <y>_n^{**}\right) = <x>_n$. and $(<x>_n \cap <y>_n^*) \cap (<x>_n \cap <y>_n^{**}) = \{n\}$. Then by [15, Theorem 3.5] both $<x>_n \cap <y>_n^*$ and $<x>_n \cap <y>_n^{**}$ are principal. Let $<x>_n \cap <y>_n^* = <a>_n$. Then $<a>_n \subseteq <x>_n$ and so $<x>_n^* \subseteq <a>_n^*$. Thus $<a>_n^* \in J$. Also, $<a>_n \subseteq <y>_n^*$ implies $<y>_n^{**} \subseteq <a>_n^*$, and so $<a>_n^* \vee <y>_n^* = L$. Now $<a>_n^* \cap <y>_n^* \cap <x>_n = <a>_n^* \cap <a>_n = \{n\}$ implies $<a>_n^* \cap <y>_n^* \subseteq <x>_n^*$. But $<x>_n^* \subseteq <y>_n^*$, $<a>_n^*$.

Hence $<a>_n^* \cap <y>_n^* = <x>_n^*$ and so $<a>_n^*$ is the required relative complement of $<y>_n^*$ in J. ■

Consider the interval $I = [n, x],\ n < x$ in a distributive lattice. For any $a \in I$ we define $(a]^{+} = \{ s \in I \mid s \wedge a = n \}$. This is of course an ideal in I and is the annihilator of a with respect to I. Dually for $b \in J = [y, n]$, we define $[b)^{+d} = \{ t \in J \mid t \vee b = n$. It is easy to check that this is a filter in J and is the dual annihilator of b with respect to J. Clearly both I and J are also n-ideals. Similarly we define (i) For any $x \in (n],\ [x)^{+d} = \{ t \leq n \mid t \vee x = n \}$ and (ii) For any $x \in (n],\ (x]^{+} = \{ t \geq n \mid t \wedge x = n \}$.

Following lemmas are needed for the proof of next two results.

Lemma 6. 1.6 *Let* L *be a distributive lattice and* $x \in (n]$ *. Let* $a < n < b$*. Then for any* $x \in L$

(i) $< x >_n^* \cap [a, n] = [a \vee (x \wedge n))^{+d}$, dual annihilator with respect to [a, n] .

(ii) $< x >_n^* \cap [n, b] = ((x \vee n) \wedge b]^{+}$, annihilator with respect to [n, b].

Proof: (i) Let $p \in < x >_n^* \cap [a, b]$. Then $a \leq p \leq n$ and $m(p, n, x) = n$.Thus

$n=(p\vee x)\wedge(p\vee n)\wedge(x\vee n)=(p\vee x)\wedge n$

$=p\vee(a\vee(x\wedge n))$, and so $p\in[a\vee(x\wedge n))^{+d}$

Conversely, let $p\in[a\vee(x\wedge n))^{+d}$. Then $p\vee a\vee(x\wedge n)=n$ and so $p\vee(x\wedge n)=n$ as $a\le p\le n$. Thus,

$n=(p\vee x)\wedge(p\vee n)=(p\vee x)\wedge n=(p\vee x)\wedge n\wedge(x\vee n)$

$=(p\vee x)\wedge(p\vee n)\wedge(x\vee v)=m(p,n,x)$, which implies $p\in<x>_n^*$ and so $p\in<x>_n^*\cap[a,n]$. This proves (i) . A dual proof of (i) proves (ii) .

Similarly we have the following result.

Lemma 6. 1.7. *Let* L *be a distributive lattice and* $n\in L$ *Then for any* $x\in L$,

(i) $<x>_n^*\cap(n]=[x\wedge n)^{+d}$ in $[n)$

and (ii) $<x>_n^*\cap[n)=(x\vee n]^+$ in $[n)$. ■

Lemma 6. 1.8. *Let* L *be a distributive lattice and* $n\in L$.

(i) *Suppose* $I=[n,x]; n<x$. *Then for any* $a,b\in I$, $(a]^+\subseteq(b]^+$ *implies* $<a>_n^*\subseteq<b>_n^*$.

(ii) *Suppose* $J=[y,n]; y<n$. *Then for any* $a,b\in J$, $[a)^{+d}\subseteq[b)^{+d}$ *implies* $<a>_n^*\subseteq<b>_n^*$

Proof : (i) Let $p \in <a>_n^*$. Then $m(p, n, a) = n$ which implies

$(a \wedge b) \vee n = n$ Now $(p \vee n) \wedge x \in I$, and

$a \wedge [(p \vee n) \wedge x] = (a \wedge p \wedge x) \vee (a \wedge x \wedge n) = (a \wedge p) \vee n = n$

This implies $(p \vee n) \wedge x \in (a]^+ \subseteq (b]^+$, and so

$(p \vee n) \wedge x \wedge b = n$. Thus, $(p \vee n) \wedge b = n$.

Therefore, $n = (p \vee n) \wedge b = (p \vee n) \wedge (b \vee n) \wedge (p \vee b) = m(p, n, b)$,

and so, $p \in <b>_n^*$. Hence $<a>_n^* \subseteq <b>_n^*$. A dual proof of above proves

(ii) ■

A lattice L with 0 is called *quasi-complemented* if for each $x \in L$ there exists

$x' \in L$ such that $x \wedge x' = 0$ and $(x \vee x']^* = (0]$, that is

$(x]^* \cap (x']^* = (0]$. This is also equivalent to the condition that for each

$x \in L$, there exists $x' \in L$ such that $(x]^{**} = (x']^*$.

Dually we can define a dual quasi-complemented lattice L with 1.

Since $F_n(L) \cong (n]^d \times [n)$, so we have

Corollary 6. 1. 9. *If* L *is a lattice with* $n \in L$, *then*

(i) $F_n(L)$ *is quasi-complemented if and only if* $(n]$ *is dual quasi-complemented and* $[n)$ *is quasi-complemented.*

(ii) $F_n(L)$ *is sectionally quasi –complemented if and only if* $(n]$ *is sectionally dual quasi-complemented and* $[n)$ *is sectionally quasi-complemented* . ■

The following theorem is a generalization of [11, Proposition 2.7].

Theorem 6. 1. 10 *Let* L *be a distributive lattice with* n *as a central elements. Then* $A_n(L)$ *is relatively complemented if and only if* $P_n(L)$ *is sectionally quasi-complemented.*

Proof : Suppose $A_n(L)$ is relatively complemented . Let I = [n, x]

.Consider $a \in I$ .Then $\langle x \rangle_n^* \subseteq \langle a \rangle_n^* \subseteq \{n\}^* = L$.

Since $\left[\langle x \rangle_n^*, L \right]$ is complemented in $A_n(L)$,

there exists $w \in L$ such that $\langle a \rangle_n^* \cap \langle w \rangle_n^* = \langle x \rangle_n^*$ and

$\langle a \rangle_n^* \veebar \langle w \rangle_n^* = L$.

Now, $L = \langle a \rangle_n^* \veebar \langle w \rangle_n^* = \left(\langle a \rangle_n^{**} \cap \langle w \rangle_n^{**} \right)^* = \left(\langle a \rangle_n \cap \langle w_n \rangle \right)^{***}$.

$= \left(\langle a \rangle_n \cap \langle w \rangle_n \right)^*$ This implies $\langle a \rangle_n \cap \langle w \rangle_n = \{n\}$, and so

$\langle a \rangle_n \cap \langle w \rangle_n \cap \langle x \rangle_n = \{ n \}$.

But $\langle w \rangle_n \cap \langle x \rangle_n = \langle (w \wedge x) \vee (w \wedge n) \vee (x \wedge n) \rangle_n = \langle (w \wedge x) \vee n \rangle_n$.

Thus , $n = (a \wedge n) \vee (a \wedge ((w \wedge x) \vee n)) \vee (((w \wedge x) \vee n) \wedge n)$

$= a \wedge ((w \wedge x) \vee n)$, where $(w \wedge x) \vee n \in I$. On the other hand

$< a >_n^* \cap < w >_n^* = < x >_n^*$ implies

$< a >_n^* \cap < w >_n^* \cap < x >_n^* = \{ n \}$ and so by

Lemma 6.1.6 , $(a]^{+} \cap ((w \wedge x) \vee n]^{+} = \{n\}$. This implies I is quasi-complemented and so $[n)$ is sectionally quasi-complemented.

A dual proof of above shows that $(n]$ is sectionally dual quasi-complemented, and so by Corollary 6. 1.9 , $P_n(L)$ is sectionally quasi-complemented.

Conversely suppose $F_n(L) = P_n(L)$ is sectionally quasi-

complemented. Since $A_n(L)$ is distributive, if suffices to prove that the interval $[<x>_n^*, L]$ is complemented for each $x \in L$. Let $<y>_n^* \in <$ $y>_n^* \in [<x>_n^*, L]$

Then $<y>_n^* = <x>_n^* \underline{\vee} <y>_n^* = <m(x, n, y)>_n^*$.

Now Consider $I = [n, x \vee n]$ in $[n)$. Then $(x \vee n) \wedge (y \vee n) \in I$. Since by Corollary 6.1.9. I is quasi-complemented, so there exists $w \in I$ such that $w \wedge (x \vee n) \wedge (y \vee n) x = n$ and

$(w]^{+} \cap ((x \vee n) \wedge (y \vee n)]^{+} = \{n\} = (x \vee n]^{+}$.

Thus by lemma 6.1.8 , $\langle w \vee ((x \vee n) \wedge (y \vee n)) \rangle_n^* = \langle x \vee n \rangle_n^*$,

and so $\langle w \rangle_n^* \cap \langle (x \vee n) \wedge (y \vee n) \rangle_n^* = \langle x \vee n \rangle_n^*$.

Dually considering the interval $[x \wedge n, n]$ in $(n]$ and using same argument there exist $v \in [x \wedge n, n]$ such that $v \vee (x \wedge n) \vee (y \wedge n) = n$ and

$<v>_n^* \cap <(x \wedge n) \vee (y \wedge n)>_n^* = <x \wedge n>_n^*$;

Then $[v, w]^* \cap \langle y \rangle_n^* = [v, w]^* \cap \langle m(x, n, y) \rangle_n^*$

$=[v, w]^* \cap [\, m(x, n, y) \wedge n,\ \ m(x, n, y) \vee n]^*$

$= \langle v \rangle_n^* \cap \langle w \rangle_n^* \cap \langle m(x,n,y) \wedge n \rangle_n^* \cap \langle m(x,n,y) \vee n \rangle_n^*$

$= \langle v \rangle_n^* \cap \langle w \rangle_n^* \cap \langle (x \wedge n) \vee (y \wedge n) \rangle_n^* \cap \langle (x \vee n) \wedge (y \vee n) \rangle_n^*$

$= \langle x \wedge n \rangle_n^* \cap \langle x \vee n \rangle_n^* = [\, x \wedge n, x \vee n\,]^* = \langle x \rangle_n^*$

Also, $[\, v, w\,]^* \veebar \langle y \rangle_n^* = [\, v, w\,]^* \veebar \langle m(x, n, y) \rangle_n$

$= ([v, w] \cap [\,(x \wedge n) \vee (y \wedge n),\ (x \vee n) \wedge (y \vee n)])^*$

$= [v \vee (x \wedge n) \vee (y \wedge n), w \wedge (x \vee n) \wedge (y \vee n)]^* = \{n\}^* = L.$

Since n is central , so $[v, w] = \langle t \rangle_n$ where $\langle t \rangle_n^* \in [\langle x \rangle_n^*, L$, which is the required relative complement of $\langle y \rangle_n^*$. ■

In [11] Cornish has proved that if L is a distributive lattice with 0 then L is quasi-complemented if and only if $A_0(L)$ is a Boolean subalgebra of $A(L)$. But we are unable to get such a result for $A_n(L)$ when $P_n(L)$ is quasi-complemented. We could not prove that there exists $d \in L$ with $\langle d \rangle_n^* = \{n\}$, when $P_n(L)$ is quasi-complemented . We leave it to the reader as an open question. "Does $A_n(L)$ possess a smallest element when $P_n(L)$ is a quasi-complemented lattice with n as a central element?"

But following the same technique of proof of theorem 6.1.10, we can establish the following result.

Theorem 6. 1. 11 *Let* L *be a distributive lattice with* n *as a central element. Then*

(i) *If* $A_n(L)$ *is Boolean then* $P_n(L)$ *is quasi-complemented.*

(ii) *If* $P_n(L)$ *is quasi-complemented and* $A_n(L)$ *has a smallest element, then* $A_n(L)$ *is Boolean.* ■

By [9] we know that a distributive lattice with 0 is a generalized Stone lattice if and only if it is both normal and sectionally quasi-complemented.

So we conclude this section with the following result. This also gives nice characterization of $P_n(L)$ which are generalized Stone.

Theorem 6.1.12. *Let* L *be a distributive lattice with* n *as a central element. Then* $P_n(L)$ *is generalized Stone if and only if* $A_n(L)$ *is a relatively complemented sublattice of* $I_n(L)$.

Proof : Suppose $P_n(L)$ is generalized Stone. Then it is normal and sectionally quasi- complemented. Thus by Proposition 6.1.2 and Theorem 6.1.10 $A_n(L)$ is a relatively complemented sublattice of $I_n(L)$.

Conversely if $A_n(L)$ is a relatively complemented sublattice of $I_n(L)$, then again by proposition 6.1.2 and Theorem 6.1.10 $P_n(L)$ is normal and sectionally quasi-complemented . Therefore, by [9, Theorem 5.7] L is generalized Boolean. ■

2. α-n-ideals in a distributive lattice

Recall that for an ideal J in a distributive lattice L with 0, $\alpha\ (J) = \{(x]^{*} | x \in J\}$, which is filter in $A_0(L)$ and conversely $\alpha^{\leftarrow}(F) = \{x, \in L \mid (x]^{*} \in F$ is an ideal in L when F is any filter in $A_0(L)$. Clearly for any ideal I, $I \subseteq \alpha^{\leftarrow}\alpha\ (I)$. An ideal I is called and α-*ideal* if $I = \alpha^{\leftarrow}\alpha\ (I)$. Now for any n-ideal J in a distributive lattice with a central element n, we define $\alpha(J) = \{<x>_n^{*} \mid x \in J$ and conversely $\alpha^{\leftarrow}(F) = \{x \in L \mid <x>_n^{*} \in F$ where F is any filter in $A_n(L)$. We start this section with the following result.

Proposition 6. 2. 1. *Let* L *be a distributive lattice and* $n \in L$ *is central. Then.*

(a) *For any* n-*ideal* J, $\alpha(J)$ *is a filter in* $A_n(L)$

(b) $\alpha^{\leftarrow}(F)$ *is an* n-*ideal in* L *where* F *is any filter of* $A_n(L)$

(c) *If* I_1, I_2 *are* n-*ideals then* $I_1 \subseteq I_2$ *implies that* $\alpha(I_1) \subseteq \alpha(I_2)$;

and if F_1, F_2 *are filters in* $A_n(L)$, *then* $F_1 \subseteq F_2$ *implies*

$\alpha^{\leftarrow}(F_1) \subseteq \overset{\leftarrow}{\alpha}(F_2)$.

(d) *For any filter* F *of* $A_n(L)$, $\alpha\ \ \alpha^{\leftarrow}(F) = F$.

(e) *The map* $I \to \alpha^{\leftarrow}\alpha(I) = [\alpha^{\leftarrow}(\alpha(I))$ *is a closure operation on the lattice of* n-*ideals. That is*

(i) $\alpha^{\leftarrow}\alpha\left(\alpha^{\leftarrow}\alpha(I)\right)=\alpha^{\leftarrow}\alpha(I)$

(ii) $I\subseteq\alpha^{\leftarrow}\alpha(I)$

(iii) $I\subseteq J$ *implies* $\alpha^{\leftarrow}\alpha(I)\subseteq\alpha^{\leftarrow}\alpha(J)$; *for any* n-*ideals* I *and* J *in* L.

Proof: (a) Let $<x>_n^*, <y>_n^* \in \alpha(J)$ with $x, y \in J$.

Then $<x>_n^* \cap <y>_n^* = \left(<x>_n \vee <y>_n\right)^* =$

$[x\wedge y\wedge n, x\vee y\vee n]^* = <t>_n^*$ as n is central . Then

$x\wedge y\wedge n = t\wedge n \le t \le t\vee n = x\vee y\vee n$ implies $t\in J$ by

convexity. Hence $<x>_n^* \cap <y>_n^* \in \alpha(J)$.

Now suppose $<x>_n^* \in \alpha(J)$, $x\in J$ and $\langle s\rangle_n^* \supseteq \langle x\rangle_n^*$ for some

$\langle s\rangle_n^* \in A_n(L)$. Then $<s>_n^* = <s>_n^* \underline{\vee} <x>_n^* = <m(s,n,x)>_n^*$

and $x\wedge n\le m(s,n,x)\le x\vee n$ implies by conversely that

$m(s,n,x) \in J$. Hence $<s>_n^* \in \alpha(J)$. Therefore $\alpha(J)$ is a filter.

(b) Since L is the largest element of $A_n(L)$, so $L\in F$. Then $L=\{n\}^*$

implies $n\in\alpha^{\leftarrow}(F)$. Now Let $x<t<y$ with $x, y\in\alpha^{\leftarrow}(F)$. Then

. $<x>_n^*, <y>_n^* \in F$. Thus, $<x>_n^* \cap <y>_n^* = [x\wedge y\wedge n,\ x\vee y\vee n]^*$

$\in F$ as F is a filter.

Moreover, $x\wedge y\wedge n\le t\wedge n\le t\vee n\le x\vee y\vee n$ implies

$<t>_n \subseteq [x \wedge y \wedge n, \quad x \vee y \vee n]$ and so

$<t>_n^* \supseteq [x \wedge y \wedge n, \quad x \vee y \vee n]^*$. Thus $<t>_n^* \in F$ as F is filter.

Therefore, $x \in \alpha^{\leftarrow}(F)$ and so $\alpha^{\leftarrow}(F)$ is convex. Now let $x \in \alpha^{\leftarrow}(F)$.

Then $x \wedge n \leq x$ implies $<x \wedge n>_n^* \supseteq <x>_n^*$. Thus $<x \wedge n>_n^* \in F$ as F is a filter, and so $x \wedge n \in \alpha^{\leftarrow}(F)$. Similarly $x \vee n \in \alpha^{\leftarrow}(F)$. Then for any $x, y \in \alpha^{\leftarrow}(F), <x>_n^* \cap <y>_n^* = [x \wedge y \wedge n, x \vee y \vee n]^* \in F$ and $[x \wedge y \wedge n, x \vee y \vee n] = <s>_n$ as n is central. Thus $s \in \alpha^{\leftarrow}(F)$ and so $s \wedge n, \quad s \vee n \in \alpha^{\leftarrow}(F)$. Therefore by convexity $s \wedge n = x \wedge y \wedge n \leq x \wedge y \leq x \vee y \leq x \vee y \vee n = s \vee n$ implies $x \wedge y, x \vee y \in \alpha^{\leftarrow}(F)$.

Hence $\alpha^{\leftarrow}(F)$ is an n-ideal of L.

(c) This is trivial .

(d) Suppose $<x>_n^* \in F$. Then $x \in \alpha^{\leftarrow}(F)$ and so $<x>_n^* \in \alpha(\alpha^{\leftarrow}(F))$.Therefore, $F \subseteq \alpha(\alpha^{\leftarrow}(F))$. Conversely, let $<x>_n^* \in \alpha(\alpha^{\leftarrow}(F))$. Then $<x>_n^* = <y>_n^*$ for some $y \in \alpha^{\leftarrow}(F)$. Thus $<y>_n^* \in F$, and so $<x>_n^* \in F$. That is $\alpha\left(\overleftarrow{\alpha}(F)\right)$, and so $F = \alpha(\alpha^{\leftarrow}(F)) \subseteq F$.

(e) (i) Since $\alpha(I)$ is a filter in $A_n(L)$, so by (d) $\alpha\, \alpha^{\leftarrow}(\alpha(I)) = \alpha(I)$. Therefore $\alpha^{\leftarrow}(\alpha\, \alpha^{\leftarrow}(\alpha(I))) = \alpha^{\leftarrow}\, \alpha(I)$. That is,
$\alpha^{\leftarrow}\, \alpha(\alpha^{\leftarrow}\, \alpha(I)) = \alpha^{\leftarrow}\, \alpha(I)$.

(ii) is obvious and (iii) follows from (c) ■

An n-ideal I is called an α-n-*ideal* if $\alpha^{\leftarrow}\ \alpha(I)=I$. Thus α-n-ideals are simply the closed elements with respect to the closure operation of proposition 6. 2.1.

Following result is a generalization of [6, Proposition 2.3] in terms of n-ideals.

Proposition 6. 2. 2. *Let* L *be a distributive lattice with* n *as a central element. Then* α-n-*ideals of* L *form a complete distributive lattice isomorphic to the lattice of filters, ordered by set inclusion, of* $A_n(L)$.

Proof : Let $\{I_i\}$ be any class of α-n-ideal of L. Then $\alpha^{\leftarrow}\ \alpha(I_i)=I_i$ for all I . By proposition 6.2.1, $\bigcap I_i \subseteq \alpha^{\leftarrow}\ \alpha\ (\bigcap I_i)$. Again $\alpha^{\leftarrow}\alpha\ (\bigcap I_i)\subseteq \alpha^{\leftarrow}\alpha\ (I_i)=I_i$ for each i. Thus $\alpha^{\leftarrow}\ \alpha\ (\bigcap I_i)\subseteq \bigcap I_i$, and so $\alpha^{\leftarrow}\ \alpha\ (\bigcap I_i)=\bigcap I_i$. Hence $\bigcap I_i$ is an α-n-ideal . Therefore, by [15, Lemma 14, P-14], the set of α-n-ideals is a complete lattice, and it is distributive as L is so. Now α is onto and both α, α are isotones by proposition 6.2.1 (c) . Moreover, for α-n-ideal I, $\alpha^{\leftarrow}\ \alpha\ (I)=I$ and by proposition 6.2.1 (d) $\alpha\ \alpha^{\leftarrow}\ (F)=F$ for any filter F of $A_n(L)$. Therefore the map α is an isomorphism from the lattice of α-n-ideal to the lattice of filters of $A_n(L)$.

Following theorem gives a nice characterization of α-n-ideals which also generalizes [11, Proposition 3.3].

Theorem 6. 2. 3. *For a central element* n *of a distributive lattice* L, *the following conditions are equivalent .*

(i) I *is an* α-n-*ideal*

(ii) *For* $x, y \in L, <x>_n^{*} = <y>_n^{*}$ *and* $x \in I$ *implies* $y \in I$.

(iii) $I = \bigcup_{x\in I} <x>_n^{**}$ *where* $\bigcup$ *is set theoretic union.*

Proof: (i) $\Rightarrow$ (ii) . Suppose I is an α-n-ideal. Then $\alpha^{\leftarrow}\alpha\ (I)=I$. Let $x, y \in L$ with $<x>_n{}^* = <y>_n{}^*$ and $x \in I$. Then $<x>_n{}^* \in \alpha(I)$ and so $<y>_n{}^* \in \alpha(I)$. This implies $y \in \alpha^{\leftarrow}\ (\alpha(I)) = I$

(ii) $\Rightarrow$ (i). Suppose (ii) holds and I is any n-ideal. $I \subseteq \alpha^{\leftarrow}\ \alpha(I)$ always holds. Thus suppose $x \in \alpha^{\leftarrow}\ \alpha\ (I)$. Then $<x>_n^* \in \alpha(I)$. This implies $<x>_n{}^* = <y>_n{}^*$ for some $y \in I$. Then by (ii), $x \in I$. Therefore, $\alpha^{\leftarrow}\alpha\ (I) \subseteq I$ and so $\alpha^{\leftarrow}\alpha\ (I) = I$; in other words I is an α-n-ideal.

(ii) $\Rightarrow$ (iii). Clearly $I \subseteq \bigcup\ <x>_n{}^{**}$. Now let $x \in I$ and $y \in <x>_n{}^{**}$. Then $<x>_n{}^* \subseteq <y>_n{}^*$. Thus

$<y>_n{}^* = <x>_n{}^* \veebar <y>_n{}^* = <m\ (x, n, y)>_n{}^*$.

Since $x \in I$ so by convexity $x \wedge n \le m(x, n, y) \le x \vee n$ implies $m\ (x, n, y) \in I$. Hence by (ii) $y \in I$ which implies $<x>_n{}^{**} \subseteq I$ and so $\bigcup_{x \in I} <x>_n{}^{**} \subseteq I$. Therefore, (iii) holds.

(iii) $\Rightarrow$ (ii). Suppose (iii) holds and $<x>_n{}^* = <y>_n{}^*$ with $x \in I$. Then $<x>_n{}^{**} = <y>_n{}^{**}$. This implies $y \in <y>_n{}^{**} = <x>_n{}^{**}$. Hence by (iii) , $y \in \bigcup_{x \in I} <x>_n{}^{**} = I$ and so (ii) holds . ■

Proposition 6. 2. 4. *For a central element* n *of a distributive lattice every minimal prime* n-*ideal is an* α-n-*ideal* .

Proof : Let P be a minimal prime n-ideal. Suppose $x \in \alpha^{\leftarrow} \alpha(P)$.

Then $< x >_n^* \in \alpha(P)$. So $< x >_n^* = < y >_n^*$ for some $y \in P$. Since P is minimal, so by 3.1.4 $< y >_n^{**} \subseteq P$. Thus, $< x >_n^{**} \subseteq P$, this implies $x \in < x >_n^{**} \subseteq P$. Therefore, $\alpha^{\leftarrow} \alpha(P) \subseteq P$. Since the reverse inclusion is trivial, so $\alpha^{\leftarrow} \alpha(P) = P$. Hence P is α-n-ideal . ■

Recall that for a prime n-ideal P of a distributive lattice L

$n(P) = \{y \in L \mid m(y, n, x) = n \text{ for some } x \in L - P$.

Clearly $n(P)$ is an n-ideal and $n(P) \subseteq P$.

Proposition 6. 2. 5. *For a prime* n-*ideal* P, $n(P)$ *is an* α-n-*ideal* .

Proof : Let $x \in \alpha^{\leftarrow} \alpha(n(P))$. Then $< x >_n^* \in \alpha(n(P))$. Thus $< x >_n^* = < y >_n^*$ for some $y \in n(P)$. Then $m(y, n, t) = n$ for some $t \in L - P$. This implies $< y >_n \cap < t >_n = \{n\}$ and so $< t >_n \subseteq < y >_n^* = < x >_n^*$. Therefore, $< x >_n^{**} \subseteq < t >_n^*$. Thus , $x \in < x >_n^{**} \subseteq < t >_n^*$ which implies $m(x, n, t) = n$ and so $x \in n(P)$.

Hence $\overleftarrow{\alpha}\ \alpha\left(n((P))\subseteq P\right.$. Since the reverse inclusion is trivial , so

$\alpha^{\leftarrow}\alpha\left(n((P))\subseteq n(P)\right.$. Therefore $n(P)$ is an α-n-ideal . ■

Following Lemma is needed to prove our next theorem. Latif in [30] have given a characterization for $P_n(L)$ to be disjunctive. Here is a slight improvement of that result.

Lemma 6. 2. 6. *For a central element* n *of a distributive lattice* L, $P_n(L)$ *is disjunctive if and only if* $<x>_n^{*}=<y>_n^{*}$ *implies* $<x>_n=<y>_n$ *for* $x, y\in L$.

Proof : Suppose $<x>_n^{*}=<y>_n^{*}$ implies $<x>_n=<y>_n$. Since for $x\in L,\ <x>_n\subseteq<y>_n^{**}$ always holds, so suppose $y\in<x>_n^{**}$. Then $<y>_n^{*}\supseteq<x>_n^{*}$

Thus, $<x>_n^{*}=<x>_n^{*}\cap<y>_n^{*}=[x\wedge y\wedge n,\quad x\vee y\vee n]^{*}=<t>_n^{*}$; as n is central . Then by the given condition, $<x>_n=<t>_n$. Thus $<x>_n=[x\wedge y\wedge n,\quad x\vee y\vee n]^{*}$ and so by convexity , $y\in<x>_n$.

Therefore $<x>_n^{**}\subseteq<x>_n$, and so $<x>_n=<x>_n^{**}$. Hence by [30] , $P_n(L)$ is disjunctive.

Conversely , Let $P_n(L)$ be disjunctive. Then for each $x \in L$,

$<x>_n = <x>_n^{**}$. Therefore, for $x, y \in L$, $<x>_n = <y>_n$ implies

$<x>_n^{**} = <y>_n^{**}$ and so $<x>_n = <y>_n$. ■

Following result is a generalization of [11, Proposition 3.4] .

Theorem 6. 2. 7. *Let* L *be a distributive lattice with* n *as a central element. Then the following condition are equivalent .*

(i) *Each prime* n-*ideal is an* α -n-*ideal*

(ii) *Each* n-*ideal is an* α -n-*ideal*

(iii) $P_n(L)$ *is disjunctive.*

Proof : (i) $\Rightarrow$ (ii) . Suppose I is any n-ideal . Then by [30, Corollary 2.2.6], $I = \bigcap\{P \mid P \supseteq I$, P prime n-ideal $\}$ .Then

$\alpha^{\leftarrow}\alpha(I) = \alpha^{\leftarrow}\alpha\ [\bigcap\{P \mid P \supseteq I]$

$= \cap\{\alpha^{\leftarrow}\alpha\ (P) \mid P \supseteq I\} = \bigcap\{P \mid P \supseteq I\} = I$ (by (i) .

Therefore (ii) holds.

(ii) $\Rightarrow$ (i) is trivial .

(ii) $\Rightarrow$ (iii) . Suppose $<x>_n^{*} = <y>_n^{*}$ for $x, y \in L$. Since by (ii) $<x>_n$ is an α -n-ideal and $x \in <x>_n$, so by theorem 6.2.3. $y \in <x>_n$. Thus $<y>_n \subseteq <x>_n$ Similarly $<x>_n \subseteq <y>_n$. Therefore $<x>_n = <y>_n$, and so by lemma 6.2.6, $P_n(L)$ is disjunctive.

(iii) $\Rightarrow$ (ii). Let I be an n-ideal. Suppose $x \in \alpha^{\leftarrow}\alpha(I)$. Then $<x>_n^* \in \alpha(I)$ and so $<x>_n^* = <y>_n^*$ for some $y \in I$. Thus by (iii), $<x>_n = <y>_n$, which implies $x \in I$. Therefore, $\alpha^{\leftarrow}\alpha(I) \subseteq I$. Since the reverse inclusion is trivial, so $\alpha^{\leftarrow}\alpha(I) = I$ and hence I is an α-n-ideal.

Proposition 6.2.2. implies that there is an order isomorphism between the prime α-n-ideals of L and the prime filter of $A_n(L)$. It is not hard to show that each α-n-ideal is an intersection of prime α-n-ideals.

Following result is well known in Lattice theory, It was proved for bounded lattices in [40] and announced in general in [39] ; an explicit proof is given in [22, P-76].

Lemma 6. 2.8. *A distributive lattice with* 0 *is relatively complemented if and only if its every prime filter is an ultra filter (proper and maximal).*

Theorem 6. 2. 9. *Let* L *be a distributive lattice with* n *as a central element. Then the following conditions are equivalent.*

(i) $P_n(L)$ *is sectionally quasi-complemented*

(ii) *Each prime* α-n-*ideals is minimal prime* n-*ideal.*

(iii) *Each* α-n-*ideal is an intersection of minimal prime* n-*ideals.*

Moreover, the above conditions are equivalent to $P_n(L)$ *being quasi-complemented if and only if there is an element* $d \in L$ *such that* $<d>_n^* = \{n\}$.

Proof : (i) $\Rightarrow$ (ii) . Suppose $P_n(L)$ is sectionally quasi-complemented. Then by theorem 6.1.10 , $A_n(L)$ is relatively complemented. Hence its every prime filter is an ultra filter. Then by proposition 6. 2.2 each prime α-n-ideals is a minimal prime n-ideal .

(ii) $\Rightarrow$ (iii). From the isomorphism between the prime α-n-ideal of L and the prime filters of $A_n(L)$, we see that each α-n-ideal is an intersection of prime α-n-ideals. This shows (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). Suppose (ii) holds. Then by proposition 6. 2. 2, each prime filter of $A_n(L)$ is maximal. Then by Lemma 6.2.7, $A_n(L)$ is relatively complemented, and so by Theorem 6.1.10, $P_n(L)$ is sectionally quasi-complemented. Last part follows from [Cornish, 11]. ■

We conclude the thesis with the following result which is a generalization of [11 , Theorem 3.7].

Theorem 6. 2. 10. *Let* L *be a distributive lattice and* $n \in L$ *is central . Then* $P_n(L)$ *is generalized Stone if and only if each prime* n-*ideal contains a unique prime* α-n-ideal.

Proof: Since minimal prime n-ideals are α-n-ideals so by the given condition, every prime n-ideal contains a unique minimal prime n-ideal. Hence by [4] $P_n(L)$ is normal . Also by the given condition each prime α-n-ideal contains a unique prime α-n-ideal. Since each minimal prime n-ideal is a prime

α-n-ideal, so each prime α-n-ideal is itself a minimal prime n-ideal. Hence by Theorem 6.2.8, $P_n(L)$ is sectionally quasi-complemented. Therefore, by [9, Theorem 6.6] $P_n(L)$ is generalized Stone.

Coversely, if $P_n(L)$ is generalized Stone, Then by [47] , each prime n-ideal contains a unique minimal prime n-ideal. Thus the result follows as each minimal prime n-ideal is a prime α-n-ideals. ■

Recommendations and Application

Conclusion and Future recommendations: From the discussions of all previous chapters it can be concluded and recommended that the concept of n-ideals can be introduced in join-semilattices. Then using these results we can study those $P_n(L)$ which are normal, relatively normal, m-normal, and relatively m-normal, where L is a join-semilattice with 0. In other words all the works of this thesis can be extended for join semilattices.

Application: Lattice theory has a lot of applications in different fields. Boolean lattice has applications in the field of hardware and software development of computer science. Also it has wide applications in networking. It can be applied to develop theories in other branches of algebra, such as group theory, Ring and Modules etc.

One of the major applications of Boolean lattices is in the switching systems, which are network of switches that involve two state devices 0 and 1 for off and on respectively.

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