STUDY ON MODULAR LATTICE AND BOOLEAN ALGEBRA.

A Thesis

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To my parents ,

who have profoundly

influenced my life

Declaration

We hereby declare that the thesis entitled " Study on modular lattice and Boolean algebra " submitted for the partial fulfillment for the Master of Philosophy degree is done by the student (Roll No. M.Phil 0051507 , Session 2000-2001) himself and is not submitted elsewhere for any other degree or diploma .

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Contents

Summary

This thesis studies extensively the nature of modular lattices and Boolean algebras. The modular lattices have been study by several authors including Abbott [2], Birkhoff [3] and Rutherford [19]. A poset is said to form a lattice if for every $a , b \in L$, $a \vee b$ and $a \wedge b$ exists in L, where $\vee$, $\wedge$ are two binary operation .A lattice L is called modular lattice if for all $a , b , c \in L$ with $a \geq b$, $a \wedge (b \vee c) = [b \vee (a \wedge c)]$. In this thesis we give several results on modular lattices which certainly extend and generalized many result in lattice theory .

In chapter one we discuss ideals , complete lattices , relatively complemented lattices and other results on lattices which are basic to this thesis . If every interval in a lattice is complemented the lattice is said to be relatively complemented .

Chapter two discusses Embeddings , Kernels and dual homomorphisms . If L , M be two lattices , a one-one homomorphism $\theta : L \longrightarrow M$ is called an embedding mapping . Also in that case we say L is embedded in M . We prove that the definition of dual meet homomorphism and dual join homomorphism are equivalent .

In chapter three we discuss on modular lattices and distributive lattices . Distributive lattices have been studied by sever author including Cignoli [4] , Cornish [5] , Cornish and Hicman [6] and Evans [7] , Nieminen [15] , [16] . Hence we prove a lattice L is distributive if and only if

$$(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \quad \forall\ a , b , c \in L$$

In chapter four we discuss Boolean algebras and Boolean functions . Previously Boolean algebras , Disjunctive Normal forms and Conjunctive Normal forms have studied by Abbott [1] . Here we extend several result on Boolean Algebras and also find the DN form of the function whose CN form is

$$f=(x\vee y\vee z)\wedge(x\vee y\vee z')\wedge(x\vee y'\vee z)\wedge(x\vee y'\vee z')\wedge(x'\vee y\vee z).$$

"Lattices , Sublattices and Complete lattices "

1.1 Introduction

In this chapter we discuss Ideals , Complete lattices and Relatively Complemented lattices .Complete lattices and semilattices have been studied by several authors Papert [18] , Rozen [20] , Varlet [22] . A lattice L is called complete lattice if for every non empty subset of L has its *Sup* and *Inf* in L . In this chapter we also proved in any lattice the distributive inequalities

(i) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

(ii) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \wedge c)$.

1.2 Relations , Lattices , Complete Lattices .

Definition (Relation) : A relation R from A to B is a subset of $A \times B$.

Example 1.2.1 : Let $A = \{ x , y \}$

$B = \{ 2 , 4 , 6 \}$

Then $R = \{ (x , 2),(x , 6),(y , 4) \}$ is a relation from A to B .

Definition (Reflexive Relation) : Let R = (A,A P(x , y)) be a relation in a set A, i.e., let R be a subset of $A \times A$. Then R is called a reflexive relation if, for every $a \in A$,

$$(a , a) \in A .$$

In other words, R is reflexive if for every element in A is related to itself .

Example 1.2.2 : Let $Y = \{1,2,3,4,5\}$ and

$R = \{(1,1),(2,2),(2,3),(3,3),(4,4),(3,4),(5,5)\}$.

Then R is a reflexive relation .

Definition (Symmetric Relation) : Let R be a subset of $A \times A$ i.e. let R be a relation of in A . Then R is called a symmetric relation if

$$(a , b) \in R \text{ implies } (b , a) \in R$$

Example 1.2.3 : Let $S = \{1,2,3\}$ and let

$$R = \{(1,2),(1,3),(2,3),(2,1),(3,1),(3,2)\}$$

Then R is a symmetric relation .

Definition (Anti-Symmetric Relation) : A relation R in a set A i.e. a subset of $A \times A$ is called an anti-symmetric relation if

$$(a , b) \in R \text{ and } (b , a) \in R \text{ implies } a = b$$

In other words , if $a \neq b$ then possibly a is related to b or possibly b is related to a but never both .

Example 1.2.4 : Let A be a family of sets, and let R be the relation in A defined by " x is a sub set of y". Then R is anti-symmetric since $C \subseteq D$ and $D \subseteq C$ implies $C = D$.

Definition (Transitive Relation): A relation R in a set A is called a transitive relation if $(a , b) \in R$ and $(b , c) \in R$ implies $(a , c) \in R$

In other words, if a is related to b and b is related to c, then a is related to c .

Example 1.2.5 : Let $B = \{ a , b , c \}$ and let

$$R = \{ (a , b),(c , b),(b , a),(a , c) \}$$

Then R is not a transitive relation since

$$(c , b) \in R \text{ and } (b , a) \in R \text{ but } (c , a) \notin R .$$

Definition (Equivalence Relation) : A relation R in a set A is an equivalence relation if

(1) R is reflexive, that is, for every $a \in A$, $(a , a) \in R$.

(2) R is symmetric, that is, $(a , b) \in R$ implies $(b , a) \in R$.

(3) R is transitive, that is, $(a , b) \in R$ and $(b , c) \in R$

implies $(a, c) \in R$.

Example 1.2.6 : Let $X = \{ a, b, c \}$ be a set and let

$R = \{ (a, a),(a, b),(a, c),(b, a),(b, b),(b, c),(c, a),(c, b),(c, c) \}$

be a relation of $A \times A$ then the relation R is an equivalence relation, since

(1) R is reflexive, $(a, a),(b, b),(c, c) \in R$

(2) R is symmetric, $(a, b),(b, a),(a, c),(c, a) \in R$ and

(3) R is transitive, $(a, c),(c, b),(a, b) \in R$.

Definition (Partially ordered set) : A non empty set P, together with a binary relation R is said to form a partially ordered set or a poset if the following conditions hold :

P1: Reflexivity : $(a, a) \in R$ for all $a \in P$.

P2: Anti-Symmetry : If $(a, b) \in R$, $(b, a) \in R$ then $a = b$ $(a, b \in P)$.

P3: Transitivity : If $(a, b) \in R$, $(b, c) \in R$ then $(a, c) \in R$ $(a, b, c \in P)$.

Example 1.2.7 : The set N of natural numbers under divisibility forms a poset .Thus here $a \leq b$ means $a \mid b$ (a divides b) .

Definition (Greatest element) : Let P be a poset . If $\exists$ an element $a \in P$ such that $x \leq a$ for all $x \in P$ then a is called greatest or unity element of P .Greatest element if it exists , will be unique .

Definition (Least element) : An element $b \in P$ will be called least or zero element of P if $b \leq x \ \forall x \in P$. It is de noted by 0 . Least element if it exists , will be unique .

Example 1.2.8 : Let $A = \{1,2,3\}$. Then $(P(A), \subseteq)$ is a poset .

Let $B = \{\phi, \{1,2\},\{2\},\{3\}\}$

Then $(B, \subseteq)$ is a poset with ϕ as least element .B has no greatest element .

Let $C = \{\{1,2\},\{2\},\{3\},\{1,2,3\}\}$.

Then C has greatest element $\{1,2,3\}$, but no least element .

If $D = \{\phi,\{1\},\{2\},\{1,2\}\}$.

Then D has both least and greatest elements namely ϕ and $\{1,2\}$.

Again $E = \{\{1\},\{2\},\{1,3\}\}$ has neither least nor greatest element .

Definition (Bounded poset) : If a poset P has least and greatest elements we call it a Bounded poset . Indeed $0 \leq x \leq u \quad \forall\, x \in P$.

Definition (Upper bound) : Let S be non empty subset of a poset P . An element $a \in P$ is called an upper bound of S if $x \leq a$ $\forall\, x \in S$.

Definition (Least upper bound) : If a is an upper bound of S s.t., $a \leq b$ for all upper bounds b of S then a is called least upper bound (l.u.b) or supremum of S . We write *Sup* S for supremum S .

It is clear that there can be more than one upper bound of a set. But *Sup*, if it exists, will be unique.

Definition (Lower bound) : An element $a \in P$ will be call a lower bound of S if $a \leq x \quad \forall\, x \in S$.

Definition (Greatest lower bound) : If a is a lower bound of S then a will called greatest lower bound (g.l.b) or Infimum S (*Inf S*) if $b \leq a$ for all lower bounds b of S .

Example 1.2.9 : Let $(Z, \leq)$ be the poset of integers

Let $S = \{ \cdots -3, -2, -1, 0, 1, 2, 3 \}$ then $Sup\, S = 3$.

Again in the poset $(R, \leq)$ of real numbers if

$S = \{x \in R \mid x < 0, x \neq 0\}$ then $Sup\, S = 0$ (and it does not belong to S).

Definition (Chain) : If P is a poset in which every two members are comparable it is called a totally ordered set or a toset or a chain. Thus if P is a chain and $x, y \in P$ then either $x \leq y$ or $y \leq x$.

Fig. 1.1

Definition (Lattice) : A poset $(L, \leq)$ is said to form a lattice if for every $a, b \in L$ $Sup\,\{a, b\}$ and $Inf\,\{a, b\}$ exist in L.

In that case, we write

$Sup\,\{a, b\} = a \vee b$ (read a join b)

$Inf\,\{a, b\} = a \wedge b$ (read a meet b)

Other notation like $a + b$ and $a \cdot b$ or $a \cup b$ and $a \cap b$ are also used for $Sup\,\{a, b\}$ and $Inf\,\{a, b\}$.

Example1.2.10: Let A be a non empty set, then the post $(P(A), \subseteq)$ of all subset of A is a lattice .Here for $X, Y \in P(A)$.

$$X \wedge Y = X \cap Y \text{ and}$$

$$X \vee Y = X \cup Y$$

As a particular case, when A= {a , b}

P(A) = {ϕ , {a} , {b} , {a , b}}

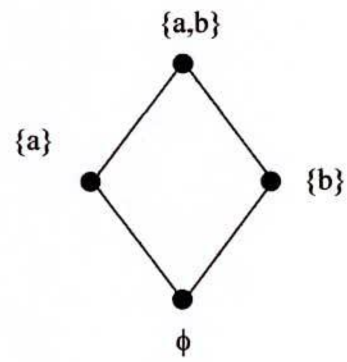

Fig. 1.2

Example 1.2.11 : The set N of all natural numbers under divisibility forms a lattice .Here

$$a \wedge b = g\,.\,c\,.\,d\,(a\,,b)$$

$$a \vee b = l\,.\,c\,.m\,(a\,,b) \qquad \text{for all } a\,,b \in N\,.$$

Example 1.2.12 : The set L = {1,2,4,5,10,20,25,50,100} of factors of 100 forms a lattice under divisibility .It is represented by the following diagram.

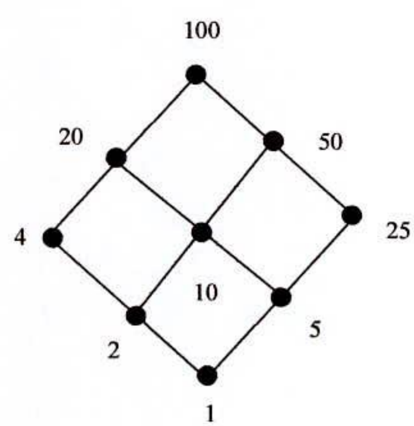

Fig. 1.3

Example 1.2.13 : Every chain is a lattice .Since any two elements x , y of chain are comparable , say $x \leq y$ we find

$$x \wedge y = Inf\{x , y\} = x \ , \quad x \vee y = Sup\ \{x ,y\} = y \ .$$

Definition (Meet-Semi Lattice) : A poset $(P, \leq)$ is called a meet-semi lattice if for all $a , b \in P$ $Inf\{a , b\}$ exists.

or

A non-empty set P together with a binary operation $\wedge$ is called a meet-semi lattice if $\forall$ a , b , c $\in$ P

(i) $a \wedge a = a$

(ii) $a \wedge b = b \wedge a$

(iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

Definition (Join-Semi Lattice) : A poset $(P, \leq)$ is called a join-semi lattice if for all $a , b \in P$ $Sup\{a , b\}$ exists.

or

A non-empty set P together with a binary operation $\vee$ is called a join-semi lattice if $\forall$ a , b , c $\in$ P

(i) $a \vee a = a$

(ii) $a \vee b = b \vee a$

(iii) $a \vee (b \vee c) = (a \vee b) \vee c$.

Definition (Sublattice) : A non-empty subset S of a lattice L is called a sublattice of L if $a , b \in S \Rightarrow a \wedge b , a \vee b \in S$.

Example 1.2.14 : Let $L = \{1,2,3,4,6,12\}$ of factors of 12 under divisibility forms a lattice. Then $A = \{1,2\}$ and $B = \{1,3\}$ are sublattice of L.

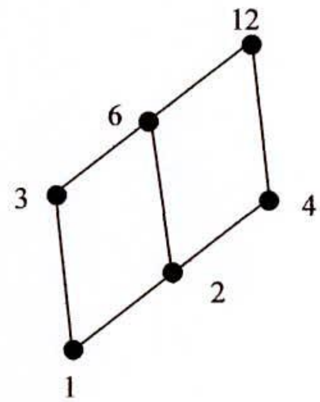

Fig. 1.4

Definition (Convex sublattice) : The subset D of the lattice L is called convex, if $a, b \in D$, $c \in L$ and $a \leq c \leq b$ imply that $c \in D$.

Example 1.2.15 : For $a, b \in L$ $a \leq b$ the interval $[a, b] = \{x \mid a \leq x \leq b\}$ is an important example of a convex sublattice.

Example 1.2.16 : For a chain C $a, b \in C$, $a \leq b$ we can also define the half-open intervals : $(a, b] = \{x \mid a < x \leq b\}$ and $[a, b) = \{x \mid a \leq x < b\}$ and the open interval : $(a, b) = \{x \mid a < x < b\}$. These are also examples of convex sublattices.

Example 1.2.17 : In the lattice $\{1,2,3,4,6,12\}$ under divisibility $\{1,6\}$ is a sublattice which is not convex as $2,3 \in [1,6]$ but $2,3 \notin \{1,6\}$. Thus $[1,6] \not\subseteq \{1,6\}$.

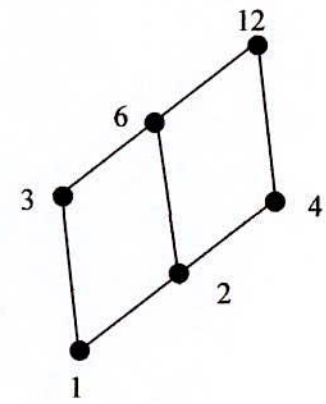

Fig. 1.5

Definition (Complete Lattice) : A lattice L is called a complete lattice if every non-empty subset of L has its *Sup* and *Inf* in L .

or

A lattice L is called a complete lattice if for any subset H of L, *Sup* H and *Inf* H exists in L.

Example 1.2.18 : Every finite lattice is complete.

Example 1.2.19 : The real interval [0,1] with usual ≤ form a complete lattice.

1.3 Ideals , Binary Operations , Dual Ideals .

Definition (Ideal) : A non-empty subset I of a lattice L is called an ideal of L if

(i) $i, j \in I \Rightarrow i \vee j \in I$

(ii) $i \in I,\ a \in L \Rightarrow a \wedge i \in I$

Example 1.3.1 : Let L = {1,2,4,8} be lattice of factors of 8 under divisibility. Then {1}, {1,2}, {1,4},{1,2,4,8} are all the ideals of L.

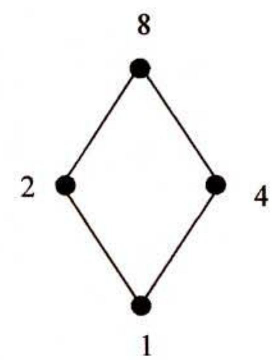

Fig. 1.6

Definition (Dual Ideal) : A non-empty subset I of a lattice L is called a dual ideal (or filter) of L if

(i) $i, j \in I \Rightarrow i \wedge j \in I$

(ii) $i \in I,\ a \in L \Rightarrow a \vee i \in I$

Example 1.3.2 : Let L = {1,2,4,8} be the lattice under divisibility.

Then A = {1,2} and {1,4} are ideals but not dual ideals.

B = {2,8} and {4,8} are dual ideals but not ideals.

C = {2,4} is neither an ideal nor a duel ideal.

Definition (Principal Ideal) : Let L be a lattice and $a \in L$ be any element. Let $(a] = \{x \mid x \leq a\}$, then $(a]$ forms an ideal of L. It is called principal ideal generated by a .

Definition (Principal dualideal) : Let L be a lattice and $a \in L$ be any element. The set $[a) = \{x \in L \mid a \leq x\}$ forms a dual ideal of L , called the principal dualideal generated by a.

Definition (Prime ideal) : An ideal P of a lattice L is called a prime ideal of L if P is properly contained in L and whenever $a \wedge b \in P$ then either $a \in P$ or $b \in P$.

or

An ideal P of a lattice L is called a prime ideal if for all $a , b \in L$, $a \wedge b \in P$ implies either $a \in P$ or $b \in P$.

Definition (Binary operation) : If a is a non-empty set then a map $f : A \times A \longrightarrow A$ is called a binary composition (or binary operation) on A .

Thus binary composition is a rule by which we combine any two member of the same set .

Multiplication is another familiar example of a binary operation on naturals or reals .

We use different symbols like $*$, o , $\oplus$ etc. for binary compositions.

If $*$ is a binary operation on a set A and $a , b \in A$ then by definition $a * b \in A$. We sometimes express this by saying that A is closed under $*$.

Example 1.3.3 : Let $A = \{0,1,2,3,4\}$. Define $\oplus$ on A by for $a, b \in A$, $a \oplus b$ means the remainder got by dividing $a+b$ by 5 . Then $\oplus$ will be a binary composition on A .The following table gives us all the values and since all the values of $a \oplus b$ for any a, b lie in A , we find A is closed under this composition .

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

In fact, the above is called addition modulo 5. One could generalize this on a set $A = \{0,1,2,\text{---------} \; n\text{-}1\}$ addition modulo n .

Definition (Algebraic Lattice) : A non-empty set L together with two binary compositions (operations) $\wedge$ and $\vee$ is said to form a lattice if $\forall\, a, b, c \in L$ the following conditions hold :

L1 : Idempotency : $a \wedge a = a$, $a \vee a = a$

L2 : Commutativity : $a \wedge b = b \wedge a$, $a \vee b = b \vee a$

L3 : Associativity : $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,

$a \vee (b \vee c) = (a \vee b) \vee c$

L4 : Absorption : $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$.

Definition (Duality) : Let O be a relation defined on a set A. Then converse of O (denoted by $\bar{O}$) is defined $a \bar{O} b \Leftrightarrow b\, O\, a$, $a, b \in A$.

Definition (Dual) : If (A, O) be a poset then the poset $(\bar{A}, \bar{O})$, where

$\bar{A} = A$ and $\bar{O}$ is converse of O is called **dual** of A.

Problem 1.3.1 : If we consider the two chains with diagram

$C_1 = \{0,1,2\}$ and $C_2 = \{0,1\}$ then $C_1 X\, C_2$ is a lattice .

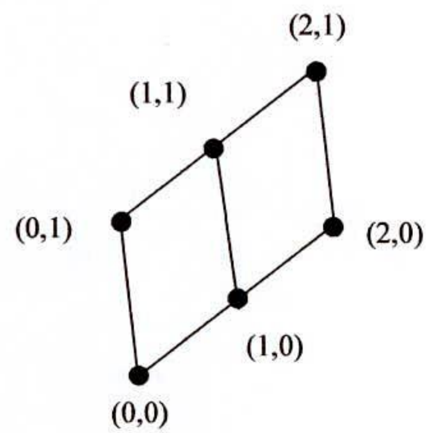

Fig. 1.7

Proof : $C_1 = \{0,1,2\}$

$C_2 = \{0,1\}$

$\therefore C_1 X\, C_2 = \{(0,0), (0,1), (1,0), (1,1), (2,0), (2,1)\}$.

Inf and *Sup* of any two elements of $C_1 X\, C_2$ lie in $C_1 X\, C_2$.

So $C_1 X\, C_2$ satisfy the conditions of lattice . Hence $C_1 X\, C_2$ is a lattice .

Example 1.3.4 :

$$(0,1) \wedge (2,0) = (0,0) \in C_1 X\, C_2$$

$$(0,1) \vee (2,0) = (2,1) \in C_1 X\, C_2$$

Theorem 1.3.2 : If L is any lattice , then for any $a , b, c \in L$, the following results hold .

(1) $a \wedge a = a$, $a \vee a = a$ (Idempotency)

(2) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$ (Commutativity)

(3) $a \wedge (b \wedge c) =(a \wedge b) \wedge c$ (Associativity)

$a \vee(b \vee c) =(a \vee b) \vee c$

(4) $a \wedge b \leq a$, $b \leq a \vee b$

(5) If $0 , u \in L$, then

$0 \wedge a = 0$, $0 \vee a = a$

$u \wedge a = a$, $u \vee a = u$.

(6) $a \wedge (a \vee b) = a$ (Absorption)

$a \vee (a \wedge b) = a$

Proof :

(1) $a \wedge a = \mathit{Inf}\{a , a\} = \mathit{Inf}\{a\} = a$.

$a \vee a = \mathit{Sup}\{a , a\} = \mathit{Sup}\{a\} = a$.

(2) $a \wedge b = \mathit{Inf}\{a , b\} = \mathit{Inf}\{b , a\} = b \wedge a$.

$a \vee b = \mathit{Sup}\{a , b\} = \mathit{Sup}\{b , a\} = b \vee a$.

(3) Let $b \wedge c = d$, then $d = \mathit{Inf}\{b , c\}$

$\Rightarrow d \leq b$, $d \leq c$

Let $e = \mathit{Inf}\{a , d\}$ then $e \leq a$, $e \leq d$

Thus $e \leq a$, $e \leq b$, $e \leq c$. (using transitivity)

Hence $e = a \wedge d = a \wedge(b \wedge c) = \mathit{Inf}\{a , b, c\}$

Therefore $a \wedge(b \wedge c) = \mathit{Inf}\{a , b, c\}$

Similarly, we can show that $(a \wedge b) \wedge c = \mathit{Inf}\{a , b, c\}$

Hence $a \wedge(b \wedge c) =(a \wedge b) \wedge c$.

Again let $b \vee c = d$, then $d = \mathit{Sup}\{b , c\}$

$\Rightarrow\ d \geq b, \quad d \geq c$

Let $e = Sup\{a, d\}$ then $e \geq a, \quad e \geq d$

Thus $e \geq a, \quad e \geq b, \quad e \geq c$ (using transitivity)

Hence $e = a \vee d = a \vee (b \vee c) = Sup\{a, b, c\}$

Similarly, we can show that $(a \vee b) \vee c = Sup\{a, b, c\}$

Hence $a \vee (b \vee c) = (a \vee b) \vee c$

(4) Follows by definitions of meet and join .

(5) Since $0 \leq x \leq u$, for all $x \in L$, the results are trivial for meet and join .

(6) $a \leq a \vee b$ By (4)

$\therefore\ a \wedge (a \vee b) = a$ [Since $a \leq b \Leftrightarrow a \wedge b = a$

$\Leftrightarrow a \vee b = b$

Again $a \wedge b \leq a$ By (4)

$\therefore\ (a \wedge b) \vee a = a$

Hence $a \vee (a \wedge b) = a$.

Problem 1.3.3 : Show that idempotent laws follow from the absorption laws .

Proof : We have $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$

Take $b = a \wedge b$ in first and we get $a \wedge (a \vee (a \wedge b)) = a$

or $a \wedge a = a$.

Similarly we can show $a \vee a = a$. •

Theorem 1.3.4 : In any lattice the distributive inequalities

(i) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

(ii) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

hold for any a , b, c .

Proof : (i) $a \wedge b \leq a$

$$a \wedge b \leq b \leq b \vee c$$

$\Rightarrow$ $a \wedge b$ is lower bound of $\{a , b \vee c\}$

$\Rightarrow$ $a \wedge b \leq a \wedge (b \vee c)$ (1)

Again $a \wedge c \leq a$

$$a \wedge c \leq c \leq b \vee c$$

$\Rightarrow$ $a \wedge c \leq a \wedge (b \vee c)$ (2)

(1) and (2) show that $a \wedge (b \vee c)$ is an upper bound of $\{a \wedge b , a \wedge c\}$

$\Rightarrow$ $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$

Hence $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$ •

(ii) $a \vee b \geq a$

$$a \vee b \geq b \geq b \wedge c$$

$\Rightarrow$ $a \vee b$ is an upper bound of $\{a , b \wedge c\}$

$\Rightarrow$ $a \vee b \geq a \vee (b \wedge c)$ (1)

Again $a \vee c \geq a$

$$a \vee c \geq c \geq b \wedge c$$

$\Rightarrow$ $a \vee c \geq a \vee (b \wedge c)$ (2)

(1) and (2) show that $a \vee (b \wedge c)$ is a lower bound of $\{a \vee b , a \vee c\}$

$\Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ ■

Theorem 1.3.5 : In any lattice L ,

$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$,

for all $a , b , c \in L$.

Proof : Since $a \wedge b \leq a \vee b$

$$a \wedge b \leq b \leq b \vee c$$

$$a \wedge b \leq a \leq c \vee a$$

We find

$$(a \wedge b) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$$

Similarly, $(b \wedge c) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

and $(c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

Hence $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ ■

Theorem 1.3.6 : The dual of a lattice is also a lattice .

Proof : Let (L, R) be a lattice and let $(\bar{L}, \bar{R})$ be its dual . Then $\bar{L} = L$ and $\bar{R}$ is converse of R . Let $x, y \in \bar{L}$ be any elements, then $x, y \in L$ and L is a lattice , $Sup\{x, y\}$ exists in L . Let it be $x \vee y$.

Then $x \, R \, (x \vee y)$

$y \, R \, (x \vee y)$

$\Rightarrow$ $(x \vee y) \, \bar{R} \, x$

$(x \vee y) \, \bar{R} \, y$

$\Rightarrow$ $x \vee y$ is a lower bound of $\{x, y\}$ in $\bar{L}$.

If z is any lower bound of $\{x, y\}$ in $\bar{L}$ then

$z \, \bar{R} \, x, \quad z \, \bar{R} \, y$

$\Rightarrow$ $x \, R \, z, \quad y \, R \, z$

$\Rightarrow$ z is an upper bound of $\{x, y\}$ in L

$\Rightarrow$ $(x \vee y) \, R \, z$ as $x \vee y = Sup\{x, y\}$ in L

$\Rightarrow$ $z \, \bar{R} \, (x \vee y)$

or that $x \vee y$ is greatest lower bound of $\{x, y\}$ in $\bar{L}$. Similarly we can show $x \wedge y$ will be $Sup\{x, y\}$ in $\bar{L}$.

Hence $\bar{L}$ is a lattice . ■

Definition (Complete Lattice) : A lattice A is called a complete lattice if every non-empty subset of A has its *Sup* and *Inf* in A .

Example 1.3.5 : Every finite lattice is complete .

Example 1.3.6 : The real interval [0 ,1] with usual $\leq$ forms a complete lattice .

Example 1.3.7 : The lattice $(Z, \leq)$ of integers is not complete as the subset

$K=\{x \in Z \mid x > 0\}$ does not have an upper bound and therefore a *Sup* in Z .

Theorem 1.3.7 : The dual of a complete lattice is a complete lattice .

Proof : Let (A, O) be a complete lattice and let $(\bar{A}, \bar{O})$ be its dual .

Then $(\bar{A}, \bar{O})$ is a lattice .

Let $\varphi \neq S \subseteq \bar{A}$ be any subset of $\bar{A}$. Since A is complete, *Sup* S and *Inf* S exists in A .

Let $a = Inf\, S$ in A

Then $a\, O\, x \quad \forall x \in A$

$\Rightarrow \quad x\, \bar{O}\, a \quad \forall x \in \bar{A}$

$\Rightarrow$ a is an upper bound of S in $\bar{A}$.

Let b be any other upper bound of S in $\bar{A}$.

Then $x\, \bar{O}\, b \quad \forall x \in \bar{A}$

$\Rightarrow \quad b\, O\, x \quad \forall x \in A$

$\Rightarrow \quad b\, O\, a$ as $a = Inf\, S$ in A

$\Rightarrow \quad a\, \bar{O}\, b$ or that a is l.u.b. of S in $\bar{A}$.

Similarly, we can show that *Sup* S in A will be *Inf* S in $\bar{A}$.

Hence $(\bar{A}, \bar{O})$ is complete . ■

Theorem 1.3.8 : If A and B are two lattices . Then the product $A \times B$ is a lattice .

Proof : Given A and B be two lattices then we have

$A \times B = \{(a, b) \mid a \in A, b \in B\}$

is a poset under the relation $\leq$ defined by

$$(a_1, b_1) \leq (a_2, b_2) \Leftrightarrow a_1 \leq a_2 \text{ in } A$$
$$b_1 \leq b_2 \text{ in } B .$$

We show $A \times B$ forms a lattice .

Let $(a_1, b_1), (a_2, b_2) \in A \times B$ be any elements .

Then $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

Since A and B are lattices, $\{a_1, a_2\}$ and $\{b_1, b_2\}$ have *Sup* and *Inf* in A and B respectively .

Let $a_1 \wedge a_2 = Inf\{a_1, a_2\}$, $b_1 \wedge b_2 = Inf\{b_1, b_2\}$

then $a_1 \wedge a_2 \leq a_1$, $a_1 \wedge a_2 \leq a_2$

$b_1 \wedge b_2 \leq b_1$, $b_1 \wedge b_2 \leq b_2$

$\Rightarrow$ $(a_1 \wedge a_2, b_1 \wedge b_2) \leq (a_1, b_1)$

$(a_1 \wedge a_2, b_1 \wedge b_2) \leq (a_2, b_2)$

$\Rightarrow$ $(a_1 \wedge a_2, b_1 \wedge b_2)$ is a lower bound of $\{(a_1, b_1), (a_2, b_2)\}$

Suppose (c, d) is any lower bound of $\{(a_1, b_1), (a_2, b_2)\}$

Then $(c, d) \leq (a_1, b_1)$

$(c, d) \leq (a_2, b_2)$

$\Rightarrow$ $c \leq a_1, c \leq a_2, d \leq b_1, d \leq b_2$

$\Rightarrow$ c is a lower bound of $\{a_1, a_2\}$ in A.

d is a lower bound of $\{b_1, b_2\}$ in B.

$\Rightarrow \quad c \le a_1 \wedge a_2 = Inf\{a_1, a_2\}$

$d \le b_1 \wedge b_2 = Inf\{b_1, b_2\}$

$\Rightarrow \quad (c, d) \le (a_1 \wedge a_2, b_1 \wedge b_2)$

or that $(a_1 \wedge a_2,\ b_1 \wedge b_2)$ is g . l . b. $\{(a_1,b_1), (a_2,b_2)\}$.

Similarly (by duality) we can say that

$(a_1 \vee a_2,\ b_1 \vee b_2)$ is l . u .b. $\{(a_1,b_1), (a_2,b_2)\}$.

Hence $A \times B$ is a lattice.

Also $\quad (a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2, b_1 \wedge b_2)$

$(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee a_2, b_1 \vee b_2)$. ■

Theorem 1.3.9 : If $(P, \le)$ is a poset with least element 0 such that every non empty subset S of P has *Sup* then P is a complete lattice.

Proof : Let S be any non empty subset of P . We need prove that *Inf* S exists.

Since 0 is the least element of P, $\quad 0 \le x \quad \forall x \in P$

and thus $\quad 0 \le s \quad \forall s \in S$

$\Rightarrow$ 0 is a lower bound of S.

subset of P and, therefore, by given condition *Sup* T exists .

Let T = set of all lower bounds of S, then T is a non empty subset of P and therefore, by given condition *Sup* T exists .

Let $k = Sup\ T$

Now $\quad s \in S \Rightarrow x \le s, \quad \forall x \in T$

$\Rightarrow$ each element of S is an upper bound of T

$\Rightarrow \quad k \le s \quad \forall s \in S$

$\Rightarrow$ k is a lower bound of S . But k being an upper bound of T means $x \leq k \quad \forall x \in T$ i.e., $x \leq k \quad \forall$ lower bounds of S

$\Rightarrow k = Inf\, S$

Hence P is a poset in which every non empty subset has *Sup* and *Inf* and thus P is a complete lattice. ■

Theorem 1.3.10 : A lattice L is a chain if and only if every non empty subset of L is a sublattice.

Proof : If the lattice is a chain then we have already shown that every non empty subset of L is a sublattice.

Conversely, let L be a lattice s.t., every non empty subset of L is a sublattice. We show L is a chain.

Let $a, b \in L$ be any elements.

Then $\{a, b\}$ being a non empty subset of L will be a sub lattice of L. Thus by definition of sublattice $a \wedge b \in \{a, b\}$

$\Rightarrow \; a \wedge b = a$ or $a \wedge b = b$

$\Rightarrow \; a \leq b$ or $b \leq a$

i.e., a, b are comparable.

Hence L is a chain. ■

Theorem 1.3.11 : Intersection of two ideals is an ideal.

Proof : Let X, Y be two ideals of a lattice L.

Since X, Y are non empty, $\exists$ some $x \in X,\ y \in Y$.

Now $x \in X,\ y \in Y \subseteq L \Rightarrow x \wedge y \in X$.

Similarly $\quad x \wedge y \in Y$

Thus $\quad X \cap Y \neq \varphi$.

Let $\quad u, v \in X \cap Y$ be any elements.

$\Rightarrow \quad u, v \in X$ and $u, v \in Y$

$\Rightarrow \quad u \vee v \in X$ and $u \vee v \in Y$ as X, Y are ideals.

$\Rightarrow \quad u \vee v \in X \cap Y$.

Again, if $a \in X \cap Y$ and $l \in L$ be any elements then $a \in X$, $a \in Y$, $l \in L$

$\Rightarrow a \wedge l \in X$ and $a \wedge l \in Y$

$\Rightarrow \quad a \wedge l \in X \cap Y$

Hence $X \cap Y$ is an ideal. ■

The result can clearly be extended to intersection of more than two ideals.

Problem 1.3.12 : Show that union of two ideals may not be an ideal.

Solution :

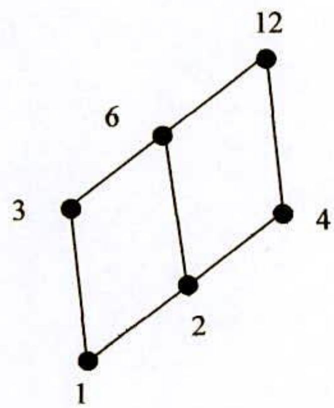

Fig. 1.8

Take $A = \{1, 2\}$, $B = \{1, 3\}$. Then A, B are ideals of the lattice $L = \{1, 2, 3, 4, 6, 12\}$ under divisibility, but $A \cup B$ is not an ideal. $2, 3 \in A \cup B$ but $2 \vee 3 = 6 \notin A \cup B$. ■

Theorem 1.3.13 : Union of two ideals is an ideal if and only if one of them is contained in the other .

Proof : One side of the theorem follows trivially . Let now A and B be two ideals of a lattice L s.t., $A \cup B$ is also an ideal of L .

Suppose $A \nsubseteq B$ and $B \nsubseteq A$

$\Rightarrow \quad \exists\ x \in A$ s.t., $x \notin B$

$\exists\ y \in B$ s.t., $y \notin A$

$\Rightarrow \quad x, y \in A \cup B$

$\Rightarrow \quad x \vee y \in A \cup B$ as $A \cup B$ is an ideal .

$\Rightarrow \quad x \vee y \in A$ or $x \vee y \in B$.

If $x \vee y \in A$, then as $y \in B \subseteq L$

$y \wedge (x \vee y) \in A$ or that $y \in A$, a contradiction .

Similarly $x \vee y \in B$ would lead us to the result that $x \in B$ which is not true .

Hence either $A \subseteq B$ or $B \subseteq A$. ■

Theorem 1.3.14 : A non empty subset I of a lattice L is an ideal if and only if

(i) $i, j \in I \Rightarrow i \vee j \in I$

(ii) $i \in I,\ x \leq i \Rightarrow x \in I$.

Proof : Let I be an ideal of a lattice L.

By definition of ideal (i) is satisfied .

Let $i \in I$, $x \leq i$, then $x = i \wedge x \in I$ (by def. of ideal).

Conversely, we need to show that $i \in I$, $a \in L \Rightarrow a \wedge i \in I$

Since $a \wedge i \leq i$ and $i \in I$

By given condition $a \wedge i \in I$

Hence I is an ideal. ■

Theorem 1.3.15 : The set of all ideals of a lattice L forms an ideal under relation $\subseteq$.

Proof : Let $I(L)$ = set of all ideals of a lattice L, then $I(L) \neq \varphi$ as $L \in I(L)$.

Clearly also $(I(L), \subseteq)$ is a poset. To show that $I(L)$ is a lattice we need find *Sup* and *Inf* of $\{A, B\}$ for any $A, B \in I(L)$. Since intersection of two ideals is an ideal and $A \cap B$ is the largest set contained in A and B it is obvious that

$$A \wedge B = \text{Inf } \{A, B\} = A \cap B$$

Again, $A \cup B$ is the smallest set containing A and B. But then $A \cup B$ may not be an ideal, so it cannot work as our $A \vee B$. We consider the set

$$X = \{x \in L \mid x \leq a \vee b \text{ for some } a \in A, b \in B\}$$

We claim $X = A \vee B$

For any $a \in A$, $a \leq a \vee b$ for any $b \in B$

$\Rightarrow \quad a \in X \ \Rightarrow A \subseteq X$

Similarly, $B \subseteq X$

Thus $X \neq \varphi$ and $A \cup B \subseteq X$.

We show X is an ideal of L.

Let $x, y \in X$ be any elements.

Then $x \leq a_1 \vee b_1$

$y \leq a_2 \vee b_2$ for some $a_1, a_2 \in A$, $b_1, b_2 \in B$

$\Rightarrow \quad x \vee y \leq (a_1 \vee b_1) \vee (a_2 \vee b_2)$

$\Rightarrow \quad x \vee y \in X$ as $a_1 \vee a_2 \in A$, $b_1 \vee b_2 \in B$.

Again, for any $x \in X$ and $l \in L$, Since $x \leq a \vee b, a \in A,\ b \in B$

we have $x \wedge l \leq x \leq a \vee b$.

$\Rightarrow$ $x \wedge l \in X$ which then is an ideal of L .

If C be any ideal of L containing A and B then $X \subseteq C$

as $x \in X$

$\Rightarrow$ $x \leq a \vee b$, for some $a \in A,\ b \in B$.

Again $a \in A \subseteq C,\ b \in B \subseteq C$ gives $a \vee b \in C$ and $x \leq a \vee b$

then yields $x \in C$.

Hence X is the smallest ideal of L containing $A \cup B$,

i.e., $X = Sup\,\{A, B\} = A \vee B$

and we have established that $(I(L), \subseteq)$ is a lattice . It is called the ideal lattice of L . ■

Problem1.3.16 : Show that an ideal of a lattice L which is also a dual ideal is the lattice itself .

Solution : Let A be an ideal as well as a dual ideal of L, then $A \subseteq L$.

We show $L \subseteq A$.

Let $l \in L$, $x \in A$ be any elements then $l \wedge x \in A$.

Again, $l \wedge x \leq l$ and, therefore, $l \in A$

$\Rightarrow$ $L \subseteq A$

Hence $A = L$.

Thus no proper subset of a lattice can be an ideal as well as a dual ideal of the lattice . ■

Theorem 1.3.17: Every convex sublattice of a lattice L is the intersection of an ideal and a dual ideal .

Proof : Let S be a convex sublattice of L

Let $A = \{ x \in L \mid \exists\, s \in S,\ x \leq s \}$. Then $A \neq \varphi$ as $S \subseteq A$.

Notice $s \leq s \quad \forall s \in S$.

We show A is an ideal of L.

Let $x, y \in A$ be any elements.

Then $\exists\ s_1, s_2 \in S$ s.t., $x \leq s_1$, $y \leq s_2$

$\Rightarrow \quad x \vee y \leq s_1 \vee s_2$

$\Rightarrow \quad x \vee y \in A$ as $s_1 \vee s_2 \in S$

Again, let $x \in A$ and $l \in L$ be any elements.

Then $x \leq s$ for some $s \in S$

Now $x \wedge l \leq x \leq s$

$\Rightarrow \quad x \wedge l \in A$

Hence A is an ideal of L.

Let $A' = \{ x \in L \mid \exists\, s \in S,\ s \leq x \}$, then by duality it follows that A' is a dual ideal of L. We show $S = A \cap A'$.

$S \subseteq A \cap A'$ (by definition of A and A')

Let $t \in A \cap A'$.

Then $t \in A$ and $t \in A'$

$\Rightarrow \quad \exists\ s1, s2 \in S$ s.t., $t \in s2$, $s1 \leq t$

i.e., $s1 \leq t \leq s2$ or that $t \in [\, s1, s2\,]$

Since S is convex sublattice, $s_1, s_2 \in S$

$[\, s_1, s_2\,] \subseteq S \Rightarrow t \in S$

$\Rightarrow A \cap A' \subseteq S$

Hence $S = A \cap A'$. ■

Theorem 1.3.18 : A lattice L is a chain if and only if all ideals in L are prime.

Proof : Let L be a chain. Let A be any proper ideal of L. If

$a \wedge b \in A$ then as a , b are in a chain, they are comparable .

Let $a \leq b$. Then $a \wedge b = a$.

Thus $a \wedge b \in A \Rightarrow a \in A \Rightarrow A$ is prime .

Conversely, let every ideal in A be prime. To show that L is a chain, let $a , b \in L$ be any elements.

Let $A = \{ x \in L \mid x \leq a \wedge b \}$ then A is easily seen to be an ideal of L. Thus A is a prime ideal.

Now $a \wedge b \in A$, A is prime, thus $a \in A$ or $b \in A$

$\Rightarrow a \leq a \wedge b$ or $b \leq a \wedge b$

$\Rightarrow a \wedge b \leq a \leq a \wedge b$

or $a \wedge b \leq b \leq a \wedge b$

$\Rightarrow a = a \wedge b$ or $b = a \wedge b$

$\Rightarrow a \leq b$ or $b \leq a$

$\Rightarrow L$ is a chain . ■

Definition (Dual Prime Ideal) : A proper dual ideal I of a lattice is called a dual prime ideal if $a \vee b \in I \Rightarrow a \in I$ or $b \in I$.

Problem 1.3.19 : Let I be a prime ideal of lattice L. Show that $L - I$ is a dual prime ideal.

Solution : Since I is not empty, $L - I$ is a proper subset of L.

Let $a , b \in L - I$. Then $a , b \in L$, $a ,b \notin I$.

$\Rightarrow a \wedge b \in L$, $a \wedge b \notin I$

(as $a \wedge b \in I \Rightarrow a \in I$ or $b \in I$ as I is prime)

$\Rightarrow a \wedge b \in L - I$

Again, let $a \in L - I$, $l \in L$

Then $a \in L$, $a \notin I$, $l \in L$

$\Rightarrow a \vee l \in L$, $a \notin I$

$\Rightarrow \quad a \vee 1 \in L,\ a \vee 1 \notin I$

(as $a \vee 1 \in I \Rightarrow a \in I$ as $a \leq a \vee 1$)

Thus $\quad a \vee 1 \in L - I$

i.e., $L - I$ is dual ideal.

Let now $a \vee b \in L - I$, then $a \vee b \in L,\ a \vee b \notin I$

$\Rightarrow \quad a, b \in L,\ a \notin I$ or $b \notin I$

(as $a, b \in I \Rightarrow a \vee b \in I$)

$\Rightarrow \quad a \in L - I$ or $b \in L - I$

or that $L - I$ is a dual prime ideal. ■

1.4 Complemented and Relatively Complemented Lattices .

Definition (Complements) : Let [a , b] be an interval in a set L .

Let $x \in [a, b]$ be any elements. If $\exists\, y \in L$ s.t.,

$$x \wedge y = a, \qquad x \vee y = b .$$

We say y is a complement of x relative to [a , b], or y is a relative complement of x in [a , b] .

Observations :

(i) If such a y exists then y lies in [a , b] as $a = x \wedge y \leq y \leq x \vee y = b$

(ii) If y is relative complement of x , x will be relative complement of y .

(iii) An element x may or may not have a relative complement . A relative complement may or may not be unique . Consider the pentagonal lattice given by the figure (1.9) .

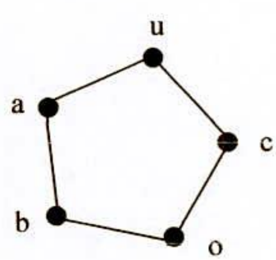

Fig. 1.9

b has no complement relative to [o , a]

a , b are both complement of c relative to [o , u]

b has only one complement c relative to [o , u]

(iv) a , b are unique complements of each

other relative to $[a, b]$

$a \wedge b = a$, $a \vee b = b$

Thus a , b are each other complements .

Let x be any other complement of a

relative to $[a, b]$

Then $a \wedge b = a = a \wedge x$

$a \vee b = b = a \vee x$

Now $b = a \vee x = (a \wedge x) \vee x = x$

Definition (Complemented) : If every element x of an interval $[a, b]$ has at least one complement relative to $[a, b]$, the interval $[a, b]$ is said to be complemented.

Definition (Relatively Complemented) : If every interval in a lattice is complemented the lattice is said to be relatively complemented.

Suppose now L is a bounded lattice. If for any $x \in L$, $\exists\, y \in L$ s.t., $x \wedge y = o$, $x \vee y = u$, y is called complement of x (we need not say relative to $[o, u]$). Further, if every element of L has a complement, we say lattice is complemented.

Thus a bounded lattice is complemented if the interval $[o, u] = L$ is complemented.

If L is a bounded lattice and is relatively complemented then L is complemented but not conversely.

Consider the pentagonal lattice,

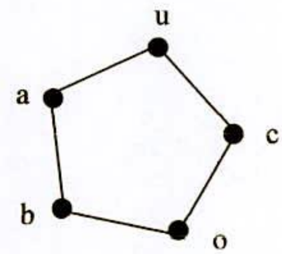

Fig. 1.10

[o , u] is complemented as a , c are each other complements

b , c are each other complements

and of course, o , u are each other complements. This lattice is not relatively complemented as b has no complement relative to [o , a] and so [o , a] is not complemented.

The lattice given by the adjacent diagram is not complemented as a has no complement (relative to [o , u]).

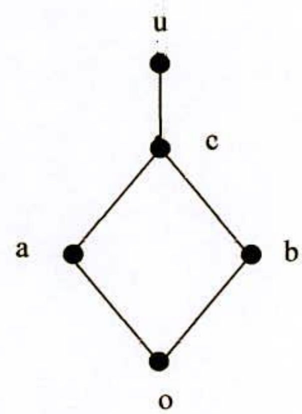

Fig. 1.11

The lattice given by the figure below is both relatively complemented as well as complemented.

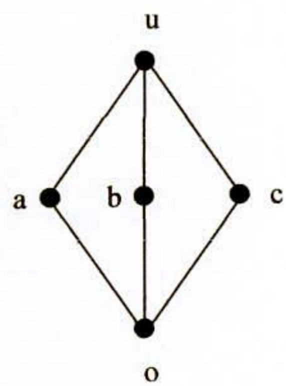

Fig. 1.12

Definition (Uniquely Complemented Lattice) : If every element of a bounded lattice L has a unique complement, we say L is uniquely complemented.

Theorem 1.4.1 : Let A be a non-empty finite set. Then (P(A), $\subseteq$) is uniquely complemented lattices.

Proof : Let $A \neq \phi$ finite set and P(A) be the power set of A .We know (P(A), $\subseteq$) forms a lattice with least element ϕ and greatest element A. Also for any X , Y $\in$ P(A)

$$X \wedge Y = X \cap Y \quad \text{and} \quad X \vee Y = X \cup Y$$

since $$X \wedge (A - X) = X \cap (A - X) = \phi$$

$$X \vee (A - X) = X \cup (A - X) = A$$

We find $A - X$ is complement of X relative to $[\phi , A]$

Thus P(A) is complemented.

Suppose Y is any complement of X , then

$$X \wedge Y = X \cap Y = \phi$$

$$X \vee Y = X \cup Y = A$$

i.e., $X \cap Y = X \cap (A - X)$

$X \cup Y = X \cup (A - X)$

$\Rightarrow$ $Y = A - X$

or that $A - X$ is uniquely complemented of X.

So $(P(A), \subseteq)$ is an uniquely complemented lattice.

Now we prove P(A) is also relatively complemented.

Consider any interval $[X, Y]$ in P(A).

Let $Z \in [X, Y]$ be any member. Then

$$Z \cap (X \cup (Y - Z)) = (Z \cap X) \cup (Z \cap (Y - Z)) = X \cup \phi = X$$

$$Z \cup (X \cup (Y - Z)) = (Z \cup X) \cup (Y - Z)) = Z \cup (Y - Z) = Y$$

Showing that $X \cup (Y - Z)$ is complemented of Z relative to $[X, Y]$. Since C was any element of any interval of P(A).

Hence P(A) is relatively complemented. ■

Theorem 1.4.2 : Two bounded lattices A and B are complemented if and only if $A \times B$ is complemented.

Proof : Let A and B be complemented and suppose o, u and o', u' are the universal bonds of A and B respectively.

Then (o, o') and (u, u') will be least and greatest elements of $A \times B$.

Let $(a, b) \in A \times B$ be any element.

Then $a \in A, b \in B$ and as A, B are complemented,

$\exists\ a' \in A, b' \in B$

s.t., $a \wedge a' = o,\ a \vee a' = u,\ b \wedge b' = o',\ b \vee b' = u'$.

Now

$$(a, b) \wedge (a', b') = (a \wedge a', b \wedge b') = (o, o')$$

$$(a, b) \vee (a', b') = (a \vee a', b \vee b') = (u, u')$$

shows that (a', b') is complement of (a, b) in $A \times B$.

Hence $A \times B$ is complemented.

Conversely, let $A \times B$ be complemented.

Let $a \in A$, $b \in B$ be any elements.

Then $(a, b) \in A \times B$ and thus has a complement, say (a', b')

Then $(a, b) \wedge (a', b') = (o, o')$, $(a, b) \vee (a', b') = (u, u')$

$\Rightarrow \quad (a \wedge a', b \wedge b') = (o, o')$, $(a \vee a', b \vee b') = (u, u')$

$\Rightarrow \quad a \wedge a' = o \quad a \vee a' = u$

$b \wedge b' = o \quad b \vee b' = u'$

i.e., a' and b' are complements of a & b respectively.

Hence A and B are complemented. ∎

Theorem 1.4.3 : Two lattices A and B are relatively complemented if and only if $A \times B$ is relatively complemented.

Proof : Let A, B be relatively complemented.

Let $[(a_1, b_1), (a_2, b_2)]$ be any interval of $A \times B$ and suppose (x, y) is any element of this interval.

Then

$$(a_1, b_1) \leq (x, y) \leq (a_2, b_2) \qquad a_1, a_2, x \in A$$
$$b_1, b_2, y \in B$$

$$\Rightarrow \qquad a_1 \leq x \leq a_2$$
$$b_1 \leq y \leq b_2$$

$$\Rightarrow \qquad x \in [a_1, a_2] \quad \text{an interval in } A$$

$y \in [b_1, b_2]$ an interval in B

Since A , B are relatively complemented, x , y have complements relative to $[a_1, a_2]$ and $[b_1, b_2]$ respectively.

Let x' and y' be these complements. Then

$$x \wedge x' = a_1, \quad y \wedge y' = b_1$$

$$x \vee x' = a_2, \quad y \vee y' = b_2$$

Now

$$(x, y) \wedge (x', y') = (x \wedge x', y \wedge y') = (a_1, b_1)$$

$$(x, y) \vee (x', y') = (x \vee x', y \vee y') = (a_2, b_2)$$

$\Rightarrow (x', y')$ is complement of (x, y) relative to $[(a_1, b_1), (a_2, b_2)]$.

Thus any interval in $A \times B$ is complemented.

Hence $A \times B$ is relatively complemented.

Conversely, let $A \times B$ be relatively complemented.

Let $[a_1, a_2]$ and $[b_1, b_2]$ be any interval in A & B.

Let $x \in [a_1, a_2]$, $y \in [b_1, b_2]$ be any elements .

Then

$$a_1 \leq x \leq a_2, \quad b_1 \leq y \leq b_2$$

$\Rightarrow \quad (a_1, b_1) \leq (x, y) \leq (a_2, b_2)$

$\Rightarrow \quad (x, y) \in [(a_1, b_1), (a_2, b_2)]$, an interval in $A \times B$.

$\Rightarrow \quad (x, y)$ has a complement, say (x', y') relative to this interval.

Thus

$$(x, y) \wedge (x', y') = (a_1, b_1)$$

$$(x, y) \vee (x', y') = (a_2, b_2)$$

$$\Rightarrow \quad (x \wedge x',\ y \wedge y') = (a_1, b_1)$$

$$(x \vee x',\ y \vee y') = (a_2, b_2)$$

$$\Rightarrow \quad x \wedge x' = a_1,\ \ x \vee x' = a_2$$

$$y \wedge y' = b_1,\ \ y \vee y' = b_2$$

$\Rightarrow$ x' is complement of x relative to $[a_1, a_2]$

y' is complement of x relative to $[b_1, b_2]$

which is turn imply that A, B are relatively complemented. ■

Theorem 1.4.4 : Dual of a complemented lattice is complemented.

Proof : Let (L, ρ) be a complemented lattice with o, u as least and greatest elements. Let $(\bar{L}, \bar{\rho})$ be the dual of (L, ρ). Then u, o are least and greatest elements of $\bar{L}$.

Let $a \in \bar{L} = L$ be any element.

Since $a \in L$, L is complemented, $\exists\ a' \in L$ s.t.,

$$a \wedge a' = o,\ a \vee a' = u \text{ in } L$$

i.e., $o = \mathit{Inf}\{a, a'\}$ in L.

$\Rightarrow$ $o\, \rho\, a,\ o\, \rho\, a'$

$\Rightarrow$ $a\, \bar{\rho}\, o,\ a'\, \bar{\rho}\, o$ in $\bar{L}$

$\Rightarrow$ o is an upper bound of $\{a, a'\}$ in $\bar{L}$

If k is any upper bound of $\{a, a'\}$ in $\bar{L}$ then $a\, \bar{\rho}\, k,\ a'\, \bar{\rho}\, k$

$\Rightarrow$ $k\, \rho\, a,\ k\, \rho\, a'$

$\Rightarrow$ $k\, \rho\, o$ as o is Inf.

$\Rightarrow$ $o\, \bar{\rho}\, k$

i.e., o is l.u.b. $\{a, a'\}$ in $\bar{L}$

i.e., $a \vee a' = o$ in $\bar{L}$

similarly, $a \wedge a' = u$ in $\bar{L}$

or that a' is complement of a in $\bar{L}$

Hence $\bar{L}$ is complemented. ■

1.5 Atoms and Covers .

Definition (Atom) : An element a in a lattice L is called an atom if it is covers o.

In other words a is an atom if and only if $a \neq 0$ and $x \wedge a = a$ or $x \vee a = 0 \quad \forall\ x \in L$.

Definition (Dual atom) : An element b is called dual atom, if u , the greatest element of the lattice covers b.

Definition (Length) : A finite chain with n elements is said to have length $n - 1$, (i.e., length is the number of 'links ' that the chain has.)

Definition (Cover) : We say a covers b if $b < a$ and there exists no c s.t., $b < c < a$.

Definition (Height or dimension) : Let L be a lattice of finite length with least element o . An element $x \in L$ is said to have height or dimension n if $l\ [\,o , x\,] = n$ and, in that case we write $h(x) = n$.

Problem 1.5.1 : Show that no ideal of a complemented lattice which is a proper sublattice can contain both an element and its complement.

Solution : Let L be a complemented lattice. Then $o , u \in L$. Let I an ideal of L such that I is a proper sub lattice of L. Suppose $\exists$ an element x in I such that its complement x' is also in I.

Then

$$x \wedge x' = o, \qquad x \vee x' = u$$

since I is sublattice, $x \wedge x'$, $x \vee x'$ are in I i.e., $o , u \in I$

Now if $l \in L$ be any element then as $u \in I$

$$l \wedge u \in I$$

$\Rightarrow l \in I \ \Rightarrow L \subseteq I$

$\Rightarrow I = L$, a contradiction. ■

Problem 1.5.2 : Let L be uniquely complemented lattice and let a be an atom in L. Show that a' the complement of a is a dual atom of L.

Solution : Since L is uniquely complemented lattice, every element has a unique complement.

Suppose a' is not a dual atom, then $\exists$ at least one x

s.t., $a' < x < u$

$\Rightarrow \quad a' \vee a \leq x \vee a$

$\Rightarrow \quad u \leq x \vee a \leq u$

$\Rightarrow \quad u = x \vee a.$

Now if $a \leq x$ then $x \vee a = x$

$\Rightarrow x = u$, not true. Again if $a \not\leq x$, then

$a \wedge x = o$ (note a is an atom)

thus $a \wedge x = o$, $a \vee x = u$

$\Rightarrow \quad x = a'$, again a contradiction.

Hence a' is a dual atom. ■

"Homomorphisms and Isomorphisms"

2.1 Introduction .

Here we discuss Homomorphisms , Isomorphisms , Meet homomorphisms and Join homomorphisms . We have prove the following problem .If L_1 , L_2 , M_1 , M_2 are lattices such that $L_1 \cong M_1$ and $L_2 \cong M_2$ then show that $L_1 \times L_2 \cong M_1 \times M_2 \cong M_2 \times M_1$.

Let $\theta : L \longrightarrow M$ be an onto homomorphism . The set $\{ x \in L \mid \theta (x) = 0' \}$ where $0'$ is the least element of M is called the Kernel of θ and is denoted by $\mathrm{Ker}\theta$. If M does not have the zero element , $\mathrm{Ker}\theta$ does not exist .

2.2 Meet and Join Homomorphisms , Isomorphisms .

Definition (Meet & Join homomorphism) : Let L and M be lattices.

A mapping $\theta : L \longrightarrow M$ is called a meet homomorphism if

$$\theta (a \wedge b) = \theta (a) \wedge \theta (b).$$

It is called a join homomorphism if

$$\theta (a \vee b) = \theta (a) \vee \theta (b).$$

Definition (Homomorphism) : If θ is both meet as well as join homomorphism, it is called a homomorphism .

A homomorphism is also sometimes called a morphism.

Definition (Isomorphism) : The map θ is also 1-1 and onto we call θ to be an isomorphism.

If θ is an isomorphism from L to L we call it an automorphism.

A homomorphism from L to L is called endomorphism.

If $\theta : L \longrightarrow M$ is onto homomorphism, we say M homomorphic image of L.

Example 2.2.1 : Let L and M be lattices

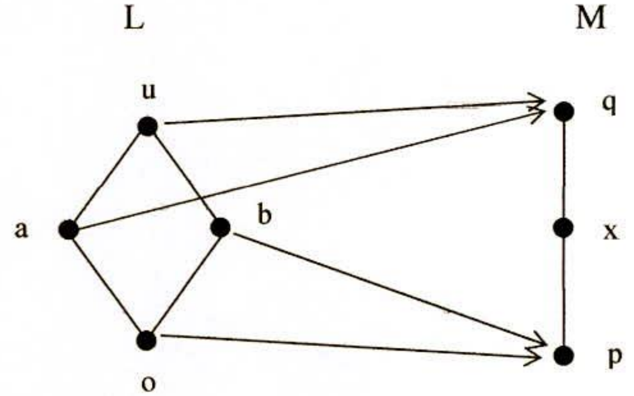

Fig. 2.1

Define $\theta : L \longrightarrow M$, s. t.,

$$\theta(o) = p \qquad \theta(a) = q$$

$$\theta(b) = p \qquad \theta(u) = q$$

Then θ is a homomorphism .

$$\theta(a \wedge b) = \theta(o) = p,\ \theta(a) \wedge \theta(b) = q \wedge p = p$$

$$\theta(a \vee b) = \theta(u) = q,\ \theta(a) \vee \theta(b) = q \vee p = q.$$

Problem 2.2.1 : If L_1, L_2, M_1, M_2 are lattices such that $L_1 \cong M_1$ and $L_2 \cong M_2$ then show that $L_1 \times L_2 \cong M_1 \times M_2 \cong M_2 \times M_1$.

Solution : Let $f : L_1 \longrightarrow M_1$ and $g : L_2 \longrightarrow M_2$ be the given isomorphisms . Define

$$\theta : L_1 \times L_2 \longrightarrow M_1 \times M_2, \text{ s. t.,}$$

$$\theta((a, b)) = (f(a), g(b))$$

Then $\theta((a, b)) = \theta((c, d))$

$\Leftrightarrow \quad (f(a), g(b)) = (f(c), g(d))$

$\Leftrightarrow \quad f(a) = f(c), \quad g(b) = g(d)$

$\Leftrightarrow \qquad a = c \ , \qquad b = d$

$\Leftrightarrow \qquad (a, b) = (c, d)$

Shows that θ is well defined 1-1 map.

Again, $\theta((a, b) \wedge (c, d)) = \theta((a \wedge c, b \wedge d))$

$$= (f(a \wedge c), g(b \wedge d))$$
$$= (f(a) \wedge f(c), g(b) \wedge g(d))$$
$$= (f(a), g(b)) \wedge (f(c), g(d))$$
$$= \theta((a, b)) \wedge \theta((c, d))$$

Similarly, $\theta((a, b) \vee (c, d)) = \theta((a, b)) \vee \theta((c, d))$

Showing thereby that θ is a homomorphism.

Finally, for any $(m_1, m_2) \in M_1 \times M_2$, since $m_1 \in M_1$ & $m_2 \in M_2$ and f, g are onto,

$\exists\ l_1 \in L_1, l_2 \in L_2$, s. t., $f(l_1) = m_1$, $g(l_2) = m_2$

and $\theta((l_1, l_2)) = (f(l_1), g(l_2)) = (m_1, m_2)$

shows that θ is onto and hence an isomorphism.

To show $M_1 \times M_2 \cong M_2 \times M_1$, we can define

$$\varphi : M_1 \times M_2 \longrightarrow M_2 \times M_1 \text{ s. t.,}$$
$$\varphi((m_1, m_2)) = (m_2, m_1)$$

It is now easy to verify that φ is an isomorphism. ■

Theorem 2.2.2 : If $\theta : L \longrightarrow M$ is onto homomorphism and $x, y \in M$ s. t., $x < y$ then $\exists\ a, b \in L$ s. t., $\theta(a) = x$, $\theta(b) = y$ and $a < b$.

Proof : Since θ is onto and $x, y \in M$, $\exists\ a, c$ in L s. t.,

$$\theta(a) = x, \quad \theta(c) = y$$

We have $\theta(a \vee c) = \theta(a) \vee \theta(c) = x \vee y = y$ as $x < y$.

And $\quad a \le a \vee c$

If $\quad a = a \vee c$ then $\theta(a) = \theta(a) \vee \theta(c) = y$

$\Rightarrow \quad x = y$ which is not true .

Thus $\quad a < a \vee c$.

Take $b = a \vee c$ and we have the result proved. ■

Theorem 2.2.3 : Homomorphic image of a relatively complemented lattice is relatively complemented .

Proof : Let $\theta : L \longrightarrow M$ be an onto homomorphism and suppose L is relatively complemented.

Let $[a', b']$ be any interval in M, since θ is onto homomorphism , $\exists$ pre images a and b for a', b' respectively such that $\theta(a) = a'$, $\theta(b) = b'$ and $a < b$ (as $a' < b'$).

Thus $[a, b]$ is an interval in L.

Let $y \in [a', b'] = [\theta(a), \theta(b)]$ be any element then as before $\exists$ a pre image x of y s .t ., $\theta(x) = y$ and $a \le x \le b$.

Now L relatively complemented implies that x has a complement x' relative to $[a, b]$,

i . e., $\quad x \wedge x' = a, \quad x \vee x' = b$

$\Rightarrow \quad \theta(x) \wedge \theta(x') = \theta(a), \; \theta(x) \vee \theta(x') = \theta(b)$

$\Rightarrow \quad y \wedge \theta(x') = a', \; y \vee \theta(x') = b'$

$\Rightarrow \quad \theta(x')$ is complement of y relative to $[a', b']$.

Thus each element in any interval in M has a complement, giving us the required result. ■

2.3 Embeddings , Kernels and Dual homomorphisms .

Definition (Embedding) : Let L , M be lattices. A one – one homomorphism $\theta : L \longrightarrow M$ is called an imbedding (embedding) mapping . Also in that case we say L is imbedded in M.

Theorem 2.3.1 : Any lattice can be imbedded into its ideal lattice.

Proof : Let I (L) be the ideal lattice of L.

Define $\theta : L \longrightarrow I(L)$, s . t. ,

$\theta(a) = (a]$, the principal ideal generated by a .

θ is then clearly well defined.

Also $\theta(a) = \theta(b)$

$\Rightarrow$ $(a] = (b]$

and $a \in (a] \Rightarrow a \in (b] \Rightarrow a \leq b$

Similarly $b \leq a$ and thus $a = b$ i.e., θ is one-one.

Again $\theta(a \wedge b) = (a \wedge b] = (a] \wedge (b] = \theta(a) \wedge \theta(b)$

$\theta(a \vee b) = (a \vee b] = (a] \vee (b] = \theta(a) \vee \theta(b)$

Hence θ is one-one homomorphism. ■

Definition (Kernel) : Let $\theta : L \longrightarrow M$ be an onto homomorphism . The set $\{ x \in L \mid \theta(x) = o' \}$ where o' is least element of M is called Kernel of θ and is denoted by Ker θ. If M does not have the zero element, Ker θ does not exist.

Theorem 2.3.2 : If $\theta : L \longrightarrow M$ is an onto homomorphism, where L , M are lattices and o' is the least element of M, then Ker θ is an ideal of L.

Proof : Since θ is onto, $o' \in M$, thus Ker $\theta \neq \varphi$ as pre image of o' exists in L.

Now $x, y \in \text{Ker}\,\theta$

$\Rightarrow \quad \theta(x) = o' = \theta(y)$

$\theta(x \vee y) = \theta(x) \vee \theta(y) = o' \vee o' = o'$

$\Rightarrow \quad x \vee y \in \text{Ker}\,\theta.$

Again $x \in \text{Ker}\,\theta$, $l \in L$, gives $\theta(x) = o'$.

Also $\theta(x \wedge l) = \theta(x) \wedge \theta(l) = o' \wedge l = o'$

$\Rightarrow \quad x \wedge l \in \text{Ker}\,\theta.$

Hence $\text{Ker}\,\theta$ is an ideal of L. ■

Theorem 2.3.3 : If $\theta : L \longrightarrow L$ be a homomorphism, where L is a complete lattice then $\exists$ some $a \in L$, s. t., $\theta(a) = a$.

Proof : Let $S = \{ x \in L \mid x \leq \theta(x) \}$.

Then $S \neq \varphi$ as $o \in S$ as $o \leq \theta(o)$ (Note $\theta(o) \in L$).

Thus S is a non empty subset of a complete lattice and therefore *Sup* S exists. Let *Sup* S = a.

Now $x \leq a \qquad \forall x \in S$

$\Rightarrow \quad \theta(x) \leq \theta(a) \qquad \forall x \in S$

$\Rightarrow \quad x \leq \theta(x) \leq \theta(a) \qquad \forall x \in S$

$\Rightarrow \quad \theta(a)$ is an upper bound of S

$\Rightarrow \quad a \leq \theta(a)$ (Def. of *Sup*)

$\Rightarrow \quad a \in S$ by def. of S and hence a is greatest element of S.

Also $a \leq \theta(a) \Rightarrow \theta(a) \leq \theta(\theta(a))$

$\Rightarrow \theta(a) \in S$ (Def. of S)

a being greatest element of S then gives $\theta(a) \leq a$

i. e., $a \leq \theta(a) \leq a$.

Hence $\theta(a) = a$, which proves our assertion. ■

Definition (Dual meet & Dual join homomorphism) : A mapping $\theta : L \longrightarrow M$ is called dual meet homomorphism if

$\theta(a \wedge b) = \theta(a) \vee \theta(b)$.

and is a called dual join homomorphism if

$\theta(a \vee b) = \theta(a) \wedge \theta(b)$.

Definition (Dual homomorphism) : It is called a dual homomorphism if it satisfies both the above conditions.

Definition (Dual Isomorphism) : A 1-1 onto dual homomorphism is called a dual isomorphism.

The reader would recall that under posets we define a dual isomorphism to be a 1-1 onto map which satisfies $a \, R \, b \Leftrightarrow \theta(b) \, R' \, \theta(a)$ where R and R' are the relations is L and M.

Theorem 2.3.4 : The definition of dual meet homomorphism & dual join homomorphism are equivalent.

Proof : To show the equivalence of two definitions,

Let $\theta : L \longrightarrow M$ be 1-1 onto s. t.,

$$\theta(a \wedge b) = \theta(a) \vee \theta(b)$$

$$\theta(a \vee b) = \theta(a) \wedge \theta(b)$$

Let $a \, R \, b$ in L

$\Rightarrow \quad a = a \wedge b$

$\Rightarrow \quad \theta(a) = \theta(a) \vee \theta(b)$ in M

$\Rightarrow \quad \theta(b) \, R' \theta(a)$

Again $\theta(b) \, R' \theta(a) \Rightarrow \theta(a) = \theta(a) \vee \theta(b) = \theta(a \wedge b)$.

$\Rightarrow \quad a = a \wedge b$ as θ is 1-1

$\Rightarrow \quad a\,R\,b \quad \text{in } L$

Conversely, let $a , b \in L$ be any elements.

Then $\quad (a \wedge b)\,R\,a, \quad (a \wedge b)\,R\,b$

$\Rightarrow \quad \theta(a)\,R'\,\theta(a \wedge b)\,,\ \theta(b)\,R'\,\theta(a \wedge b)$

or that $\theta(a \wedge b)$ is an upper bound of $\{\theta(a), \theta(b)\}$

Suppose $y \in M$ is any upper bound of $\{\theta(a), \theta(b)\}$ then since θ is onto, $\exists\ x \in L$ s.t., $\theta(x) = y$.

Now $\theta(x)$ is an upper bound of $\{\theta(a), \theta(b)\}$ gives

$$\theta(a)\,R'\,\theta(x), \quad \theta(b)\,R'\,\theta(x)$$

$\Rightarrow \quad x\,R\,a,\ x\,R\,b$

$\Rightarrow \quad x$ is a lower bound of $\{a , b\}$

$\Rightarrow \quad x\,R\,(a \wedge b), \qquad a \wedge b = \mathit{Inf}\{a , b\}$

$\Rightarrow \quad \theta(a \wedge b)\,R'\,\theta(x) = y$

i. e., $\quad \theta(a \wedge b)$ is a least upper bound of $\{\theta(a), \theta(b)\}$

i. e., $\quad \theta(a \wedge b) = \theta(a) \vee \theta(b)$.

Similarly we can show $\theta(a) \wedge \theta(b) = \theta(a \vee b)$.

Hence the two definitions are equivalent. ■

"Modular Lattices and Distributive Lattices"

3.1 Introduction .

Modular lattices and Distributive lattices have been studied extensively by many authors including Cigonli- [4] , Cornish [5] , Evans [7] , [8] and Nieminen [15] , [16] . A lattice L is called a modular lattice if for all $a , b , c \in L$ with $a \geq b$

$a \wedge (b \vee c) = [b \vee (a \wedge c)]$

In this chapter we also prove any non modular lattice L contains a sub-isomorphic with the pentagonal lattice . Two intervals [a , b] and [c , d] of a lattice are called transposed if $b \wedge c = a$ and $b \vee c = d$.

3.2 Modular Lattices .

Definition (Modular lattice) : A lattice L is called a modular lattice if

$\forall\, a , b , c \in L$, with $a \geq b$

$a \wedge (b \vee c) = [b \vee (a \wedge c)]$

(Dual of a modularity will read as

For $a , b , c \in L$ with $a \leq b$, $a \vee (b \wedge c) = b \wedge (a \vee c)$

Hence dual of a modular lattice is modular)

Example 3.2.1 : The lattice given by the following diagram are modular .

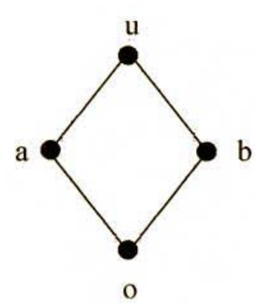

Fig. 3.1

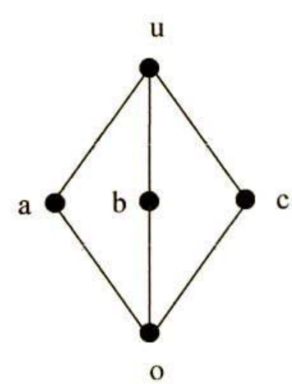

Fig. 3.2

Example 3.2.2 : A chain is a modular lattice .

Theorem 3.2.1 : A sublattice of a modular lattice is modular.

Proof : Let S be a sublattice of a modular lattice L. If $a, b, c \in S$ with $a \geq b$ then as $S \subseteq L$, $a, b, c \in L$ and, therefore,

$$a \wedge (b \vee c) = b \vee (a \wedge c).$$

Since S is closed under $\wedge$ and $\vee$ this result holds in S and hence S is modular. ■

Problem 3.2.2 : Show that a lattice of length two is modular.

Solution : If a lattice L has length 2, then $l[o, u] = 2$ and thus $l[a, b] \leq 2$ for any $a, b \in L$, $a \leq b$.

Let $a, b, c \in L$ be three elements s.t., $a > b$ and c is not comparable with a or b. Then $b \wedge c < b < a < a \vee c$ (at no place equality holds). Which shows $l[b \wedge c, a \vee c] \geq 3$, which is not possible. Thus we cannot find any triplet a, b, c in L s.t., $a > b$ and c is not comparable with a or b. Hence L is modular. ■

Theorem 3.2.3 : Homomorphic image of a modular lattice is modular.

Proof : Let $\theta : L \longrightarrow M$ be an onto homomorphism and suppose L is modular. Let $x, y, z \in M$ be three elements with $x > y$. Since θ is onto homomorphism, $\exists\ a, b, c \in L$ s.t.,$\theta(a) = x$, $\theta(b) = y$, $\theta(c) = z$, where $a > b$.

Now L is modular, $a, b, c \in L$, $a > b$, thus we get

$$a \wedge (b \vee c) = b \vee (a \wedge c)$$

Now
$$\begin{aligned} x \wedge (y \vee z) &= \theta(a) \wedge (\theta(b) \vee \theta(c)) \\ &= \theta(a) \wedge (\theta(b \vee c)) \\ &= \theta(a \wedge (b \vee c)) \end{aligned}$$

$$= \theta (b \vee (a \wedge c))$$

$$= \theta (b) \vee \theta (a \wedge c)$$

$$= \theta (b) \vee [\theta (a) \wedge \theta (c)]$$

$$= y \vee (x \wedge z)$$

Hence M is modular. ■

Theorem 3.2.4 : Two lattices L and M are modular if and only if $L \times M$ is modular.

Proof : Let L and M be modular.

Let $(a_1 , b_1) , (a_2 , b_2) , (a_3 , b_3) \in L \times M$ be three elements with $(a_1 , b_1) \geq (a_2 , b_2)$.

Then $a_1 , a_2 , a_3 \in L$, $a_1 \geq a_2$

$b_1 , b_2 , b_3 \in M$, $b_1 \geq b_2$

and since L and M are modular, we get

$$a_1 \wedge (a_2 \vee a_3) = a_2 \vee (a_1 \wedge a_3)$$

$$b_1 \wedge (b_2 \vee b_3) = b_2 \vee (b_1 \wedge b_3)$$

Thus

$$\begin{aligned} (a_1 , b_1) \wedge [(a_2 , b_2) \vee (a_3 , b_3)] &= (a_1 , b_1) \wedge (a_2 \vee a_3 , b_2 \vee b_3) \\ &= (a_1 \wedge (a_2 \vee a_3) , b_1 \wedge (b_2 \vee b_3)) \\ &= (a_2 \vee (a_1 \wedge a_3) , b_2 \vee (b_1 \wedge b_3)) \\ &= (a_2 , b_2) \vee (a_1 \wedge a_3 , b_1 \wedge b_3) \\ &= (a_2 , b_2) \vee [(a_1 , b_1) \wedge (a_3 , b_3)] \end{aligned}$$

Hence $L \times M$ is modular.

Conversely, let $L \times M$ be modular.

Let $a_1 , a_2 , a_3 \in L$, $a_1 \geq a_2$

$b_1 , b_2 , b_3 \in M$, $b_1 \geq b_2$

then $(a_1 , b_1) , (a_2 , b_2) , (a_3 , b_3) \in L \times M$ and

$$(a_1, b_1) \geq (a_2, b_2).$$

Since $L \times M$ is modular , we fiend

$(a_1, b_1) \wedge [(a_2, b_2) \vee (a_3, b_3)] = (a_2, b_2) \vee [(a_1, b_1) \wedge (a_3, b_3)]$

or $(a_1, b_1) \wedge (a_2 \vee a_3, b_2 \vee b_3) = (a_2, b_2) \vee (a_1 \wedge a_3, b_1 \wedge b_3)$

or $(a_1 \wedge (a_2 \vee a_3), b_1 \wedge (b_2 \vee b_3) = (a_2 \vee (a_1 \wedge a_3), b_2 \vee (b_1 \wedge b_3))$

$\Rightarrow$ $$a_1 \wedge (a_2 \vee a_3) = a_2 \vee (a_1 \wedge a_3)$$

$$b_1 \wedge (b_2 \vee b_3) = b_2 \vee (b_1 \wedge b_3)$$

$\Rightarrow$ L and M are modular. ■

Theorem 3.2.5 : A lattice L is modular if and only if I (L) , the ideal lattice of L is modular.

Proof : Let L be modular .

Let $A, B, C \in I(L)$ be three members s . t., $B \subseteq A$.

We show $A \cap (B \vee C) = B \vee (A \cap C)$

Let $x \in A \cap (B \vee C)$ be any element .

Then $x \in A$ and $x \in B \vee C$.

$\Rightarrow$ $x \in A$ and $x \leq b \vee c$ for some $b \in B$, $c \in C$

Since $b \in B \subseteq A$, $x \vee b \in A$. Let $x \vee b = a$

Now $x \leq b \vee c$, $x \leq a \Rightarrow x \leq a \wedge (b \vee c)$

$\Rightarrow x \leq b \vee (a \wedge c)$ as $a \geq b$ & L is modular.

Again, $a \wedge c \leq a$, $a \in A \Rightarrow a \wedge c \in A$

$a \wedge c \leq c$, $c \in C \Rightarrow a \wedge c \in C$

Thus $a \wedge c \in A \cap C$ and as $b \in B$ we find $x \in B \vee (A \cap C)$

i . e., $A \cap (B \vee C) \subseteq B \vee (A \cap C)$

$B \vee (A \cap C) \subseteq A \cap (B \vee C)$ follows by modular inequality, or to prove independently, let $y \in B \vee (A \cap C)$.

Then $y \leq b \vee k$ where $b \in B$, $k \in A \cap C$

Thus $y \leq b \vee k$, $(b \in B \subseteq A , k \in A \Rightarrow b \vee k \in A)$

$\Rightarrow \quad y \in A$

Also $y \leq b \vee k$, $b \in B , k \in C \Rightarrow y \in B \vee C$

i . e., $\quad y \in A \cap (B \vee C)$

Showing that $B \vee (A \cap C) \subseteq A \cap (B \vee C)$

Hence $A \wedge (B \cup C) = B \vee (A \cap C)$ or that $I (L)$ is modular.

Conversely, let $I (L)$ be modular. Since L can be imbedded into $I (L)$, it is isomorphic to a sublattice of $I (L)$. This sublattice must be modular as $I (L)$ is modular. Hence L is modular. ■

Theorem 3.2.6 : If a , b are any elements of a modular lattice L then

$$[a \wedge b , a] \cong [b , a \vee b] .$$

Proof : We know that an interval in a lattice is a sublattice . We establish the isomorphism .

Define a map $\psi : [a \wedge b , a] \longrightarrow [b , a \vee b]$, s . t.,

$$\psi (x) = x \vee b , \qquad x \in [a \wedge b , a].$$

Then ψ is well defined as

$$x \in [a \wedge b , a] \Rightarrow a \wedge b \leq x \leq a$$

$$\Rightarrow (a \wedge b) \vee b \leq x \vee b \leq a \vee b$$

$$\Rightarrow b \leq x \vee b \leq a \vee b$$

$$\Rightarrow x \vee b \in [b , a \vee b]$$

Also $\quad x_1 = x_2 \Rightarrow x_1 \vee b = x_2 \vee b$

$\Rightarrow \psi (x_1) = \psi (x_2)$.

ψ is one – one as

Let $\quad \psi (x_1) = \psi (x_2) \quad$ [So $x_1 , x_2 \in [a \wedge b , a]$]

Then $\quad x_1 \vee b = x_2 \vee b$

$\Rightarrow \quad a \wedge (x_1 \vee b) = a \wedge (x_2 \vee b)$

$\Rightarrow \; x_1 \vee (a \wedge b) = x_2 \vee (a \wedge b)$ using modularity, $a \geq x_1, x_2$

$\Rightarrow \quad x_1 = x_2$ as $a \wedge b \leq x_1, x_2$

ψ is onto as

Let $y \in [b, a \vee b]$ be any element. We show $a \wedge y$ is the required pre image.

$$y \in [b, a \vee b] \Rightarrow b \leq y \leq a \vee b$$

$$\Rightarrow a \wedge b \leq a \wedge y \leq a \wedge (a \vee b)$$

$$\Rightarrow a \wedge y \in [a \wedge b, a]$$

Also $\quad \psi(a \wedge y) = (a \wedge y) \vee b.$

So we need show $\quad y = (a \wedge y) \vee b.$

Now $\quad y \leq a \vee b \Rightarrow y \wedge (a \vee b) = y$

$\Rightarrow y = y \wedge (b \vee a) = b \vee (y \wedge a)$ (using modularity)

Hence ψ is onto.

Again, $x_1 \leq x_2 \Rightarrow x_1 \vee b \leq x_2 \vee b$

$\Rightarrow \psi(x_1) \leq \psi(x_2)$

And $\quad \psi(x_1) \leq \psi(x_2)$

$\Rightarrow \quad x_1 \vee b \leq x_2 \vee b$

$\Rightarrow \quad a \wedge (x_1 \vee b) \leq a \wedge (x_2 \vee b)$

$\Rightarrow \quad x_1 \vee (a \wedge b) \leq x_2 \vee (a \wedge b)$

$\Rightarrow \quad x_1 \leq x_2$

Thus $\quad x_1 \leq x_2 \Leftrightarrow \psi(x_1) \leq \psi(x_2).$

Hence ψ is an isomorphism. ■

Definition (Transposed) : Two intervals $[a, b]$ and $[c, d]$ of a lattice are called transposed if $b \wedge c = a$ and $b \vee c = d$.

Theorem 3.2.7 : Any non modular lattice L contains a sublattice isomorphic with the pentagonal lattice.

Proof : Since L is non modular $\exists$ at least three elements a , b , c $a \geq b$ s . t., $a \wedge (b \vee c) \neq b \vee (a \wedge c)$.

In view of the remarks of definition, we must have $a > b$, and as in any lattice the modular lattice inequality

$(a \geq b, a \wedge (b \vee c) \geq b \vee (a \wedge c))$ holds,

we get $a \wedge (b \vee c) > b \vee (a \wedge c)$.

Consider the chain

$$a \wedge c \leq b \vee (a \wedge c) < a \wedge (b \vee c) \leq b \vee c \qquad \text{-----} (1)$$

We show at all place, strict inequality holds.

Suppose $a \wedge c = b \vee (a \wedge c)$

Then $\quad b \leq a \wedge c \qquad (x = y \vee x \Rightarrow y \leq x)$

$\Rightarrow \quad b \vee c \leq (a \wedge c) \vee c$

$\Rightarrow \quad b \vee c \leq c \leq b \vee c$

$\Rightarrow \quad b \vee c = c$

$\Rightarrow \quad a \wedge (b \vee c) = a \wedge c$, a contradiction to (1)

Thus $\quad a \wedge c < b \vee (a \wedge c)$. Similarly $a \wedge (b \vee c) < b \vee c$.

Hence chain (1) becomes .

$$a \wedge c < b \vee (a \wedge c) < a \wedge (b \vee c) < b \vee c \qquad \text{---------------} (2)$$

Consider now the chain

$$a \wedge c \leq c \leq b \vee c$$

As seen above $b \vee c = c$ leads to a contradiction and similarly $a \wedge c = c$ would give a contradiction.

Hence $\quad a \wedge c < c < b \vee c \qquad \text{----------------} (3)$

We thus have two chains (2) and (3) with same end points.

We show c does not lie in chain (2) . For this it is sufficient to prove that c is not comparable with $a \wedge (b \vee c)$.

Suppose $a \wedge (b \vee c) \leq c$

Then $a \wedge (a \wedge (b \vee c)) \leq a \wedge c$

$\Rightarrow$ $a \wedge (b \vee c) \leq a \wedge c$ a contradiction to (2)

Again, if $a \wedge (b \vee c) > c$

then as $a \geq a \wedge (b \vee c)$

We find $a > c$ which gives $a \wedge c = c$, a contradiction to (3).

Hence the chain (2) and (3) form a pentagonal subset

$$S = \{ a \wedge c,\ b \vee (a \wedge c), a \wedge (b \vee c), b \vee c, c \} \text{ of } L.$$

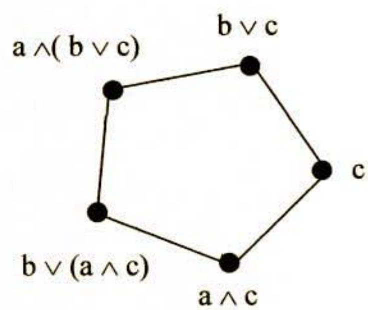

Fig. 3.3

We show now this pentagonal subset is a sublattice. For that meet and join of any two elements of S should lie inside S. Meet and join of any two comparable elements being one of them is clearly in S.

Now $[a \wedge (b \vee c)] \wedge c = a \wedge [(b \vee c) \wedge c] = a \wedge c \in S$

Also $[a \wedge (b \vee c)] \vee c \geq [b \vee (a \wedge c)] \vee c$ by (2)

$= b \vee [(a \wedge c) \vee c] = b \vee c$

and $a \wedge (b \vee c) \leq b \vee c$ gives

$$(a \wedge (b \vee c)) \vee c \leq (b \vee c) \vee c = b \vee c$$

Thus $[a \wedge (b \vee c)] \vee c = b \vee c \in S.$

Similarly, we can show $[b \vee (a \wedge c)] \vee c = b \vee c \in S$

$[b \vee (a \wedge c)] \wedge c = a \wedge c \in S$

Hence S forms a sublattice of L. ■

Theorem 3.2.8 : A lattice L is modular if and only if for $a, b, c \in L$, the three relations $a \geq b$, $a \wedge c = b \wedge c$, $a \vee c = b \vee c$ imply $a = b$.

Proof : Let L be modular and suppose $a, b, c \in L$ are such that $a \geq b$, $a \wedge c = b \wedge c$, $a \vee c = b \vee c$.

Then $a = a \wedge (a \vee c) = a \wedge (b \vee c) = b \vee (a \wedge c)$

$= b \vee (b \wedge c) = b.$ (using modularity and absorption)

Conversely, suppose the condition holds. We want to show that L is modular. Suppose L is not modular. Then by above theorem $\exists$ a pentagonal sublattice

$\{a \wedge c, b \vee (a \wedge c), a \wedge (b \vee c), b \vee c, c\}$ in L

where $a \wedge (b \vee c) > b \vee (a \wedge c)$

Put $a \wedge (b \vee c) = x$ and $b \vee (a \wedge c) = y$ then $x > y$.

Thus we have the three relations

$$x \geq y, \ x \wedge c = y \wedge c \ (= a \wedge c)$$

$$x \vee c = y \vee c \ (= a \vee c).$$

Thus by given condition, we must have $x = y$ which implies

$a \wedge (b \vee c) = b \vee (a \wedge c)$ a contradiction.

Hence L must be modular. ■

Theorem 3.2.9 : A lattice L is modular if and only if no interval $[x, y]$ of L contains an element which has two different comparable complements relative to $[x, y]$.

Proof : Suppose L is modular. Suppose $[x, y]$ is an interval in L such that an element c in $[x, y]$ has two comparable complements a, b $(a \geq b)$ relative to $[x, y]$.

Then $$c \wedge a = c \wedge b \ (= x)$$
$$c \vee a = c \vee b \ (= y)$$

Thus $$a = a \wedge (a \vee c) = a \wedge (b \vee c) = b \vee (a \wedge c)$$
$$= b \vee (b \wedge c)$$
$$= b .$$

i. e., no interval can contain an element which has two different comparable complements relative to the interval.

Conversely, let the given condition hold. Suppose L is not modular.

Then L contains a pentagonal sublattice

$$\{ a \wedge c, b \vee (a \wedge c), a \wedge (b \vee c), b \vee c, c \}$$

Put $$a \wedge (b \vee c) = t,$$
$$b \vee (a \wedge c) = r, \text{ then } t > r$$

Also $$t \wedge c = r \wedge t = a \wedge c = x \quad (\text{say})$$
$$t \vee c = r \vee t = b \vee c = y \quad (\text{say})$$
$$t = t \wedge c \leq t \vee c = y$$
$$\Rightarrow \quad x \leq y$$

i . e., $\exists$ an interval $[x, y]$ in L which has an element c

$$[x = t \wedge c \leq c \leq t \vee c = y],$$

with two different comparable complements t and r relative to

$[x, y]$ a contradiction.

Hence L is modular. ■

Theorem 3.2.10 : A complemented modular lattice is relatively complemented.

Proof : Let L be a complemented modular lattice.

Let $[a, b]$ be any interval in L and $x \in [a, b]$ be any element. Since L is complemented, x has a complement, say, x'.

Then $x \wedge x' = o$, $x \vee x' = u$, $a \leq x \leq b$.

Take $y = a \vee (b \wedge x')$

Then
$$\begin{aligned} x \wedge y &= x \wedge [a \vee (b \wedge x')] \\ &= a \vee (x \wedge (b \wedge x')) \quad [\text{ as } x \geq a, \text{ L is modular }] \\ &= a \vee (b \wedge x \wedge x') \\ &= a \vee (b \wedge o) = a \vee o = a. \end{aligned}$$

$$\begin{aligned} x \vee y &= x \vee [a \vee (b \wedge x')] \\ &= (x \vee a) \vee (b \wedge x') \\ &= x \vee (b \wedge x') \\ &= b \wedge (x \vee x') \qquad [\text{ as } b \geq x, \text{ L is modular }] \\ &= b \wedge u \\ &= b. \end{aligned}$$

Hence $y = a \vee (b \wedge x')$ is relative complement of x in $[a, b]$ proving our assertion. ■

3.3 Distributive Lattice .

Definition (Distributive Lattice) : A lattice L is called a distributive lattice if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad \forall\ a , b , c \in L$$

Example 3.3.1 : The lattice $(P (X) , \subseteq)$ is a distributive lattice as

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) .$$

Example 3.3.2 : A chain is a distributive lattice .

Let a , b , c be any three members of a chain , then any two of these are comparable .

Suppose $a \le b , a \ge c , b \le c$

then $a \le b \le c \le a \Rightarrow a = b = c$.

Thus $a \wedge (b \vee c) = a = (a \wedge b) \vee (a \wedge c)$

If $a \le b , a \ge c , c \le b$

then $c \le a , a \le b , c \le b$

thus $a \wedge (b \vee c) = a \wedge b = a$

$(a \wedge b) \vee (a \wedge c) = a \vee c = a$.

* * A distributive lattice is always modular . Converse is not true as the lattice

M_5 given by

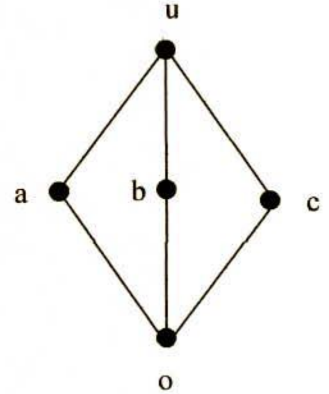

Fig. 3.4

is not distributive , but modular . Notice

$$a \wedge (b \vee c) = a , \quad \text{whereas}$$

$$(a \wedge b) \vee (a \wedge c) = o$$

Theorem 3.3.1: A lattice L is distributive if and only if

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \qquad \forall \ a, b, c \in L .$$

Proof : Let L be distributive .

$$\begin{aligned} \text{Now} \quad (a \vee b) \wedge (a \vee c) &= [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c] \\ &= a \vee [(a \vee b) \wedge c] \qquad [\text{absorption}] \\ &= a \vee [(a \wedge c) \vee (b \wedge c)] \\ &= [a \vee (a \wedge c)] \vee (b \wedge c) \\ &= a \vee (b \wedge c) \end{aligned}$$

Conversely, let $a , b , c \in L$ be any three elements , then

$$\begin{aligned} (a \wedge b) \vee (a \wedge c) &= [(a \wedge b) \vee a] \wedge [(a \wedge b) \vee c] \\ &= a \wedge [(a \wedge b) \vee c] \\ &= a \wedge [(c \vee a) \wedge (c \vee b)] \\ &= [a \wedge (c \vee a)] \wedge (c \vee b) \\ &= a \wedge (c \vee b) \\ &= a \wedge (b \vee c) \end{aligned}$$

Hence L is distributive . ■

Theorem 3.3.2 : A lattice L is distributive if and only if

$$(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \qquad \forall \ a, b, c \in L$$

Proof : Let L be a distributive lattice .

$$\begin{aligned} (a \vee b) \wedge (b \vee c) \wedge (c \vee a) &= \{ a \wedge [(b \vee c) \wedge (c \vee a)] \} \vee \\ &\qquad \{ b \wedge [(b \vee c) \wedge (c \vee a)] \} . \\ &= [\{ a \wedge (c \vee a) \} \wedge (b \vee c)] \vee [\{ b \wedge (b \vee c) \} \wedge (c \vee a)] \end{aligned}$$

$=[a\wedge(b\vee c)]\vee[b\wedge(c\vee a)]$

$=(a\wedge b)\vee(a\wedge c)\vee(b\wedge c)\vee(b\wedge a)$

$=(a\wedge b)\vee(b\wedge c)\vee(c\wedge a)$.

Conversely, we first show that L is modular.

Let x, y, z be any three elements of L, with $x\geq y$.

Then $x\wedge(y\vee z)=[x\wedge(x\vee z)]\wedge(y\vee z)$ (absorption)

$=(x\vee y)\wedge(x\vee z)\wedge(y\vee z)$ $(x\geq y)$

$=(x\vee y)\wedge(y\vee z)\wedge(z\vee x)$

$=(x\wedge y)\vee(y\wedge z)\vee(z\wedge x)$

$=(y\vee(y\wedge z))\vee(z\wedge x)$ $(x\geq y)$

$=y\vee(x\wedge z)$

i. e., L is modular.

Now for any $a, b, c \in L$

$a\wedge(b\vee c)=[a\wedge(a\vee c)]\wedge(b\vee c)$

$=[a\wedge(a\vee b)\wedge(a\vee c)]\wedge(b\vee c)$

$=a\wedge[(a\vee b)\wedge(b\vee c)\wedge(c\vee a)]$

$=a\wedge[(a\wedge b)\vee(b\wedge c)\vee(c\wedge a)]$

$=a\wedge[(b\wedge c)\vee((a\wedge b)\vee(c\wedge a))]$

Now using modularity, $a\geq a\wedge b, a\geq c\wedge a$ gives

$a\geq(a\wedge b)\vee(c\wedge a)$ we get

$a\wedge(b\vee c)=[(a\wedge b)\vee(c\wedge a)]\vee[(b\wedge c)\wedge a]$

$=(a\wedge b)\vee[(c\wedge a)\vee[(c\wedge a)\wedge b]]$

$=(a\wedge b)\vee(c\wedge a)$

Hence L is distributive. ■

Theorem 3.3.3 : Homomorphic image of a distributive lattice is distributive.

Proof : Let $\theta : L \longrightarrow M$ be an onto homomorphism where L is a distributive lattice .

Let $x, y, z \in M$ be any elements . Since θ is onto , $\exists$ $a, b, c \in L$ s. t., $\theta(a) = x, \theta(b) = y, \theta(c) = z$

Now
$$\begin{aligned} x \wedge (y \vee z) &= \theta(a) \wedge [\theta(b) \vee \theta(c)] \\ &= \theta(a) \wedge (\theta(b \vee c)) \\ &= \theta(a \wedge (b \vee c)) \\ &= \theta((a \wedge b) \vee (a \wedge c)) \\ &= \theta(a \wedge b) \vee \theta(a \wedge c) \\ &= (\theta(a) \wedge \theta(b)) \vee (\theta(a) \wedge \theta(c)) \\ &= (x \wedge y) \vee (x \wedge z) . \end{aligned}$$

Shows M is distributive . ■

"Boolean Algebras and Boolean Functions"

4.1 Introduction .

A complemented distributive lattice is called a Boolean lattice . Let $(A, \wedge, \vee, ')$ be a Boolean algebra . Expressions involving members of A and the operations $\wedge$, $\vee$ and complementation are called Boolean expressions . Any function specifying these Boolean expressions is called a Boolean function . A Boolean function is said to be in disjunctive normal form (DN form) in n variables $x_1, x_2, x_3,$ -------------, x_n if it can be written as join of terms of the type

$f_1(x_1) \wedge f_2(x_2) \wedge f_3(x_3) \wedge$ ------------ $\wedge f_n(x_n)$. where $f_i(x_i) = x_i$ for all $i = 1, 2, 3,$ ---------------, n and no two terms are same .

4.2 Boolean Lattices , Boolean Subalgebras .

Definition (Boolean lattice) : A complemented distributive lattice is called a Boolean lattice .

Since compliments are unique in a Boolean lattice we can regard a Boolean lattice as an Boolean algebra with two binary operations $\wedge$ and $\vee$ and one unary operation $'$. Boolean lattices so considered are called Boolean algebras . In other words, by a Boolean Algebra , we mean a system consisting of a non empty set L together with two binary operations $\wedge$ and $\vee$ and a unary operation $'$, satisfying ($\forall$ a , b , c $\in$ L)

(i) $a \wedge a = a, \ a \vee a = a$

(ii) $a \wedge b = b \wedge a, \ a \vee b = b \vee a$

(iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c, \ a \vee (b \vee c) = (a \vee b) \vee c$

(iv) $a \wedge (a \vee b) = a, \ a \vee (a \wedge b) = a$

(v) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(vi) $\forall\, a \in L,\ \exists\ a' \in L$, s . t., $a \wedge a' = o,\ a \vee a' = u$

where o , u are elements of L satisfying

$o \le x \le u \quad \forall\ x \in L$

(a' will be unique and is the complement of a)

Example 4.2.1 : Let S be a non empty set , then (P (S) , $\subseteq$) we know from a distributive lattice and each element has a complement. Thus

(P (S) , $\subseteq$) is a Boolean lattice .

Example 4.2.2 : Let S = Set of factors of 30 , then L forms a Boolean lattice under divisibility .

Example 4.2.3 : Let A = { o , a , b, u } . Define $\wedge, \vee$ and complementation $'$ by

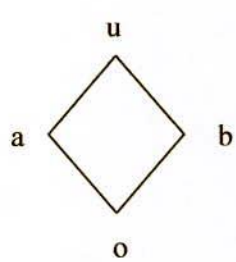

Fig. 4.1

$\wedge$	o	a	b	u
o	o	o	o	o
a	o	a	o	a
b	o	o	b	b
u	o	a	b	u

$\vee$	o	a	b	u
o	o	a	b	u
a	a	a	u	u
b	b	u	b	u
u	u	u	u	u

	$'$
o	u
a	b
b	a
u	o

Then A forms a Boolean algebra under these operations

$\wedge, \vee, '$.

Definition (Boolean subalgebra) : A sub algebra (or a Boolean sub algebra) is a non empty subset S of a Boolean algebra L s. t.,

$a, b \in S \Rightarrow a \wedge b,\ a \vee b,\ a' \in S$.

We thus realize that a sub algebra differs from a sublattice in as much as it is closed under complementation also . Notice that if $[a, b]$ be an interval in a Boolean algebra L , where $a > o$, then $[a, b]$ is a sub lattice of L , but is not a sub algebra as

$$a \in [a, b] \Rightarrow a' \in [a, b]$$
$$\Rightarrow a \wedge a' \in [a, b]$$
$$\Rightarrow o \in [a, b]$$

which is not possible as $a > o$.

Hence a Boolean sublattice may not be a Boolean sub algebra . (The converse being , of course , true)

Problem 4.2.1 : Show that a non empty subset S of a Boolean algebra is a sub algebra if it is closed under $\vee$ and complementation .

Solution : We need prove that for any $a, b \in S$, $a \wedge b \in S$

Now $(a \wedge b)' = a' \vee b' \in S$

$\Rightarrow$ $(a \wedge b) = \left((a \wedge b)'\right)' \in S$

similarly , one can show that S would be a sub algebra if it is closed under $\wedge$ and complementation . ■

Theorem 4.2.2 : In a Boolean algebra , the following results hold

(i) $(a')' = a$

(ii) $(a \wedge b)' = a' \vee b'$ [De Morgan's law]

(iii) $(a \vee b)' = a' \wedge b'$ [De Morgan's law]

(iv) $a \leq b \Leftrightarrow a' \geq b'$

(v) $a \leq b \Leftrightarrow a \wedge b' = o \Leftrightarrow a' \vee b = u$

Proof: (i) Let $(a')' = a''$, then

$$a \wedge a' = o, \quad a \vee a' = u$$

$$a' \wedge a'' = o, \; a' \vee a'' = u$$

$$\Rightarrow \quad a \wedge a' = a'' \wedge a', \; a \vee a' = a'' \vee a'$$

$$\Rightarrow \quad a'' = a$$

(ii) We have $(a \wedge b) \wedge (a' \vee b')$

$$= [(a \wedge b) \wedge a'] \vee [(a \wedge b) \wedge b']$$

$$= [(a \wedge a') \wedge b] \vee [a \wedge (b \wedge b')]$$

$$= [o \wedge b] \vee [a \wedge o]$$

$$= o \vee o$$

$$= o$$

$$(a \wedge b) \vee (a' \vee b') = (a' \vee b') \vee (a \wedge b)$$

$$= [(a' \vee b') \vee a] \wedge [(a' \vee b') \vee b]$$

$$= [(a' \vee a) \vee b'] \wedge [a' \vee (b' \vee b)]$$

$$= (u \vee b') \wedge (a' \vee u)$$

$$= u \wedge u$$

$$= u$$

Hence $(a \wedge b)' = a' \vee b'$

(iii) Similar as (ii)

(iv) $a \leq b \;\Rightarrow\; a = a \wedge b$

$$\Rightarrow a' = (a \wedge b)' = a' \vee b'$$

$$\Rightarrow b' \leq a'$$

$$b' \leq a' \Rightarrow b'' \geq a''$$

$$\Rightarrow b \geq a$$

(v) $a \le b \Rightarrow a \wedge b' \le b \wedge b' \Rightarrow o \le a \wedge b' \le o \Rightarrow a \wedge b' = o$.

Again let $a \wedge b' = o$.

Then $a = a \wedge u = a \wedge (b \vee b') = (a \wedge b) \vee (a \wedge b')$

$$= (a \wedge b) \vee o = (a \wedge b)$$

$\Rightarrow \quad a \le a \wedge b.$

Second result follows similarly. ∎

Problem 4.2.3 : If A, B, C lattices such that $B \cong C$, then

$$A \times B \cong A \times C$$

Solution : Let $f : B \longrightarrow C$ be the given isomorphism.

Define $\theta : A \times B \longrightarrow A \times C$, s.t.,

$$\theta((a, b)) = (a, f(b))$$

then since $\theta((a, b)) = \theta((c, d))$

$\Leftrightarrow \quad (a, f(b)) = (c, f(d))$

$\Leftrightarrow \quad a = c, \quad f(b) = f(d)$

$\Leftrightarrow \quad a = c, \quad b = d$ (f being well defined $1-1$ map)

$\Leftrightarrow \quad (a, b) = (c, d)$

We find θ is a well defined $1-1$ map.

Again, for any $(x, y) \in A \times C$, as $y \in C$, $f : B \longrightarrow C$ is onto, $\exists \; b \in B$, s.t., $f(b) = y$.

Now $\theta((x, b)) = (x, f(b)) = (x, y)$ and thus θ is onto.

Finally,

$$\theta((a, b) \wedge (c, d)) = \theta(a \wedge c, b \wedge d) = (a \wedge c, f(b \wedge d))$$

$$= (a \wedge c, f(b) \wedge f(d))$$

$$= (a, f(b)) \wedge (c, f(d))$$

$$= \theta((a,b)) \wedge \theta((c,d))$$

Similarly, $\theta((a,b) \vee (c,d)) = \theta((a,b) \vee \theta((c,d))$.

Hence θ is an isomorphism . ∎

Theorem 4.2.4 : Let L be a relatively complemented lattice with least element o . Then any ideal of L can equal the Kernel corresponding to at most one congruence relation .

Proof : Let C be any congruence relation on L . Let $a, b \in L$ be any two elements .Then $a \wedge b \in [o, a \vee b]$ and as L is relatively complemented , $a \wedge b$ has a complement , say p, relative to $[o, a \vee b]$. Thus

$$a \wedge b \wedge p = o, \ (a \wedge b) \vee p = a \vee b \text{ or } p \leq a \vee b$$

Let K_c be the kernel corresponding to C , then

$$K_c = \{ x \in L \mid x \, C \, o \}$$

We claim $(a \vee b) C (a \wedge b)$ if and only if $p \in K_c$

Let $(a \vee b) C (a \wedge b)$ then as $p C p$

$$p \wedge (a \vee b) C p \wedge (a \wedge b) \Rightarrow p C o \Rightarrow p \in K_c$$

Again $p \in K_c \Rightarrow p C o$

$$\Rightarrow p \vee (a \wedge b) C o \vee (a \wedge b) \text{ as } (a \wedge b) \vee (a \wedge b)$$

$$\Rightarrow (a \vee b) C (a \wedge b).$$

Hence $(a \vee b) C (a \wedge b) \Leftrightarrow p \in K_c$

Then $a C b \Leftrightarrow p \in K_c$

Suppose now I is any ideal of L such that it equals the Kernels K_{c1} and K_{c2} corresponding to two congruence relations C_1 and C_2 over L.

Then $I = K_{c1}, I = K_{c2} \Rightarrow K_{c1} = K_{c2}$.

Let $a, b \in L$ be any elements , then p exists and

$$a\, C_1\, b \Leftrightarrow p \in K_{c1} \Leftrightarrow p \in K_{c2} \Leftrightarrow a\, C_2\, b$$

i . e., $C_1 = C_2$ which proves our assertion . ■

Theorem 4.2.5 : If L is a Boolean algebra then any ideal of L equals the Kernel corresponding to one and only one congruence relation over L .

Proof : Since L is distributive and has zero , then any ideal equals the Kernel corresponding to at least one congruence relation .

Again , since L is relatively complemented the ideal cannot equal Kernels corresponding to more than one congruence relation . Hence any ideal will equal Kernel corresponding to just one congruence relation . ■

4.3 Rings , Boolean Rings , Boolean Functions .

Definition (Ring) : A non-empty set R together with two binary operations , additions (denoted " + ") and multiplication (denoted by " · ") is called a ring if it is satisfied the following laws :

1. Associative law of addition :

   $(a+b)+c=a+(b+c) \qquad \forall\, a, b, c \in R$

2. Existence of additive identity zero :

   $\exists\ 0 \in R \ \Rightarrow a+0=0+a, \qquad \forall\ a \in R$

3. Existence of additive inverse :

   $a \in R \Rightarrow \exists\ -a \in R \ \Rightarrow a+(-a)=(-a)+a=0, \ \forall\, a \in R$

4. Commutative law of addition :

   $a+b=b+a \qquad \forall\ a, b \in R$

5. Associative law of multiplication :

   $(a.b).c=a.(b.c) \qquad \forall\ a, b, c \in R$

6. Distributive laws :

   (i) Left : $a.(b+c)=a.b+a.c \ \ \forall\, a, b, c \in R$

   (ii) Right : $(a+b).c=a.c+b.c \ \ \forall\, a, b, c \in R$

Definition (Ring with unity) : A ring R is called a ring with unity if there exists an element $1 \neq 0 \in R$ such that $a.1=1.a=a$, $\forall\, a \in R$ where 1 is called the multiplicative identity or multiplicative unity .

Definition (Commutative ring) : A ring R is called commutative ring if under the binary operation of multiplication $a.b=b.a$ $\forall\ a, b \in R$.

Definition (Ring with zero divisor) : A ring R is called with zero divisors if there exists at least two elements a and b of R such that $a \,.\, b = 0$ where $a \neq 0$ and $b \neq 0$.

Definition (Boolean ring) : A ring R is called a Boolean ring if

$$a^2 = a \qquad \forall \ a \in R .$$

Theorem 4.3.1 : Every Boolean algebra is a Boolean ring with unity .

Proof : A Boolean ring is a ring is which $x^2 = x \ \ \forall x$.

Let $(A , \wedge , \vee , ')$ be a Boolean algebra .

Define two operations + and . on A by

$$a \,.\, b = a \wedge b$$

$$a + b = (a \wedge b') \vee (a' \wedge b) \qquad a , b \in A$$

Then + and $\cdot$ are clearly binary compositions on A .

To show that $< A , + , \cdot >$ forms a Boolean ring , we verify all the conditions in the definitions .

Let $a , b , c \in A$ be any members .

$$a + b = (a \wedge b') \vee (a' \wedge b) = (b \wedge a') \vee (b' \wedge a) = b + a$$

$$(a + b) + c = [(a + b) \wedge c'] \vee [(a + b)' \wedge c]$$

$$= [\{ (a \wedge b') \vee (a' \wedge b) \} \wedge c'] \vee [\{ (a \wedge b') \vee (a' \wedge b) \}' \wedge c]$$

$$= [(a \wedge b' \wedge c') \vee (a' \wedge b \wedge c')] \vee [(a \wedge b')' \wedge (a' \wedge b)' \wedge c]$$

$$= [(a \wedge b' \wedge c') \vee (a' \wedge b \wedge c')] \vee [(a' \vee b) \wedge (a \vee b') \wedge c]$$

$$= [(a \wedge b' \wedge c') \vee (a' \wedge b \wedge c')] \vee [\{ (a' \vee b) \wedge a \} \vee \{ a' \vee b) \wedge b'\} \wedge c]$$

$$= [(a \wedge b' \wedge c') \vee (a' \wedge b \wedge c')] \vee [\{ (a' \wedge a) \vee (b \wedge a) \vee (a' \wedge b') \vee (b \wedge b')\} \wedge c]$$

$$= (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c') \vee [\{ (b \wedge a) \vee (a' \wedge b') \} \wedge c]$$

$$= (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c') \vee [(b \wedge a \wedge c) \vee (a' \wedge b' \wedge c)]$$

$= (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c') \vee (b \wedge a \wedge c) \vee (a' \wedge b' \wedge c)$

Since the resulting value is symmetric in a, b, c it will also be equal to $(b+c)+a = a+(b+c)$ (by commutativity of $+$)

Hence $+$ is associative .

Again $a + 0 = (a \wedge u) \vee (a' \wedge 0) = a = 0 + a$.

Also $a + a = (a \wedge a') \vee (a' \wedge a) = 0$.

Thus $(A, +)$ forms an abelian group .

Since $a \cdot b = a \wedge b$ and $\wedge$ is commutative and associative .

We find $\cdot$ is also commutative and associative .

Again, $a(b+c) = a \wedge (b+c) = a \wedge [(b \wedge c') \vee (b' \wedge c)]$

$$= (a \wedge b \wedge c') \vee (a \wedge b' \wedge c)$$

$$\begin{aligned} ab + ac &= (a \wedge b) + (a \wedge c) \\ &= [(a \wedge b) \wedge (a \wedge c)'] \vee [(a \wedge b)' \wedge (a \wedge c)] \\ &= [(a \wedge b) \wedge (a' \wedge c')] \vee [(a' \vee b') \wedge (a \wedge c)] \\ &= [(a \wedge b \wedge a') \vee (a \wedge b \wedge c') \vee (a \wedge c \wedge a') \\ &\qquad \vee (a \wedge c \wedge b')] \\ &= (a \wedge b \wedge c') \vee (a \wedge b' \wedge c) \end{aligned}$$

Hence $a(b+c) = ab + ac$

Similarly, $(b+c)a = ba + ca$.

Finally , since $a \cdot u = a \wedge u = a = u \wedge a = u \cdot a$.

We find $(A, +, \cdot)$ forms a commutative ring with unity u .

Also as $a \cdot a = a \wedge a = a \quad \forall a$

we gather that A forms a Boolean ring . ■

Theorem 4.3.2 : Every Boolean ring with unity is a Boolean algebra .

Proof : Let $< A, +, \cdot >$ be a Boolean ring with unity .

We defined two operations $\wedge$ and $\vee$ on A by

$$a \wedge b = a \cdot b$$

$$a \vee b = a + b + a\,b$$

Then since $\cdot$ is commutative (a Boolean ring is commutative) and associative we find $\wedge$ is commutative and associative .

Again $a \vee a = a + a + a\,a = (a + a)\,a = 0 + a$

(In a Boolean Ring $a + a = 0 \;\; \forall\, a$, where 0 is zero of the ring)

Also $a \vee b = a + b + a\,b = b + a + b\,a = b \vee a$

$$\begin{aligned}(a \vee b) \vee c &= (a \vee b) + c + (a \vee b) \cdot c \\ &= (a + b + a\,b) + c + (a + b + a\,b)\,c \\ &= a + b + a\,b + c + a\,c + b\,c + a\,b\,c\end{aligned}$$

Since , $a \vee (b \vee c) = (b \vee c) \vee a$ (by commutativity of $\vee$)

By symmetry , $(b \vee c) \vee a = b + c + b\,c + a + b\,a + c\,a + a\,b\,c$

Hence $\vee$ is associative .

Finally to check absorption , we find

$$\begin{aligned}a \wedge (a \vee b) &= a (a + b + a\,b) \\ &= a^2 + a\,b + a^2\,b \\ &= a + a\,b + a\,b \\ &= a + 2\,a\,b \\ &= a\end{aligned}$$

(as $x + x = 0 \;\; \forall\, x$)

$$\begin{aligned}a \vee (a \wedge b) = a \vee a\,b &= a + a\,b + a\,a\,b \\ &= a + 2\,a\,b \\ &= a\end{aligned}$$

Thus A is a lattice .

We leave distributively for the reader to verify . Let now $a \in A$ be any element . We show it has a complement , namely , $a + 1$ (where 1 is unity of ring A)

Now $a \wedge (a + 1) = a (a + 1)$

$$= a^2 + a$$

$$= a + a$$

$$= 0$$

$$a \vee (a + 1) = a + a + 1 + a (a + 1)$$

$$= 2a + 1 + a + a$$

$$= 1 + 2a$$

$$= 1$$

Showing that $a' = a + 1$

Notice , in the ring A , $0 \cdot a = 0 \quad \forall \quad a \in A$

(0 being zero of ring)

$\Rightarrow \quad 0 \wedge a = 0 \quad \forall \ a \in A$.

Again $1 \cdot a = a \quad \forall a$

i . e., $1 \wedge a = a \quad \forall a \in A$

Thus 0 and 1 are least and greatest elements of the lattice A . ■

Definition (Boolean function) : Let $(A , \wedge , \vee , ')$ be a Boolean algebra . Expressions involving members of A and the operations $\wedge$, $\vee$ and complementation are called Boolean expressions or Boolean polynomials. For example , $x \vee y'$, x , $x \wedge 0$ etc. are all Boolean expressions . Any function specifying these Boolean expressions is called a Boolean function. Thus if $f (x , y) = x \wedge y$ then f is the Boolean function and $x \wedge y$ is the Boolean expression (or value of the function f) . Since it is normally the functional value (and not the function) that we are interested in , we call these expressions the Boolean functions .

4.4 Disjunctive normal forms , Complete Disjunctive normal forms

Definition (Disjunctive normal form) : A Boolean function (expression) is said to be in disjunctive normal form (DN form) in n variables $x_1, x_2, x_3, \text{---------}, x_n$ if it can be written as join of terms of the type

$$f_1(x_1) \wedge f_2(x_2) \wedge f_3(x_3) \wedge \text{-------------} \wedge f_n(x_n)$$

where $f_i(x_i) = x_i$ or x'_i, for all $i = 1, 2, 3, \text{-----------} n$

and no two terms are same . Also , 1 and 0 are said to be in disjunctive normal form .

Definition (Minterms or minimal polynomials) : The terms of type

$$f_1(x_1) \wedge f_2(x_2) \wedge f_3(x_3) \wedge \text{-------------} \wedge f_n(x_n)$$

are called minterms or minimal polynomials . (A normal form is also called a canonical form) .

For instance , $(x \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z)$ is in disjunctive normal form (in three variables) and each of the brackets is a minterms .

Problem 4.4.1 : Put the function

$f = [(x \wedge y')' \vee z'] \wedge (x' \vee z)'$ in the DN form .

Solution : We have

$$f = [(x' \vee y'') \vee z'] \wedge (z' \wedge x'')$$

$$= (x' \vee y \vee z') \wedge (z' \wedge x)$$

$$= (x' \wedge z' \wedge x) \vee (y \wedge z' \wedge x) \vee (z' \wedge z' \wedge x)$$

$$= 0 \vee (x \wedge y \wedge z') \vee (x \wedge z')$$

$$= (x \wedge y \wedge z') \vee [(x \wedge z') \wedge (y \vee y')] \text{ (Note this step)}$$

$$= (x \wedge y \wedge z') \vee [(x \wedge z' \wedge y) \vee (x \wedge z' \wedge y')]$$

$$= (x \wedge y \wedge z') \vee (x \wedge y' \wedge z'). \quad \blacksquare$$

Definition (Complete disjunctive normal form) : If a disjunctive normal form in n variables contains all the 2^n minterms then it is called the complete disjunctive normal form in n variables .

Example 4.4.1 : For example , $(x \wedge y) \vee (x' \wedge y) \vee (x \wedge y') \vee (x' \wedge y')$ is the complete disjunctive normal form in two variables.

Problem 4.4.2 : Write the function $x \vee y'$ in the disjunctive normal form in three variables x , y , z .

Solution : We have

$$x \vee y' = [x \wedge (y \vee y') \wedge (z \vee z')] \vee [y' \wedge (x \vee x') \wedge (z \vee z')]$$

$$= [\{(x \wedge y) \vee (x \wedge y')\} \wedge (z \vee z')] \vee [\{(y' \wedge x) \vee (y' \wedge x')\} \wedge (z \wedge z')$$

$$= (x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (y' \wedge x \wedge z) \vee (y' \wedge x \wedge z') \vee (y' \wedge x' \wedge z) \vee (y' \wedge x' \wedge z').$$

$$= (x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z'). \blacksquare$$

Problem 4.4.3 : Find the Boolean expression for the function f given by

$$f(x, y, z) = \begin{cases} 1 & \textit{When} \quad x = z = 1, y = 0 \\ & \qquad x = 1, y = z = 0 \\ 0 & \qquad \textit{otherwise} \end{cases}$$

Proof : The function is specified by the minterms

$(x \wedge y' \wedge z)$ and $(x \wedge y' \wedge z')$

i.e., the function in the DN form is

$$(x \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \qquad \blacksquare$$

Example 4.4.2 : Let $A = \{ 0, 1 \}$ and $f : A^2 \longrightarrow A$, be defined by

$$f(x, y) = (x \wedge y) \vee (x' \wedge y) \vee (x \wedge y') \vee (x' \wedge y')$$

i.e., f is in complete DN form . We calculate all values of $f(x, y),\ x, y \in A$.

Now $f(0, 0) = (0 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) = 1$

$f(1, 0) = (1 \wedge 0) \vee (0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) = 1$

$f(0, 1) = (0 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) = 1$

$f(1, 1) = (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0) = 1$

(Note $x = 0 \Leftrightarrow x' = 1$)

We thus notice that in each case , one minterm is $1 \wedge 1 = 1$ and all others are zero . Also the resulting value of $f(x, y)$ is always 1.

If we go through similar process , with a function f which is in complete DN form in three variables x , y , z we will get the same result . we can generalize this result .

Example 4.4.3 : Let $A = \{ 0, 1 \}$ and $f : A^3 \longrightarrow A$ be the function defined by $f(x, y, z) = x \wedge (y \vee z)$, then the functional values of f are given by

$f(0, 0, 0) = 0 \wedge (0 \vee 0) = 0$

$f(1, 0, 0) = 1 \wedge (0 \vee 0) = 0$

$f(0, 1, 0) = 0 \wedge (1 \vee 0) = 0$

$f(0, 0, 1) = 0 \wedge (0 \vee 1) = 0$

$f(1, 1, 0) = 1 \wedge (1 \vee 0) = 1$

$$f(1,0,1) = 1 \wedge (0 \vee 1) = 1$$

$$f(0,1,1) = 0 \wedge (1 \vee 1) = 0$$

$$f(1,1,1) = 1 \wedge (1 \vee 1) = 1$$

which we sometimes write in the tabular form as

x	y	z	f(x,y,z)
0	0	0	0
1	0	0	0
0	1	0	0
0	0	1	0
1	1	0	1
1	0	1	1
0	1	1	0
1	1	1	1

Problem 4.4.4.: Find the Boolean expression that defines the function f given by

$$f(0,0,0) = 0$$

$$f(0,1,0) = 1$$

$$f(0,0,1) = 0$$

$$f(0,1,1) = 0$$

$$f(1,0,0) = 1$$

$$f(1,0,1) = 1$$

$$f(1,1,0) = 0$$

$$f(1,1,1) = 1$$

Solution : We consider those values of $f(x, y, z)$ which are equal to 1. The minterms corresponding to $f(0,1,0)$, $f(1,0,0)$, $f(1,0,1)$ and $f(1,1,1)$ will be

$$(x' \wedge y \wedge z'), (x \wedge y' \wedge z'), (x \wedge y' \wedge z), (x \wedge y \wedge z)$$

Hence the function in DN form is

$$f(x, y, z) = (x' \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee$$
$$(x \wedge y' \wedge z) \vee (x \wedge y \wedge z)$$

which can be simplified

$$= (x' \wedge y \wedge z') \vee x \wedge [(y' \wedge z') \vee (y' \wedge z) \vee (y \wedge z)]$$
$$= (x' \wedge y \wedge z') \vee x \wedge [\{y' \wedge (z' \vee z)\} \vee (y \wedge z)]$$
$$= (x' \wedge y \wedge z') \vee x \wedge [y' \vee (y \wedge z)]$$
$$= (x' \wedge y \wedge z') \vee x \wedge [(y' \vee y) \wedge (y' \vee z)]$$
$$= (x' \wedge y \wedge z') \vee x \wedge (y' \vee z)$$
$$= (x' \wedge y \wedge z') \vee [(x \wedge y') \vee (x \wedge z)].$$ ■

Example 4.4.4 : Complete DN form in 2 variable is

$$(x \wedge y) \vee (x' \wedge y) \vee (x \wedge y') \vee (x' \wedge y')$$

Let $f = (x \wedge y)$ [any one DN form]

then $f' = (x \wedge y)' = x' \vee y'$

$$= [x' \wedge (y \vee y')] \vee [y'(x \vee x')]$$
$$= (x' \wedge y) \vee (x' \wedge y') \vee (y' \wedge x) \vee (y' \wedge x')$$
$$= (x' \wedge y) \vee (x \wedge y') \vee (x' \wedge y').$$

Thus what we gather form here is that if we pick up any DN form from the compete DN form then complement of that DN form will contain the ' left out ' terms in the complete DN form .

Take for instance, $p = (x \wedge y) \vee (x' \wedge y)$

then $p' = [(x \wedge y) \vee (x' \wedge y)]'$

$$= (x \wedge y)' \wedge (x' \wedge y)'$$

$$= (x' \vee y') \wedge (x \vee y')$$

$$= (x \wedge x') \vee y'$$

$$= y'$$

$$= y' \wedge (x \vee x')$$

$$= (y' \wedge x) \vee (y' \wedge x')$$

the ' left out ' terms in the complete DN form .

Problem 4.4.5 : In a Boolean algebra , show that

$$f(x, y) = [x \wedge f(1, y)] \vee [x' \wedge f(0, y)]$$

Solution : We know that any function f (in 2 variables) in complete DN form is

$$f(x, y) = (x \wedge y) \vee (x' \wedge y) \vee (x \wedge y') \vee (x' \wedge y')$$

$$= [x \wedge (y \vee y')] \vee [x' \wedge (y \wedge y')] \text{ ------ } (1)$$

Put $x = 1$, $x' = 0$ and we get

$$f(1, y) = [1 \wedge (y \vee y')] \vee [0 \wedge (y \wedge y')]$$

$$= y \vee y'$$

Again , by putting $x = 0$, $x' = 1$ we get

$$f(0, y) = y \vee y'$$

Thus (1) gives

$$f(x . y) = [x \wedge f(1, y)] \vee [x' \wedge f(0, y)] \qquad \blacksquare$$

4.5 Conjunctive Normal Forms .

Definition (Conjunctive Normal Form) : A Boolean function f is said to be in **conjunctive normal form** (CN form) in n variables

$x_1, x_2, x_3,$ -------x_n

if f is meet of terms of the type

$$f_1(x_1) \vee f_2(x_2) \vee \text{---------------} f_n(x_n)$$

where $f_i(x_i) = x_i$ or x_i' for all $I = 1, 2, 3$ ------------n

and no two terms are same . Also 0 and 1 are said to be in CN form .

Problem 4.5.1 : Put the function

$f = [(x \wedge y') \vee z'] \wedge (x' \vee z)'$ in the CN form .

Solution : We have

$$\begin{aligned}
f &= [(x' \vee y) \vee z'] \wedge (x \wedge z') \\
&= (x' \vee y \vee z') \wedge [(x \wedge z') \vee (y \wedge y')] \\
&= (x' \vee y \vee z') \wedge \{[(x \wedge z') \vee y] \wedge [(x \wedge z') \vee y']\} \\
&= (x' \vee y \vee z') \wedge [(x \vee y) \wedge (z' \vee y) \wedge \\
&\qquad\qquad (x \vee y') \wedge (z' \vee y') \\
&= (x' \vee y \vee z') \wedge [\{x \vee y \vee (z \wedge z')\} \wedge \\
&\qquad \{(z' \vee y) \vee (x \wedge x')\} \wedge \{(x \vee y') \vee (z \wedge z')\} \wedge \\
&\qquad\qquad \{(z' \vee y') \vee (x \wedge x')\} \\
&= (x' \vee y \vee z') \wedge (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge \\
&\qquad (z' \vee y \vee x) \wedge (z' \vee y \vee x') \wedge (x \vee y' \vee z) \wedge \\
&\qquad\qquad (x \vee y' \vee z') \wedge (z' \vee y' \vee x) \wedge (z' \vee y' \vee x') \\
&= (x \vee y \vee z) \wedge (x' \vee y \vee z') \wedge (x \vee y \vee z') \\
&\qquad \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (x' \vee y' \vee z').
\end{aligned}$$

■

Problem 4.5.2 : Put the function $x \wedge (y \vee z)$ in the CN form .

Solution : $x \wedge (y \vee z) = [x \vee (y \wedge y')] \wedge [(y \vee z) \vee (x \wedge x')]$

$= (x \vee y) \wedge (x \vee y') \wedge (y \vee z \vee x) \wedge (y \vee z \vee x')$

$= (x \vee y) \vee (z \wedge z') \wedge (x \vee y') \vee (z \wedge z') \wedge (y \vee z \vee x)$

$\wedge (x' \vee y \vee z)$

$= (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z')$

$\wedge (x \vee y \vee z) \wedge (x' \vee y \vee z)$

$= (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z')$

$\wedge (x' \vee y \vee z)$. ■

Problem 4.5.3 : Find the DN form of the function whose CN form is

$f = (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge$

$(x \vee y' \vee z') \wedge (x' \vee y \vee z)$

Solution : We know $f = (f')'$. Thus

$f = [\{(x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge$

$(x \vee y' \vee z') \wedge (x' \vee y \vee z)\}']'$

$= [(x \vee y \vee z)' \vee (x \vee y \vee z')' \vee (x \vee y' \vee z)' \vee$

$(x \vee y' \vee z')' \vee (x' \vee y \vee z)']'$ (by De Morgan's law)

$= [(x' \wedge y' \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z') \vee$

$(x' \wedge y \wedge z) \vee (x \wedge y' \wedge z')]'$ (by De Morgan's law)

$= (x \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z')$. ■

Note : By similar steps we can find the CN form of a function from its DN form

Problem 4.5 4 : Find the CN form of the function

$f = (x \wedge (y' \vee z)) \vee z'$ and then find its DN form from it .

Solution : $f=(x\wedge(y'\vee z))\vee z'$

$=(x\vee z')\wedge((y'\vee z)\vee z')$

$=x\vee z'$

$=(x\vee z')\vee(y\wedge y')$

$=(x\vee y\vee z')\wedge(x\vee y'\vee z')$

Now $f=(f')'=[\{(x\vee y\vee z')\wedge(x\vee y'\vee z')\}']'$

$=[(x\vee y\vee z')'\vee(x\vee y'\vee z')]'$

$=[(x'\wedge y'\wedge z)\vee(x'\wedge y\wedge z)]'$

$=(x\wedge y\wedge z)\vee(x'\wedge y\wedge z')\vee(x\wedge y\wedge z')$

$\vee(x\wedge y'\wedge z)\vee(x\wedge y'\wedge z')\vee(x'\wedge y'\wedge z')$

If we wish , we can get DN form independently , as

$f=x\vee z'=[x\wedge(y\vee y')]\vee[z'\wedge(x\vee x')]$

$=(x\wedge y)\vee(x\wedge y')\vee(z'\wedge x)\vee(z'\wedge x')$

$=(x\wedge y)\wedge(z\vee z')\vee(x\wedge y')\wedge(z\vee z')\vee$

$(z'\wedge x)\wedge(y\vee y')\vee(z'\wedge x')\wedge(y\vee y')$

$=(x\wedge y\wedge z)\vee(x\wedge y\wedge z')\vee(x\wedge y'\wedge z)$

$\vee(x\wedge y'\wedge z')\vee(z'\wedge x\wedge y)\vee(z'\wedge x\wedge y')$

$\vee(z'\wedge x'\wedge y)\vee(z'\wedge x'\wedge y')$

$=(x\wedge y\wedge z)\vee(x\wedge y\wedge z')\vee(x\wedge y'\wedge z)\vee$

$(x\wedge y'\wedge z')\vee(x'\wedge y\wedge z')\vee(x'\wedge y\wedge z)$. ■

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