On Some Asymptotic Solutions of Fourth Order Critically Damped Nonlinear Systems

M. Phil. THESIS

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<u>Dedicated to My</u>

Beloved Parents

and

Affectionate Daughter

Declaration

I hereby declare that this thesis entitled "On Some Asymptotic Solutions of Fourth Order Critically Damped Non-linear Systems" submitted for the partial fulfillment for the degree of Master of Philosophy is done by myself under the supervision of Dr. Fouzia Rahman and Dr. M. Ali Akbar as supervisor and co-supervisor respectively and is not submitted elsewhere for any other degree or diploma.

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The author

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Abstract

Krylov and Bogoliubov introduced a perturbation method named "asymptotic averaging method". The method was developed only to obtain the periodic solution of second order nonlinear differential systems. Now the method is used to obtain the solutions of oscillatory, damped oscillatory, over-damped, critically damped and more critically damped systems with of second, third, fourth etc. order nonlinear differential equations by imposing some restrictions to make the solutions uniformly valid. The method of Krylov and Bogoliubov has been improved and justified by Bogoliubov and Mitropolskii. In this dissertation, we have modified and extended the Krylov-Bogoliubov-Mitropolskii (KBM) method to investigate the fourth order critically damped and more critically damped nonlinear systems. We have imposed some restrictions on the eigenvalues to determine the unknown functions which are related to the variational equations. To get the solutions of the variational equations, we have replaced the variables by their corresponding linear values. For justification of the solution obtained by the extended KBM method, we have compared the results to those obtained by the fourth order Runge-Kutta method.

Introduction

The subject of differential equation not only is one of the most beautiful parts of mathematics, but it is also an essential tool for modeling many physical situations such as spring mass system, resistor-capacitor-inductor circuits, bending of beams, chemical reactions, pendulums, the motion of the rotating mass around another body and so forth. These equations have also demonstrated their usefulness in ecology, economics and biology. That is a large number of problems in engineering and science can be formulated in the form of differential equation. Therefore the solution of such problems lies essentially in solving the corresponding differential equations. The differential equations may be linear or nonlinear, autonomous or non-autonomous. Practically, numerous differential equations involving physical phenomena are nonlinear. In many cases it is possible to replace such a nonlinear equation by a related linear equation, which approximates the actual non linear equation closely enough to give useful results. The method of small oscillations is a well-known example of the linearization of problems, which are essentially nonlinear. However, such a "linearization" is not always feasible; and when it is not, the original nonlinear equation itself must be considered. Methods of solutions of linear differential equations are comparatively easy and highly developed. Whereas, very little of a general character is known about nonlinear equations. The study of nonlinear equations is generally confined to a variety of rather special cases, and one must resort to various methods of approximation.

Van der pol first paid attention to the new (self-excited) oscillations and indicated that their existence is inherent in the nonlinearity of the differential equations characterizing the process. This nonlinearity appears, thus, as the very essence of these phenomena and by linearizing the differential equation in the sense of the method of small oscillations, one simply eliminates the possibility of investigating such problems. Thus it is necessary to deal with the nonlinear problems directly instead of evading them by dropping the nonlinear terms. To unravel nonlinear differential equations there exist some methods. Among the methods, the method of perturbations, *i.e.*, asymptotic expansions in terms of a small parameter are foremost. Perturbation methods have recently received much attention as methods for accurately and quickly computing numerical solutions of dynamic, stochastic, economic equilibrium models, both single-agent or rational-expectations models and multi-agent or game-theoretic models. A perturbation method is based on the following aspects: The equations to be solved are sufficiently "smooth" or sufficiently differentiable a number of times in the required regions of variables and parameters.

In this thesis, we shall discuss problems that can be described by the dynamical systems of the fourth order nonlinear autonomous differential equations with small nonlinearities by use of the Krylov-Bogoliubov-Mitropolski (KBM) method. The method was developed only to obtain the periodic solutions of second order nonlinear differential equations. Now the method is used to obtain the solutions of oscillatory, damped oscillatory, critically damped, more critically damped and non-oscillatory systems with second, third, fourth etc. order nonlinear differential equations by imposing some restrictions to make the solutions uniformly valid.

An important approach to the study of such nonlinear oscillations is the small parameter expansion. Two widely spread methods in this theory are mainly used in the literature; One is averaging asymptotic method of KBM and the other is multi-time scale method.

In the KBM method the solution starts with the solution of linear equation (sometimes called the generating solution of the linear equation), only using the amplitude and phase of the solution of the linear differential equation which are assumed time dependent functions instead of constants. This method introduces an additional condition on the first derivative of the generating solution for determining the solution of a second order equation.

KBM demanded that the asymptotic solutions are free from secular terms. These assumptions are definitely valid for second and third order equations But for the fourth order equation the correction terms sometimes contain secular terms, although the solution is generated by the classical KBM asymptotic method. For this reason, the traditional solutions fail to explain the proper situation of the systems.

In order to avoid the appearances of secular terms and obtain the desired results, we need to impose some additional conditions. The main objective of this thesis is to find out these limitations and determine the proper solutions under some special conditions. The results may be used in mechanics, physics, chemistry, plasma physics, circuit and control theory, population dynamics etc.

Chapter 1

The Survey and the Proposal:

1.1 The Survey:

The behavior of many physical systems is adequately described by linear differential and algebraic equations, and the solution for the simulation response is usually a straightforward procedure and well established. However, systems whose response must be described by non linear equations may present special difficulties.

During the last several decades a number of Russian scientists, like, Mandelstam and Papalexi [39], Krylov and Bogoliubov [31], Bogoliubov and Mitropolskii [13] worked jointly and investigated nonlinear mechanics. An important aspect of various perturbation methods is their relationship with each other. Among them, Krylov and Bogoliubov are certainly to be found most active. In most treatments of nonlinear oscillations by perturbation methods only periodic oscillations are treated, transients are not considered. Krylov and Bogoliubov [31] have introduced a new perturbation method to discuss transients. They considered primarily equations of the form

$$\ddot{x} + \omega^2 x = \varepsilon\, f\,(x, \dot{x}, t, \varepsilon) \tag{1.1}$$

where ε is a small positive parameter and f is a power series in ε, where coefficients are polynomials in $x, \dot{x},\ \sin t$ and $\cos t$. The method of Krylov and Bogoliubove (KB) starts with the solution of the linear equation, assuming that in the nonlinear case, the amplitude and phase in the solution of the linear equation are time dependent functions rather than constants. This procedure introduces an additional condition on the first derivative of the assumed solution for determining the solution.

Extensive uses have been made and some important works are done by Stoker [92], Mc Lachlan [40], Minorsky [43], Nayfeh [50, 51] and Bellman *et al.* [11].

The method of Krylov and Bogoliubov is an asymptotic method in the sense that $\varepsilon \to 0$. An asymptotic series itself may not be convergent, but for a fixed number of terms, the approximate solution tends to the exact solution as ε tends to zero. It is noted that the term asymptotic is frequently used in the theory of oscillations also in the sense that $\varepsilon \to \infty$. But in this case the mathematical method is quite different.

In general, f contains neither ε nor t thus the equation (1.1) can be written as

$$\ddot{x} + \omega^2 x = \varepsilon f\,(x, \dot{x}) \tag{1.2}$$

When $\varepsilon = 0$, the equation (1.2) reduces to linear equation and its solution is

$$x = a\cos(\omega t + \varphi) \tag{1.3}$$

where a and φ are arbitrary constants to be determined using initial conditions.

When $\varepsilon \neq 0$, but is sufficiently small, then Krylov and Bogoliubov assumed that the solution of (1.2) is still given by (1.3) together with the derivative of the form.

$$\dot{x} = -a\omega \sin(\omega t + \varphi) \tag{1.4}$$

where a and φ are functions of t, rather than being constants. In this case the solution of (1.2) is

$$x = a(t)\cos(\omega t + \varphi(t)) \tag{1.5}$$

and the derivative of the solution is

$$\dot{x} = -\,a(t)\omega \sin(\omega t + \varphi(t)) \tag{1.6}$$

Differentiating the assumed solution (1.5) with respect to t, we obtain

$$\dot{x} = \dot{a}\cos\psi - a\omega\sin\psi - a\dot{\varphi}\sin\psi, \quad \psi = \omega t + \varphi \tag{1.7}$$

Comparing (1.4) and (1.7), we obtain

$$\dot{a}\cos\psi - a\,\dot{\varphi}\sin\psi = 0 \tag{1.8}$$

Again differentiating (1.6) with respect to t, we have

$$\ddot{x} = -\dot{a}\,\omega\sin\psi - a\,\omega^2\cos\psi - a\omega\dot{\varphi}\cos\psi \tag{1.9}$$

Substituting the value of $\ddot{x}$ from (1.9) in to the equation (1.2) and using equations (1.5) – (1.6), we obtain.

$$\dot{a}\,\omega\sin\psi + a\omega\dot{\varphi}\cos\psi = -\varepsilon\, f(a\cos\psi, -a\omega\sin\psi) \tag{1.10}$$

Solving (1.8) and (1.10) for $\dot{a}$ and $\dot{\varphi}$, yields

$$\dot{a} = -\frac{\varepsilon}{\omega}\sin\psi\; f\,(a\cos\psi, -a\omega\sin\psi) \tag{1.11}$$

$$\dot{\varphi} = -\frac{\varepsilon}{a\omega}\cos\psi\; f\,(a\cos\psi, -a\omega\sin\psi) \tag{1.12}$$

It is seen that the original equation (1.2) of the second order and provides a system of equations, (1.11) and (1.12), each of the first order. The interesting feature of this transformation lies in the fact that these first-order equations are now written in terms of the amplitude a and phase φ as dependent variables

From the form of the right sides of equations (1.11) and (1.12), it is seen that both $\dot{a}$ and $\dot{\varphi}$ are periodic functions of time. From the fact that the right-hand terms of these equations contain a small parameter ε, one can conclude that both a and φ, being periodic, and are functions which vary slowly during one period $T = \dfrac{2\pi}{\omega}$ as trigonometric functions are involved.

It is reasonable, therefore, to consider a and φ as constant during a period T. It is possible to transform equation (1.11) and (1.12) into more convenient form. For this purpose, expanding $\sin\psi\, f(a\cos\psi, -a\omega\sin\psi)$ and $\cos\psi\, f(a\cos\psi, -a\omega\sin\psi)$ in Fourier series in the total phase ψ, the first approximate solution of (1.2), by averaging (1.11) and (1.12) over one period is

$$
\begin{aligned}
\dot{a} &= -\frac{\varepsilon}{2\pi\,\omega}\int_0^{2\pi}\sin\psi\, f(a\cos\psi,\, a\omega\sin\psi)\; d\psi \\
\dot{\varphi} &= -\frac{\varepsilon}{2\pi\,\omega a}\int_0^{2\pi}\cos\psi\, f(a\cos\psi,\, a\omega\sin\psi)\, d\psi
\end{aligned}
\qquad (1.13)
$$

where a and φ are independent of time under the integrals.

The first approximation in the form in which they were originally obtained by Krylov and Bogoliubov [31] and in which they are generally used in applications.

Later, this technique has been amplified and justified mathematically by Bogoliubov and Mitropolskii [13], and extended to non-stationary vibrations by Mitropolskii [44]. They assumed the solution of the nonlinear differential equation (1.2) in the form

$$
x = a\cos\psi + \varepsilon\, u_1\,(a,\psi) + \varepsilon^2\, u_2\,(a,\psi) + \dots\dots\dots + \varepsilon^n\, u_n\,(a,\psi) + O(\varepsilon^{n+1}) \qquad (1.14)
$$

where, u_k, $k = 1, 2, \dots, n$ are periodic functions of ψ with a period 2π, and the quantities a and ψ are functions of time t, defined by

$$
\begin{aligned}
\dot{a} &= \varepsilon\, A_1(a) + \varepsilon^2\, A_2\,(a) + \dots\dots + \varepsilon^n A_n\,(a) + O(\varepsilon^{n+1}) \\
\dot{\psi} &= \omega + \varepsilon\, B_1(a) + \varepsilon^2 B_2\,(a) + \dots\dots + \varepsilon^n B_n\,(a) + O(\varepsilon^{n+1})
\end{aligned}
\qquad (1.15)
$$

where u_k, A_k and B_k, $(k = 1, 2, \dots, n)$ are to be chosen in such a way that the equation (1.14) and (1.15) satisfy the differential equation (1.2). Since there are no restrictions in choosing the functions A_k and B_k, that generate the arbitrariness in the definitions of the functions u_k (Bogoliubov and Mitropolskii [13]). To remove this arbitrariness, the following additional conditions are imposed

$$\begin{aligned} &\int_0^{2\pi} u_k\,(a,\psi)\cos\psi\, d\psi = 0, \\ &\int_0^{2\pi} u_k\,(a,\psi)\sin\psi\, d\psi = 0. \end{aligned} \tag{1.16}$$

These conditions guarantee the absence of secular terms in all successive approximations. Differentiating (1.14) twice with respect to t, substituting $x, \dot{x}$ *and* $\ddot{x}$, using the relations (1.15) and equating the coefficients of ε^k, ($k = 1, 2, \cdots, n$) one obtains recursive systems

$$\omega^2 \left(\frac{\partial^2 u_k}{\partial \psi^2} + u_k \right) = f^{(k-1)}\,(a, \psi) + 2\omega\left(aB_k \cos\psi + A_k \sin\psi\right), \tag{1.17}$$

where

$$f^{(0)}\,(a,\psi) = f\,(a\cos\psi,\, -a\omega\sin\psi),$$

$$\begin{aligned} f^{(1)}\,(a,\psi) = {} & u_1\, f_x\,(a\cos\psi,\, -a\omega\sin\psi) + \left(A_1\cos\psi - aB_1\sin\psi + \omega\frac{\partial u_1}{\partial\psi}\right) \\ & \times f_{\dot{x}}\,(a\cos\psi,\, -a\omega\sin\psi) + \left(aB_1^2 - A_1\frac{dA_1}{da}\right)\cos\psi + \left(2A_1 B_1 - aA_1\frac{dB_1}{da}\right)\sin\psi \\ & - 2\omega\left(A_1\frac{\partial^2 u_1}{\partial a\,\partial\psi} + B_1\frac{\partial^2 u_1}{\partial\psi^2}\right) \end{aligned} \tag{1.18}$$

Here $f^{(k-1)}$ is a periodic function of ψ with period 2π depending also on the amplitude a. Therefore, $f^{(k-1)}$ as well as u_1 can be expanded in a Fourier series as

$$\begin{aligned} f^{(k-1)}\,(a,\psi) &= g_0^{(k-1)}(a) + \sum_{n=1}^{\alpha}\left(g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi\right) \\ u_k\,(a,\psi) &= v_0^{(k-1)}(a) + \sum_{n=1}^{\alpha}\left(v_n^{(k-1)}\cos n\psi + w_n^{(k-1)}(a)\sin n\psi\right) \end{aligned} \tag{1.19}$$

where

$$g_0^{(k-1)} = \frac{1}{2\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\,d\psi$$

$$g_n^{(k-1)} = \frac{1}{\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\cos n\psi\,d\psi \tag{1.20}$$

$$h_n^{(k-1)} = \frac{1}{\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\sin n\psi\,d\psi, \quad n \geq 1$$

Here $v_1^{(k-1)} = w_1^{(k-1)} = 0$ for all values of k, since both integrals of (1.16) vanish.

Substituting these values into the equation (1.17), yield.

$$\begin{aligned}&\omega^2 v_0^{(k-1)}(a) + \sum_{n=1}^{\alpha}\omega^2(1-n^2)[v_n^{(k-1)}(a)\cos n\psi + w_n^{(k-1)}(a)\sin n\psi]\\ &= g_0^{(k-1)}(a) + (g_1^{(k-1)}(a) + 2\omega a B_k)\cos n\psi + (h_1^{(k-1)}(a) + 2\omega A_k)\sin\psi\\ &+ \sum_{n=2}^{\alpha}[g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi]\end{aligned} \tag{1.21}$$

Now equating the coefficients of the harmonics of the same order, give

$$g_1^{(k-1)}(a) + 2\omega a B_k = 0, \qquad h_1^{(k-1)}(a) + 2\omega A_k = 0, \quad v_0^{(k-1)}(a) = \frac{g_0^{(k-1)}(a)}{\omega^2},$$

$$v_n^{(k-1)}(a) = \frac{g_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad w_n^{(k-1)}(a) = \frac{h_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \; n \geq 1 \tag{1.22}$$

These are the sufficient conditions to obtain the desired order of approximation. For the first order approximation, we have

$$\begin{aligned}A_1 &= -\frac{h_1^{(0)}(a)}{2\omega} = -\frac{1}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, -aw\sin\psi)\sin\psi\,d\psi,\\ B_1 &= -\frac{g_1^{(0)}}{2a\omega} = -\frac{1}{2\pi a\omega}\int_0^{2\pi} f(a\cos\psi, -aw\sin\psi)\cos\psi\,d\psi.\end{aligned} \tag{1.23}$$

Thus, the variational equations (1.15) become

$$\dot{a} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\, d\psi,$$
$$\dot{\psi} = \omega - \frac{\varepsilon}{2\pi a\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi\, d\psi. \tag{1.24}$$

We see that the equations of (1.24) are similar to the equations in (1.13). Therefore the first order solution obtained by Bogoliubov and Mitropotskii [13] is identical to the original solution obtained by KB [31]. In the second method, higher order solution can be found easily. The unknown function u_1 called correction term, is obtained from (1.22) &(1.19) as

$$u_1 = \frac{g_0^{(1)}(a)}{\omega^2} + \sum_{n=2}^{\alpha} \frac{g_n^{(1)}(a)\cos n\psi + h_n^{(1)}(a)\cos n\psi}{\omega^2\ (1-n^2)} \tag{1.25}$$

The solution (1.14) together with u_1 is known as the first order improved solution in which a and ψ are obtained from (1.24). If the values of the functions A_1 and B_1 are substituted form (1.23) into (1.18), the function $f^{(1)}$, and in the similar manner, the functions A_1, B_1 and u_2 can be found. Therefore the determination of the higher order approximation is complete.

Somewhat different nonlinear phenomena occur when the amplitude of the dependent variable of a dynamical system is less or greater than unity. The damping is negative when the amplitude is less than unity and the damping is positive when the amplitude is greater than unity. The governing equation having these phenomena is

$$\ddot{x} - \varepsilon\,(1 - x^2)\,\dot{x} + x = 0 \tag{1.26}$$

This equation is known as Van der Pol [94] equation. This equation has very extensive field of application in connection with self-excited oscillations in electron-tube circuits.

The method of KB is very similar to that of Van der Pol and related to it. Van der Pol applies the method of variation of constants to the basic solution $x = a\cos\omega t + b\sin\omega t$ of $\ddot{x} + \omega^2 x = 0$, on the other hand KB apply the same method to the basic solution $x = a\cos(\omega t + \varphi)$ of the same equation. Thus in the KB method the varied constants are a and φ, while in the Van der Pol's method the constants are a and b. The method of KB seems more interesting form the point of view of applications, since it deals directly with the amplitude and phase of the quasi-harmonic oscillation.

Volosov [95] and Museenkov [49] also obtained higher order effects.

The KB method has been extended by Kruskal [30] to solve the fully nonlinear differential equation

$$\ddot{x} = F(x, \dot{x}, \ddot{x}) \tag{1.27}$$

The solution of this equation is based on recurrent relations and is given as the power series of the small parameter.

Cap [26] has studied nonlinear systems of the form

$$\ddot{x} + \omega^2 f(x) = \varepsilon F(x, \dot{x}) \tag{1.28}$$

He solved this equation by using elliptic functions in the sense of Krylov and Bogoliubov. Later, the method of Krylov-Bogoliubov-Mitropolskii (KBM) has been extended by Popov [55] to damped nonlinear systems

$$\ddot{x} + 2k\dot{x} + \omega^2 x = \varepsilon f(x, \dot{x}) \tag{1.29}$$

where $-2k\dot{x}$ is the linear damping force and $0 < k < \omega$. It is noteworthy that, because of the importance of the Popov's method [55] in the physical systems, involving damping force, Mendelson [41] and Bojadziev [24] rediscovered Popov's results. In case of damped nonlinear systems the first equation of (1.15) has been replaced by

$$\dot{a} = -ka + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots\ldots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}) \tag{1.30}$$

Murty *et al.* [47] found a hyperbolic type asymptotic solution of an over-damped system represented by the nonlinear differential equation (1.29) in the sense of KBM method; *i. e*. in the case $k > \omega$.They used hyperbolic function, $\cosh\varphi$ or $\sinh\varphi$ instead of the harmonic function, which is used in [13, 31, 41, 55]. In the case of oscillatory or damped oscillatory process $\cosh\varphi$ may be used arbitrarily for all kinds of initial conditions. But in case of non-oscillatory systems $\cosh\varphi$ or $\sinh\varphi$ should be used depending on the given set of initial conditions (Bojadziev and Edwards [23], Murty *et al.* [47], Murty [48]). Murty and Deekshatulu [46] developed a simple analytical method to obtain the time response of second order nonlinear over-damped systems with small nonlinearity represented by the equation (1.29), based on the Krylov-Bogolibov method of variation of parameters. Shamsul [82] extended the KBM method to find solutions of over-damped nonlinear systems, when one root becomes much smaller than the other root. Murty [48] has presented a unified KBM method for solving the nonlinear systems represented by the (1.29) which cover the undamped, damped and overdamped cases. Bojadziev and Edwards [23] investigated solutions of oscillatory and non-oscillatory systems represented by (1.29) when k and ω are slowly varying functions of time t. Arya and Bojadziev [9, 10] examined damped oscillatory systems and time-dependent oscillating systems with slowly varying parameters and delay. Shamsul *et al.* [72] extended the Krylov-Bogoliubov-Mitropolskii method to certain non-oscillatory nonlinear systems with varying coefficients. Later, Shamsul [84] has unified the KBM method for solving *n*-th order nonlinear differential equation with varying coefficients. Sattar [63] has developed an asymptotic method to solve a second order critically damped nonlinear system represented by (1.29). He has found the asymptotic solution of the system (1.29) in the form

$$x = a(1+\psi) + \varepsilon u_1(a,\psi) + \cdots + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}) \qquad (1.31)$$

where a is defined in the equation (1.30) and ψ is defined by

$$\dot{\psi} = 1 + \varepsilon\, C_1(a) + \varepsilon^2\, C_2\,(a) + \ldots.. + \varepsilon^n C_n\,(a) + O(\varepsilon^{n+1}), \tag{1.32}$$

Shamsul [69] has developed a new perturbation technique to find approximate analytical solution of both second order over-damped and critically damped nonlinear systems. Later, he [78] extended the method to *n*-th order nonlinear differential systems. Shmsul [79, 85] has also extended the KBM method for certain non-oscillatory nonlinear systems when the eigenvalues of the unperturbed equation are real and non-positive. Shamsul [71] has presented a new perturbation method based on the work of the Krylov-Bogoliubov-Mitropolskii method to find approximate solutions of second order nonlinear systems with large damping. Shamsul *et al.* [74] investigated perturbation solution of a second order time-dependent nonlinear system based on the modified Krylov-Mitropolskii method.

Making use of the KBM method Bojadziev [14] has investigated solutions of nonlinear damped oscillatory systems with small time lag. Bojadzive [19] has also found solutions of damped forced nonlinear vibrations with small time delay. Bojadziev [20], Bojadziev and Chan [21] applied the KBM method to problems of population dynamics. Bojadziev [22] used the KBM method to investigate solutions of nonlinear biological and biochemical systems. Lin and Khan [35] have also used the KBM method to some biological problems. Proskurjakov [56], Bojadziev *et al.* [15] have investigated periodic solutions of nonlinear systems by the KBM and Poincare method, and compared the two solutions. Bojadziev and Lardner [16, 17] have investigated monofrequent oscillations in mechanical systems including the case of internal resonance, governed by hyperbolic differential equation with small nonlinearities. Bojadziev and Lardner [18] have also investigated solution for a certain hyperbolic partial differential equation with small nonlinearity and large time delay included into both unperturbed and perturbed parts of the equation.

Osiniskii [52], first extended the KBM method to a third order nonlinear differential equation

$$\dddot{x} + k_1 \ddot{x} + k_2 \dot{x} + k_3 x = \varepsilon f(x, \dot{x}, \ddot{x}) \tag{1.33}$$

where ε is a small positive parameter and f is a nonlinear function. Osiniskii assumed that the asymptotic solution is in the form

$$x = a + b\cos\psi + \varepsilon u_1(a,b,\psi) + \cdots + \varepsilon^n u_n(a,b,\psi) + o(\varepsilon^{n+1}) \tag{1.34}$$

where each $u_{k,}$ $k = 1,2,\ldots,n$ is a periodic function of ψ with period 2π and, a, b and ψ are functions of time , given by

$$\begin{aligned}
\dot{a} &= -\lambda a + \varepsilon\ A_1(a) + \varepsilon^2\ A_2\ (a) + \ldots.. + \varepsilon^n A_n\ (a) + O(\varepsilon^{n+1}) \\
\dot{b} &= -\mu b + \varepsilon\ B_1(b) + \varepsilon^2\ B_2\ (b) + \ldots.. + \varepsilon^n B_n\ (b) + O(\varepsilon^{n+1}) \\
\dot{\psi} &= \omega + \varepsilon\ C_1(b) + \varepsilon^2 C_2\ (b) + \ldots.. + \varepsilon^n C_n\ (b) + O(\varepsilon^{n+1})
\end{aligned} \tag{1.35}$$

where $-\lambda, -\mu \pm \omega$ are the eigenvalues of the equation (1.31) when $\varepsilon = 0$.

Osiniskii [53] has also extended the KBM method to a third order nonlinear partial differential equation with initial friction and relaxation. Mulholland [45] studied nonlinear oscillations governed by a third order differential equation. Lardner and Bojadziev [33] investigated nonlinear damped oscillations governed by a third order partial differential equation. They introduced the concept of "couple amplitude" where the unknown functions A_k, B_k and C_k depend on both the amplitudes a and b. Rauch [57] has studied oscillations of a third order nonlinear autonomous system. Bojadziev [24], Bojadziev and Hung [25] developed a technique by using the method of KBM to investigate a weakly nonlinear mechanical system with strong damping. Sattar [64] has extended the KBM asymptotic method for three-dimensional over-damped nonlinear systems. First, Shamsul and Sattar [66] developed a method to solve third order critically damped autonomous nonlinear differential systems. Shamsul [77] redeveloped the method presented in [66] to find approximate solutions of critically damped nonlinear systems in the presence of different damping forces by considering different sets of variational equations. Later, he unified the KBM method for solving critically damped nonlinear systems [91]. Shamsul and Sattar [70] studied time dependent third order oscillating systems with damping based on an extension of the asymptotic method of Krylov-Bogoliubov-Mitropolskii. Shamsul [82], Shamsul *et al.* [89] has developed a simple method to obtain the time response of some order over-damped nonlinear systems together with slowly varying coefficients under some special conditions. Later, Shamsul [78], Shamsul and Bellal [83] have extended the method presented in [82] to obtain the time response of n-th order $(n \geq 2)$, over-damped systems. Shamsul [81] has also developed a method for obtaining non-oscillatory solution of third order nonlinear systems. Shamsul and Sattar [67] presented a unified KBM method for solving third order nonlinear systems. Shamsul [75] has also presented a unified Krylov-Bogoliubov-Mitropolskii method, which is not the formal form of the original KBM method, for solving n-th order nonlinear systems. The solution contains some unusual variables. Yet this solution is very important. Shamsul [87] has also presented a modified and compact form of the Krylov-Bogoliubov-Mitropolskii unified method for solving a n-th order nonlinear differential equation. The formula presented in [87] is compact, systematic and practical, and easier then that of [75]. Shamsul [88] developed a general formula based on the extended Krylov-Bogoliubov-Mitropolskii method, for obtaining asymptotic solution of an n-th order time dependent quasi linear differential equation with damping. Bojadziev [24], Bojadziev and Hung [25] used at least two trial solutions to investigate time dependent differential systems; one is for resonant case and the other is for the non-resonant case. But Shamsul [88] used only one set of variational equations, arbitrarily for both resonant and non-resonant cases. Shamsul *et al.* [90] presented a general form of the KBM method for solving nonlinear partial differential equations. Raymond and Cabak [58] examined the effects of internal resonance on impulsive forced nonlinear systems with two-degree-of-freedom. Ali Akbar *et al.* [2, 3] found asymptotic solution of fourth order over-damped and under-damped nonlinear systems based on the work of [75]. Ali Akbar *et al.* [4] also developed a simple technique for obtaining certain over-damped solution of an n-th order nonlinear differential equation. Ali Akber *et al.* [5] presented the KBM unified method for solving n-th order nonlinear systems under some special conditions including the case of internal resonance. Ali Akbar *et al.* [7] also developed perturbation theory for fourth order nonlinear systems with large damping. Ali Akbar *et al.* [6] developed an asymptotic method for fourth order more critically damped nonlinear systems.

1.2 The Proposal:

We propose a perturbation system to solve fourth order nonlinear differential equations

$$x^{(4)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = -\varepsilon\, f(x, \dot{x}, \ddot{x}, \dddot{x})$$

where ε is the small positive parameter; k_1, k_2, k_3, k_4 are constants, and f is the given nonlinear function.

The Krylov-Bogoliubov-Mitropolskii (KBM) method for solving fourth order critically damped nonlinear systems is presented in **Chapter 2**. In **Chapter 3** we have investigated solutions of fourth order more critically damped nonlinear systems.

Chapter 2

Asymptotic Solutions of Fourth Order Critically Damped Nonlinear Systems Under Some Special Conditions

2.1 Introduction

Most of the well-known perturbations methods *e. g.*, Struble's method [93] Krylov-Bogoliubov-Mitropolskii (KBM) method [13, 31], and multiple time-scale method [50] was originally formulated to find periodic solution of second order nonlinear differential equations with small nonlinearities,

$$\ddot{x} + \omega^2 x = -\varepsilon\ f(x, \dot{x}),\ \varepsilon << 1 \tag{2.1}$$

Several authors extended these methods to investigate similar nonlinear differential equation with a strong linear damping effect, $-2k\dot{x},\ k = O(1)$, modeled by the following equation

$$\ddot{x} + 2k\dot{x} + \omega^2 x = -\varepsilon\ f(x, \dot{x}) \tag{2.2}$$

Popov [55] was familiar among them, who extended the KBM method and investigated the damped oscillatory case of equation (2.2). Owing to physical importance of this method, Mendelson [41] rediscovered the Popov's results. Murty *et al.* [47] have developed an asymptotic method based on the method of Bogoliubov to obtain the response of nonlinear over-damped system. Murty [48] also presented a unified method for solving equation (2.2). Such a unified solution is a general one and covers the three cases; *i. e.*, under-damped, undamped and over-damped cases. It is noted that, the unified solution represent the original KBM solution [13, 31] as the limit $k \to 0+$. Sattar [63] has found an asymptotic solution of a second order critically damped nonlinear system. Sattar [64] also studied third order over-damped system. Shamsul [80] investigated some special over-damped systems whose eigenvalues are in integral multiple. Shamsul [77] also studied a third order critically damped nonlinear system whose unequal eigenvalues are in integral multiple. Shamsul and Sattar [66] have extended Bogoliubov's asymptotic method to a third order critically damped nonlinear system. Shamsul and Sattar [67] have also presented a unified method for obtaining approximate solutions of third order damped and over-damped oscillatory nonlinear systems based on the KBM method.

Murty *et al.* [47] also extended the KBM method to solve fourth order over-damped nonlinear systems. But their method was too much complex and laborious. Ali Akbar *et al.* [2] again presented an asymptotic method for fourth order over-damped nonlinear systems which is simple and easier than the method presented by Murty *et al.* but the results obtained by Ali Akbar *et al.* method is same as the results obtained by Murty *et al.* method. Later, Ali Akbar *et al.* [3] extended the method presented in [2] for fourth order damped oscillatory systems. Ali Akbar *et al.* [4] also presented a simple technique for obtaining certain over-damped solutions of an n-th order nonlinear differential equation. Rokibul *et al.* [60] have extended the KBM method fourth order critically damped nonlinear systems.

In this chapter, we have extended the KBM method for solving fourth order critically damped nonlinear differential systems which is different from the technique presented by Rokibul *et al.* [60]. The solutions obtained by the presented method show good coincidence with those obtained by numerical method.

2.2 The method

Let us consider the following fourth order weakly nonlinear ordinary differential equation

$$x^{(4)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = -\varepsilon\, f(x,\dot{x},\ddot{x},\dddot{x}) \tag{2.3}$$

where $x^{(4)}$ represents the fourth derivative of x with respect to t, over dots are used for the first, second and third derivatives with respect to *t*; ε is the small parameter; k_1, k_2, k_3, k_4 are constants, and f is the given nonlinear function. Since the system is critically damped, so the eigenvalues are real, negative and two of them are equal. Suppose the four eigenvalues are $-\lambda_1$, $-\lambda_2$, $-\lambda_3$ and $-\lambda_4$, where two of the eigenvalues say $-\lambda_1$ and $-\lambda_2$ are equal. When $\varepsilon = 0$, the equation (2.3) becomes linear and the solution of the linear equation is

$$x(t,0) = \left(a_{1,0} + t\, a_{2,0}\right)\, e^{-\lambda_2 t} + a_{3,\,0}\, e^{-\lambda_3 t} + a_{4,\,0}\, e^{-\lambda_4 t} \tag{2.4}$$

where $a_{j,\,0}$ $(j = 1, 2, 3, 4)$ are constants of integration.

When $\varepsilon \neq 0$, following Shamsul [75] a solution of the equation (2.3) is sought in the form

$$x(t,\varepsilon) = (a_1(t) + t\, a_2(t))e^{-\lambda_2 t} + a_3(t)\, e^{-\lambda_3 t} + a_4(t)\, e^{-\lambda_4 t} + \varepsilon\, u_1(a_1,a_2,a_3,a_4,t) + \dots \tag{2.5}$$

where each a_j, $(j = 1, 2, 3, 4)$ satisfy the first order differential equation

$$\dot{a}_j(t) = \varepsilon\, A_j(a_1,a_2,a_3,a_4,t) + \dots \tag{2.6}$$

Confining to only a first few terms $1, 2, 3, \dots, n$ in the series expansion of (2.5) and (2.6), we evaluate the functions $u_j, A_j,\ j = 1, 2, 3, \dots, n$, such that $a_j(t),\ j = 1, 2, 3, \dots, n$, appearing in (2.5) and (2.6) satisfy the given differential equation (2.3) with an accuracy of order ε^{n+1}. In order to determine these unknown functions it is customary in KBM method that the correction terms, u_j must exclude terms (known as secular terms) which make them large. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a lower order, usually the first (Murty *et al.* [47]).

Now differentiating the equation (2.5) four times with respect t, substituting the value of x and the derivatives $\dot{x}, \ddot{x}, \dddot{x}, x^{(4)}$ in the original equation (2.3), utilizing the relation presented in (2.6) and finally equating the coefficients of ε, we obtain

$$\begin{aligned}
& e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\left(\frac{\partial A_1}{\partial t}+2A_2+t\frac{\partial A_2}{\partial t}\right) \\
& +e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)A_3 \\
& +e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)A_4 \\
& +\left(\frac{\partial}{\partial t}+\lambda_2\right)^2\left(\frac{\partial}{\partial t}+\lambda_3\right)\left(\frac{\partial}{\partial t}+\lambda_4\right)u_1=-f^{(0)}(a_1,a_2,a_3,a_4,t)
\end{aligned} \tag{2.7}$$

where $f^{(0)}(a_1,a_2,a_3,a_4,t)=f(x_0,\dot{x}_0,\ddot{x}_0,\dddot{x}_0)$

and $x_0=(a_1+a_2\,t)e^{-\lambda_2 t}+a_3\,e^{-\lambda_3 t}+a_4\,e^{-\lambda_4 t}$.

Now, $f^{(0)}$ can be expanded in a Taylor's series (Murty and Deekshatulu [46]) in the form

$$\begin{aligned}
f^{(0)} = & \sum_{j,k,l,m=0}^{\infty} F_{0,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t} \\
& +(a_1+a_2\,t)\sum_{j,k,l,m=0}^{\infty} F_{1,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t} \\
& +(a_1+a_2\,t)^2\sum_{j,k,l,m=0}^{\infty} F_{2,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t} \\
& +(a_1+a_2\,t)^3\sum_{j,k,l,m=0}^{\infty} F_{3,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}+\ldots
\end{aligned} \tag{2.8}$$

Substituting the value of $f^{(0)}$ from equation (2.8) into equation (2.7), we obtain

$$
\begin{aligned}
&e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\left(\frac{\partial A_1}{\partial t}+2A_2+t\frac{\partial A_2}{\partial t}\right)\\
&+e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)A_3+e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)A_4\\
&+\left(\frac{\partial}{\partial t}+\lambda_2\right)^2\left(\frac{\partial}{\partial t}+\lambda_3\right)\left(\frac{\partial}{\partial t}+\lambda_4\right)u_1=-\sum_{j,k,l,m=0}^{\infty}F_{0,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}\\
&-(a_1+a_2t)\sum_{j,k,l,m=0}^{\infty}F_{1,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}\\
&-(a_1+a_2t)^2\sum_{j,k,l,m=0}^{\infty}F_{2,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}\\
&-(a_1+a_2t)^3\sum_{j,k,l,m=0}^{\infty}F_{3,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}+\ldots
\end{aligned}
\tag{2.9}
$$

Bogoliubov and Mitropolskii [13], Krylov and Bogoliubov [31], Sattar [63], Shamsul [69, 77, 81] Shamsul and Sattar[66] imposed the condition that u_1 cannot contain the fundamental terms (the solution presented in equation (2.4) is called generating solution and its terms are called fundamental terms) of $f^{(0)}$. *i. e.*, the terms $(a_1+t\ a_2)^0$ and $(a_1+t\ a_2)^1$. Therefore equation (2.9) can be separated for the unknown functions u_1 and A_1, A_2, A_3, A_4 in the following way:

$$
\begin{aligned}
&e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\left(\frac{\partial A_1}{\partial t}+2A_2+t\frac{\partial A_2}{\partial t}\right)\\
&+e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)A_3+e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)A_4\\
&=-\sum_{j,k,l,m=0}^{\infty}F_{0,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}\\
&-(a_1+a_2t)\sum_{j,k,l,m=0}^{\infty}F_{1,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}
\end{aligned}
\tag{2.10}
$$

and

$$\left(\frac{\partial}{\partial t}+\lambda_2\right)^2\left(\frac{\partial}{\partial t}+\lambda_3\right)\left(\frac{\partial}{\partial t}+\lambda_4\right)u_1$$
$$=-(a_1+a_2\,t)^2\sum_{j,k,l,m=0}^{\infty}F_{2,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t} \tag{2.11}$$
$$-(a_1+a_2\,t)^3\sum_{j,k,l,m=0}^{\infty}F_{3,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}-\ldots$$

Now, equating the coefficients of t^0 and t^1 from both sides of the equation (2.10), we obtain

$$e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\left(\frac{\partial A_1}{\partial t}+2A_2\right)$$
$$+e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)A_3+e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)A_4 \tag{2.12}$$
$$=-\sum_{j,k,l,m=0}^{\infty}F_{0,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}$$
$$-a_1\sum_{j,k,l,m=0}^{\infty}F_{1,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}$$

and

$$e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\frac{\partial A_2}{\partial t}$$
$$=-a_2\sum_{j,k,l,m=0}^{\infty}F_{1,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t} \tag{2.13}$$

Solving equation (2.13), we obtain

$$A_2=\sum_{j,k,l,m=0}^{\infty}\frac{a_2\,F_{1,m}(a_1,a_2,a_3,a_4)e^{-((j-1)\lambda_2+k\lambda_3+l\lambda_4)t}}{\left((j-1)\lambda_2+k\lambda_3+l\lambda_4\right)\left(\lambda_2+(k-1)\lambda_3+l\lambda_4\right)\left(j\lambda_2+k\lambda_3+(l-1)\lambda_4\right)} \tag{2.14}$$

Substituting the value of A_2 from equation (2.14) into equation (2.12), we obtain

$$e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\frac{\partial A_1}{\partial t}+e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)A_3$$

$$+e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)A_4$$

$$=-\sum_{j,k,l,m=0}^{\infty} F_{0,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t} \tag{2.15}$$

$$-a_1\sum_{j,k,l,m=0}^{\infty} F_{1,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}$$

$$-2e^{-\lambda_2 t}\sum_{j,k,l,m=0}^{\infty}\frac{a_2\,F_{1,m}(a_1,a_2,a_3,a_4)e^{-((j-1)\lambda_2+k\lambda_3+l\lambda_4)t}}{((j-1)\lambda_2+k\lambda_3+\lambda_4)}$$

Now it is not easy to solve the equation (2.15) for the unknown functions A_1, A_3 and A_4, if the nonlinear function f and the eigen values $-\lambda_2, -\lambda_3, -\lambda_4$ of the linear equation of (2.3) are not specified. When these are specified the values of A_1, A_3 and A_4 can be found subject to the condition that the coefficient in the solution of A_1, A_3 and A_4 do not become large (Ali Akbar *et al.* [4], Shamsul [78, 80, 82]).

For this reason, we have considered that the relations $\lambda_2 \approx 3\lambda_3$ and $\lambda_2 \approx 2\lambda_3 + 2\lambda_4$ exist among the eigenvalues. These relations are important, since under these relations the coefficients in the solution of A_1, A_3 and A_4 do not become large.

Equation (2.11) is a fourth order inhomogeneous linear differential equation. When the nonlinear function f is specified, we can find the particular solution of the equation (2.11) for the unknown function u_1 by well-known operator method.

Since $\dot{a}_1, \dot{a}_2, \dot{a}_3, \dot{a}_4$ are proportional to the small parameter ε, so they are slowly varying functions of time t. Hence their rate of change are very small *i. e.*, they are almost constant. Therefore, it is plausible to replace a_1, a_2, a_3, a_4 by their respective values obtained in the linear case (*i. e.* the values of a_1, a_2, a_3, a_4 obtained when $\varepsilon = 0$) in the right hand side of (2.6). This replacement was first made by Murty *et al.* [47, 48]

to solve similar type of nonlinear equations. Thus substituting the values of A_1, B_1 and C_1 into the equation (2.6) and integrating, we obtain

$$\left.\begin{aligned} a_1 &= a_{1,0} + \varepsilon \int_0^t A_1(a_{1,0}, a_{2,0}, a_{3,0}, a_{4,0}, t)\, dt \\ a_2 &= a_{2,0} + \varepsilon \int_0^t A_2(a_{1,0}, a_{2,0}, a_{3,0}, a_{4,0}, t)\, dt \\ a_3 &= a_{3,0} + \varepsilon \int_0^t A_3(a_{1,0}, a_{2,0}, a_{3,0}, a_{4,0}, t)\, dt \\ a_4 &= a_{4,0} + \varepsilon \int_0^t A_4(a_{1,0}, a_{2,0}, a_{3,0}, a_{4,0}, t)\, dt \end{aligned}\right\} \tag{2.16}$$

Substituting the values of a_1, a_2, a_3, a_4 and u_1 in the equation (2.5), we shall get the complete solution of (2.3).

Thus the determination of the first order approximate solution is completed.

The method can be carried out for higher order nonlinear systems in the same way.

2.3 Example

As an example of the above method, we have considered the Duffing equation type nonlinear system

$$x^{(4)} + k_1 \ddot{x} + k_2 \ddot{x} + k_3 \dot{x} + k_4 x = -\varepsilon x^3, \tag{2.17}$$

Here, $f = x^3$ and $x_0 = (a_1 + a_2 t) e^{-\lambda_2 t} + a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t}$.

Therefore,

$$\begin{aligned} f^{(0)} &= a_3^3 e^{-3\lambda_3 t} + 3 a_3^2 a_4 e^{-(2\lambda_3 + \lambda_4)t} + 3 a_3 a_4^2 e^{-(\lambda_3 + 2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t} \\ &+ 3(a_1 + a_2 t)\left(a_3^2 e^{-2\lambda_3 t} + 2 a_3 a_4 e^{-(\lambda_3 + \lambda_4)t} + a_4^2 e^{-2\lambda_4 t}\right) \\ &+ 3(a_1 + a_2 t)^2 \left(a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t}\right) + e^{-3\lambda_2 t} (a_1 + a_2 t)^3 \end{aligned} \tag{2.18}$$

Therefore, comparing the equation (2.8) and (2.18), we obtain

$$\sum_{j,k,l,m=0}^{\infty} F_{0,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}$$
$$= a_3^3 e^{-3\lambda_3 t} + 3a_3^2\, a_4\, e^{-(2\lambda_3+\lambda_4)t} + 3a_3\, a_4^2\, e^{-(\lambda_3+2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t}$$

$$\begin{aligned}&\sum_{j,k,l,m=0}^{\infty} F_{1,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t}\\ &= 3\left(a_3^2\, e^{-2\lambda_3 t} + 2a_3\, a_4\, e^{-(\lambda_3+\lambda_4)t} + a_4^2\, e^{-2\lambda_4 t}\right)\end{aligned} \tag{2.19}$$

$$\sum_{j,k,l,m=0}^{\infty} F_{2,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t} = 3\left(a_3\, e^{-\lambda_3 t} + a_4\, e^{-\lambda_4 t}\right)$$

$$\sum_{j,k,l,m=0}^{\infty} F_{3,m}(a_1,a_2,a_3,a_4)e^{-(j\lambda_2+k\lambda_3+l\lambda_4)t} = e^{-3\lambda_2 t}$$

Therefore, equations (2.11)-(2.13) respectively become

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\lambda_2\right)^2\left(\frac{\partial}{\partial t}+\lambda_3\right)\left(\frac{\partial}{\partial t}+\lambda_4\right)u_1 = -(a_1+a_2\, t)^2\, 3\left(a_3\, e^{-\lambda_3 t} + a_4\, e^{-\lambda_4 t}\right)\\ &-(a_1+a_2\, t)^3 e^{-3\lambda_2 t}\end{aligned} \tag{2.20}$$

$$\begin{aligned}&e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\left(\frac{\partial A_1}{\partial t}+2\, A_2\right)\\ &+e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)A_3\\ &+e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)A_4\\ &= -\{a_3^3 e^{-3\lambda_3 t} + 3a_3^2\, a_4\, e^{-(2\lambda_3+\lambda_4)t} + 3a_3\, a_4^2\, e^{-(\lambda_3+2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t}\}\\ &-3a_1\left(a_3^2\, e^{-2\lambda_3 t} + 2a_3\, a_4\, e^{-(\lambda_3+\lambda_4)t} + a_4^2\, e^{-2\lambda_4 t}\right)\end{aligned} \tag{2.21}$$

$$\begin{aligned}&e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\frac{\partial A_2}{\partial t}\\ &= -3a_2\left(a_3^2\, e^{-2\lambda_3 t} + 2a_3\, a_4\, e^{-(\lambda_3+\lambda_4)t} + a_4^2\, e^{-2\lambda_4 t}\right)\end{aligned} \tag{2.22}$$

Therefore, solving equation (2.22), we obtain

$$A_2 = q_1\, a_2\, a_3^2 e^{-2\lambda_3 t} + q_2\, a_2\, a_3\, a_4\, e^{-(\lambda_3+\lambda_4)t} + q_3\, a_2\, a_4^2\, e^{-2\lambda_4 t}, \tag{2.23}$$

where

$q_1 = 3/2\lambda_3(\lambda_2+\lambda_3)(\lambda_2+2\lambda_3-\lambda_4)$, $\qquad q_2 = 6/(\lambda_2+\lambda_3)(\lambda_2+\lambda_4)(\lambda_3+\lambda_4)$,

$q_3 = 3/2\lambda_4(\lambda_2+\lambda_4)(\lambda_2-\lambda_3+2\lambda_4)$.

Now substituting the value of A_2 from equation (2.23) into equation (2.21), we obtain

$$
\begin{aligned}
& e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\frac{\partial A_1}{\partial t}+e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)A_3 \\
& +e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)A_4=-\{a_3^3 e^{-3\lambda_3 t}+3a_3^2\,a_4\,e^{-(2\lambda_3+\lambda_4)t} \\
& +3a_3\,a_4^2\,e^{-(\lambda_3+2\lambda_4)t}+a_4^3 e^{-3\lambda_4 t}\}-3a_1\{a_3^2\,e^{-2\lambda_3 t}+2a_3\,a_4\,e^{-(\lambda_3+\lambda_4)t} \\
& +a_4^2\,e^{-2\lambda_4 t}\}-2e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\{q_1 a_2 a_3^2 e^{-2\lambda_3 t} \\
& +q_2\,a_2\,a_3\,a_4\,e^{-(\lambda_3+\lambda_4)t}+q_3\,a_2\,a_4^2\,e^{-2\lambda_4 t}\}.
\end{aligned} \tag{2.24}
$$

To separate the equation (2.24) for determining the unknown functions A_1 A_3 and A_4, we consider the most important relations $\lambda_2 \approx 3\lambda_3$ and $\lambda_2 \approx 2\lambda_3 + 2\lambda_4$ (Ali Akbar *et al.* [4], Shamsul [78, 80, 82]). exist among the eigenvalues and $\lambda_1 = \lambda_2$ for critical damping. Under these conditions, we obtain

$$
\begin{aligned}
& e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_4\right)\frac{\partial A_1}{\partial t} \\
& =-\{a_3^3 e^{-3\lambda_3 t}+3a_1\,a_3^2\,e^{-(\lambda_2+2\lambda_3)t}+6a_1\,a_3\,a_4\,e^{-(\lambda_2+\lambda_3+\lambda_4)t} \\
& +3a_1 a_4^2\,e^{-(\lambda_2+2\lambda_4)t}\}-\left\{\frac{3a_2 a_3^2}{\lambda_3}e^{-(2\lambda_3+\lambda_2)t}+\frac{12\,a_2\,a_3\,a_4}{(\lambda_3+\lambda_4)}e^{-(\lambda_2+\lambda_3+\lambda_4)t}\right. \\
& \left.+\frac{3a_2\,a_4^2}{\lambda_4}\,e^{-(2\lambda_4+\lambda_2)t}\right.
\end{aligned} \tag{2.25}
$$

$$
e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)A_3=-\{3a_3\,a_4^2\,e^{-(\lambda_3+2\lambda_4)t}+a_4^3 e^{-3\lambda_4 t}\} \tag{2.26}
$$

$$
e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_2\right)^2\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)A_4=-3a_3^2\,a_4\,e^{-(2\lambda_3+\lambda_4)t} \tag{2.27}
$$

The particular solutions of equations (2.25)-(2.27) respectively become

$$A_1 = p_1 a_3^3 e^{-(3\lambda_3-\lambda_2)t} + p_2\, a_1\, a_3^2\, e^{-2\lambda_3 t} + p_3\, a_1\, a_3\, a_4\, e^{-(\lambda_3+\lambda_4)t} + p_4\, a_1 a_4^2\, e^{-2\lambda_4 t} + p_5\, a_2 a_3^2 e^{-2\lambda_3 t} + p_6\, a_2\, a_3\, a_4\, e^{-(\lambda_3+\lambda_4)t} + p_7\, a_2\, a_4^2\, e^{-2\lambda_4 t} \tag{2.28}$$

$$A_3 = r_1\, a_3\, a_4^2\, e^{-2\lambda_4 t} + r_2\, a_4^3\, e^{-(3\lambda_4-\lambda_3)t} \tag{2.29}$$

$$A_4 = s_1\, a_3^2\, a_4\, e^{-2\lambda_3 t} \tag{2.30}$$

where

$p_1 = 1/2\lambda_3(\lambda_2 - 3\lambda_3)(\lambda_4 - 3\lambda_3)$, $p_2 = 3/2\lambda_3(\lambda_2 + \lambda_3)(\lambda_2 + 2\lambda_3 - \lambda_4)$,

$p_3 = 6/(\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)$, $p_4 = 3/2\lambda_4(\lambda_2 + \lambda_4)(\lambda_2 - \lambda_3 + 2\lambda_4)$,

$p_5 = 3/2\lambda_3(\lambda_2 + \lambda_3)(\lambda_2 + 2\lambda_3 - \lambda_4)$, $p_6 = 12/(\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)$,

$p_7 = 3/2\lambda_4(\lambda_2 + \lambda_4)(\lambda_2 - \lambda_3 + 2\lambda_4)$, $r_1 = 3/(\lambda_3 + \lambda_4)(\lambda_2 - \lambda_3 - 2\lambda_4)^2$,

$r_2 = 1/2\lambda_4(\lambda_2 - 3\lambda_4)^2$, $s_1 = 3/(\lambda_3 + \lambda_4)(\lambda_2 - 2\lambda_3 - \lambda_4)^2$,

The solution of the equation (2.20) is

$$\begin{aligned} u_1 &= l_1\, a_1\, a_3^2\, e^{-(\lambda_2+2\lambda_4)t} + l_2\, a_2\, a_3^2\, e^{-(\lambda_3+2\lambda_4)t} + l_3\, a_3^3\, e^{-3\lambda_4 t} \\ &+ a_1\, a_3\, a_4 e^{-(\lambda_2+2\lambda_4)t}(2\, l_1\, t + l_4) + a_2\, a_3\, a_4 e^{-(\lambda_2+2\lambda_4)t}(2\, l_2\, t + l_5) \\ &+ a_3^2\, a_4 e^{-(\lambda_2+2\lambda_4)t}(3\, l_3\, t + l_6) + a_1 a_4^2\, e^{-(\lambda_2+2\lambda_4)t}(l_1 t^2 + l_7\, t + l_8) \\ &+ a_2 a_4^2\, e^{-(\lambda_3+2\lambda_4)t}(l_2 t^2 + l_9\, t + l_{10}) + a_3 a_4^2\, e^{-3\lambda_4 t}(3\, l_3\, t^2 + l_{11} t + l_{12}) \\ &+ a_4^3\, e^{-3\lambda_4 t}(l_3 t^3 + l_{13} t^2 + l_{14} t + l_{15}) \end{aligned} \tag{2.31}$$

where

$l_1 = -3/\{2\lambda_4(\lambda_2 + 2\lambda_4 - \lambda_3)(\lambda_2 + \lambda_4)^2\}$,

$l_2 = -3/\{2\lambda_4(\lambda_3 + 2\lambda_4 - \lambda_2)(\lambda_3 + \lambda_4)^2\}$,

$l_3 = -1/\{4\lambda_4^2(3\lambda_4 - \lambda_2)(3\lambda_4 - \lambda_3)\}$,

$l_4 = 2\, l_1 \left(\dfrac{2}{(\lambda_2 + \lambda_4)} + \dfrac{1}{(\lambda_2 + 2\lambda_4 - \lambda_3)} + \dfrac{1}{2\lambda_4} \right)$,

$$l_5 = 2\, l_2 \left(\frac{2}{(\lambda_3 + \lambda_4)} + \frac{1}{(\lambda_3 + 2\lambda_4 - \lambda_2)} + \frac{1}{2\lambda_4} \right),$$

$$l_6 = 3\, l_3 \left(\frac{2}{(3\lambda_4 - \lambda_2)} + \frac{1}{(3\lambda_4 - \lambda_3)} + \frac{1}{\lambda_4} \right),$$

$$l_7 = l_1 \left(\frac{4}{(\lambda_2 + \lambda_4)} + \frac{2}{(\lambda_2 + 2\lambda_4 - \lambda_3)} + \frac{1}{\lambda_4} \right),$$

$$l_8 = l_1 \left(\begin{array}{l} \frac{4}{(\lambda_2 + \lambda_4)} + \frac{6}{(\lambda_2 + \lambda_4)^2} + \frac{2}{(\lambda_2 + 2\lambda_4 - \lambda_3)^2} \\ + \frac{2}{\lambda_4(\lambda_2 + \lambda_4)} + \frac{1}{\lambda_4(\lambda_2 + 2\lambda_4 - \lambda_3)} + \frac{1}{2\lambda_4^2} \end{array} \right),$$

$$l_9 = l_2 \left(\frac{4}{(\lambda_3 + \lambda_4)} + \frac{2}{(\lambda_3 + 2\lambda_4 - \lambda_2)} + \frac{1}{\lambda_4} \right),$$

$$l_{10} = l_2 \left(\begin{array}{l} \frac{4}{(\lambda_3 + \lambda_4)} + \frac{6}{(\lambda_3 + \lambda_4)^2} + \frac{2}{(\lambda_3 + 2\lambda_4 - \lambda_2)^2} \\ + \frac{2}{\lambda_4(\lambda_3 + \lambda_4)} + \frac{1}{\lambda_4(\lambda_3 + 2\lambda_4 - \lambda_2)} + \frac{1}{2\lambda_4^2} \end{array} \right),$$

$$l_{11} = 3\, l_3 \left(\frac{2}{\lambda_4} + \frac{2}{(3\lambda_4 - \lambda_2)} + \frac{2}{(3\lambda_4 - \lambda_3)} \right),$$

$$l_{12} = 3\, l_3 \left(\begin{array}{l} \frac{3}{2\lambda_4^2} + \frac{2}{\lambda_4(3\lambda_4 - \lambda_2)} + \frac{2}{\lambda_4(3\lambda_4 - \lambda_3)} + \frac{2}{(3\lambda_4 - \lambda_2)^2} \\ + \frac{2}{(3\lambda_4 - \lambda_3)^2} + \frac{2}{(3\lambda_4 - \lambda_2)(3\lambda_4 - \lambda_3)} \end{array} \right),$$

$$l_{13} = l_3 \left(\frac{3}{\lambda_4} + \frac{3}{(3\lambda_4 - \lambda_2)} + \frac{3}{(3\lambda_4 - \lambda_3)} \right),$$

$$l_{14} = l_3 \left(\begin{array}{l} \frac{9}{2\lambda_4^2} + \frac{6}{\lambda_4(3\lambda_4 - \lambda_2)} + \frac{6}{\lambda_4(3\lambda_4 - \lambda_3)} + \frac{6}{(3\lambda_4 - \lambda_2)^2} \\ + \frac{6}{(3\lambda_4 - \lambda_3)^2} + \frac{6}{(3\lambda_4 - \lambda_2)(3\lambda_4 - \lambda_3)} \end{array} \right),$$

$$l_{15} = l_3\{\frac{3}{\lambda_4^3} + \frac{9}{2\lambda_4^2(3\lambda_4-\lambda_2)} + \frac{9}{2\lambda_4^2(3\lambda_4-\lambda_3)} + \frac{6}{\lambda_4(3\lambda_4-\lambda_2)^2}$$
$$+ \frac{6}{\lambda_4(3\lambda_4-\lambda_3)^2} + \frac{6}{(3\lambda_4-\lambda_2)^3} + \frac{6}{(3\lambda_4-\lambda_3)^3} + \frac{6}{(3\lambda_4-\lambda_2)^2(3\lambda_4-\lambda_3)}$$
$$+ \frac{6}{(3\lambda_4-\lambda_2)(3\lambda_4-\lambda_3)^2} + \frac{6}{\lambda_4(3\lambda_4-\lambda_2)(3\lambda_4-\lambda_3)}\}$$

Substituting the values of A_1, A_2, A_3 and A_4 from equation (2.23), (2.28)-(2.30) into (2. 6), we obtain

$$\begin{aligned}
\dot{a}_1 &= \varepsilon\{p_1 a_3^3 e^{-(3\lambda_3-\lambda_2)t} + p_1 a_1 a_3^2 e^{-2\lambda_3 t} \\
&\quad + p_3 a_1 a_3 a_4 e^{-(\lambda_3+\lambda_4)t} + p_4 a_1 a_4^2 e^{-2\lambda_4 t} + p_5 a_2 a_3^2 e^{-2\lambda_3 t} \\
&\quad + p_6 a_2 a_3 a_4 e^{-(\lambda_3+\lambda_4)t} + p_7 a_2 a_4^2 e^{-2\lambda_4 t}\}, \\
\dot{a}_2 &= \varepsilon\{q_1 a_2 a_3^2 e^{-2\lambda_3 t} + q_2 a_2 a_3 a_4 e^{-(\lambda_3+\lambda_4)t} + q_3 a_2 a_4^2 e^{-2\lambda_4 t}\}, \\
\dot{a}_3 &= \varepsilon\{r_1 a_3 a_4^2 e^{-2\lambda_4 t} + r_2 a_4^3 e^{-(3\lambda_4-\lambda_3)t}\}, \\
\dot{a}_4 &= \varepsilon s_1 a_3^2 a_4 e^{-2\lambda_3 t}.
\end{aligned} \tag{2.32}$$

Since $\dot{a}_1, \dot{a}_2, \dot{a}_3, \dot{a}_4$ are proportional to the small parameter ε, so they are slowly varying functions of time t. Hence they are almost constant; therefore we can solve equation (2.32) by assuming a_1, a_2, a_3 and a_4 are constants (Murty and Deekshatulu [46], Murty *et al.* [47]) in the right hand sides of (2.32). Thus the solutions of equation (2.32) is

$$
\begin{aligned}
a_1 &= a_{1,0} + \varepsilon\{p_1\, a_{3,0}^3 \frac{1-e^{-(3\lambda_3-\lambda_2)t}}{3\lambda_3-\lambda_2} + p_2\, a_{1,0}\, a_{3,0}^2 \frac{1-e^{-2\lambda_3 t}}{2\lambda_3} \\
&+ p_3\, a_{1,0}\, a_{3,0}^3\, a_{4,0} \frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4} + p_4\, a_{1,0}\, a_{4,0}^3 \frac{1-e^{-2\lambda_4 t}}{2\lambda_4} \\
&+ p_5\, a_{2,0}\, a_{3,0}^2 \frac{1-e^{-2\lambda_3 t}}{2\lambda_3} + p_6\, a_{2,0}\, a_{3,0}\, a_{4,0} \frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4} \\
&+ p_7\, a_{2,0}\, a_{4,0}^2 \frac{1-e^{-2\lambda_4 t}}{2\lambda_4}\}, \\
a_2 &= a_{2,0} + \varepsilon\{q_1 a_{2,0}\, a_{3,0}^2 \frac{1-e^{-2\lambda_3 t}}{2\lambda_3} + q_2\, a_{2,0}\, a_{3,0}\, a_{4,0} \frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4} \\
&+ q_3\, a_{2,0}\, a_{4,0}^2 \frac{1-e^{-2\lambda_4 t}}{3\lambda_4-\lambda_3}\}, \\
a_3 &= a_{3,0} + \varepsilon\{r_1\, a_{3,0}\, a_{4,0}^2 \frac{1-e^{-2\lambda_4 t}}{2\lambda_4} + r_2\, a_{4,0}^3 \frac{1-e^{-2\lambda_4 t}}{2\lambda_4}\}, \\
a_4 &= a_{4,0} + \varepsilon\, s_1\, a_{3,0}^2\, a_{4,0} \frac{1-e^{-2\lambda_3 t}}{2\lambda_3}
\end{aligned}
\qquad (2.33)
$$

Therefore, we obtain the first approximate solution of the equation (2.17) is

$$x(t,\varepsilon) = (a_1 + a_2\, t)e^{-\lambda_2 t} + a_3\, e^{-\lambda_3 t} + a_4\, e^{-\lambda_4 t} + \varepsilon\, u_1 \qquad (2.34)$$

where a_1, a_2, a_3, a_4 are given by the equation (2.33) and u_1 is given by the equation (2.31).

2.4 Results and Discussion

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we sometimes compare the approximate solution to the numerical solution. With regard to such a comparison concerning the presented asymptotic solution obtained by the KBM method of this chapter, we refer the works of Murty *et al* [47].

For the imposed conditions $\lambda_2 \approx 3\lambda_3$, $\lambda_2 \approx 2\lambda_3 + 2\lambda_4$ and $\lambda_1 = \lambda_2$, in this chapter we have computed $x(t,\varepsilon)$ by equation (2.34) in which a_1, a_2, a_3, a_4 are evaluated by the equation (2.33) and u_1 is evaluated by the equation (2.31) for different sets of initial conditions and for various values of t. A second solution of (2.17) is computed by fourth order Runge-Kutta method, and compared with the approximate analytical solutions. The approximate analytic solutions and numerical solutions are plotted in the figures (From Fig. 2.1 to Fig. 2.5). From figures we see that our approximate analytical solutions show good coincidence with numerical solutions.

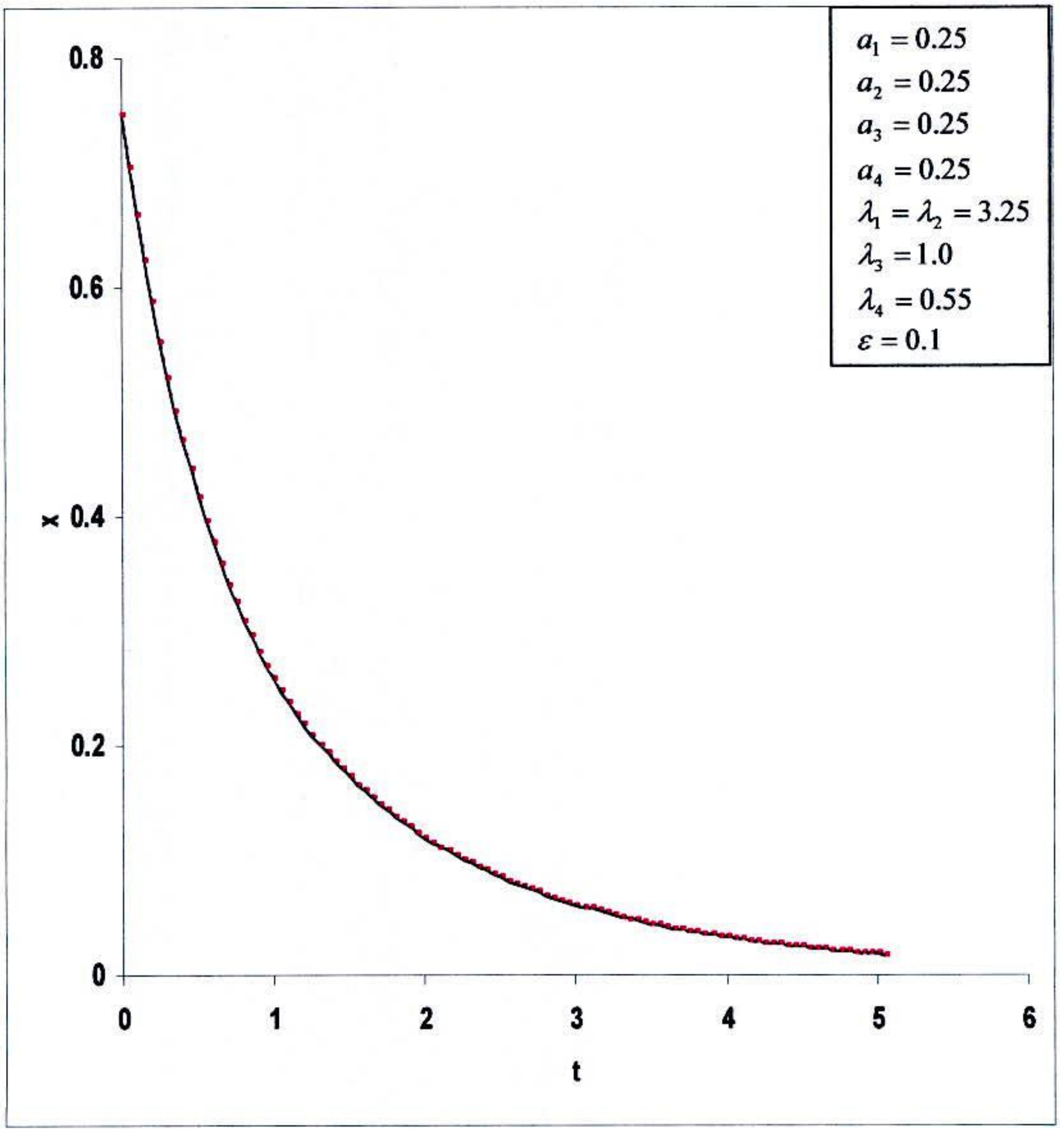

Fig. 2.1: Analytic solution in solid line — and solution by a fourth order Runge-Kutta method in dashed line - - -.

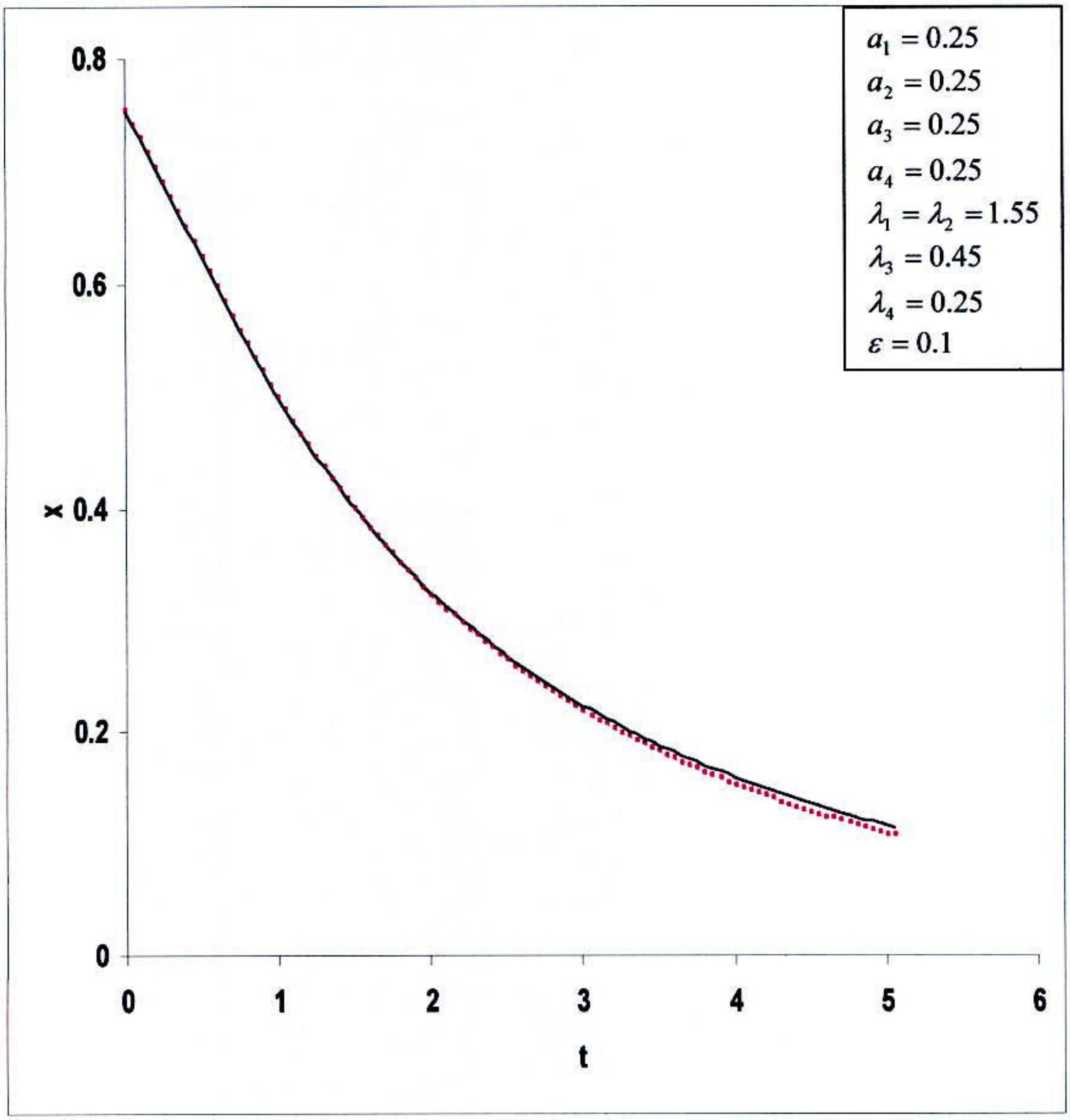

Fig. 2.2: Analytic solution in solid line — and solution by a fourth order Runge-Kutta method in dashed line - - -.

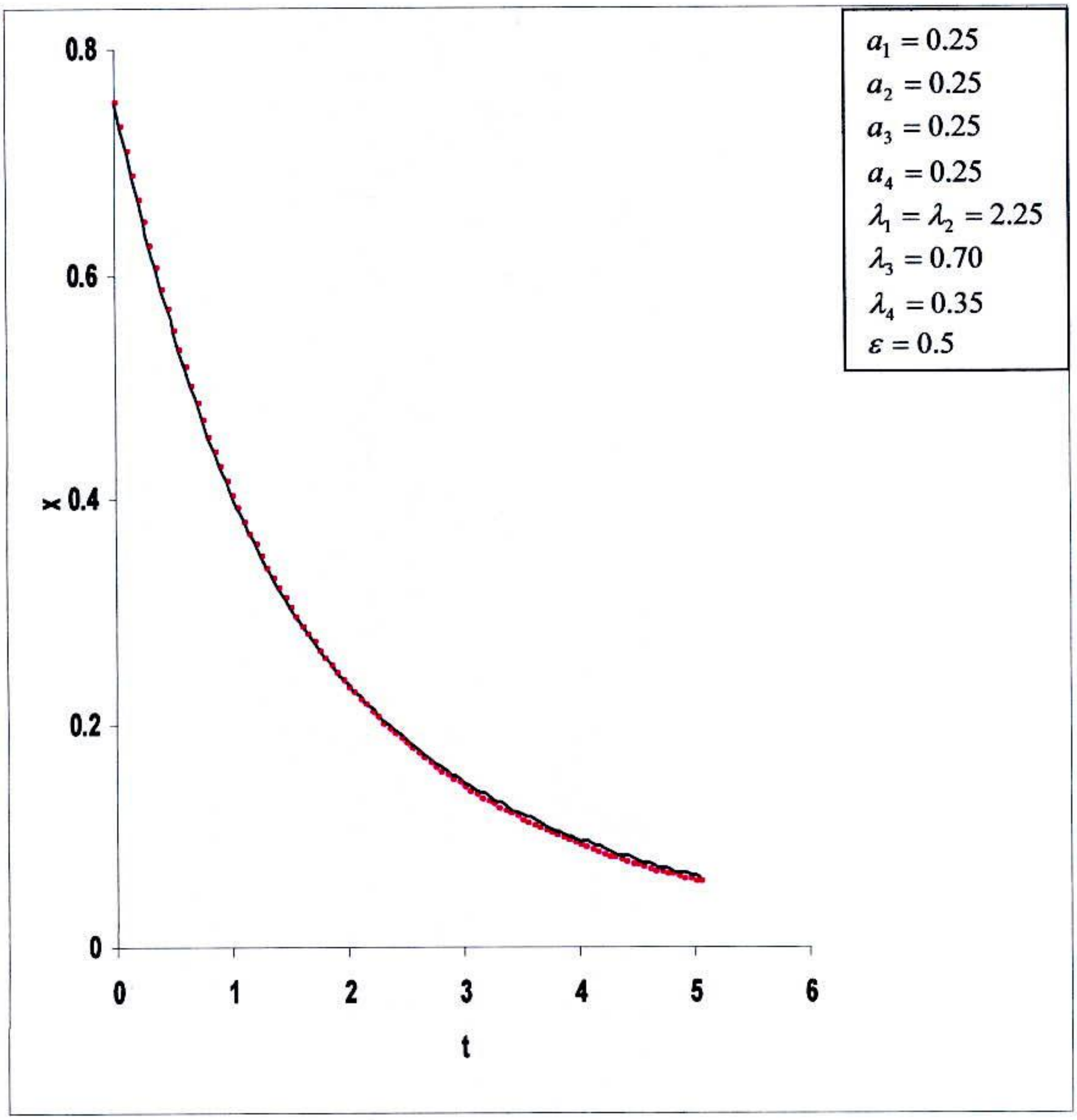

Fig. 2.3: Analytic solution in solid line — and solution by a fourth order Runge-Kutta method in dashed line - - -.

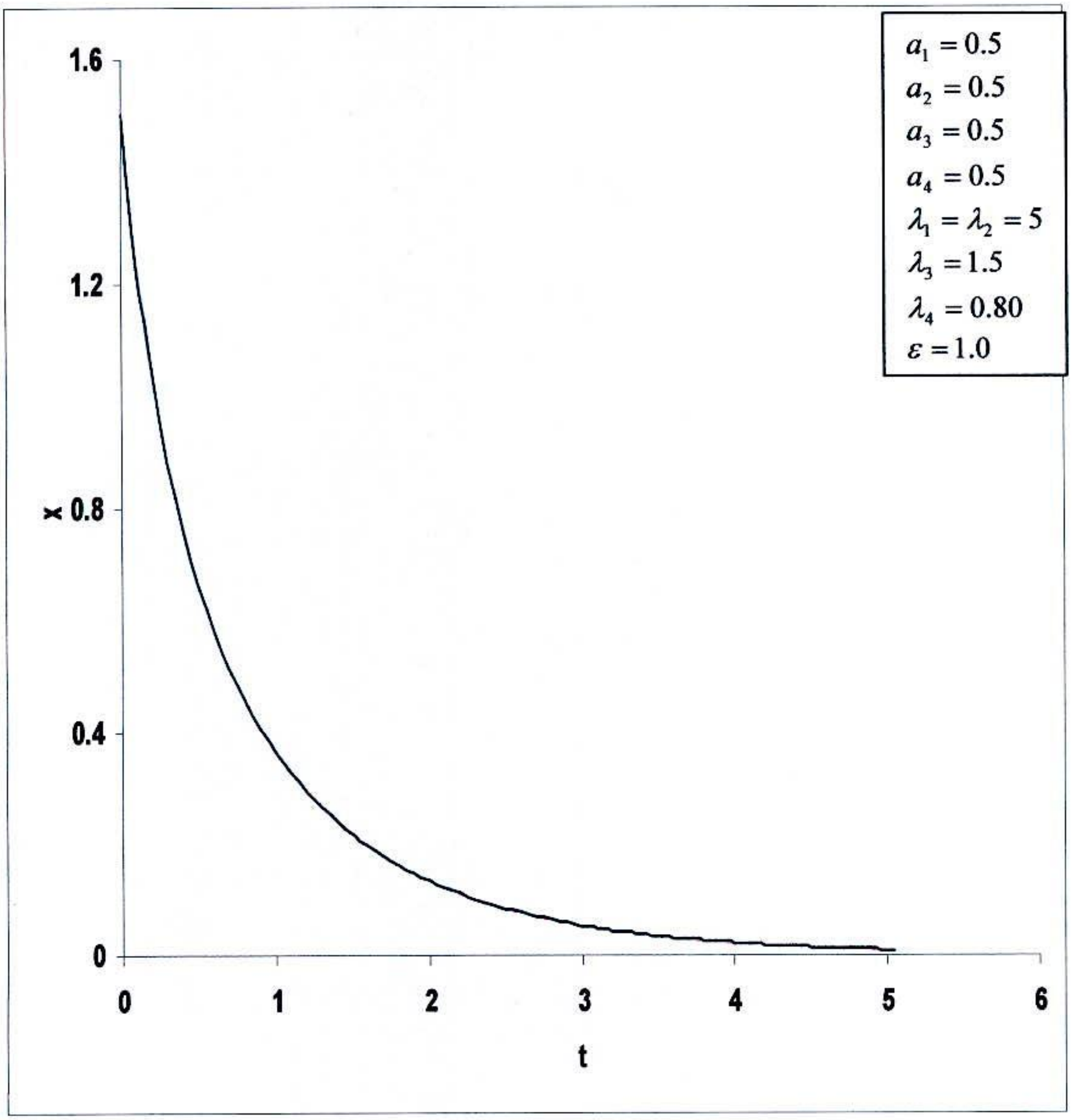

Fig. 2.4: Analytic solution in solid line — and solution by a fourth order Runge-Kutta method in dashed line - - -..

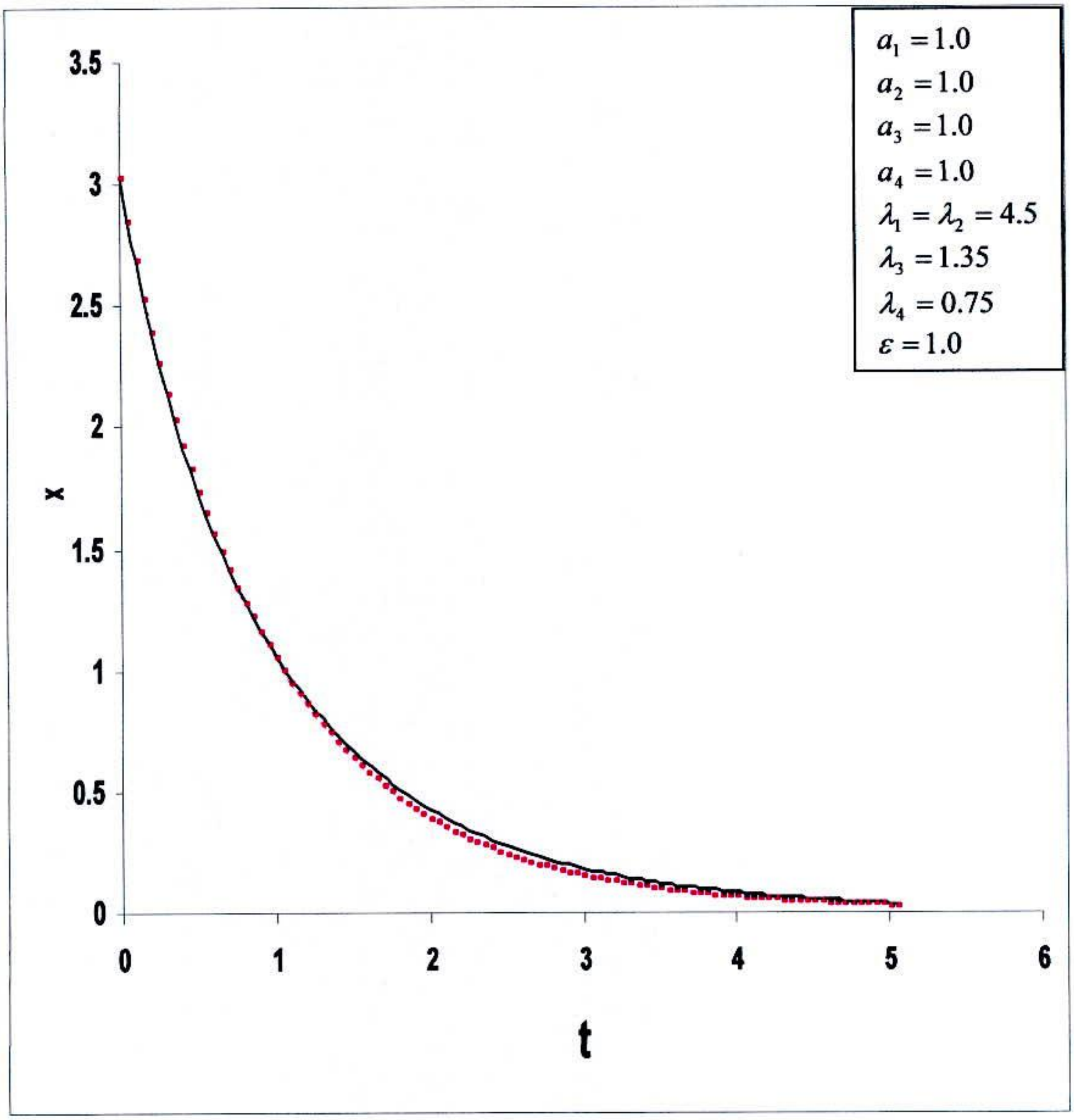

Fig. 2.5: : Analytic solution in solid line — and solution by a fourth order Runge-Kutta method in dashed line - - -..

2.5 Conclusion

The Krylov-Bogoliubov-Mitropolskii method has been extended to solve fourth order critically damped nonlinear systems under some special conditions. The method is important when the relations $\lambda_2 \approx 3\lambda_3$, $\lambda_2 \approx 2\lambda_3 + 2\lambda_4$ and $\lambda_1 = \lambda_2$ exist among the eigenvalues. The solutions obtained by this method show good agreement with those obtained by numerical method.

Chapter 3

Asymptotic Solutions of Fourth Order More Critically Damped Nonlinear Systems Under Some Special Conditions

3.1 Introduction

Krylov-Bogoliubov-Mitropolskii (KBM) [13, 31] method is a widely used tool to study nonlinear differential systems with small nonlinearities. Originally, the method was developed for obtaining periodic solutions of second order nonlinear systems with small nonlinearities. Later, the method has been extended by Popov [55] to damped oscillatory nonlinear systems. Owing to physical importance of damped oscillatory nonlinear systems, Mendelson [41] rediscovered the Popov's results. Murty *et al.* [47] also extended the KBM method for obtaining second and fourth order over-damped nonlinear systems. Sattar [63] has found an asymptotic solution of a second order critically damped nonlinear systems. Shamsul [69] presented a new asymptotic technique for second order over-damped and critically damped nonlinear systems. Shamsul [68] has generalized the KBM asymptotic method. Sattar [64] also studied third order over-damped nonlinear systems. Shamsul [77] studied a third order critically damped nonlinear systems whose unequal eigenvalues are in integral multiple. Shamsul and Sattar [66] have extended the Bogoliubov's asymptotic method to a third order critically damped nonlinear systems. Shamsul and Sattar [67] also presented a unified KBM method for obtaining the approximate solutions of third order damped, undamped and over-damped systems. Rokibul *et al.* [59] developed a new technique for solving third order critically damped nonlinear systems.

Murty *et al.* [47] has also extended the KBM method to solve fourth order over-damped nonlinear systems. But their method is too much complex and laborious. On the other hand Ali Akbar *et al.* [2] have found an asymptotic solution of fourth order over-damped nonlinear systems which is simple and easier than the method presented by Murty *et al.* [47] but the results obtained by [2] is same as the results obtained by [47]. Later, Ali Akbar *et al.* [3] extended the method presented in [2] to damped oscillatory nonlinear systems. Ali Akbar *et al.* [4] have also presented a simple technique for obtaining certain over-damped solutions of an *n*-th order nonlinear differential equation. Rokibul *et al.* [60] have presented a technique for obtaining the solutions of fourth order critically damped nonlinear systems.

In the present chapter, a fourth order more critically damped nonlinear system is considered and an asymptotic solution is found by extending the KBM method. The results obtained by the presented method agree with those obtained by numerical method nicely.

3.2 The method

Consider a fourth order weakly nonlinear system governed by the ordinary differential equation

$$x^{(4)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = -\varepsilon f(x,\dot{x},\ddot{x},\dddot{x}) \tag{3.1}$$

where $x^{(4)}$ denote the fourth derivative of x with respect to t, and over dots are used to denote first, second and third derivatives; $k_1,\ k_2, k_3, k_4$ are constants, ε is the small parameter and $f(x,\dot{x},\ddot{x},\dddot{x})$ is the given nonlinear function. As the equation is fourth order, so, we shall get four real negative eigenvalues, where three of the eigenvalues are equal because the system is critically damped. Suppose the eigenvalues are $-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4$ and since the system is more critically damped, so we assume that $\lambda_1 = \lambda_2 = \lambda_3$.

When $\varepsilon = 0$, the equation (3.1) becomes linear and the solution of the corresponding linear equation is

$$x(t,0) = \left(a_1 + a_2\, t + a_3\, t^2\right) e^{-\lambda_3 t} + a_4\, e^{-\lambda_4 t} \tag{3.2}$$

where $a_j,\;\; j = 1, 2, 3, 4$ are constants of integration.

When $\varepsilon \neq 0$, following Shamsul [75], a solution of the equation (3.1) is sought in the form

$$x(t,\varepsilon) = \left(a_1(t) + a_2(t)\; t + a_3(t)\; t^2\right) e^{-\lambda_3 t} + a_4(t)\; e^{-\lambda_4 t} + \varepsilon\, u_1(a_1, a_2, a_3, a_4, t) + \ldots \tag{3.3}$$

where each $a_j(t),\;\; j = 1, 2, 3, 4$ are functions of t and satisfy the first order differential equation

$$\dot{a}_j(t) = \varepsilon\, A_j(a_1, a_2, a_3, a_4, t) + \ldots \tag{3.4}$$

Confining to only a first few terms 1, 2, 3, ...,n in the series expansion of (3.3) and (3.4), we evaluate the functions $u_j, A_j,\;\; j = 1, 2, 3, \ldots, n$, such that $a_j(t),\;\; j = 1, 2, 3, \ldots, n$, appearing in (3.3) and (3.4) satisfy the given differential equation (3.1) with an accuracy of order ε^{n+1}. In order to determine these unknown functions it is customary in KBM method that the correction terms, u_j must exclude terms (known as secular terms) which make them large. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a lower order, usually the first (Murty *et al.* [47]).

Now differentiating the equation (3.3) four times with respect t, substituting the value of x and the derivatives $\dot{x}, \ddot{x}, \dddot{x}, x^{(4)}$ in the original equation (3.1), utilizing the relation presented in (3.4) and finally equating the coefficients of ε, we obtain

$$\begin{aligned} & e^{-\lambda_3 t}\left(\frac{\partial}{\partial t} - \lambda_3 + \lambda_4\right)\left\{\frac{\partial^2 A_1}{\partial t^2} + 3\frac{\partial A_2}{\partial t} + 6\; A_3 + t\left(\frac{\partial^2 A_3}{\partial t^2} + 6\frac{\partial A_3}{\partial t}\right) + t^2\,\frac{\partial^2 A_3}{\partial t^2}\right\} \\ & + e^{-\lambda_4 t}\left(\frac{\partial}{\partial t} - \lambda_4 + \lambda_3\right)^3 A_4 + \left(\frac{\partial}{\partial t} + \lambda_3\right)^3 \left(\frac{\partial}{\partial t} + \lambda_4\right) u_1 = -f^{(0)}(a_1, a_2, a_3, a_4, t) \end{aligned} \tag{3.5}$$

where $f^{(0)}(a_1,a_2,a_3,a_4,t)=f(x_0,\dot{x}_0,\ddot{x}_0,\dddot{x}_0)$

and $x_0=\left(a_1(t)+a_2(t)\,t+a_3(t)\,t^2\right)e^{-\lambda_3 t}+a_4(t)\,e^{-\lambda_4 t}$.

Now, $f^{(0)}$ can be expanded in a Taylor's series (Murty and Deekshatulu [46]) of the form

$$\begin{aligned} f^{(0)} = &\sum_{i,j,k=0}^{\infty} F_{0,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} + t\sum_{i,j,k=0}^{\infty} F_{1,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} \\ &+t^2\sum_{i,j,k=0}^{\infty} F_{2,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} + t^3\sum_{i,j,k=0}^{\infty} F_{3,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} + \ldots \end{aligned} \tag{3.6}$$

Substituting the value of $f^{(0)}$ from equation (3.6) into equation (3.5), we obtain

$$\begin{aligned} &e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)\left\{\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial A_2}{\partial t}+6\,A_3+t\left(\frac{\partial^2 A_2}{\partial t^2}+6\frac{\partial A_3}{\partial t}\right)+t^2\,\frac{\partial^2 A_3}{\partial t^2}\right\} \\ &+e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)^3 A_4+\left(\frac{\partial}{\partial t}+\lambda_3\right)^3\left(\frac{\partial}{\partial t}+\lambda_4\right)u_1 \\ &=-\sum_{i,j,k=0}^{\infty} F_{0,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} - t\sum_{i,j,k=0}^{\infty} F_{1,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} \\ &-t^2\sum_{i,j,k=0}^{\infty} F_{2,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} - t^3\sum_{i,j,k=0}^{\infty} F_{3,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} - \ldots \end{aligned} \tag{3.7}$$

According to KBM [13, 31], Sattar [63] and Shamsul [66, 69, 77, 81], u_1 does not contain the fundamental terms (the solution presented in equation (3.2) is called generating solution and its terms are called fundamental terms) of $f^{(0)}$. Therefore equation (3.7) can be separated in the following way:

$$\begin{aligned} &e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)\left\{\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial A_2}{\partial t}+6\,A_3+t\left(\frac{\partial^2 A_2}{\partial t^2}+6\frac{\partial A_3}{\partial t}\right)+t^2\,\frac{\partial^2 A_3}{\partial t^2}\right\} \\ &+e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)^3 A_4 = -\sum_{i,j,k=0}^{\infty} F_{0,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} \\ &-t\sum_{i,j,k=0}^{\infty} F_{1,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} - t^2\sum_{i,j,k=0}^{\infty} F_{2,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} \end{aligned} \tag{3.8}$$

and

$$\left(\frac{\partial}{\partial t}+\lambda_3\right)^3\left(\frac{\partial}{\partial t}+\lambda_4\right)u_1=-t^3\sum_{i,j,k=0}^{\infty}F_{3,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t}+\cdots \tag{3.9}$$

Now equating the coefficients of t^0, t^1 and t^2 from both sides of the equation (3.8), we obtain

$$e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)\frac{\partial^2 A_3}{\partial t^2}=-\sum_{i,j,k=0}^{\infty}F_{2,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} \tag{3.10}$$

$$e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)\left(\frac{\partial^2 A_2}{\partial t^2}+6\frac{\partial A_3}{\partial t}\right)=-\sum_{i,j,k=0}^{\infty}F_{1,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} \tag{3.11}$$

$$\begin{aligned}&e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial A_2}{\partial t}+6\,A_3\right)\\&+e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)^3 A_4=-\sum_{i,j,k=0}^{\infty}F_{0,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t}\end{aligned} \tag{3.12}$$

Solving equation (3.10), we obtain

$$A_3=\sum_{i,j=0}^{\infty}\frac{F_2(a_1,a_2,a_3,a_4)e^{-((i-1)\lambda_3+j\lambda_4)t}}{((i-1)\lambda_3+j\lambda_4)^2\left(i\lambda_3+(j-1)\lambda_4\right)} \tag{3.13}$$

Substituting the value of A_3 from equation (3.13) into equation (3.11), and integrating, we obtain

$$\begin{aligned}A_2=&\,6\sum_{i,j,k=0}^{\infty}\frac{F_{2,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t}}{((i-1)\lambda_3+j\lambda_4)^2\left(\lambda_3+(j-1)\lambda_4\right)^2}\\&-\sum_{i,j,k=0}^{\infty}\frac{F_{1,k}(a_1,a_2,a_3,a_4)\,e^{-(i\lambda_3+j\lambda_4)t}}{((i-1)\lambda_3+j\lambda_4)^2\left(\lambda_3+(j-1)\lambda_4\right)}\end{aligned} \tag{3.14}$$

Now, substituting the value of A_3 from equation (3.13) and the value of A_2 from the equation (3.14) into the equation (3.12), we obtain

$$
\begin{aligned}
&e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)\frac{\partial^2 A_1}{\partial t^2}+e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)^3 A_4 \\
&=-6e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)\sum_{i,j,k=0}^{\infty}\frac{F_{2,k}(a_1,a_2,a_3,a_4)\,e^{-((i-1)\lambda_3+j\lambda_4)t}}{((i-1)\lambda_3+j\lambda_4)^2\,(i\lambda_3+(j-1)\lambda_4)} \\
&-3e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)\frac{\partial}{\partial t}\{6\sum_{i,j,k=0}^{\infty}\frac{F_{2,k}(a_1,a_2,a_3,a_4)\,e^{-(i\lambda_3+j\lambda_4)t}}{((i-1)\lambda_3+j\lambda_4)^2\,(i\lambda_3+(j-1)\lambda_4)^2} \\
&-\sum_{i,j,k=0}^{\infty}F_{1,k}(a_1,a_2,a_3,a_4)\,e^{-(i\lambda_3+j\lambda_4)t}\}-\sum_{i,j,k=0}^{\infty}F_{0,k}(a_1,a_2,a_3,a_4)\,e^{-(i\lambda_3+j\lambda_4)t}
\end{aligned}
\tag{3.15}
$$

Now, we have a single equation (3.15) for obtaining two unknown functions A_1, and A_4. So, we need to impose some restrictions to split the equation (3.15) for determining the unknown functions A_1, and A_4 (Shamsul [77, 78, 80, 82]). In this chapter, we have imposed the restriction that, if $i \geq j$; the term $e^{-(i\lambda_3+j\lambda_4)t}$ balance with A_1, or if $j > i$ the term $e^{-(i\lambda_3+j\lambda_4)t}$ balance with A_4, when $\lambda_3 > \lambda_4$. For the sake of definiteness of the eigenvalues, it is possible to have the relation $\lambda_3 > \lambda_4$ between them. This restriction is important, since under this restriction the coefficients of A_1, and A_4 do not become large (the principle of the KBM method is that the coefficients of A_1, A_2, A_3 and A_4 must be small) as well as the unknown functions A_1, and A_4 can be determined very quickly. This restriction has another importance, that, the solution is also useful in the case of more critically (when three eigenvalues are equal) damped systems. This restriction is not used in previous papers [2, 59, 60, 63, 69, 78, 80, 82, 83].

The values of A_1, A_2, A_3 and A_4 will be obtained in terms of a_1, a_2, a_3, a_4 and *t*; and since $\dot{a}_1$, $\dot{a}_2$, $\dot{a}_3$, $\dot{a}_4$ are proportional to the small parameter ε, so they are slowly varying functions of time *t*. Hence their rates of change are very small *i.e.* they are almost constant. Therefore, it is plausible to replace a_1, a_2, a_3 and a_4 by their respective values obtained in the linear case (*i. e.* the values of a_1, a_2, a_3 and a_4 obtained when $\varepsilon = 0$) in the right hand side of (3.4).

This replacement was first made by Murty *et al.* [47] to solve similar type nonlinear equations. Thus, by substituting the values of A_1, A_2, A_3 and A_4 into equation (3.4) and integrating we will get the values of a_1, a_2, a_3 and a_4.

Equation (3.9) is a non-homogeneous linear differential equation, so it can be solved for u_1 by well known operator method.

Substituting the values of a_1, a_2, a_3, a_4 and u_1 in equation (3.3), we shall get the complete solution of (3.1).

Thus the determination of the first order approximate solution is completed. The method can be carried out for higher order nonlinear systems in the same way.

3.3 Example

As an example of the above method, we have considered the Duffing equation type nonlinear system

$$x^{(4)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = -\varepsilon x^3, \tag{3.16}$$

Here, $f = x^3$ and $x_0 = \left(a_1(t) + a_2(t)\, t + a_3(t)\, t^2\right) e^{-\lambda_3 t} + a_4(t)\, e^{-\lambda_4 t}$.

Therefore,

$$\begin{aligned} f^{(0)} &= a_1^3 e^{-3\lambda_3 t} + 3a_1^2 a_4\, e^{-(2\lambda_3+\lambda_4)t} + 3a_1\, a_4^2\, e^{-(\lambda_3+2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t} \\ &+ t\left(3a_1^2 a_2\, e^{-3\lambda_3 t} + 6a_1\, a_2\, a_4\, e^{-(2\lambda_3+\lambda_4)t} + 3a_2\, a_4^2 e^{-(\lambda_3+2\lambda_4)t}\right) \\ &+ t^2\left(3a_1\, a_2^2\, e^{-3\lambda_3 t} + 3a_1^2 a_3\, e^{-3\lambda_3 t} + 3a_2^2 a_4\, e^{-(2\lambda_3+\lambda_4)t} + 6a_1\, a_3\, a_4\, e^{-(2\lambda_3+\lambda_4)t}\right. \\ &\left.+ 3a_3\, a_4^2 e^{-(\lambda_3+2\lambda_4)t}\right) + t^3\left(a_2^3\, e^{-3\lambda_3 t} + 6a_1\, a_2\, a_3\, e^{-3\lambda_3 t} + 6a_2\, a_3\, a_4\, e^{-(2\lambda_2+\lambda_4)t}\right) \\ &+ t^4\left(3a_2^2\, a_1\, e^{-3\lambda_3 t} + 3a_1\, a_3^2\, e^{-3\lambda_3 t} + 3\, a_3^2\, a_4\, e^{-(2\lambda_3+\lambda_4)t}\right) + 3t^5 a_2\, a_3^2\, e^{-3\lambda_3 t} + t^6 a_3^3\, e^{-3\lambda_3 t} \end{aligned} \tag{3.17}$$

Comparing equations (3.6) and (3.17), we obtain

$$\begin{aligned} &\sum_{i,j,k=0}^{\infty} F_{0,k}(a_1, a_2, a_3, a_4)\, e^{-(i\lambda_3+j\lambda_4)t} \\ &= a_1^3 e^{-3\lambda_3 t} + 3a_1^2 a_4\, e^{-(2\lambda_3+\lambda_4)t} + 3a_1\, a_4^2\, e^{-(\lambda_3+2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t} \end{aligned} \tag{3.18}$$

$$\sum_{i,j,k=0}^{\infty} F_{1,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t}$$
$$= 3a_1^2 a_2\, e^{-3\lambda_3 t} + 6a_1\, a_2\, a_4\, e^{-(2\lambda_3+\lambda_4)t} + 3a_2\, a_4^2 e^{-(\lambda_3+2\lambda_4)t} \tag{3.19}$$

$$\sum_{i,j,k=0}^{\infty} F_{2,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} = 3a_1\, a_2^2\, e^{-3\lambda_3 t} + 3a_1^2 a_3\, e^{-3\lambda_3 t}$$
$$+ 3a_2^2 a_4\, e^{-(2\lambda_3+\lambda_4)t} + 6a_1\, a_3\, a_4\, e^{-(2\lambda_3+\lambda_4)t} + 3a_3\, a_4^2 e^{-(\lambda_3+2\lambda_4)t} \tag{3.20}$$

$$\sum_{i,j,k=0}^{\infty} F_{3,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} = a_2^3\, e^{-3\lambda_3 t} + 6a_1\, a_2\, a_3\, e^{-3\lambda_3 t} + 6a_2\, a_3\, a_4\, e^{-(2\lambda_2+\lambda_4)t} \tag{3.21}$$

$$\sum_{i,j,k=0}^{\infty} F_{4,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} = 3a_2^2\, a_3\, e^{-3\lambda_3 t} + 3a_1\, a_3^2\, e^{-3\lambda_3 t} + 3\, a_3^2\, a_4\, e^{-(2\lambda_3+\lambda_4)t} \tag{3.22}$$

$$\sum_{i,j,k=0}^{\infty} F_{5,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} = 3a_2\, a_3^2\, e^{-3\lambda_3 t} \tag{3.23}$$

$$\sum_{i,j,k=0}^{\infty} F_{6,k}(a_1,a_2,a_3,a_4)e^{-(i\lambda_3+j\lambda_4)t} = a_3^3\, e^{-3\lambda_3 t} \tag{3.24}$$

Therefore, equations (3.9)-(3.12) respectively become

$$\left(\frac{\partial}{\partial t}+\lambda_3\right)^3 \left(\frac{\partial}{\partial t}+\lambda_4\right)u_1 = -\{t^3(a_2^3\, e^{-3\lambda_3 t} + 6a_1\, a_2\, a_3\, e^{-3\lambda_3 t} + 6a_2\, a_3\, a_4\, e^{-(2\lambda_2+\lambda_4)t})$$
$$+ t^4(3a_2^2\, a_3\, e^{-3\lambda_3 t} + 3a_1\, a_3^2\, e^{-3\lambda_3 t} + 3\, a_3^2\, a_4\, e^{-(2\lambda_3+\lambda_4)t}) + 3t^5\, a_2\, a_3^2\, e^{-3\lambda_3 t} + t^6 a_3^3\, e^{-3\lambda_3 t}\} \tag{3.25}$$

$$e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)\frac{\partial^2 A_3}{\partial t^2} = -\{3a_1\, a_2^2\, e^{-3\lambda_3 t} + 3a_1^2 a_3\, e^{-3\lambda_3 t}$$
$$+ 3a_2^2 a_4\, e^{-(2\lambda_3+\lambda_4)t} + 6a_1\, a_3\, a_4\, e^{-(2\lambda_3+\lambda_4)t} + 3a_3\, a_4^2 e^{-(\lambda_3+2\lambda_4)t}\} \tag{3.26}$$

$$e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)\left(\frac{\partial^2 A_2}{\partial t^2}+6\frac{\partial A_3}{\partial t}\right)$$
$$= -\{3a_1^2 a_2\, e^{-3\lambda_3 t} + 6a_1\, a_2\, a_4\, e^{-(2\lambda_3+\lambda_4)t} + 3a_2\, a_4^2 e^{-(\lambda_3+2\lambda_4)t}\} \tag{3.27}$$

$$e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}-\lambda_3+\lambda_4\right)\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial A_2}{\partial t}+6\, A_3\right)+e^{-\lambda_4 t}\left(\frac{\partial}{\partial t}-\lambda_4+\lambda_3\right)^3 A_4$$
$$= -\{a_1^3 e^{-3\lambda_3 t} + 3a_1^2 a_4\, e^{-(2\lambda_3+\lambda_4)t} + 3a_1\, a_4^2\, e^{-(\lambda_3+2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t}\} \tag{3.28}$$

Therefore, solving equation (3.26), we obtain

$$A_3 = r_1(a_1a_2^2 + a_1^2a_3)e^{-2\lambda_3 t} + r_2(a_2^2\,a_4 + 2a_1\,a_3\,a_4)e^{-(\lambda_3+\lambda_4)t} + r_3\,a_3\,a_4^2\,e^{-2\lambda_4 t}, \tag{3.29}$$

where

$$r_1 = 3/4\lambda_3^2(3\lambda_3 - \lambda_4), \qquad r_2 = 3/2\lambda_3(\lambda_3+\lambda_4)^2, \qquad r_3 = -3/4\lambda_4^2(\lambda_3+\lambda_4).$$

Now substituting the value of A_3 from equation (3.29) into equation (3.27), we obtain

$$\begin{aligned}
&e^{-\lambda_3 t}\left(\frac{\partial}{\partial t} - \lambda_3 + \lambda_4\right)\frac{\partial^2 A_2}{\partial t^2} = -6\lambda_3(3\lambda_3 - \lambda_4)r_1(a_1a_2^2 + a_1^2a_3)e^{-3\lambda_3 t}\\
&-6\lambda_3(\lambda_3+\lambda_4)r_2(a_2^2\,a_4 + 2a_1\,a_3\,a_4)e^{-(2\lambda_3+\lambda_4)t} - 6\,\lambda_4^2(\lambda_3+\lambda_4)r_3\,a_3\,a_4^2\,e^{-3\lambda_4 t}\\
&-\{3a_1^2a_2\,e^{-3\lambda_3 t} + 6a_1\,a_2\,a_4\,e^{-(2\lambda_3+\lambda_4)t} + 3a_2\,a_4^2e^{-(\lambda_3+2\lambda_4)t}\}
\end{aligned} \tag{3.30}$$

Now, integrating equation (3.30), we obtain

$$\begin{aligned}
A_2 &= q_1(a_1a_2^2 + a_1^2a_3)e^{-2\lambda_3 t} + q_2(a_2^2\,a_4 + 2a_1\,a_3\,a_4)e^{-(\lambda_3+\lambda_4)t} + q_3\,a_3\,a_4^2\,e^{-3\lambda_4 t}\\
&+ q_4\,a_1^2a_2\,e^{-2\lambda_3 t} + q_5\,a_1\,a_2\,a_4\,e^{-(\lambda_3+\lambda_4)t} + q_6\,a_2\,a_4^2e^{-2\lambda_4 t}
\end{aligned} \tag{3.31}$$

where

$$q_1 = 9/4\lambda_3^3(3\lambda_3 - \lambda_4), \qquad q_2 = 9/\lambda_3(\lambda_3+\lambda_4)^3, \qquad q_3 = 9/4\lambda_4^3(\lambda_3+\lambda_4),$$

$$q_4 = 3/4\lambda_3^2(3\lambda_3 - \lambda_4), \qquad q_5 = 3/\lambda_3(\lambda_3+\lambda_4)^2, \qquad q_6 = 3/4\lambda_4^2(\lambda_3+\lambda_4).$$

Now substituting the value of A_3 from equation (3.29), and the value A_2 from equation (3.31) into equation (3.28), we obtain

$$\begin{aligned}
&e^{-\lambda_3 t}\left(\frac{\partial}{\partial t} - \lambda_3 + \lambda_4\right)\frac{\partial^2 A_1}{\partial t^2} + e^{-\lambda_4 t}\left(\frac{\partial}{\partial t} - \lambda_4 + \lambda_3\right)^3 A_4\\
&= -3\,e^{-\lambda_3 t}\left(\frac{\partial}{\partial t} - \lambda_3 + \lambda_4\right)\frac{\partial}{\partial t}\{q_1(a_1a_2^2 + a_1^2a_3)e^{-2\lambda_3 t} + q_2(a_2^2\,a_4 + 2a_1\,a_3\,a_4)e^{-(\lambda_3+\lambda_4)t}\\
&+ q_3\,a_3\,a_4^2\,e^{-2\lambda_4 t} + q_4\,a_1^2a_2\,e^{-2\lambda_3 t} + q_5\,a_1\,a_2\,a_4\,e^{-(\lambda_3+\lambda_4)t} + q_6\,a_2\,a_4^2e^{-2\lambda_4 t}\}\\
&-6e^{-\lambda_3 t}\left(\frac{\partial}{\partial t} - \lambda_3 + \lambda_4\right)\{r_1(a_1a_2^2 + a_1^2a_3)e^{-2\lambda_3 t} + r_2(a_2^2\,a_4 + 2a_1\,a_3\,a_4)e^{-(\lambda_3+\lambda_4)t}\\
&+ r_3\,a_3\,a_4^2\,e^{-2\lambda_4 t}\} - \{a_1^3e^{-3\lambda_3 t} + 3a_1^2a_4\,e^{-(2\lambda_4+\lambda_3)t} + 3a_1\,a_4^2\,e^{-(\lambda_3+2\lambda_4)t} + a_4^3e^{-3\lambda_4 t}\}
\end{aligned} \tag{3.32}$$

To separate the equation (3.32) for determining the unknown functions A_1 and A_4, we have impose the restriction that, if $i \geq j$; the term $e^{-(i\lambda_3 + j\lambda_4)t}$ balance with A_1, or if $j > i$ the term $e^{-(i\lambda_3 + j\lambda_4)t}$ balance with A_4, when $\lambda_3 > \lambda_4$. Under these restrictions, we obtain

$$\begin{aligned}
&e^{-\lambda_3 t}\left(\frac{\partial}{\partial t} - \lambda_3 + \lambda_4\right)\frac{\partial^2 A_1}{\partial t^2} = -6\, q_1\, \lambda_3\,(3\lambda_3 - \lambda_4)(a_1 a_2^2 + a_1^2 a_3)e^{-3\lambda_3 t} \\
&-6\, q_2\, \lambda_3\,(\lambda_3 + \lambda_4)(a_2^2\, a_4 + 2a_1\, a_3\, a_4)e^{-(2\lambda_3 + \lambda_4)t} - 6\, q_4\, \lambda_3\,(3\lambda_3 - \lambda_4)\, a_1^2 a_2\, e^{-3\lambda_3 t} \\
&-6\, q_5\, \lambda_3\,(\lambda_3 + \lambda_4) a_1\, a_2\, a_4\, e^{-(2\lambda_3 + \lambda_4)t} + 6 r_1\,(3\lambda_3 - \lambda_4)\,(a_1 a_2^2 + a_1^2 a_3)e^{-3\lambda_3 t} \\
&+12\, r_2\, \lambda_3(a_2^2\, a_4 + 2a_1\, a_3\, a_4)e^{-(2\lambda_3 + \lambda_4)t} - a_1^3 e^{-3\lambda_3 t} - 3a_1^2 a_4\, e^{-(2\lambda_3 + \lambda_4)t}
\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}
&e^{-\lambda_4 t}\left(\frac{\partial}{\partial t} - \lambda_4 + \lambda_3\right)^3 A_4 = -6\, q_3\, \lambda_4\,(\lambda_3 + \lambda_4)\, a_3\, a_4^2\, e^{-3\lambda_4 t} \\
&-6\, q_6\, \lambda_4\,(\lambda_3 + \lambda_4) a_2\, a_4^2 e^{-(\lambda_3 + 2\lambda_4 t)} - 6\, r_3\,(\lambda_3 + \lambda_4)\, a_3\, a_4^2\, e^{-(\lambda_3 + 2\lambda_4)t} \\
&-\{3a_1\, a_4^2\, e^{-(\lambda_3 + 2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t}\}
\end{aligned} \tag{3.34}$$

The particular solutions of equations (3.33)-(3.34) respectively become

$$\begin{aligned}
&A_1 = p_1\,(a_1 a_2^2 + a_1^2 a_3)e^{-2\lambda_3 t} + p_2\,(a_2^2\, a_4 + 2a_1\, a_3\, a_4)e^{-(\lambda_3 + \lambda_4)t} \\
&+ p_3\, a_1^2 a_2\, e^{-2\lambda_3 t} + p_4\, a_1\, a_2\, a_4\, e^{-(\lambda_3 + \lambda_4)t} + p_5\,(a_1 a_2^2 + a_1^2 a_3)e^{-2\lambda_3 t} \\
&+ p_6\,(a_2^2\, a_4 + 2a_1\, a_3\, a_4)e^{-(\lambda_3 + \lambda_4)t} + p_7\, a_1^3 e^{-2\lambda_3 t} + p_8\, a_1^2 a_4\, e^{-(\lambda_3 + \lambda_4)t}
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
A_4 &= s_1\, a_3\, a_4^2\, e^{-2\lambda_4 t} + s_2\, a_2\, a_4^2 e^{-(\lambda_3 + \lambda_4 t)} + s_3\, a_3\, a_4^2\, e^{-(\lambda_3 + \lambda_4)t} \\
&\quad + s_4\, a_1\, a_4^2\, e^{-(\lambda_3 + \lambda_4)t} + s_5\, a_4^3 e^{-2\lambda_4 t}
\end{aligned} \tag{3.36}$$

where

$p_1 = 27/8\lambda_3^4(3\lambda_3 - \lambda_4)$, $p_2 = 27/\lambda_3(\lambda_3 + \lambda_4)^4$, $p_3 = 9/\lambda_3^3(3\lambda_3 - \lambda_4)$,

$p_4 = 9/\lambda_3(\lambda_3 + \lambda_4)^3$, $p_5 = -9/8\lambda_3^3(3\lambda_3 - \lambda_4)$, $p_6 = -18/\lambda_4^2(\lambda_3 + \lambda_4)^3$,

$p_7 = 1/8\lambda_3^2(3\lambda_3 - \lambda_4)$, $p_8 = 1/2\lambda_3(\lambda_3 + \lambda_4)^2$, $s_1 = -27/2\lambda_4^2(\lambda_3 - 3\lambda_4)^3$,

$s_2 = 9/16\lambda_4^4$, $s_3 = -9/16\lambda_4^5$, $s_4 = 3/8\,\lambda_4^3$, $s_5 = -1/(\lambda_3 - 3\lambda_4)^3$.

The solution of the equation (3.25) is

and

$$
\begin{aligned}
u_1 &= l_1\, a_1\, a_3^2\, e^{-(\lambda_1+2\lambda_3)t} + l_2\, a_2\, a_3^2\, e^{-(\lambda_2+2\lambda_3)t} + l_3\, a_3^3\, e^{-3\lambda_3 t} \\
&+ a_1\, a_3\, a_4 e^{-(\lambda_1+2\lambda_3)t}(2\, l_1 t + l_4) + a_2\, a_3\, a_4 e^{-(\lambda_1+2\lambda_3)t}(2\, l_2 t + l_5) \\
&+ a_3^2\, a_4 e^{-(\lambda_1+2\lambda_3)t}(3 l_3\, t + l_6) + a_1 a_4^2\, e^{-(\lambda_1+2\lambda_3)t}(l_1\, t^2 + l_7\, t + l_8) \\
&+ a_2 a_4^2\, e^{-(\lambda_2+2\lambda_3)t}(l_2\, t^2 + l_9 t + l_{10}) + a_3 a_4^2\, e^{-3\lambda_3 t}(3 l_3 t^2 + l_{11} t + l_{12}) \\
&+ a_4^3\, e^{-3\lambda_3 t}(l_3\, t^3 + l_{13} t^2 + l_{14} t + l_{15})
\end{aligned} \tag{3.37}
$$

where

$$l_1 = -3/\{2\lambda_3(\lambda_1 + 2\lambda_3 - \lambda_2)(\lambda_1 + \lambda_3)^2\}, \qquad l_2 = -3/\{2\lambda_3(\lambda_2 + 2\lambda_3 - \lambda_1)(\lambda_2 + \lambda_3)^2\},$$

$$l_3 = -1/\{4\lambda_3^2(3\lambda_3 - \lambda_1)(3\lambda_3 - \lambda_2)\}, \qquad l_4 = 2l_1\left(\frac{2}{(\lambda_1 + \lambda_3)} + \frac{1}{(\lambda_1 + 2\lambda_3 - \lambda_2)} + \frac{1}{2\lambda_3}\right),$$

$$l_5 = 2l_2\left(\frac{2}{(\lambda_2 + \lambda_3)} + \frac{1}{(\lambda_2 + 2\lambda_3 - \lambda_1)} + \frac{1}{2\lambda_3}\right),\ l_6 = 3\, l_3\left(\frac{2}{(3\lambda_3 - \lambda_1)} + \frac{1}{(3\lambda_3 - \lambda_2)} + \frac{1}{\lambda_3}\right),$$

$$l_7 = l_1\left(\frac{4}{(\lambda_1 + \lambda_3)} + \frac{2}{(\lambda_1 + 2\lambda_3 - \lambda_2)} + \frac{1}{\lambda_3}\right),$$

$$l_8 = l_1\left(\begin{aligned}&\frac{4}{(\lambda_1 + \lambda_3)} + \frac{6}{(\lambda_1 + \lambda_3)^2} + \frac{2}{(\lambda_1 + 2\lambda_3 - \lambda_2)^2} \\ &+ \frac{2}{\lambda_3(\lambda_1 + \lambda_3)} + \frac{1}{\lambda_3(\lambda_1 + 2\lambda_3 - \lambda_2)} + \frac{1}{2\lambda_3^2}\end{aligned}\right),$$

$$l_9 = l_2\left(\frac{4}{(\lambda_2 + \lambda_3)} + \frac{2}{(\lambda_2 + 2\lambda_3 - \lambda_1)} + \frac{1}{\lambda_3}\right),$$

$$l_{10} = l_2\left(\begin{aligned}&\frac{4}{(\lambda_2 + \lambda_3)} + \frac{6}{(\lambda_2 + \lambda_3)^2} + \frac{2}{(\lambda_2 + 2\lambda_3 - \lambda_1)^2} \\ &+ \frac{2}{\lambda_3(\lambda_2 + \lambda_3)} + \frac{1}{\lambda_3(\lambda_2 + 2\lambda_3 - \lambda_1)} + \frac{1}{2\lambda_3^2}\end{aligned}\right),$$

$$l_{11} = 3l_3\left(\frac{2}{\lambda_3} + \frac{2}{(3\lambda_3 - \lambda_1)} + \frac{2}{(3\lambda_3 - \lambda_2)}\right),$$

$$l_{12} = 3\, l_3 \begin{pmatrix} \frac{3}{2\lambda_3^2} + \frac{2}{\lambda_3(3\lambda_3 - \lambda_1)} + \frac{2}{\lambda_3(3\lambda_3 - \lambda_2)} + \frac{2}{(3\lambda_3 - \lambda_1)^2} \\ + \frac{2}{(3\lambda_3 - \lambda_2)^2} + \frac{2}{(3\lambda_3 - \lambda_1)(3\lambda_3 - \lambda_2)} \end{pmatrix},$$

$$l_{13} = l_3 \left(\frac{3}{\lambda_3} + \frac{3}{(3\lambda_3 - \lambda_1)} + \frac{3}{(3\lambda_3 - \lambda_2)} \right),$$

$$l_{14} = l_3 \begin{pmatrix} \frac{9}{2\lambda_3^2} + \frac{6}{\lambda_3(3\lambda_3 - \lambda_1)} + \frac{6}{\lambda_3(3\lambda_3 - \lambda_2)} + \frac{6}{(3\lambda_3 - \lambda_1)^2} \\ + \frac{6}{(3\lambda_3 - \lambda_2)^2} + \frac{6}{(3\lambda_3 - \lambda_1)(3\lambda_3 - \lambda_2)} \end{pmatrix},$$

$$\begin{aligned} l_{15} = l_3 \{ & \frac{3}{\lambda_3^3} + \frac{9}{2\lambda_3^2(3\lambda_3 - \lambda_1)} + \frac{9}{2\lambda_3^2(3\lambda_3 - \lambda_2)} + \frac{6}{\lambda_3(3\lambda_3 - \lambda_1)^2} \\ & + \frac{6}{\lambda_3(3\lambda_3 - \lambda_2)^2} + \frac{6}{(3\lambda_3 - \lambda_1)^3} + \frac{6}{(3\lambda_3 - \lambda_2)^3} + \frac{6}{(3\lambda_3 - \lambda_1)^2(3\lambda_3 - \lambda_2)} \\ & + \frac{6}{(3\lambda_3 - \lambda_1)(3\lambda_3 - \lambda_2)^2} + \frac{6}{\lambda_3(3\lambda_3 - \lambda_1)(3\lambda_3 - \lambda_2)} \} \end{aligned}$$

Substituting the values of A_1, A_2, A_3 and A_4 from equations (3.35), (3.31), (3.29) and (3.36) into (3.4), we obtain

$$\begin{aligned} \dot{a}_1 = \varepsilon \{ & p_1 (a_1 a_2^2 + a_1^2 a_3) e^{-2\lambda_3 t} + p_2\, (a_2^2\, a_4 + 2 a_1\, a_3\, a_4) e^{-(\lambda_3 + \lambda_4)t} \\ & + p_3\, a_1^2 a_2\, e^{-2\lambda_3 t} + p_4\, a_1\, a_2\, a_4\, e^{-(\lambda_3 + \lambda_4)t} + p_5 (a_1 a_2^2 + a_1^2 a_3) e^{-2\lambda_3 t} \\ & + p_6 (a_2^2\, a_4 + 2 a_1\, a_3\, a_4) e^{-(\lambda_3 + \lambda_4)t} + p_7\, a_1^3 e^{-2\lambda_3 t} + p_8\, a_1^2 a_4\, e^{-(\lambda_3 + \lambda_4)t} \}, \\ \dot{a}_2 = \varepsilon \{ & q_1 (a_1 a_2^2 + a_1^2 a_3) e^{-2\lambda_3 t} + q_2 (a_2^2\, a_4 + 2 a_1\, a_3\, a_4) e^{-(\lambda_3 + \lambda_4)t} \\ & + q_3\, a_3\, a_4^2\, e^{-3\lambda_4 t} + q_4\, a_1^2 a_2\, e^{-2\lambda_3 t} + q_5\, a_1\, a_2\, a_4\, e^{-(\lambda_3 + \lambda_4)t} + q_6\, a_2\, a_4^2 e^{-2\lambda_4 t} \}, \\ \dot{a}_3 = \varepsilon \{ & r_1 (a_1 a_2^2 + a_1^2 a_3) e^{-2\lambda_3 t} + r_2 (a_2^2\, a_4 + 2 a_1\, a_3\, a_4) e^{-(\lambda_3 + \lambda_4)t} + r_3\, a_3\, a_4^2\, e^{-2\lambda_4 t} \}, \\ \dot{a}_4 = \varepsilon \{ & s_1\, a_3\, a_4^2\, e^{-2\lambda_4 t} + s_2\, a_2\, a_4^2 e^{-(\lambda_3 + \lambda_4 t)} + s_3\, a_3\, a_4^2\, e^{-(\lambda_3 + \lambda_4) t} \\ & + s_4\, a_1\, a_4^2\, e^{-(\lambda_3 + \lambda_4) t} + s_5\, a_4^3 e^{-2\lambda_4 t} \}. \end{aligned} \tag{3.38}$$

Since a_1, a_2, a_3, a_4 are proportional to the small parameter ε, so they are slowly varying functions of time *t*. Therefore, we can solve equation (3.38) by assuming a_1, a_2, a_3 and a_4 are constants in the right hand sides of (3.38). This assumption was first made by Murty *et al.* [46, 47]. Thus the solutions of equation (3.38) are

$$a_1 = a_{1,0} + \varepsilon \left\{ p_1 (a_{1,0}\, a_{2,0}^2 + a_{1,0}^2 a_{3,0}) \frac{1-e^{-2\lambda_3 t}}{2\lambda_3} + p_2 (a_{2,0}^2\, a_{4,0} + 2\, a_{1,0}\, a_{3,0}\, a_{4,0}) \frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4} \right.$$

$$+ p_3\, a_{1,0}^2\, a_{2,0} \frac{1-e^{-2\lambda_3 t}}{2\lambda_3} + p_4\, a_{1,0}\, a_{2,0}\, a_{4,0} \frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4} + p_5 (a_{1,0}\, a_{2,0}^2 + a_{1,0}^2 a_{3,0}) \frac{1-e^{-2\lambda_3 t}}{2\lambda_3}$$

$$\left. + p_6 (a_{2,0}^2\, a_{4,0} + 2 a_{1,0}\, a_{3,0}\, a_{4,0}) \frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4} + p_7\, a_{1,0}^3 \frac{1-e^{-2\lambda_3 t}}{2\lambda_3} + p_8\, a_{1,0}^2\, a_{4,0} \frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4} \right\}$$

$$a_2 = a_{2,0} + \varepsilon \left\{ q_1 \left(a_{1,0}\, a_{2,0}^2 + a_{1,0}^2\, a_{3,0}\right) \frac{1-e^{-2\lambda_3 t}}{2\lambda_3} + q_2 \left(a_{4,0}\, a_{2,0}^2 + 2 a_{1,0}\, a_{3,0} a_{4,0}\right) \frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4} \right.$$

$$+ q_3\, a_{3,0}\, a_{4,0}^2 \frac{1-e^{-3\lambda_4 t}}{3\lambda_4} + q_4\, a_{1,0}^2\, a_{2,0} \frac{1-e^{-2\lambda_3 t}}{2\lambda_3} + q_5\, a_{1,0}\, a_{2,0} a_{4,0} \frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4} \tag{3.39}$$

$$\left. + q_6 a_{2,0} a_{4,0}^2 \frac{1-e^{-2\lambda_4 t}}{2\lambda_4} \right\}$$

$$a_3 = a_{3,0} + \varepsilon \{ r_1 (a_{1,0} a_{2,0}^2 + a_{1,0}^2 a_{3,0}) \frac{1-e^{-2\lambda_3 t}}{2\lambda_3}$$

$$+ r_2 (a_{2,0}^2\, a_{4,0} + 2 a_{1,0}\, a_{3,0}\, a_{4,0}) \frac{1-e^{-(\lambda_3+\lambda_4)t}}{(\lambda_3+\lambda_4)} + r_3\, a_{3,0}\, a_{4,0}^2 \frac{1-e^{-2\lambda_4 t}}{2\lambda_4} \},$$

$$a_4 = \varepsilon \left\{ s_1\, a_{3,0}\, a_{4,0}^2 \frac{1e^{-2\lambda_4 t}}{2\lambda_4} + s_2\, a_{2,0}\, a_{4,0}^2 \frac{1e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4} + s_3\, a_{3,0}\, a_{4,0}^2 \frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4} \right.$$

$$\left. + s_4\, a_{1,0}\, a_{4,0}^2 \frac{1e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4} + s_5\, a_{4,0}^3 \frac{1-e^{-2\lambda_4 t}}{2\lambda_4} \right\}.$$

Therefore, we obtain the first approximate solution of the equation (3.16) as

$$x(t,\varepsilon) = (a_1 + a_2\, t + a_3 t^2) e^{-\lambda_3 t} + a_4\, e^{-\lambda_4 t} + \varepsilon\, u_1 \tag{3.40}$$

where a_1, a_2, a_3, a_4 are given by the equations (3.39) and u_1 is given by the equation (3.37).

3.4 Results and Discussion

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we sometimes compare the approximate solution to the numerical solution. With regard to such a comparison concerning the presented asymptotic solution obtained by the KBM method of this chapter, we refer the works of Murty *et al* [47].

We have computed $x(t,\varepsilon)$ by equation (3.40) in which a_1, a_2, a_3, a_4 are evaluated by the equation (3.39) and u_1 is evaluated by the equation (3.37) with different sets of initial conditions and for various t. The approximate analytic solutions and numerical solutions are plotted in the figures (From Fig. 3.1 to Fig. 3.5). From figures we see that our approximate analytical solutions show good coincidence with numerical solutions.

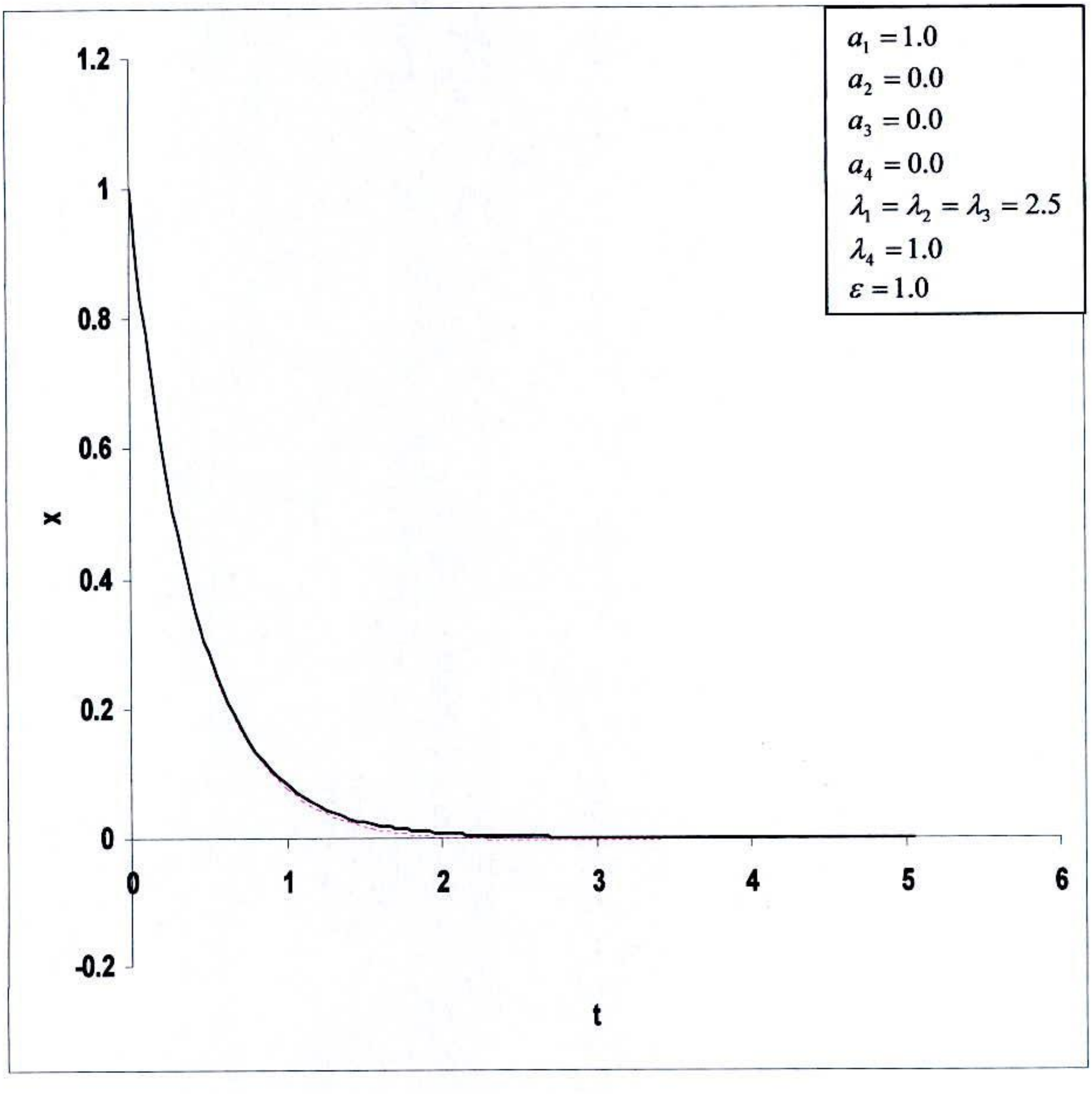

Fig. 3.1: Analytic solution in solid line — and solution by a fourth order Runge-Kutta method in dashed line - - -.

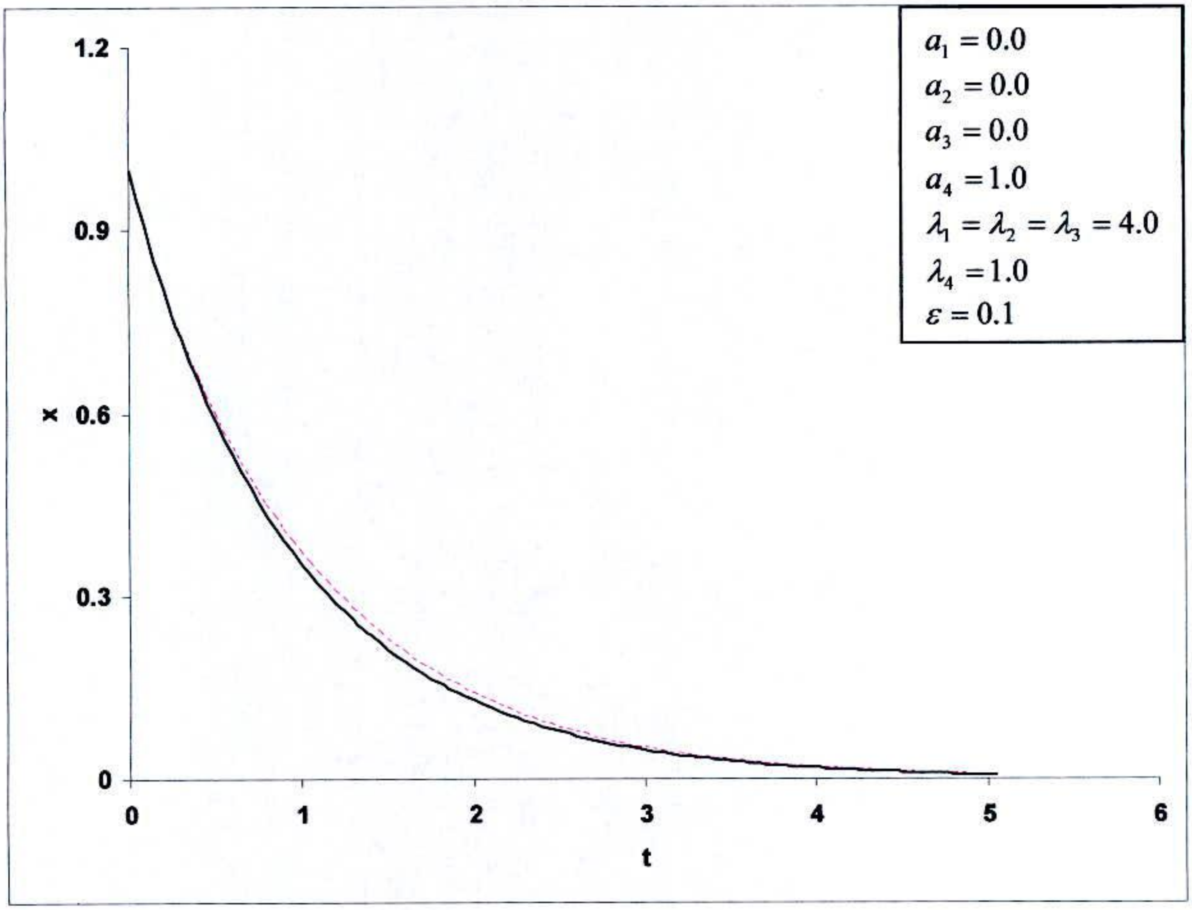

Fig. 3.2 Analytic solution in solid line — and solution by a fourth order Runge-Kutta method in dashed line - - -.

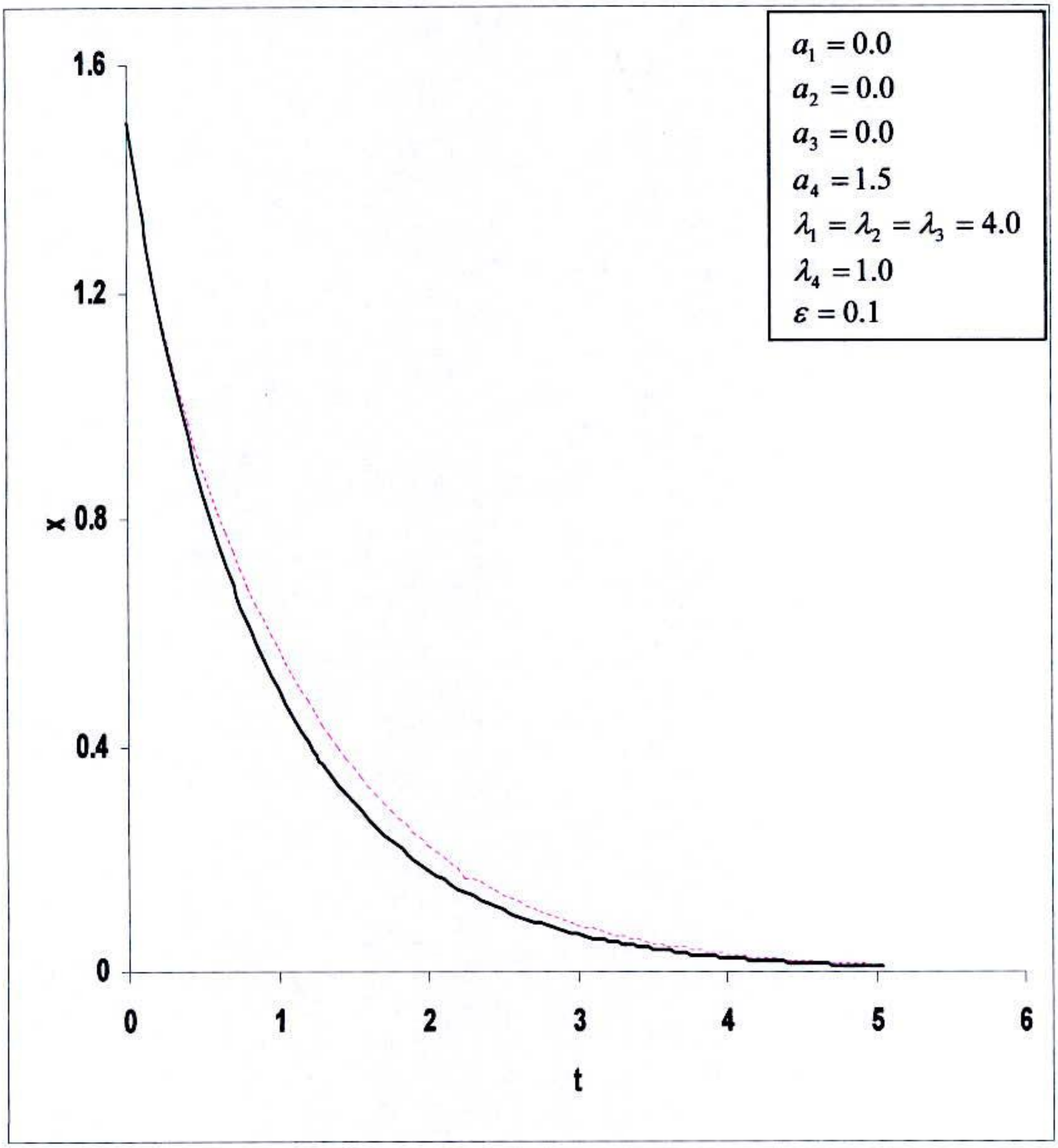

Fig. 3.3: Analytic solution in solid line — and solution by a fourth order Runge-Kutta method in dashed line - - -.

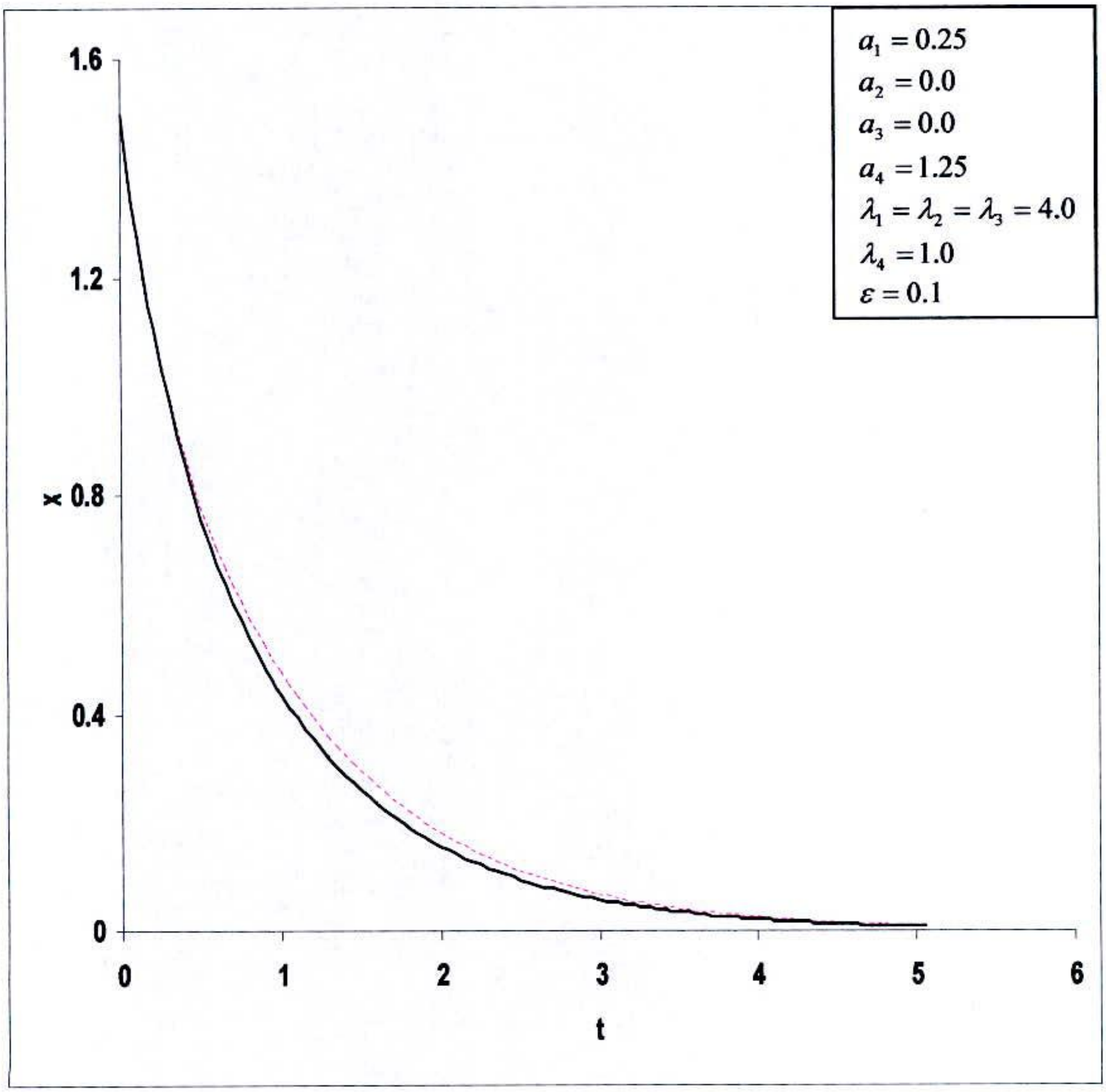

Fig. 3.4: Analytic solution in solid line — and solution by a fourth order Runge-Kutta method in dashed line - - -.

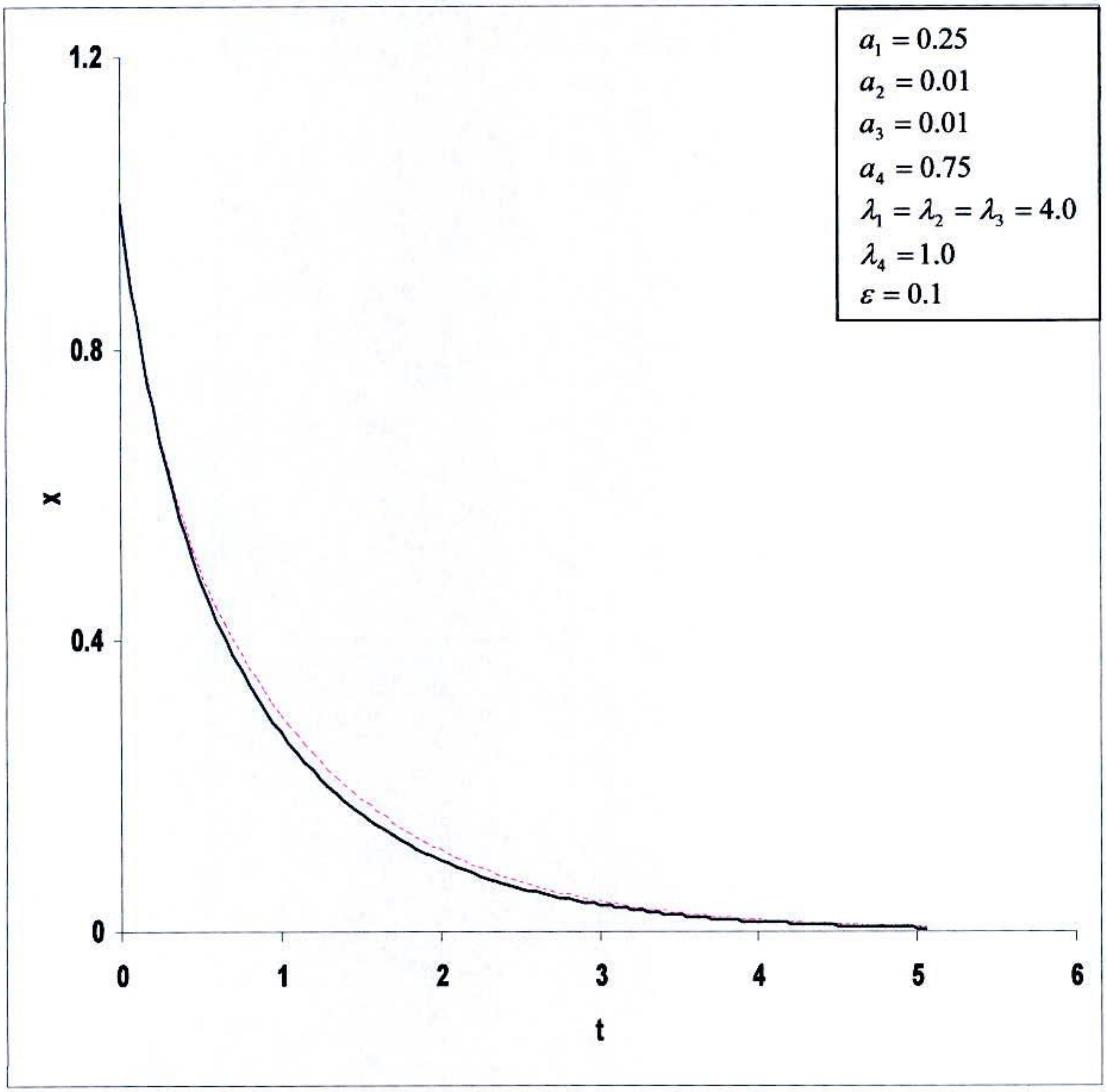

Fig. 3.5: Analytic solution in solid line — and solution by a fourth order Runge-Kutta method in dashed line - - -.

3.5 Conclusion

The Krylov-Bogoliubov-Mitropolskii method has been extended for solving fourth order more critically damped nonlinear systems. For different sets of damping forces as well as for different sets of initial conditions, the solutions obtained by the present method coincidence with those obtained by numerical method nicely. The solutions are also useful for strongly more critically damped nonlinear systems.

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