A Study of Sectionally Pseudocomplemented Lattice
&
Boolean Algebra

A thesis submitted in the partial fulfillment of the requirements for the degree of Masters of Philosophy in Mathematics

by

Md. Nazmul Hasan
Roll No: 0451552

Khulna University of Engineering & Technology
Khulna 9203, Bangladesh

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Dedicated to my

Beloved parents

&

Respectable teachers

DECLARATION

This is to certify that the thesis work entitled "A Study of Sectionally Pseudocomplemented Lattice and Boolean Algebras" has been carried out by Md. Nazmul Hasan in the Department of Mathematics Khulna University of Engineering & Technology Khulna, Bangladesh. The above research work or any part of the work has not been submitted anywhere for the award of any degree or diploma.

Signature of the Supervisor
Professor Dr.Md. Abul Kalam Azad
Department of Mathematics
Khulna University of Engineering & Technology
Khulna, Bangladesh.

Signature of the Students
Md. Nazmul Hasan
Roll No: 0451552

APPROVAL

This is to certify that the thesis work submitted by Md. Nazmul Hasan entitled "A Study of Sectionally Pseudo Complemented Lattice & Boolean Algebras" has been approved by the Board of the Examiners for the partial fulfillment of the requirements for the degree of Masters of Philosophy in the Department of Mathematics Khulna University of Engineering & Technology, Khulna, Bangladesh inJune-2009.

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PUBLICATIONS

1. Book: Ayub,Elias, **Nazmul** (2009); Lattice Theory and Boolean Algebra, Titas Publications, Dhaka, Bangladesh, ISBN-984-8759-00-0.
2. Book: Elias, Hafiz, **Nazmul** (2011); Abstract Algebra, Titas Publications, Dhaka, Bangladesh, ISBN-984-8760-00-0.
3. Research Paper: **Md. Abul kalam Azad, Md. Nazmul Hasan, Md. Zaidur Rahman** (2009); Switching circuit in Boolean Algebra, Daffodil International University, Dhaka, Bangladesh.vol-5, Issue-2, July 2010.
4. Research paper: **Nazmul** (2009); A case study on Garbage Management of Dhaka city Problem and Solution; National Academy for Education and Management (NAEM), Dhaka, Bangladesh.

ABASTRACT

This thesis studies the nature of a sectionally pesudocomplemented lattice and Boolean Algebra. Lattice theory is a part of Mathematics. Boolean algebra and Boolean function is an important of lattice theory. A nonempty set P together with a binary relation R is said to form a partially order set or a poset if the following conditions hold:

(i) Reflexivity (ii) Anti-symmetry (iii) Transitivity.

A poset $(L,\leq)$ is said to form a lattice if for every $a,b \in L$ if $a \vee b$ and $a \wedge b$ exist in L. A lattice is said to be complemented lattice if every element has a complement.

Let L be a bounded distributive lattice, let $a \in L$ an element $a^{*} \in L$ is called a pseudocomplement of a in L if the following conditions holds: (i) $a \wedge a^{*} = 0$ (ii) $\forall x \in L$, $a \wedge x = 0$ implies that $x \leq a^{*}$.

A complement distributive lattice is called a Boolean lattice. Since complements are unique in a Boolean lattice as an algebra with two binary operations $\wedge$ and $\vee$ and one unary operation $'$. Boolean lattices so considered are called Boolean algebra. Moreover we can discuss on relatively pseudocomplemented Lattices. In this thesis, we have given several results on sectionnally (relatively) pseudocomplemented lattices which certainly extended and generalized many results in lattice theory.

In chapter one is to outline and fix the notation for some of the concepts of lattices which are basic to this thesis. Some more definitions and formulate results on a orbitrary lattices for later use. We have considered this section as the base and background for the study of subsequent sections. For the background material in Lattice theory we have refered the readers to the of G. Birkoff [14] G. Gratzer [15] and V.K. Khanna [24] and several authors.

In chapter two, we have given a description of difference classes of lattices. We have also studied normal lattices and distributive quasi-complemented lattices. Generalized stone lattices have been studied by H. Lakser [16,17], K.B Lee [20] and many other authors.

We have given a characterization of minimal prime ideals of a sectionally pseudocomplemented lattices. Then we have shown that a distributive Lattice L with 0 is generalized stone if and only if it is both normal and sectionally quasi-complemented.

In chapter three introduces the concept of relative annihilators in lattices. Relative annihilators in lattices were studied by several authors including Mandelker [21] and Verlet [22]. B.A. Davey [1] has used the annihilators in studying relatively normal lattices. Here we have studied the relative annihilators in lattices. In terms of relative annihilators, we have characterized modular and distributive lattices. Relatively stone lattices have been studied by several authors including Mandelker [26], T.P. Speed [23] Gratzer and Schmidt [15]. Here we use given several characterizations of relatively stone lattices, which are certainly the generalization of above authors work. We have also shown for a distributive lattice L in which every closed interval in pseudocomplemented is relatively stone if and only if any two incomparable prime ideals of L are comaximal.

In chapter four, we have studied lattices with the greatest element 1 where on each interval $[a,1]$ an antitone bijection is defined. We have characterized these lattices by means of two induced binary operations proving that the resulting algebras form a variety.

The congruence properties of this variety and the properties of the underlying lattices are investigated. We have shown that this variety contains a single minimal subquasi variety join-lattices, whose principal filters are Boolean lattices, were used by J.C. Abbott [18]. We have introduced a further generalization of this concept, defining the notion of a lattice with sectionally antitone bijection. We have also introduced Residuated Lattices studied by Ward and Diworth [26] and several authors. Two mono graph contain a compendium on residuated lattices. They are that by Blyth and Janowitz [2]. In this paper we will compare a certain modification of a residuated Lattice.

In chapter five, It is shown that every directoid equipped with sectionally switching mappings can be represented as a certain implication algebra. The concept of directoid was introduced by J. Jezek and R. Quackenbush [19] in the sake to axiomatize algebraic structures defined by on upward directed ordered set. In certain sense, directoids generalize semilattices.

In chapter six. We have studied switching Mapping introduced by Chajda and Emanovsky [3]. A mapping f of $[a,1]$ onto itself is called switching mapping if $f(a)=1$ and for $x \in [a,1], a \neq x \neq 1$.In the section$[q,1]$ is determined by that of $[p,1]$, we say that the compatibility condition for antitony switching mappings and connection with complemention in sections have been shown.

CONTENTS

Chapter One

First concepts of Lattices

Introduction: The intention of this section is to outline and fix the notation for some of the concepts of lattices which are basic to this thesis. Some more definitions and formulate some results on arbitrary lattices for latter use. We consider this section as the base and background for the study of subsequent sections. For the background material in lattice theory we refer the reader to the text of G. Birkhoff [14], G. Gratzer [13] and Vijay K. Khanna [24].

1.1 Preliminaries

Some definitions with examples:

Set: Any collection of objects which are related to each other.

Finite set: A set is finite if it consists of a specific number of different elements

Example 1.1.2 Let A be the set of months of a year. Then A is a finite set.

Infinite set: A set is infinite if it does not consist of a specific number of different elements.

Example 1.1.3 Let $A = \{1,3,5,7,...........\}$. Then A is infinite.

Comparable: Two set A and B are said to be comparable if $A \subset B$ $or\, B \subset A$ i.e. if one set is a subset of the other.

Example 1.1.4 Let $A = \{1,3,6,9\}$ and $B = \{1,3,6,9,12,...........\}$. Then A and B are comparable i.e. $A \subset B$.

Empty set: A set having no element is called the empty set or null or void set and denoted by ϕ.

Example 1.1.5 Let A is a set having no element. i.e. $A = \phi$

Line diagram: If $A \subset B$ then we write B on a higher level then A and connect them by a line.

Fig-1.1

If $A \subset B$ and $B \subset C$

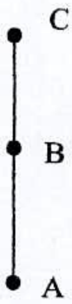

Fig-1.2

Example 1.1.6 Let $A = \{3\}, B = \{4\}$ and $C = \{3,4\}$ then the line diagram of A, B and C.

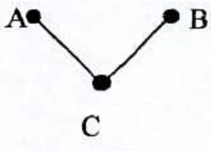

fig-1.3

Power set: The family of all the subsets of any set is called the power set of X. Notice, since ϕ is contained in every set, $\phi \in p(X)$ we denote the power set of X is $p(X)$.

Example 1.1.7 Let $X = \{a,b\}$ then $P(X) = \{\{a,b\}, \{a\}, \{b\}, \varphi\}$

Disjoint set: If the set A and B have no common elements. i.e. no element of A is in B and no element is in A, then we say A and B are disjoint set.

Example 1.1.8 Let $A = \{a,b\}$ and $B = \{c,d\}$, then A and B are disjoint set, since $A \cap B = \varphi$.

Theorem 1.1.1 Let A and B be two sets which are not comparable. Construct the line diagram A, B and $A \cap B$

Proof: Since $A \cap B \subset A$ and $A \cap B \subset B$ so $A \cap B$ is a subset of both A and B

Accordingly, we have the following line diagram,

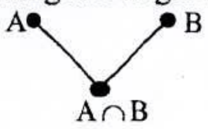

Fig-1.4

Function: Let A and B be two sets, a relation $R : A \to B$ is called a function if each element of A is assigned to a unique element of B

Example 1.1.9 $f(x) = x^3 + 1$ is a function.

Domain and Co-domain: If the relation $R : A \to B$ is a function then the set A is called domain and the set B is called co-domain.

Example 1.1.10 $A = \{1,2,3\} \to B\{2,4,6,8\} : f(x) = 2x$ then the set A is domain and B is co-domain.

One-one function: Let f be a function from A to B the function. Then f is said to be one-one function if every element of A is assigned to single element of B

Example 1.10 $f(x) = x^3 + 1$ is a one-one function .

Onto function: Let f be function from A to B then the function f is said to be onto function if every element of B is assigned.

Product function: Let f be a function from A to B and let g be a function of B the co-domain of f, into C The new function is called a production function or composite function of f and g and it is denote by $g \circ f$ or (gf)

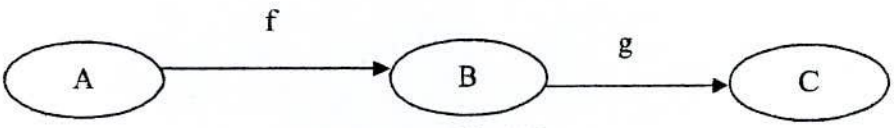

Fig-1.5

Relation: A relation R from A to B is subset of $A \times B$

Example 1.1.13 Let $X = \{x, y, z\}$ and $Y = \{a, b\}$,then

$R = \{(x,a),(x,b),(y,a),(y,b),(z,a),(z,b)\}$ is a relation from A to B

Equivalence Relation: A relation R in a set A is an equivalence relation if

a. R is a reflexive: $(a,a) \in R \forall a \in A$

i.e. $aRa \ \forall a \in R$

b. R is symmetric: $(a,b) \in R$ then $a,b \in R$

i.e. $aRb \Rightarrow bRa$

c. R is transitive: $(a,b),(b,c) \in R$ then $a,b,c \in R$

i.e. $aRb, bRc \Rightarrow aRc$

Example 1.1.14 Let $A = \{1,2,3\}$ be a set and

$R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(2,3)\}$ be a relation of R then the relation is an equivalent relation. Since

a. R **is reflexive,** $(1,1),(2,2),(3,3) \in R$

b. R **is symmetric,** $(1,2),(2,1),(1,3),(3,1) \in R$

c. R **is transitive,** $(2,1),(1,3),(2,3) \in R$

Partially order relation: A relation R in a set A is an partially order relation if

a. **Reflexivity**: $aRa \quad \forall a \in R$

b. Transitivity: $aRb, \ bRc$ then $aRc \quad \forall a,b,c \in R$

c. Anti-symmetry: $aRb,\ bRc$ then $a = b \quad \forall a, b \in R$

Example 1.1.15 On a set $A = \{1,2,3\}$

$\therefore R = \{(1,1),(2,2),(3,3)\}$ is reflexive, anti-symmetry and transitive.

Quasi order relation: A relation R in a set A is quasi order relation if

a. R is a reflexive: $a, a \in R \quad \forall a \in A$

i.e. $aRa \quad \forall a \in R$

b. R **is transitive**: $(a,b),(b,c) \in R$ then $a, c \in R$

i.e. $aRb,\ bRc \Rightarrow aRc$

Example 1.1.16 Let $A = \{1,2,3\}$ be a set and $R = \{(1,1),(2,2),(3,3),(2,1),(1,3),(2,3)\}$

a. R **is reflexive,** $(1,1),(2,2),(3,3) \in R$

b. R **is transitive,** $(2,1),(1,3),(2,3) \in R$

Totaliy order set: If P is poset in which every two members are comparable it is called a totally order set or toset or a chain.

Example 1.1.17 If P is a chain and $x, y \in P$ then $x \leq y$ or $y \leq x$. Clearly also if x, y are distinct element of a chain then either $x \leq y$ or $y \leq x$

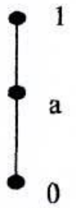

fig-1.6

Greatest element of a poset: Let P be a poset. If $\exists$ an element $a \in P$ s.t. $x \leq a \ \forall x \in P$ then a is called greatest or unit element of P . Greatest element if exist, will be unique.

Example 1.1.18 Let $A = \{1,2,3\}$ then $(p(A), \subseteq)$ is a poset.

Let $B = \{\{1,2\},\{2\},\{1\},\{3\},\{1,2,3\}\}$ then $(B,\subseteq)$ is a poset with $\{1,2,3\}$ as greatest element.

Least element of a poset: Let P be a poset. If $\exists$ an element $x \in P$ s.t. $b \leq x\ \forall x \in P$ then b is called least or zero element of P. Least element if exist, will be unique.

Example 1.1.19 Let $A = \{1,2,3\}$ then $(P(A),\subseteq)$ is a poset. Let

$B = \{\phi\{1,2\},\{2\},\{1\},\{3\},\{1,2,3\}\}$ then $(B,\subseteq)$ is a poset with φ as least element.

Maximal element: An element a in a poset P is called maximal element of P if $a < x$ for no $x \in P$.

Example 1.1.20 In the poset $A = \{2,3,4,6\}$ under divisility 4 and 6 are both maximal element.

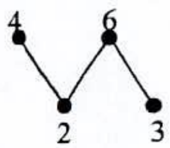

Fig-1.7

Minimal element: An element b in a poset P is called minimal element of P if $a < x < b$ for no $x \in P$.

Example 1.1.21 In the poset $A = \{2,3,4,6\}$ under divisility 2 and 3 are both minimal

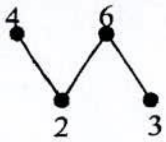

Fig-1.8

Upper bound of a set: Let S be a nonempty subset of a poset P. An element $a \in P$ is called an upper bound of S if $x \leq a\ \forall x \in S$

Example 1.1.22 $< L;\wedge,\vee,0,1 >$ is bounded lattice.

Least upper bound of a set: If a is an upper bound of S s.t. $a \leq b$ for all upper bound b of S then a is called least upper bound (l.u.b) or supremum of S. We write sup S for supremum S.

Lower bound of a set: An element $a \in P$ will be called a lower bound of S if $a \leq x \quad \forall x \in S$.

Greatest lower bound of a set: If a is a lower bound of S s.t. $b < a$ for all lower bounds b of S then a is called greatest lower bound (g.l.b) or infemum of S. We write inf S for infemum S.

Lattice: A poset $(L, \leq)$ is said to form a lattice if for every $a,b \in L$ sup$\{a,b\}$ and inf$\{a,b\}$ exist in L. In that case, we write,

$\sup\{a,b\} = a \vee b$ (read a join b)

$\inf\{a,b\} = a \wedge b$ (read a meet b)

Other notation like $a+b$ and ab or $a \cup b$ and $a \cup b$ are also used for sup$\{a,b\}$ and inf $\{a,b\}$

Example 1.1.27 The set, $L = \{1,2,3,4,6,12\}$ of factors of 12 under divisibility forms a lattice. It is represented by the following diagram:

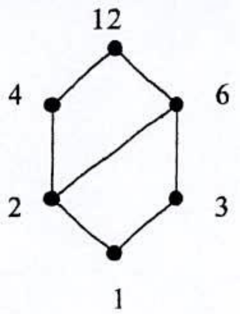

fig-1.9

Theorem 1.1.2 If S is a non empty finite subset of a poset P. Then S has sup. and inf.

Proof: Let $(p,\leq)$ be a lattice. Let S be any non-empty finite subset of P

Case-1: S has single element a, then $\inf S = \sup S = a$.

Case-2: S has two elements a,b then by the definition of lattice, supS and infS exist.

Case-3: S has three elements say, $S = \{a,b,c\}$. Since by the definition of lattice any two elements of P have sup and inf.

We take, $d = \inf\{a,b\}, e = \inf\{c,d\}$

We show $e = \inf\{a,b,c\}$

By the definition of d and e,

$d \le a, d \le b, e \le c, e \le d$ thus $e \le a, e \le b, e \le c$

$\Rightarrow e$ is a lower bound of $\{a,b,c\}$

If f is any lower bound of $\{a,b,c\}$ then. $f \le a, f \le b, f \le c$.

$f \le a, f \le b,$ and $d = \inf\{a,b\}$ gives $f \le d$

$f \le c, f \le d,$ and $e = \inf\{a,b\}$ gives $f \le e$

Hence $e = \inf\{a,b,c\} = \inf S$

Similarly $\sup S$ exists.

The result can similarly be extended to any finite number of elements in S.

Indeed Inf $S = \inf\{.....\inf\{a_1, a_2\}, a_3\}.......a_n\}$

If $S = \{a_1, a_2, a_3, ...a_n\}$

Conversely, the result holds trivially as when every non-empty finite subset has sup. and inf. □

Theorem 1.1.3 Let L be a lattice, then for any $a,b,c \in L$ the following results hold:

i. $a \wedge a = a,\ a \vee a = a$ (Idempotency)

ii. $a \wedge b = b \wedge a,\ a \vee b = b \vee a$ (Commutative)

iii. $a \wedge (b \wedge c) = (a \wedge b) \wedge c,\ a \vee (b \vee c) = (a \vee b) \vee c$ (Associativity)

iv. $a \wedge b \le a,\ b \le a \vee b$

v. $a \le b \Leftrightarrow a \wedge b = a$ (Consistency)

$\Leftrightarrow a \wedge b = b$

vi. If $0, u \in L$ then

$0 \wedge a = 0,\ 0 \vee a = a$

$u \wedge a = a,\ u \vee a = a = u$

vii. $a \wedge (a \vee b) = a$ (absorption)

$f \leq a, f \leq b,$ and

viii. $a \leq b, c \leq d$

$\Rightarrow a \wedge c \leq b \vee d$

$\therefore a \vee c \leq b \vee d$

In particular case,

$a \leq b \Rightarrow a \wedge x \leq b \wedge x$

$a \vee x \leq b \vee x \quad \forall x \in L$

Proof: We proof results for the meet operation, similarly we can prove the results for join operation.

i. $a \wedge a = \inf\{a, a\} = \inf\{a\} = a$

ii. $a \wedge b = \inf\{b, a\} = b \wedge a$

iii. Let $b \wedge c = d$ then $d = \inf\{b, c\}$

$\Rightarrow d \leq b, d \leq c$

Again let $e = \inf\{a, d\}$, then $e \leq a,\ e \leq d$

thus $e \leq a, e \leq b, e \leq c$ (by using transitivity)

Now $e = a \wedge d = a \wedge (b \wedge c) = \inf\{a, b, c\}$

Similarly we can show that $(a \wedge b) \wedge c = \inf\{a, b, c\}$

$\therefore a \wedge (b \wedge c) = (a \wedge b) \wedge c$

iv. Since any two elements a, b of a chain are comparable, say $a \leq b$, we find $a \wedge b = \inf\{a,b\} = a$

v. $a \leq b,\ a \leq a$ (by reflexivity)

$\Rightarrow a$ is a lower bound of $\{a,b\}$ and therefore $a = a \wedge b, \quad \therefore a \wedge b = a$

vi. Since $0 \leq x \leq u, \forall x \in L$

vii. $a \leq a \vee b$ by(iv)

$\therefore a \wedge (a \vee b) = a$ by(v)

$\therefore a \vee b \leq a$ by (iv)

$\therefore (a \vee b) \vee = a$ by(v) .

viii. $a \wedge c \leq a \leq b$

$a \wedge c \leq c \leq b$

Thus $a \wedge c$ is lower bound of $\{b,d\}$

Hence $a \wedge c \leq b \wedge b$, the glb$\{b,d\}$

Also then. $a \leq b, X \leq X \Rightarrow a \wedge x \leq b \wedge x$ □

Theorem 1.1.4 In any lattice L the distributive inequalities

i. $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

ii. $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ hold for any $a,b,c \in L$.

Proof: $a \wedge b \leq a$

$a \wedge b \leq b \vee c$

$\Rightarrow a \wedge b$ is lower bound of $\{a, b \vee c\}$

$\Rightarrow a \wedge b \leq a \wedge (b \vee c)$..................$(i)$

Again $a \wedge c \leq a$

$a \wedge c \leq c \leq b \vee c$

$\Rightarrow a \wedge c \leq a \wedge (b \vee c)$.................$(ii)$

(i) and (ii) show that $a \wedge (b \vee c)$ is an upper bound of $\{a \wedge b, a \vee c\}$

$\Rightarrow a \wedge (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$. □

Similarly we can prove the other inequality.

Note: The above are also called semi-distributive lattice.

Theorm 1.1.5 In any lattice L,

$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (c \vee a)$, for all $a,b,c \in L$

Proof: Since $a \wedge b \leq a \vee b$

$a \wedge b \leq b \vee c$

$a \wedge b \leq a \leq c \vee a$

we find $(a \wedge b) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

Similarly, $(b \wedge c) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

and $(c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

Hence $(a \wedge b) \vee (b \wedge c) \vee (c \vee a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ □

1.2 Algebraic Lattice

Algebraic Lattice: A non-empty set L together with two binary compositions (operations) $\wedge$ (meet) and $\vee$ (join) is said to form a algebraic lattice if for all $a,b,c \in L$ the following conditions are hold:

(i) Idempotency: $a \wedge a = a, a \vee a = a, a \in L$

(ii) Commutativity: $a \wedge b = b \wedge a, a \vee b = b \vee a, a, b \in L$

(iii) Associativity: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

$a \vee (b \vee c) = (a \vee b) \vee c \quad a,b,c \in L$

(iv) Absorption: $a \wedge (a \wedge b) = a$

$a \vee (a \wedge b) = a \quad a,b \in L$

Theorem 1.2.1 Show that a poset $(L,\wedge,\vee)$ is a lattice iff $(L,\wedge,\vee)$ is algebraic lattice.

Proof: Suppose L is a non-empty set

So that $a \wedge b = \inf\{a,b\}$ and $a \wedge b = \sup\{a,b\}$

Then $a \wedge a = \inf\{a,a\}$ and $a \vee a = \sup\{a,a\} = a$

So $\wedge$ and $\vee$ are idempotent.

$a \wedge b = \inf\{a,b\} = \inf\{b,a\} = b \wedge a$

$a \vee b = \sup\{a,b\} = \sup\{b,a\} = b \vee a$

So $\wedge$ and $\vee$ are commutative.

Next $a \wedge (b \wedge c) = \inf\{a, b \wedge c\} = \inf\{a, \inf\{b,c\}\}$

$= \inf\{\inf\{a,b\}, c\} = \inf\{a \wedge b, c\} = (a \wedge b) \wedge c$

$a \vee (b \vee c) = \sup\{a, b \vee c\} = \sup\{a, \sup\{b,c\}\}$

$= \sup\{\sup\{a,b\}, c\} = \sup\{a \vee b, c\} = (a \vee b) \vee c$

So $\wedge$ and $\vee$ are associative.

Finally, $a \wedge (b \vee c) = a \wedge \sup\{a,b\} = \inf\{a, \sup\{a,b\}\} = a$

$a \vee (b \wedge c) = a \vee \inf\{a,b\} = \sup\{a, \inf\{a,b\}\} = a$

Hence $\wedge$ and $\vee$ satisfy two absorption identity.

So $(L,\wedge,\vee)$ is a lattice.

Conversly, Since $\wedge$ is idempotent i.e, $a \wedge a = a \forall a \in L$

So $a \leq a$

$\therefore$ $\leq$ is reflexive. Since $\wedge$ is commutative

$\therefore$ $a \wedge a = a, a \vee a = a, a \in L$

$\Rightarrow a = b$ [since $a \wedge b = a$]

So $\leq$ is anti-symmetric.

Let $a \leq b$ and $b \leq c$. Then $a = a \wedge b, b = b \wedge c$

$\Rightarrow a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c$

$\Rightarrow a = a \wedge c \Rightarrow a \geq c$. So, $\geq$ is transitive.

$\therefore$ $(L, \leq)$ is a poset. □

Problem 1.2.1 Non-empty subset of every chain is sublattice.

Solution: Let L be a chain and if s be a non-empty subset of L. If a,b be two elements of S i.e., $a,b \in S$

$\Rightarrow a,b \in L \Rightarrow a,b$ are comparable.

Again let $b \geq a$ i.e, $a \leq b$

Then $a \wedge b = a \in S$. And $a \vee b = b \in S$. □

1.3 Semi lattice, Convex Lattice, Bounded Lattice and Complete Lattice

Meet semi Lattice: *A* non empty set P together with a binary operation $\wedge$ (meet) is called a meet semi Lattice if for all $a,b,c \in P$,

i. Idempotency: $a \wedge a = a$

ii. Commutativity: $a \wedge b = b \wedge a$

iii. Associativity: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Join semi lattice: *A* non empty set P together with a binary operation $\vee$ (join) is called a meet semi lattice

if for all $a,b,c \in P$,

i. Idempotency: $a \vee a = a$

ii. Commutativity: $a \vee b = b \vee a$

iii. Associativity: $a \vee (b \vee c) = (a \vee b) \vee c$

Theorem 1.3.1 A and B two lattices then prove that $A \times B$ is also a lattice.

Proof: Given that A and B two lattices then is a poset under the relation $\leq$ defined by

$(x_1, y_1) \leq (x_2, y_2)$

$\Leftrightarrow x_1 \leq x_2$ in A, $y_1 \leq y_2$ in B.

We show that $A \times B$ forms a lattice.

Let $(x_1, y_1), (x_2, y_2) \in A \times B$ be any element.

Then $(x_1, x_2) \in A$ and $(y_1, y_2) \in B$

Since A and B are lattices.

So $\{x_1, x_2\}$ and $\{y_1, y_2\}$ have sup and inf in A and B respectively.

Let $x_1 \wedge x_2 = \inf\{x_1, x_2\}$ and $y_1 \wedge y_2 = \inf\{y_1, y_2\}$

Then $x_1 \wedge x_2 \leq x_1, x_1 \wedge x_2 \leq x_2, y_1 \wedge y_2 \leq y_1, y_1 \wedge y_2 \leq y_2$

$\Rightarrow (x_1 \wedge x_2, y_1 \wedge y_2) \leq (x_1, y_1) \Rightarrow (x_1 \wedge x_2, y_1 \wedge y_2) \leq (x_2, y_2)$

$\Rightarrow (x_1 \wedge x_2, y_1 \wedge y_2)$ is a lower bound of $\{(x_1, y_1), (x_2, y_2)\}$

Suppose (z, w) is any lower bound of $\{(x_1, y_1), (x_2, y_2)\}$

Then

$$(z, w) \leq (x_1, y_1)$$
$$(z, w) \leq (x_2, y_2)$$

$\Rightarrow z \leq x_1,\ z \leq x_2,\ w \leq y_1,\ w \leq y_2$

z is a lower bound of $\{x_1, x_2\}$ in A

and w is a lower bound of $\{y_1, y_2\}$ in B

$\Rightarrow z \leq x_1 \wedge x_2 = \inf\{x_1, x_2\}$

$w \leq y_1 \wedge y_2 = \inf\{y_1, y_2\} \Rightarrow (z, w) \leq (x_1 \wedge x_2, y_1 \wedge y_2)$

Or that $x_1 \wedge x_2, y_1 \wedge y_2)$ is g.l.b $\{(x_1, y_1), (x_2, y_2)\}$

Similarly we can say that,

$(x_1 \wedge x_2, y_1 \wedge y_2)$ is a least upper bound of $\{(x_1, y_1), (x_2, y_2)\}$

Hence $A \times B$ is a lattice. □

Convex Lattice: A subset K of a lattice L is called a convex lattice if

$a, b \in k, c \in L, a \leq c \leq b$ imply that $c \in K$.

Convex sublattice: A subset K of a Lattice L is called a convex sublattice if for

all $a, b \in K, [a \wedge b, a \vee b] \subseteq K$

Example 1.3.2 In the lattice $\{1,2,3,4,6,12\}$ under divisibility $\{1,6\}$ is a sublattice which is

not convex as $2,3 \in [1,6]$ but $2,3 \notin \{1,6\}$

Diagrammatically the lattice $\{1,2,3,4,6,12\}$ can be represented by the figure 1.9.

Theorem 1.3.2 A sublattice of a lattice L is a convex sublattice iff for all

$x, y \in K (x \leq y), [x, y] \subseteq K$.

Proof: Let k be a convex sublattice of L and $x, y \in k(x \leq y)$, be any elements, then by

the definition

$[x \wedge y, x \vee y] \subseteq K$

$[x, y] \subseteq K$ as $x \leq y \Rightarrow x \wedge y = x$

$x \leq y \Rightarrow x \vee y = y$

Conversely, let $[x, y] \subseteq K \quad \forall x, y (x \leq y)$

Let $x, y \in K$ be a sub lattice.

Also are comparable. $\therefore [x \wedge y, x \vee y] \subseteq K$ □

Bounded Lattice: A lattice with a largest and a smallest element is called a bounded lattice. Smallest element is denoted by zero and the largest element is denoted by one.

Complete Lattice: A lattice L is called complete lattice if for its every sub set K, both $\sup K$, and $\inf K$ exists in L.

Finite Lattice: A lattice L is called finite lattice if it contains a finite number of elements.

Example 1.3.4 Let $L = \{1,2,5,10\}$ be a lattice under divisibility. Here in the lattice the finite number of element in L. So, L is finite lattice.

1.4 Ideal

Ideal of a Lattice: A non empty set I of a lattice L is called an ideal of L iff

i. $a,b \in I \Rightarrow a \vee b \in I$

ii. $a \in I, i \in I \Rightarrow a \wedge i \in I$

Example 1.4.1 Let $L = \{1,2,5,10\}$ be a lattice of factors of 10 under divisibility. Then $\{1\}$, $\{1,2\}$, $\{1,5\}$, $\{1,2,5,10\}$ are all the ideals of L.

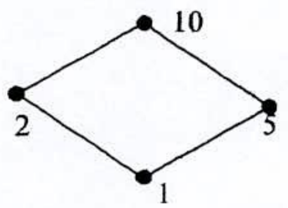

Fig: 10

Prime Ideal: An ideal P of L is called a prime ideal if for any $x, y \in L, x \wedge y \in P$ implies $x \in P$ or $y \in P$

Example 1.4.2 Let $L = \{1,2,3,4,6,12\}$ of factors of 12 under divisibility forms a Lattice then $\{1,2,4\}$ be a prime ideal of L (**figure 1.9**).

Principal Ideal: An ideal which generated by a single element is called principal ideal.

Example 1.4.3 Let $(a] = \{x/x \leq a\}$ then the ideal $(a]$ is generated by the element a.

Hence $(a]$ is principal ideal.

Filter or dual ideal: A non empty subset I of a lattice L is called dual ideal of L if

i. $a, b \in I$ implies that $a \wedge b \in I$

ii. $d \in I, a \in L$ implies that $d \wedge a \in I$

Example 1.4.4 Let $L = \{1,2,5,10\}$ be the lattice under divisibility.

Then $\{10\}$, $\{5,10\}$, $\{2,10\}$ are all dual ideals of a lattice L.

Theorem 1.4.1 If L is a chain then prove that every ideal of a lattice L is prime.

Proof: First suppose that every ideal of L is prime.

Now we are to show that L is a chain.

Let $a, b \in L$ then $a \wedge b \in L$

Now consider the ideal $I = (a \wedge b)$

By hypothesis I is prime.

Now $a \wedge b \in i \Rightarrow$ either $a \in I$ or $b \in I$

$\Rightarrow$ either $a \leq a \wedge b$ or $b \leq a \wedge b$

$\Rightarrow$ either $a = a \wedge b$ or $b = a \wedge b$ $\Rightarrow$ either $a \leq b$ or $b \leq a$

$\Rightarrow$ L is chain.

Conversely, let L be a chain and P be an ideal of L, we are to show that P is prime.

Let $x, y \in L$ with $x \wedge y \in P$. Since L is chain.

Then either $x \leq y$ or $y \leq x$

$\Rightarrow$either $x \wedge y = x$ or $x \wedge y = y$ $\Rightarrow$either $x \in P$ or $y \in P$

$\Rightarrow$ P is a prime ideal of L. □

Theorem 1.4.2 Suppose K and I be non-empty subset of a lattice L.

i) I is an ideal iff for all $x, y \in I$ $x \vee y \in L$ and for all $x \in I$, implies $t \leq I$

ii) $(K] = [x \in L / x \leq K_1 \vee K_2 \vee \ldots\ldots\ldots .\vee K_n$ for some $K_1, K_2, \ldots, K_n \in K\}$

iii. $(a] = \{x \in L / x \leq a$.

proof: i. Suppose I is an ideal.

So I is a sublattice and so for all $x, y \in I, x \vee y \in L$

Now let $x \in I, t \leq x$ implies $t \leq L$

Then $t = t \wedge x \in L$

Conversely, suppose I has the stated properties,

Let $x, y \in I$ then $x \wedge y \leq x$ implies $x \wedge y \in I$

i.e. I is a sub lattice.

Now suppose $i \in I, x \in I$ then $i \wedge x \leq I$

Implies $i \wedge x \leq I$

Therefore I is an ideal. □

ii. Let $x, y \in k_1 \vee k_2 \vee \ldots.. \wedge k_n$ for some $k_1, k_2, \ldots.., k_n \in K$

then, $x \leq k_1 \vee k_2 \vee \ldots.. \wedge k_n$ for some $k_1, k_2, \ldots.., k_n \in K$

$y \leq k_1 \vee k_2 \vee \ldots.. \wedge k_n$ for some $k_1, k_2, \ldots.., k_n \in K$

so $x \vee y \leq k_1 \vee k_2 \vee \ldots.. \wedge k_n \vee k_1 \vee k_2 \vee \ldots\ldots.. \vee k_n$

$\therefore$ $x, y \in k_1 \vee k_2 \vee \ldots.. \wedge k_n$ for some $k_1, k_2, \ldots.., k_n \in K$

If $x \in k_1 \vee k_2 \vee \ldots.. \wedge k_n$ for some $k_1, k_2, \ldots.., k_n \in K$

and $t \leq L$ with $t \leq x$, then

$x \leq k_1 \vee k_2 \vee \ldots.. \wedge k_n$ for some $k_1, k_2, \ldots.., k_n \in K$

and $t \leq x \leq k_1 \vee k_2 \vee \ldots.. \wedge k_n$ implies

$t \in k_1 \vee k_2 \vee \ldots.. \wedge k_n$ for some $k_1, k_2, \ldots.., k_n \in K$

Hence $k_1 \vee k_2 \vee ...... \vee k_n$ is an ideal, which contain K . □

1.5 Complemented Lattice

Complemented Lattice: In a bounded lattice L, a is a complement of b if $a \wedge b = 0$ and $a \wedge b = I$. A complemented lattice is a bounded lattice in which every element has a complement.

Example 1.5.1 Let $[a,b]$ be an interval in a lattice L . Let $x \in [a,b]$ be any element. If there exists $y \in b$ such that $x \wedge y = a,\ x \vee y = b$.

We say y is a complement of x relative to $[a,b]$, or y is relative complement of x in $[a,b]$. In every element x of an interval $[a,b]$ has at least one complement relative to $[a,b]$, the interval $[a,b]$ is called complemented. Further, If every interval in a lattice is complemented, the lattice is said to be relatively complemented.

Theorem 1.5.1 If L_1 and L_2 are relatively complemented, then Cartesian product is also relatively complemented.

Proof: Since L_1 and L_2 be relatively complemented.

Let $[(x_1, y_1), (x_2, y_2)]$ be any interval of $L_1 \times L_2$ and suppose (a,b) is any element of this interval. Then $(x_1, y_1) \leq (a,b) \leq (x_2, y_2)$; $x_1, x_2, a \in L_1,\ y_1, y_2, b \in L_2$.

Implies that $x_1 \leq a \leq x_2 \quad y_1 \leq b \leq y_2$

Implies that $a \in [x_1, x_2]$ an interval in L_1 and $b \in [y_1, y_2]$ be an interval in L_2

Since L_1, L_2 are relatively complemented, (a,b) have complements relative to $[x_1, x_2]$ and $[y_1 y_2]$ respectively.

Let a' and b' be these complements, then $a \wedge a' = x_1,\ a \vee a' = x_2$

$$b \wedge b' = y_1,\ b \vee b' = y_2$$

Now $(a,b) \wedge (a',b') = (a \wedge a', b \wedge b') = (x_1, x_2)$

$(a,b) \vee (a',b') = (a \vee a', b \vee b') = (x_1, x_2)$

i.e., (a',b') is complement of (a,b) relative to $[(x_1,y_1),(x_2,y_2)]$, thus any interval in $L_1 \times L_2$ is complemented.

Hence $L_1 \times L_2$ is relatively complemented.

Conversely, Let $L_1 \times L_2$ be relatively complemented.

Let $[x_1, x_2]$ and $[y_1, y_2]$ be any intervals in L_1 and L_2.

Let $a \in [x_1, x_2]$, $b \in [y_1, y_2]$ be any elements then $x_1 \leq a \leq x_2$, $y_1 \leq b \leq y_2$

Implies that $(x_1, x_1) \leq (a,b) \leq (x_2, y_2)$

Implies that $(a,b) \in [(x_1,y_1),(x_2,y_2)]$ an interval in $L_1 \times L_2$

Implies that (a,b) has a complement,

Say, (a',b') relative to this interval.

Thus, $(a,b) \wedge (a',b') = (x_1, y_1)$

$$(a,b) \vee (a',b') = (x_1, y_1)$$

Implies that $(a \wedge a', b \wedge b') = (x_1, y_1)$

$$(a \vee a', b \vee b') = (x_1, y_1)$$

Implies that $(a \wedge a' = x_1,\ a \vee a' = x_2$

$$(b \wedge b' = y_1,\ b \vee b' = y_2,$$

Implies that a' is complement of a relative to $[x_1,x_2]$, b' is complement of b relative to $[y_1, y_2]$.

Hence L_1 and L_2 are relatively complemented. □

Theorem 1.5.2 Dual complemented lattice is always complemented.

Proof: Let (L,ρ) be a complemented lattice with $(0,1)$ as least and greatest elements.

Let (L,ρ), be the dual of (L,ρ), then $(0,1)$ are least and greatest element of L.

Let $a \in L = L$ be any element.

Since $a \in L$, L is complemented,

$\exists\ a' \in L\ \exists$ s.t. $a \wedge a' = 0, a \vee a'$ in L

i.e. $0 = \inf\{a, a'\}$ in L

$\Rightarrow 0 \rho a,\ 0 \rho a' \Rightarrow a \rho 0,\ a' \rho 0$ in L

$\Rightarrow$ 0 is the upper bound of $\{a, a'\}$ in L

then $a \rho k,\ a' \rho k \Rightarrow k \rho a,\ k \rho a'$

$\Rightarrow k \rho 0$ as 0 is infimum. $\Rightarrow 0 \rho k$

i.e., 0 is l.u.b $\{a, a'\}$ in L i.e., $a \vee a'$ in L

Similarly, $a \wedge a' = 1$ in L

or that a' is complement of a in L

Hence L is complemented. □

Theorem 1.5.3 A complemented distributive lattice is relatively complemented.

Proof: Let L be a complemented distributive lattice.

Let $[a,b]$ be any interval in L and $x \in [a,b]$ be any element.

Since 1 is complemented x has a complemented ,

say x' then $x \wedge x' = 0$ and $x \vee x' = u$

$a \le x \le b,\ a \le x \le b$, take $y = (a \vee x') \wedge b$

then $x \wedge y = x \wedge [(a \vee x') \wedge b] = [(x \wedge a) \vee (x \wedge x')] \wedge b$

$= [(x \wedge a) \vee 0] \wedge b = (x \wedge a) \wedge b = a \wedge b = a.$

and $x \vee y = x \vee [(a \vee x') \wedge b] = [(x \wedge a) \vee (x \wedge x')] \wedge b$

$= [(x \wedge a) \vee 0] \wedge b = (x \wedge a) \wedge b = a \wedge b = a.$

$= (x \vee (a \vee x')) \wedge (x \vee b] = ((x \wedge a) \vee x') \wedge b = u \wedge b = b$

Hence y is relatively complemented of x in $[a,b]$. □

Chapter Two

An extension of sectionally pseudocomplement Lattice

Introduction: In this chapter we study sectionally antitone and residuated lattices.Firstly, lattices with the greatest element 1 where on each interval [a, 1] an antitone bijection is defined. We characterize these lattices by means of two induced binary operations proving that the resulting algebras form a variety. We show that this variety contains a single minimal subquasi variety Join-lattices, whose principal filters are Boolean lattices, were used by. J.C. Abbott [18] for a characterization of the logic connective implication in the classical propositional logic. These lattices also have the property that on each principal filter of them an antitone involution is defined. Motivated by this observation, the notion of a lattice with sectionally antitone involutions was defined in [4] and [5]. In this paper we introduce a further generalization of this concept, defining the notion of a lattice with sectionally antitone bijections. Our aim is to obtain by means of these lattices 'nice' algebraic structures, i.e. a variety of algebras characterized by 'nice' congruence properties.

Secondly, Residuated lattices were introduced by Ward and Dilworth [26] and studied by several authors. Two monographs contain a compendium on residuated lattices. They are that by Blyth and Janowitz [2] (where it is renamed as a residuated Abelian semi-group with a unit). In this short note we will compare a certain modification of a residuated lattice with already introduced concepts (see [2, 8]).

2.1 Lattices with sectionally antitone bijections

Let $A = (A, \vee, \wedge, *, \circ, 1)$ be a lattice with the greatest element 1. For each $a \in L$ the interval $[a,1]$ (with respect to the induced order) will be called a section. We say that L is a lattice with sectionally antitone bijections if for each $a \in L$ there exists a bijection a of $[a,1]$ into itself such that $x \le y \Leftrightarrow f_a(y) \le f_a(x)$, for all $x, y \in [a,1]$ of course, the inverse f_a^{-1} of f_a is also an antitone bijection on $[a,1]$. If each f_a is an involution, i.e. $f_a^{\,2}(x) = a$ for all $x \in [a,1]$ then L is called a lattice with sectionally antitone involutions (see [5]). Given a lattice L with sectionally antitone bijections, we can introduce two new binary operations on L as follows:

$x \circ y = f_y(x \vee y)$ and $x * y = f_y^{\,-1}(x \vee y)$..........$(p)$

Since $x \vee y \in [y,1]$, $*$ and $\circ$ are everywhere defined operations on the set L.

Conversely, one can check immediately that for any a L and $x \in [a,1]$

$f_a(x) = x \circ a$ and $f^{-1}(x) = x * a$...........(A)

Clearly, if all the mappings $f(a)$ are involutions, then $x \circ y = x * y$ for all $x, y \in L$

(Since $f_a = f_a^{\,-1}$ for each $a \in L$).

The following Lemmas are extension of [5] and theorems are shown in semilattice.

Lemma 2.1.1 Let L be a lattice with sectionally antitone bijections $*$ and $\circ$ be operations defined by (p). Then

(1) $x \circ x = x * x = 1, x \circ 1 = 1, 1 \circ x = 1 * x = x$

(2) $(x \circ y) * y = (x * y) \circ y = (y \circ x) * x = (y * x) \circ x$

(3) $(((x \circ y) * y) \circ z) \circ (x \circ z) = (((x \circ y) * y) * z) \circ (x * z) = 1$

Proof: Suppose $a, b \in L$ and $a \leq b$ Then

$$\left.\begin{cases} a \circ b = f_b(a \vee b) = f_b(b) = 1 \quad and \\ a * b = f_b^{-1}(a \vee b) = f_b^{-1}(b) = 1 \end{cases}\right\} \text{.............(Q)}$$

Hence $x \circ x = x * x = 1$ and $x \circ 1 = x * 1 = 1$. We also obtain $1 \circ x = f_x(1) = x$ and $1 * x = f_x^{\,-1}(1) = x.$

Thus (1) is satisfied.

(2) $(x \circ y) * y = f_y^{\,-1}(f_y(x \vee y) \vee y) = f_y^{\,-1}(f_y(x \vee y)) = x \vee y$ since $f_y(x \vee y) \geq y$ and hence $f_y(x \vee y) \vee y = f_y(x \vee y).$

Analogously, we can check $(x * y) \circ y = x \vee y, (y \circ x) * x = x \vee y,$ *and* $(y * x) \circ x = x \vee y.$

(3) As $(x \circ y) * y = x \vee y.$ We get $((x \circ y) * y) \circ z = f_z(x \vee y \vee z).$ Further, $(x \circ z) = f_z(x \vee z).$

However, $x \vee y \leq x \vee y \vee z$. and f_z is antitone, thus

$((x \circ y) * y) \circ z = f_z(x \vee y \vee z) \leq f_z(x \vee z) = x \circ z$.

Analogously, we prove $((x \circ y) * y) * z = f_z^{-1}(x \vee y \vee) \leq f_z^{-1}(x \vee z) = x * z$.

By (Q) we obtain (3) immediately. □

Theorem 2.1.1 Let $A = (A, \vee, \wedge, *, \circ, 1)$ be an algebra of type (2, 2, 0) satisfying the identities (1) and (2). Define a binary relation $\leq$ on A as follows:

$a \leq b$ if and only if $a \circ b = 1$ .... (R)

Then the following assertions are equivalent:

(i) The algebra A satisfies identity (3).

(ii) For any $x, y, z \in A$ the implications

(4) $(x \circ z) = f_z(x \vee z)$. and $x \leq y \Rightarrow y * z \leq x * z$. are satisfied.

(iii) $\leq$ is a partial order on A and $(A, \leq)$ is a lattice with the greatest element 1, where $a \vee b = (a \circ b) * b$ and for any $a \in A$ the maps $f \leq f_a(x) = x \circ a$, $f_a^{-1} = x * a$ are antitone bijections on $[a,1]$.

Proof: $(i) \Rightarrow (ii)$. Suppose $x \leq y$. Then using (i), (R) and (3) we obtain:

$(y \circ z) \circ (x \circ z) = ((1 * y) \circ z)(x \circ z) = (((x \circ y) * y) \circ z) \circ (x \circ z) = 1$,

and hence $y \circ z \leq x \circ z$.

Analogously, we obtain:

$(y * z) \circ (x * z) = ((1 * y) * z)(x * z) = (((x \circ y) * y) * z) \circ (x * z) = 1$, whence $y * z \leq x * z$.

$(ii) \Rightarrow (iii)$. Assume that (1), (2) and (4) are satisfied. First we prove that the relation $\leq$ defined by (R) is a partial order.

Due to (1), $\leq$ is reflexive. Suppose $x \leq y$. and $y \leq x$. Then $x \circ y = 1$. and $y \circ x = 1$. hence by (1) and (2),

$x = 1 * x = (y \circ x) * x = (x \circ y) * y = 1 * y = y$,

thus $\leq$ is anti-symmetrical.

Suppose $x \leq y$ and $y \leq z$. Then we get $y \circ z = 1$ by (R), and $x \circ z \geq y \circ z$, by (4). Hence we obtain $x \circ z = 1$, *i.e.*, $x \leq z$. Thus $\leq$ is transitive, i.e., it is a partial order.

As for any $x \in A$ we have $x \circ 1 = 1$ we get $x \leq 1$ for all $x \in A$. Therefore,

$x = 1 \circ z \leq x \circ z,$ for all $x, z \in A$ and hence

$z \circ (x \circ z) = 1,$ for all $x, z \in A$ ....(S)

Define $a \vee b = (a \circ b) * b$ for all $a, b \in A.$ Then (2) and (S) implies

$a \circ ((a \circ b) * b) = (a \circ ((b * a) \circ a) = 1$ and

$b \circ ((a \circ b) * b) = b \circ ((a * b) \circ b) = 1,$

thus $a \leq a \vee b$ and $b \leq a \vee b.$

Suppose now $a \leq c$ and $b \leq c$ for some $c \in A.$ Then $b \circ c = 1$ and

$c = 1 * c = (b \circ c) * c = (c \circ b) * b$ by (2). This gets

$((a \circ b) * b) \circ c = ((a \circ b) * b) \circ ((c \circ b) * b).$

Due to (4) we infer $a \leq c \Rightarrow c \circ b \leq a \circ b \Rightarrow (a \circ b) * b \leq (c \circ b) * b$ and hence

$((a \circ b) * b) \circ ((c \circ b) * b) = 1, i.e., ((a \circ b) * b) \circ c = 1$ proving $a \vee b \leq c.$ Thus $a \vee b$

is $\sup\{a, b\}$ w.r.t $\leq$.

Now consider $a \in A, f_a, f_a^{-1}$ defined by (A) and $x \in [a, 1].$ Then

$f_a^{-1}(f_a(x)) = (x \circ a) * a = x \vee a = x$ and

$f_a(f_a^{-1}(x)) = (x * a) \circ a = x \vee a = x,$

thus f_a and f_a^{-1} are bijections on $[a, 1].$ (and inverses each of other).

For $x, y \in [a, 1]$ with $x \leq y$ we have by (4)

$f_a(y) = y \circ a \leq x \circ a = f_a(x)$ and

$f_a^{-1}(y) = y * a \leq x * a = f_a^{-1}(x),$

therefore f_a and f_a^{-1} are antitone bijections.

$(iii) \Rightarrow (i).$ By the assumptions of (iii) $(A, \leq)$ is a join semilattice with sectionally antitone bijections. Take any $x, y \in A.$ Since

$f_y(x \vee y) = (x \vee y) \circ y = ((x \circ y) * y) \circ y = f_y(f_y^{-1}(x \circ y)) = x \circ y$ and

$f_y^{-1}(x \vee y) = (x \vee y) * y = ((x * y) \circ y) * y = f_y^{-1}(f_y(x * y)) = x * y,$

$*$ *and* $\circ$ can be also defined using relation (p). By applying Lemma 2.1.1, we obtain that the algebra $A = (A, \vee, \wedge, *, \circ, 1)$ satisfies the identity (3) . □

2.2 Sectionally residuated Lattice

At first, we recall the basic concept:

By a residual lattice is meant an algebra $L = (L,\vee,\wedge,*,\circ,0,1)$ such that

(i) $L = (L,\vee,\wedge,0,1)$ is a bounded lattice,

(ii) $L = (L,*,1)$ is a commutative monoid,

it satisfies the so-called adjointness property: $(x \vee y) * z = y$ if and only if $y \leq z \leq x \circ y$

Let us note (see, e. g., [1]) that $x \vee y$ is the greatest element of the set $(x \vee y) * z = y$

Moreover, if we consider $x * y = x \wedge y$, then $x \circ y$ is the relative pseudocomplement of x with respect to y, i. e., for $* = \wedge$ residuated lattices are just relatively pseudo-complemented lattices. It is well known that every relatively pseudocomplemented lattice is distributive. An extension of relative pseudocomplementation for the non-distributive case was already involved [8,9]. The identities characterizing sectionally pseudocomplemented lattices are presented in [18], i. e., the class of these lattices is a variety in the signature $\{\vee,\wedge,\circ,1\}$. We are going to apply a similar approach for the adjointness property.

Definition 1. A lattice $L = (L,\vee,\wedge,1)$ with the greatest element 1 is sectionally pseudocomplemented if each interval [y, 1] is a pseudocomplemented lattice. From now on, denote by $x \vee y$ the pseudocomplement of $x \vee y$ in the interval [y, 1]. Naturally, $x \vee y \in [y,1]$ thus $L = (L;\vee,\wedge,1)$ is sectionally pseudocomplemented if and only if '$\circ$' is an (every where defined) operation on L.

Definition 2. An algebra $L = (L;\vee,\wedge,*,\circ,1)$ is called a sectionally residuated lattice if

(i) $L = (L,\vee,\wedge,0,1)$ is a lattice with the greatest element 1;

(ii) $L = (L,*,1)$ is a commutative monoid;

(iii) It satisfies the sectional adjointness property: $(x \vee y) * z = y$ if and only if $y \leq z \leq x \circ y$

Lemma 2.2.1 Let $L = (L;\vee,\wedge,*,\circ,1)$ be a sectionally residuated lattice. Then $x * y$ is the greatest element of the set $\{z; (x \vee y) * z = y\}$

This immediately yields the following facts:

$(x \vee y) * (x \circ y) = y$,....................(1)

$(x \vee y) * y = y,$ ....(2)

$y \leq x \circ y,$ ....(3)

Lemma 2.2.2 Let $L = (L;\vee,\wedge,*,\circ,1)$ be a sectionally residuated lattice. Then $x \leq y,$ if and only if $x \circ y = 1$

Proof: Suppose $x \leq y,$ then $x \vee y = y,$ and, by Lemma 1, $x \circ y$ is the greatest element of the set $\{z; y * z = y\}$. By definition 2, $y * 1 = 1$ thus $x \circ y = 1$. Conversely, suppose $x \circ y = 1$. Then, by (1), we have $y = (x \vee y) * (x \circ y) = (x \vee y) * 1 = x \vee y$ whence $x \leq y$. □

Lemma 2.2.3 In a sectionally residuated lattice, the following identities are satisfied: $x \circ x = 1, x \circ 1 = 1, 0 \circ x = 1,$ *and* $1 \circ x = x$

Proof: The first three identities follow directly by Lemma 2.2.2. Further, by Lemma 2.2.1, $1 \circ x$ is the greatest element of the set $\{z; 1 * z = x\} = \{x\}$ *thus* $1 \circ x = x)$. □

Lemma 2.2.4 In a sectionally residuated lattice, $a * b = a$ if and only if $a = b$

Proof: Putting $x = y = a,$ and $z = b$ in the sectional adjointness property, the assumption $a * b = a$ yields $(a \vee a) * b,$ *iff* $a \leq b \leq a \circ a = 1$ thus $a \leq b$

Conversely, $a \leq b$ implies by Lemma 2.2.3 $a \leq b \leq 1 = a \circ a$ and, by sectional adjointness, $a * b = (a \vee a) * b = a$. □

Applying Lemma 2.2.2 and Lemma 2.2.4, we get

Corollary 1. In a sectionally residuated lattice,

(a) $x * y = x$ x y = x if and only if $x \circ y = 1$;

(b) $x * x = x$

Lemma 2.2. 5 In a sectionally residuated lattice, $x \wedge y \leq x * y$.

Proof: By (3) we have $x \wedge y \leq x \circ (x \wedge y)$. Applying sectional adjointness, we infer $x * (x \wedge y) = (x \vee (x \wedge y)) * (x \wedge y)$x and, analogously, $y * (x \wedge y) = x \wedge y.$ Hence, by Corollary 1 (b),

$x * y * (x \wedge y) = x * (x \wedge y) * y * (x \wedge y) = (x \wedge y) * (x \wedge y) = x \wedge x$

and, by Lemma 2.2.4, $x \wedge y \leq x * y.$ □

Theorem 2.2.1 Let $L = (L;\vee,\wedge,*,\circ,1)$ be a sectionally residuated lattice. Then it is a sectionally pseudocomplemented lattice.

Proof: Replacing y by $x \wedge y$ in the sectional adjointness property, we obtain

$x * z = x \wedge y$ *iff* $x \wedge y \leq z \leq x \circ (x \wedge y)$.

However, $x \circ (x \wedge y)$ is the greatest element of the set

$\{t; (x \vee (x \wedge y)) * t = x \wedge y\} = \{t; x * t = x \wedge y\}$.

By Lemma 2.2.5, $x \wedge t \leq x * t = x \wedge y$, thus the greatest t of this property satisfies $t \geq y$.

Thus $y \leq x \circ (x \wedge y)$, *i.e.*, $x \wedge y \leq y \leq x \circ (x \wedge y)$

and, by the sectional adjointness, $x * y = (x \wedge (x \vee y)) * y = x \wedge y$.

Hence, $x \circ y$ is the pseudocomplement of $x \vee y$ in the interval $[y, 1]$. □

Chapter Three

Relative annihilators in Lattices

Introduction: Through this chapter we will be concerned with the relative annihilators in lattices. For $a,b \in L$, we define $\langle a,b \rangle = \{x / a \wedge x \leq b\}$. According to [23], $\langle a,b \rangle$ is known as an annihilators of a relative to b or simply relative annihilator. It is very easy to see that in presence of distributivity $\langle a,b \rangle$ is an ideal of L. Relative annihilators in lattices have been studied by many authors including Mandelker [21] and T.P. Speed [23]. Also B.A. Davey [1] has used the annihilators in studying relative normal lattices. We also include characterizations of modular and distributive lattices in terms of relative annihilators. Then we have generalized some of the results of Mandelker [21] on relative annihilators. We have shown that in a distributive lattice L. $\langle a,b \rangle \vee \langle b,a \rangle = L$ for all $a,b \in L$ if and only if the filters containing any given prime filter form a chain. For the background material in lattice theory see Gratzer [13], Mandelker [21], T.P. Speed [23] and Gratzer and Schmidt [15] have studied relatively stone lattices. In section two we have introduced the notion of relatively stone lattices and generalises several results of [13], [21], [23].

3.1 Some characterizations of relative annihilators in Lattice

Modular lattice: A lattice L is called a modular lattice iff for all $a,b,c \in L$ with $a \geq b$

$a \wedge (b \vee c) = b \vee (a \wedge c)$

Example: The following diagrams are modular,

i. If $a = b$ then the above definition becomes

$a \wedge (b \vee c) = a \wedge (a \vee c) = a$

$a \vee (b \wedge c) = a \vee (a \wedge c) = a$

ii. If $c \geq b$

Then $a \geq b, c \geq b \Rightarrow a \wedge \vee c \geq b, a \wedge c \geq b$

Thus $a \wedge (b \vee c) = a \wedge c$

$b \wedge (a \vee c) = a \wedge c$

iii. For $a,b,c \in L$, with $a \geq b$

$a \vee (b \wedge c) = b \wedge (a \vee c)$

Hence dual of a modular lattice is modular.(see picture 3.1)

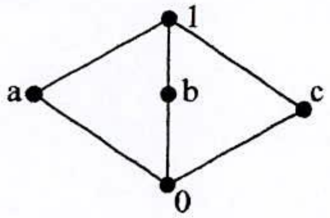

Fig: 3.1

Distributive Lattice:

A lattice L is called distributive lattice if

$a \wedge (b \vee c) = (a \wedge b) \vee a \wedge c) \qquad \forall a,b,c \in L$

i. If $a \leq b,\ a \geq c,\ b \leq c$,

then $a \leq b \leq c \Rightarrow a = b = c$

Thus, $a \wedge (b \vee c) = a = (a \wedge b) \vee (a \wedge c)$

ii. If $a \leq b,\ a \leq c,\ c \leq b$

Then, $a \leq b,\ a \geq b,\ c \leq b$

Thus $a \wedge (b \vee c) = a \wedge b = a$

$(a \wedge b) \vee (a \wedge c) = a \vee c = a$

Hence a chain is always a distributive lattice.

iii. A distributive lattice is always modular.

Boolean Algebra: A non empty set $< L, \wedge, \vee, ', 0, 1 >$ with the binary operations $\wedge, \vee$ unary operation and nullary operations 0,1 is called a Boolean algebra if it satiesfy the following conditions :

i) $a \wedge a = a, \quad a \vee a = a \quad \forall\ a \in L$

ii) $a \wedge b = b \wedge a, \quad a \vee b = b \vee a \quad \forall\ a,b \in L$

iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c, \quad a \vee (b \vee c) = (a \vee b) \vee c \quad \forall\ a,b,c \in L$

iv) $a \wedge (a \vee b) = a, \quad a \vee (a \wedge b) = a \quad \forall\ a,b \in L$

v) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \forall \; a,b,c \in L$

vi) There exists $0 \in L, 1 \in L$ such that $a \vee 0 = a \quad a \wedge 1 = a \;\; \forall \; a \in L$

vii) Each $a \in L, \; a' \in L$ such that $a \wedge a' = 0, \;\; a \vee a = 1$

viii) $0' = 1, \;\; 1' = 0$

ix) $(a \wedge b)' = a' \vee b', \qquad (a \vee b)' = a' \wedge b'$

Lemma 3.1.1 A lattice L is distributive iff

$\forall x, y, z \in L, \;\; t \wedge ((x \wedge y) \vee (x \wedge z)) = (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$

Proof: Suppose L is distributive, then obviously,

$t \wedge ((x \wedge y) \vee (x \wedge z)) = (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$

Conversely, suppose L has the given property. Let $a, b, c \in L$ with $b \vee c$ exists.

Set $t = b \vee c$, then

$a \wedge (b \vee c) = a \wedge ((t \wedge b) \vee (t \wedge c))$

$= (a \wedge t \wedge b) \vee (a \wedge t \wedge c) = (a \wedge b) \vee (a \wedge c)$

Therefore L is distributive. □

Notice that a lattice L is modular if for all $x, y, z \in L$ with $z \leq x$ and whenever $y \vee z$ exists then $x \wedge (y \vee z) = (x \wedge y) \vee z$

We can also easily characterize modular lattices by the following result.

Lemma 3.1.2 A lattice L is modular iff for all $t, x, y, z \in L$ with $z \leq x$,

$x \wedge ((t \wedge y) \vee (t \wedge z)) = (x \wedge t \wedge y) \vee (t \wedge z)$.

Proof: Suppose L is modular. Then obviously,

$x \wedge ((t \wedge y) \vee (t \wedge z)) = (x \wedge t \wedge y) \vee (t \wedge z)$.

Conversely, suppose L has the given property,

Let, $a, b, c \in L$ with $c \leq a$ and $b \vee c$ exists.

Set, $t = b \vee c$, then

$a \wedge (b \vee c) = a \wedge ((t \wedge b) \vee (t \wedge c)) = (a \wedge t \wedge b) \vee (t \wedge c) = (a \wedge b) \vee c$.

Therefore L is modular. □

Theorem 3.1.3 A lattice is modular iff whenever $b \leq a$, if $t \wedge x \in b$ and $t \wedge y \in (a, b)$ for any, $t \in L$ then $(t \wedge y) \vee (t \wedge x) \in (a, b)$.

Proof: Suppose L is modular, since $(t \wedge y) \vee (t \wedge x) \in (a,b)$. So $a \wedge t \wedge y \leq b$

also, $t \wedge y \leq b \leq a$

Thus by modularity of L,

$a \wedge ((t \wedge y) \vee (t \wedge y)) = (a \wedge t \wedge y) \vee (t \wedge x) \leq b,$

and so, $(t \wedge x) \vee (t \wedge y) \in (a,b)$

Conversely, let the given condition holds,

Suppose $t, x, y, z \in L$ with $z \vee x$ then $(t \wedge z) \vee (t \wedge x \wedge y) \leq x$.

Also, $t \wedge x \wedge y \leq (t \wedge z) \vee (t \wedge x \wedge y)$ implies $t \wedge y \in (x, (t \wedge z) \vee (t \wedge x \wedge y))$.

Then by hypothesis, $(t \wedge z \vee (t \wedge y) \in (x, (t \wedge z) \vee (t \wedge x \wedge y))$.

This implies $x \wedge ((t \wedge y) \vee (t \wedge z)) \leq (t \wedge x \wedge y) \vee (t \wedge z)$.

Since the reverse inequality is trivial, so by Lemma 3.1.3 L is modular. □

Theorem 3.1.4 Suppose L is a lattice. Then the following conditions are equivalent:

i. L is distributive.

ii. (a,b) is an ideal for all $a,b \in L$

iii. (a,b) is an ideal whenever $b \leq a$

Proof: Since (i) implies (ii) and (iii) are trivial,

We shall prove only (iii) implies (i).

Suppose (iii) holds. Let. $t, x, y, z \in L$

Then $(t \wedge x \wedge y) \vee (t \wedge x \wedge z) \leq x$ implies

$(x, (t \wedge x \wedge y) \vee (t \wedge x \wedge z))$ is an ideal.

Again $(t \wedge x \wedge y) \leq (t \wedge z \wedge y) \vee (t \wedge x \wedge z)$

Implies $(t \wedge y) \in (x, (t \wedge x \wedge y) \vee (t \wedge x \wedge z))$

Similarly, $(t \wedge z) \in (x, (t \wedge x \wedge y) \vee (t \wedge x \wedge z))$

Hence $(t \wedge y) \vee (t \wedge z) \in (x, (t \wedge x \wedge y) \vee (t \wedge x \wedge z))$

Thus $x \wedge ((t \wedge y) \vee (t \wedge z)) \in (x, (t \wedge x \wedge y) \vee (t \wedge x \wedge z))$

Since the reverse inequality is trivial,

So $x \wedge ((t \wedge y) \vee (t \wedge z)) = (x, (t \wedge x \wedge y) \vee (t \wedge x \wedge z))$

Therefore by Lemma 3.1.1 L is distributive. □

Lemma 3.1.5 In any distributive lattice L, each of the following conditions on a given filter F implies the next.

(i) For all $a,b \in L$, there exists an element $x \in F$ such that $a \wedge x$ and $b \wedge x$ are comparable.

(ii) The filters containing F form a chain.

(iii) F is prime.

(iv) F contains a prime filter.

Proof: (i) implies (ii). Suppose (i) holds. If (ii) fails then there exists non-comparable filters G and H containing F. Choose elements $a \in F - G$ and $b \in F - G$. Then by (i) there exists $x \in F$ such that $a \wedge x$ and $b \wedge x$ are comparable. Suppose. $a \wedge x \leq b \wedge x$. Since $x \in F - G$, so $a \wedge x \in G$

Then $a \wedge x \leq b$ implies $b \in G$, which gives a contradiction.

Therefore (ii) holds.

(ii) implies (iii). Suppose (ii) holds. Let $a,b \in L$ with $a \vee b$ exists and $a \vee b \in F$. Let $G = F \vee [a)$ and. $H = F \vee [b)$.

By (ii), either $G \subseteq H$ or $H \subseteq G$. Suppose $G \subseteq H$. Then $a \in H$, and so $a = x \wedge b$ for some $x \in F$. Since $x, a \vee b \in F$, so $x \wedge (a\vee) \in F$. Thus by distributivity of L,

$(x \wedge a) \vee (x \wedge b) = (x \wedge a) \vee a = a \in F$.

Therefore F is prime. (iii) implies (iv) is trial. □

Theorem 3.1.6 For as distributive lattice L the identity $(a,b) \vee (b,a) = L$ for all $a,b \in L$ holds if and only if

(i) For all $a,b \in L$, there exists an element $x \in F$ such that $a \wedge x$ and $b \wedge x$ are comparable.

(ii) The filters containing F form a chain.

(iii) F is prime.

(iv) F contains a prime filter.

Proof: Suppose the identity holds. We only to show that (iv) implies (i). Let $a,b \in L$. Suppose P is a prime filter contained in F. Choose $z \in P$.

Since $(a,b) \vee (b,a) = L$, so $z = x \vee y$ for some $x \in (a,b)$ and. $y \in (b,a)$.

Since P is prime, either $x \in P$ or $y \in P$.

Suppose $x \in P$.

Then $x \in F$, and $x \in (a,b)$ implies $a \wedge x \leq b$ and so $a \wedge x \leq b \wedge x$.

Therefore (i) holds.

Conversely, suppose all the conditions are equivalent. Let there exists $a,b \in L$.

Such that $I = (a,b) \vee (b,a)$ is proper ideal of L.

Then there exists a prime filter P disjoint from I.

Then by (iii) implies (i), there exists $x \in P$ such that $a \wedge x$ and $b \wedge x$ are comparable. Suppose, $a \wedge x \leq b$ implies $x \in (a,b)$ which is a contradiction as. $P \cap I = \Phi$. Therefore, $(a,b) \vee (b,a) = L$. □

3.2 Relatively stone Lattices

We start this section with the following characterization of relatively stone lattices, which is generalization. A pseudocomplemented lattice L is called a stone lattice if for each $x \in L,\ x^{*} \vee x^{**} = I$. We call a distributive lattice L a relatively stone lattice if each closed interval $[x,y]$ with $x < y (x, y \in L)$ is a stone lattice. Two prime ideals P and Q of a lattice L are said to be comaximal if $P \vee Q = L$.

The following characterization of relatively stone lattices, which is a generalization of [15, Theorem 5]

Theorem 3.2.1 Suppose L is a distributive lattice in which every closed interval is pseudocomplemented. Then the following conditions are equivalent:

(i) L is relatively stone.

(ii) For all $x, y \in L,\ (x,y) \vee (y,x) = L$

Proof: (i) implies (ii). Suppose L is stone. Let $x, y \in L$. For any $a \in L$ consider $I = [x \wedge y \wedge a, a]$ in L. Let * denotes the pseudocomplement in I.

Now, $x \wedge y \wedge a = (x \wedge a) \wedge (y \wedge a)$

since I stone, so $a = (x \wedge y \wedge a)^{*} = ((x \wedge a) \wedge (y \wedge a))^{*} = (x \wedge a)^{*} \vee (y \wedge a)^{*}$

thus $a = r \vee s$ where $r = (x \wedge a)^{*}, s = (y \wedge a)^{*}$

then $x \wedge a \wedge r = y \wedge a \wedge s = x \wedge y \wedge a$

since $r,s \le a$ we have $x \wedge y \wedge a = x \wedge r = y \wedge s$.

This implies $x \wedge r \le y$ and $y \wedge s \le x$ and so $a = r \vee s \in (x,y) \vee (y,x)$

Hence (ii) holds.

(ii) Implies (i). Let $[a,b]$ be any closed interval in L and let * denotes pseudocomplemente in $[a,b]$.Let $x \in [a,b]$

By hypothesis $(x^*,x^{**}) \vee (x^{**},x^*) = L$. Hence $b = r \vee s$ for some $r \in (x^*,x^{**})$ and $s \in (x^{**},x^*)$. Since $a,r,s \le b,$ so by the upper bound property $a \vee r, a \vee s$ exists. Now $r \wedge x^* \le x^{**}$ and $r \wedge x^{**} \le x^*$. Thus, $x^* \wedge (a \vee r) \le x^{**}$. Moreover $x^* \wedge (a \vee r) \le x^*$ is obvious. Hence $x \wedge (a,r) \le x^{**} \wedge x^* = a$ Since $a \vee r \in [a,b]$ so $a \vee r \le x^{**}$. Similarly $a \vee s \le x^*$. Hence $b = (a \vee r) \vee (a \vee s) \le x^* \vee x^{**} \le b$. This implies $x^* \vee x^{**} = b$ and so $[a,b]$ is a stone Lattice.

In other words, L is relatively stone. □

Definition: A filter F of a lattice L is called meet irreducible if $F = G \wedge H$ implies either $F = G$ or $F = H$ where G and H are filters of L .

Theorem 3.2.2 Let L be a distributive lattice. A filter F of L is prime if and only if it is meet irreducible.

Proof: Suppose F is prime and $F = G \wedge H$ for some filters G and H of L. If $G \ne F$. Then there exists $g \ne F$. Suppose $h \in H$. Then for any $f \in F, g \wedge f \in F, g \wedge f \in G$ and $h \wedge f \in H$. Hence $(g \wedge f) \vee (h \wedge f) \in G \wedge H = F$.

But $g \wedge f \notin F$ as $g \notin G$. Since F is prime so $h \wedge f \in F$ which implies $h \in F$ This implies $H \subseteq F$. As $F \subseteq H$ is obvious, so $F = H$. Therefore F is meet irreducible.

Conversely, suppose F is meet irreducible. Let $a,b \in L$ such that $a \vee b$ exists and $a \vee b \in F$. Set $G = F \vee [a)$ and $H = F \vee [b)$ clearly, $F \subseteq G \wedge H$. Now, let $x \in G \wedge H$ then for some $f_1, f_1 \in F$. Hence, $x \ge f_1 \wedge f_2 \wedge a \ge f_1 \wedge f_2 \wedge b$ Put $f = f_1 \wedge f_1$ then we get $x \ge f \wedge a, x \ge f \wedge b$ which implies that $x \ge (f \wedge a) \vee (f \wedge b)$.

Now $x \ge (f \wedge a) \vee (f \wedge b) = f \wedge (a \vee b),$ as L is distributive and $a \vee b$ exists. Therefore, $(f \wedge a) \vee (f \wedge b) \in F$ as $a \vee b \in F$. Hence $x \in F$.

Therefore, $G \wedge H \subseteq F$ and so $G \wedge H = F$. Since F is meet irreducible, so either $G = F$ or $H = F$, that is either $a \in F$ or $b \in F$. Hence F is prime. □

Following theorem generalizes a result of [17].

Theorem 3.2.4 In a distributive lattice L, the following conditions are equivalent.

(i) Any proper filter which contains a prime filter is prime.

(ii) For any pair of non-comparable prime ideals P and Q, $P \vee Q = L$.

Proof: (i) implies (ii). Let L be a distributive lattice and let P and Q be two non-comparable prime ideals in L such that $P \vee Q \neq L$. Then there exists a prime filter F disjoint from the ideal $P \vee Q, L - P$ and $L - Q$ are non-comparable prime filters. Such that $(L - P) \wedge (P - Q) = G \supset F$, where G is a filter and by assumption (i), G prime, which is impossible. Because, the theorem 3.2.3, G is meet-irreducible.

Hence for any pair of non-comparable prime ideals G and $Q, P \vee Q = L$

(ii) implies (i). Let L be a distributive lattice and let there exists a prime filter P and a non-prime proper filter G such that $F \subset G$. Thus, G is not meet irreducible. Then there exists filters $A \neq G$ and $B \neq G$ such that $G = A \cap B$.

So we can find two elements a and b such that $a \in A, a \notin B$ and $b \in B, b \notin A$.

Then there exists a prime filter A_1 containing A and disjoint from $(b]$ and prime filter B_1 containing B and disjoint from $(a]$. A_1 and B_1 contain G and are non-comparable.

Thus by assumption (ii), $(L - A_1) \vee (L - B_1) = L$.

Which would imply that any element of F is the join of two elements not belonging to F, hence a contradiction. □

Following result is due to [13, Theorem 2.7]

Theorem 3.2.5 For any distributive lattice L the following conditions are equivalent:

(i) For all $a, b \in, < a, b > \vee < b, a >= L$.

(ii) The filters containing any given filter form a chain.

Proof: (i) $\Leftrightarrow$ (ii) holds by theorem 3.1.6 and 3.2.5.

Theorem 3.2.6 Suppose L be a distributive lattice in which every closed interval is pseudocomplemented. Then the following conditions are equivalent:

(i) L is relatively stone.

(ii) The set of all prime ideals contained in a prime ideal is a chain.

(iii) Any two incomparable prime ideals are comaximal.

(iv) The set of all prime filters of L containing a prime filter is a chain.

(v) Any proper filter which contains a prime filter is prime.

(vi) L_F is a chain for each prime filter F of L.

Proof: (i) implies (ii). Suppose (i) hold. Then by Theorem 3.2.1, $<x,y> \vee <y,x>= L$ for all $x,y \in L$. If (ii) does not hold, then there exists prime ideals P,Q,R with $P \supseteq Q,R$; and Q and R are incomparable. Let $x \in Q-R$ and $y \in R-Q$.

Then $<x,y> \subseteq R$ and, y,x >⊆ Q. Thus $L =<x,y> \vee <y,x> \subseteq Q \vee R \subseteq P \neq L$ which is a contradiction. Hence (ii) holds.

(ii)⇔ (iii) and (ii)⇔(iv) are trivial.

(iii)⇔ (v) holds by theorem 3.2.4.

(iii) implies (vi). Suppose (iv) holds. Then the prime filters of L_F form a chain for any prime filter of L. But, in a distributive lattice if the set of prime filters form a chain, then the lattice itself is a chain.

Therefore L_F is a chain for each prime filter F of L.

implies. Let F be any prime filter of L. By (vi) L_F is chain, and so for x,y in L, we have either $\psi_F(x) \leq \psi_F(y)$ or $\psi_F(y) \leq \psi_F(x)$. In either case,

$<\psi_F(x),\psi_F(y)> \vee <\psi_F(y),\psi_F(x)>= L_F$

i.e. $\psi_F(<x,y> \vee <y,x>) = \psi_F(L)$, and so by the principle of localization,

$(<x,y> \vee <y,x>) = L.$ Hence by Theorem 3.2.1. L is relatively stone. □

We have given several characterizations of these F which are filters in a relatively stone lattice L. Then we have proved that L_F is relatively stone.

Theorem 3.2.7 If F is a filter in a relatively stone lattice L. Then L_F is relative stone.

Proof: Suppose L is relatively stone. Let $\psi_F(x), \psi_F(y) \in L_F$..

Then $< \psi_F(x), \psi_F(y) > \vee < \psi_F(y), \psi_F(x) >= \psi_F < x, y > \vee \psi_F < y, x >$

$= \psi_F[< x, y > \vee < y, x >]$.

$\psi_F(L) = L_F$ as L is relatively stone.

Hence L_F is relatively stone. □

We conclude this section with the following examples.

Notice that both the lattices are relatively pseudocomplemented. In lattice of figure 3.1, notice that $(a], (b]$ and $(c]$ are only prime ideals. Here both $(a]$ and $(b]$ are incomparable with $(c]$. Moreover, $(a] \vee (c] = (b] \vee (c] = L_1$, therefore L_1 is relatively stone. But for lattices of figure 3.2, observe that $(a]$ and $(b]$ are incomparable prime ideals. But $(a] \vee (b] \neq L_2$. Therefore, L_2 is not relatively stone.

Also notice that though L_1 is relatively stone, it is not generalized stone as $0 \notin L_1$. □

3.3 Relative annihilators in normal Lattices

Mandelker [21] has characterized distributive lattices L, in which $< a,b > \vee < b,a >= L$ identically for all a, b in L, as those lattices in which the filters containing any given prime filter form a chain. Surely, in such lattices every prime filter must be contained in unique minimal filter. Hence bounded distributive lattices L in which $< a,b > \vee < b,a >= L$ identically for all a, b in L are the examples of normal lattices.

The following conditions are equivalent:

1. L is normal.
2. Each prime filter in L is contained in unique maximal filter.
3. Any two minimal prime ideals in L are comaximal.

Cornish [25, theorem 3.7] a characterization of a normal lattice in terms of relative annihalitor ideals, is obtained in the following:

Theorem 3.3.1 Let L is a distributive lattice then the following conditions are equivalent in L.

(i) Every prime filter in L is contained in a unique maximal filter.

(ii) $<a,b> \vee <b,a> = L$ identically for $a,b \in L$ with $a \wedge b = 0$.

(iii) For any prime filter P in L and for any $a,b \in L$ with $a \wedge b = 0$ there exists x in L such that $a \wedge x$ and $b \wedge x$ are comparable.

Proof: $(i) \Rightarrow (ii)$ Let $<a,b> \vee <b,a> = I(\neq L)$ with $a \wedge b = 0$. By Stone's theorem, $a \wedge b = 0$ are exists a prime filter P disjoint with I. Consider the filter $P \vee [a)$. If $b \in P[a)$ then $b = t \wedge a$ for some $t \in P$ will imply $t \in <a,b>$ and hence $t \in I \cap P = \phi$, a contradiction. Therefore $b \notin P \vee [a)$ proves that $P \vee [a)$ is a proper filter. Let M_1 be a maximal filter containing $PV[a)$. Similarly, there exists a maximal filter, say M_2, containing the proper filter $PV[b)$. As $a \wedge b = 0, b \notin M_1$ and $a \notin M_2$. Hence $M_1 \neq M_2$. Thus the prime filter P is contained in two distinct maximal filters M_1 and M_2 contradicting the assumption. Therefore $<a,b> \vee <b,a> = L$ for $a,b \in L$ with $a \wedge b = 0$.

$(ii) \Rightarrow (iii)$ Let P be any prime filter and a,b are in L with $a \wedge b = 0$. By (ii) $<a,b> \vee <b,a> = L$. For any $t \in P$ we have, $t = x \wedge y$ for some $x \in <a,b>$ and $y \in <b,a>$. As $x \vee y \in P$ and P is prime, $x \in P$ or $y \in P$. Without loosing generally, assume that $x \in P$. Then by choice of $x, a \wedge x \leq b$ will imply $a \wedge x \leq b \wedge x$ and the implication follows.

$(iii) \Rightarrow (i)$ Let P be a prime filter such that $P \subseteq M_1$ and $P \subseteq M_2$ where M_1 and M_2, distinct maximal filters in L. Let $a \in M_1$ such that $a_1 \notin M_2$. But then there exists $a_2 \in M_2$ such that $a_1 \wedge a_2 = 0$. By assumption (iii) there exists $x \in P$ such that $a_1 \wedge x$ and $a_2 \wedge x$ are comparable.

Assume without loss of generality $a_1 \wedge x \leq a_2 \wedge x$. As $a_1 \wedge x \in M_1$ implies $a_2 \wedge x \in M_1$ we get $a_2 \in M_1$. But then $0 = a_1 \wedge a_2 \in M_1$, contradiction the maximality of M_1. Hence the prime filter P must be contained in a unique maximal filter. □

Note that each prime filter L is contained in unique maximal filter if and only if each minimal prime filter in L is contained in a unique maximal filter. Using this property we get, the following theorem.

Theorem 3.3.2 Let L is a distributive lattice then the following conditions are equivalent in L.

(i) Any two distinct minimal prime ideals are equivalent in L.

(ii) For any two distinct maximal filters M_1 and M_2 in L there exists $a_1 \notin M_1$ and $a_2 \notin M_2$ with $a_1 \wedge a_2 = 1$.

(iii) For any maximal filter M, M is the unique maximal filter containing the filter $W(M) = \{x \in / x \vee y = 1\}$ for some $y \notin M$

Proof: $(i) \Rightarrow (ii)$ Let M_1 and M_2 be any two distinct maximal filter containing the filters in L. By (i) $(L/M_1) \vee (L/M_2) = L.$ As $l \in L$ there exists $a_1 \notin M_1$ and $a_2 \notin M_2$ with $a_1 \vee a_2 = 1$ and the implication follows.

$(ii) \Rightarrow (iii)$ Let $W(M) \subseteq M_1$ for some maximal filter $M_1 = M_2$. Hence by (ii) there exists $a \notin M$ and $b \notin M_1$ such that $a_1 \vee a_2 = 1$. But then $b \in W(M) \subseteq M_1$; a contradiction. Hence $(ii) \Rightarrow (iii)$.

$(iii) \Rightarrow (i)$ Let F be a minimal prime filter contained in two distinct maximal filters M_1 and M_2 in L. As F is minimal. $F = W(F)$ and hence $W(F) \subseteq M_1$ and $W(F) \subseteq M_2$. But $F \subseteq M_1 \Rightarrow W(M_1) \subseteq W(F) \subseteq M_2$; a contradiction.

Thus each minimal prime in L is contained in a unique maximal filter. □

3.4 Relatively complemented Lattices

Atom: An element a in a lattice L is called an atom if it covers 0. In other words a is an atom iff $a \neq 0$ and $x \wedge a = a$ *or* $x \wedge a = 0 \quad \forall\, x \in L.$

Dual atom: An element b is called dual atom if u, the greatest element of the lattice covers b.

Theorem: 3.4.1 Let L be a lattice then the following implications hold:

(i) L is a Boolean algebra $\Rightarrow L$ is a relatively complemented.

(ii) L is relatively complemented $\Rightarrow L$ is sectionally complemented.

(iii) L is finite and sectionally complemented $\Rightarrow$ every non-zero element a of L is a join of finitely many atoms.

Proof: Let L be a Boolean algebra and let $a \le x \le b$. Define $y := b \wedge (a \vee x')$ then y is a complement of x in $[a,b]$,

Since $x \wedge y = x \wedge (b \wedge (a \vee x')) = x \wedge (a \vee x')$

$= (x \wedge a) \vee (x \wedge x') = x \wedge a = a$

and $x \vee y = x \vee (b \wedge (a \vee x')) = x \vee ((b \wedge a) \vee (b \wedge x'))$

$= x \vee (b \wedge x') = (x \vee b) \wedge (x \vee x') = b \wedge 1 = b.$

Thus L is relatively complemented.

(ii) If L is relatively complemented, then every $[a,b]$ is complemented; thus every interval $[a,b]$ is complemented i.e. L is sectionally complemented.

(iii) Let $\{p_1, \dots\dots\dots\dots, p_n\}$ be the set of atoms $\le a \in L$ and let $b = p_1 \vee \dots\dots\dots \vee p_n$.

Now $b \le a$, and if we suppose that $b \ne a$, then b has non-zero complement, say c in $[0,a]$. Let p be an atom $\le c$, then $p \in \{p_1, \dots\dots\dots\dots, p_n\}$ and thus $p = p \wedge b \le c \wedge b = 0$, which is contradiction. Hence $a = b = p_1 \vee \dots\dots\dots \vee p_n$. □

Finite Boolean algebra can be characterized of [17] as follows:

Theorem: 3.4.2 (Representation theorem) Let B be a finite Boolean algebra, and let A denote the set of all atoms in B. Then B is isomorphic to $P(A)$

i.e., $(B,\wedge,\vee) \cong (P(A),\cap,\cup)$

Proof: Let $v \in B$ be an arbitrary element and let $A(v) = \{a \in / a \le v\}$. Then $A(v) \subseteq A$.

Define $h : B \to P(A); v \to A(v)$

We show that h is Boolean isomorphic. First we show that h is a Boolean homomorphism.

For an atom a and for $v, w \in V$ we have

$a \in A(v \wedge w) \Leftrightarrow a \le v \wedge w \Leftrightarrow a \le v$ and $a \le w \Leftrightarrow a \in A(v) \cap A(w)$,

which proves $h(v \wedge w) = h(v) \cap h(w)$.

Similarly,

$a \in A(v \vee w) \Leftrightarrow a \le v \vee w \Leftrightarrow a \le v$

or $a \le w \Leftrightarrow a \in A(v) \cup A(w)$

Finally, $a \in A(v') \Leftrightarrow a \le v' \Leftrightarrow a \wedge v' = 0 \Leftrightarrow a \le / v \Leftrightarrow a \in A \setminus A(v)$;

Hence, the second equivalent. Note that $h(0) = \phi$ and 0 is the unique element which is mapped to ϕ. Since B is finite, we are able to verify that h is bijective. We know that every $v \in B$ can be expressed as join of finitely many atoms: $v = a_1 \vee \ldots\ldots\ldots\ldots \vee a_n$ with all atoms $a_i \le v$. Let $h(v) = h(w), i.e., A(v) = A(w)$. Then $a_i \in (v)$ and $a_i \in (w)$.

Therefore $a_i \le w$ and thus $v \le w$. Reversing the roles of v and w yields $v = w$, and this shows that h is injective.

To show that h is surjective we verify that for each $c \in P(A)$. There is a some $v \in B$ such that $h(v) = c$. Let $c = \{c_1, c_2, \ldots\ldots\ldots\ldots, c_n\}$ and $v = c_1 \vee c_2 \vee \ldots\ldots\ldots\ldots \vee c_n$.

Then $A(v) \supseteq c$, hence $h(v) \supseteq c$.

Conversely, if $a \in h(v)$, Then a is an atom with $a \le v = c_1 \vee c_2 \vee \ldots\ldots\ldots\ldots \vee c_n$.

Therefore $a \le c_i$, for some $i \in \{1, 2 \ldots\ldots\ldots\ldots, n\}$,

so $a = c_i \in c$. Altogether this implies $h(v) = A(v) = c$ □

Chapter Four

Pseudocomplemented Lattices

Introduction: In lattice theory there are different classes of lattices known as variety of lattices. Distributive pseudocomplemented lattice is one of the large variety. Throughout this chapter we discuss pseudocomplemented lattice. Pseudocomplemented lattice have been introduced by H. Lakser [16,17], K.B.Lee [20] and several author. In this chapter we have studied pseudocomplemented lattices and generalization of several results.

4.1 **Pseudocomplemented Lattice**

Pseudocomplemented: Let L be a bounded distributive lattice, let $a \in L$ an element $a^* \in L$ is called a pseudocomplement of a in L if the following conditions holds: (i) $a \wedge a^* = 0$ (ii) $\forall x \in L$, $a \wedge x = 0$ implies that $x \leq a^*$.

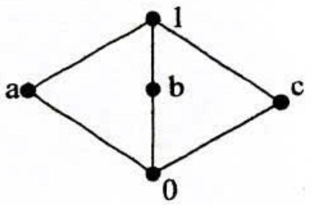

fig-4.1

Pseudocomplemented Lattice: A bounded lattice L is called a pseudocomplemented lattice if its every element has a pseudocomplement.

Example 4.1.1

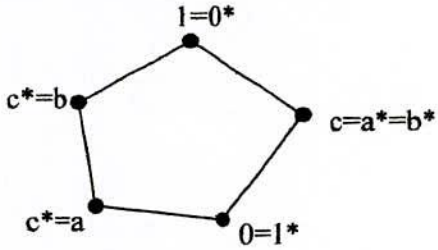

fig-4.2

The lattice $L = \{0, a, b, c, 1\}$ shown by the fig (4.2) is pseudocomplemented.

Lattice with pseudocomplementation: An algebra, $< L, \wedge, \vee, *, 0, 1 >$ where $\wedge, \vee$ are binary operation * is a unary operation and 0,1 are nullary operations is called lattice with pseudocomplemention if $< L, \wedge, \vee, *, 0, 1 >$ is bounded lattice, i.e. $\forall\ a \in L$ there exists a^* such that $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \wedge a^* = x, \forall\ x \in L$.

Pseudocomplemented distributive lattice: A bounded distributive lattice L is called a pseudocomplemented distributive lattice if its every element has pseudo complemented.

Example 4.1.2

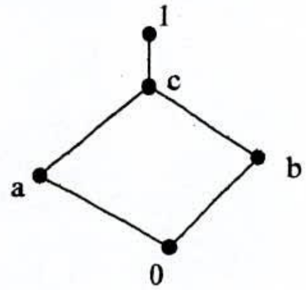

fig-4.3

Consider the finite distributive lattice of fig. (4.3). As a distributive lattice it has twenty five sublattices and eight congruence's; as a lattice with pscomplementation has three sub-algebras and five congruence's.

Sublattice: As a lattice L:

$\{0\},\{a\},\{b\},\{c\},\{1\},\{0,b\},\{0,c\},\{0,1\},\{0,a,b,c\},L,\{a,c\},\{a,c,1\},\{b,c\},\{a,1\},\{b,1\},$
$\{b,c,1\},\{c,1\},\{0,a,1\},\{0,b,1\},\{0,c,1\},\{0,a,c\},\{0,b,c\},\{0,a,c,1\},\{0,b,1\} = 25$

As a lattice with pseudocomplementation: $\{0,1\}, L, \{0,c,1\}$

Congruence: As a lattice:

$\omega = \{0\},\{a\},\{b\},\{c\},\{1\}$

$\tau = \{0,a,b,c,1\}$

$\phi = \{0,a\},\{b,c\},\{1\}$

$\phi = \{0,a\},\{b,c,1\}$

$\psi = \{0,b\},\{a,c\},\{1\}$

$\xi = \{0,b\},\{a,c,1\}$

$\varsigma = \{0,a,b,1\},\{1\}$

$\eta = \{c,1\},\{a\},\{b\},\{0\}$

Congruence as a lattice with pseudocomplementation, $\omega,\tau,\varphi,\xi,\eta$

The following theorem is an extension of a fundamental result in lattice theorem [13]. Though our proof include for the convenience of the reader .

Theorem 4.1.1 A pseudocomplemented and distributive lattice L such that

$S(L) = \{a/a^* = 0\}$ and $D(L) = \{a/a^* = 0\}$ Then for $a,b \in L$

i. $a \wedge a^* = 0$

ii. $a \leq b$ implies that $a^* \geq b^*$

iii. $a \leq a^{**}$

iv. $a^* = a^{***}$

v. $(a \vee b)^* = a^* \wedge b^*$

vi. $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

vii. $a \wedge b = 0$ $iff a^{**} \wedge b^{**} = 0$

viii. $a \wedge (a \wedge b)^* = a \wedge b^*$

ix. $0^* = 1$ and $1^* = 0$

x. $a \in S(L)$ *iff* $a = a^{**}$

xi. $a,b \in S(L)$ implies that $a \wedge b \in S(L)$

xii. $\sup(L)\{a,b\} = (a \vee b)^* = (a^* \wedge b^*)^*$

xiii. $0,1 \in D(L)$ and $S(L) \cap D(L) = \{1\}$

xiv. $a,b \in D(L)$ implies that $a \wedge b \in D(L)$

xv. $a \in D(L)$ and $a \leq b$ implies that $b \in D(L)$

xvi. $a \vee a^* \in D(L)$

xvii. $x \to x^{**}$ is a meet-homomorphism of L onto $S(L)$

Proof: *i.* By the definition of pseudocomplement

$a \wedge a^* = 0 \quad \forall a \in L$

ii. For $b \wedge b^* = 0$ and $a \leq b \Rightarrow a \wedge b^* = 0$

which implies that $a \geq b^*$

iii. By the definition of pseudocomplement $a \wedge a^* = a^* \wedge a = 0$

similarly $a^* \wedge (a^*)^* \Rightarrow a^* \wedge a^{**} = 0$ and $a \wedge a^* = 0$

$\Rightarrow a \leq a^{**}$

hence $a \leq a^{**}$

iv. We get,

$a \leq a^{**}$ implies that $a \geq a^{**}$............$(A)[by(ii)]$.

Again $a \leq a^{**} = 0$

i.e., $a^{**} \wedge a^* = 0$.

Similarly, $a^{**} \wedge (a^{**})^* = 0$ implies that $a^{**} \wedge a^{***} = 0$

and $a^{**} \wedge a^* = 0$ implies that $a \leq a^{**}$............(B)

From (A) and (B) we get,

$a^* = a^{***}$ hence $a^* = a^{***}$

v. We have,

$(a \vee b) \wedge (a^* \wedge b^*) = (a \wedge a^* \wedge b^*) \vee (b \wedge a^* \wedge b^*) = (0 \wedge b^*) \vee (a^* \wedge 0)$ $by(i)$

$= 0 \vee 0 = 0$

Let $(a \vee b) \wedge x = 0$.

Implies that $(a \wedge x) \vee (b \wedge x) = 0$

Implies that $a \wedge x = 0$ and $b \wedge x = 0$ Implies that $x \leq a^*$ and $x \leq b^*$

Implies that $x \leq a^* \wedge b^*$

Therefore $a^* \wedge b^*$ is the pseudocomplement of $a \vee b$

vi. Let $a, b \in L$ implies that $a^*, b^* \in L$ implies that $a^{**}, b^{**} \in S(L)$

implies that $a^{**} \wedge b^{**} \in S(L)$

But $a^{**} \wedge b^{**}$ is the smallest element of $S(L)$ containing $a \wedge b$

So $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

vii. If $a \wedge b$ by (vi)

Then $a^{**} \wedge b^{**} = (a \wedge b)^{**} = 0^{**} = 0$

So $a^{**} \wedge b^{**} = 0.$

Conversely, if $a^{**} \wedge b^{**} = 0.$ by $(ii) a \leq a^{**}, b \leq b^{**}$ (ii) $\forall a, b \in L$

Then $a \wedge b \leq a^{**} \wedge b^{**} = 0.$ $\therefore a \wedge b = 0$

Hence $a \wedge b = 0$ iff $a^{**} \wedge b^{**} = 0.$

viii. Since $a \wedge b \leq b$ so $(a \wedge b)^* \geq b^*$ and so

$a \wedge (a \wedge b)^* \geq a \wedge b^*$ ....(A)

Again $(a \wedge b) \wedge (a \wedge b)^* = 0$. Implies that $(a \wedge (a \wedge b)^*) = 0,$

Therefore $a \wedge (a \wedge b)^* \leq b^*$. Implies that $a \wedge a \wedge (a \wedge b)^* \leq a \wedge b^*$.

Implies that $a \wedge (a \wedge b)^* \leq a \wedge b^*$ ....(B)

From (A) and $(B) \Rightarrow$ $a \wedge (a \wedge b)^* = a \wedge b^*$

Hence $a \wedge (a \wedge b)^* = a \wedge b^*$.

xi. We have $0 \wedge x = 0 \forall \in L$ and $0 \wedge 1 = 0$

But $x \leq 1$ $\forall x \in L$

Hence $0^* = 1$ Again, $0^* = 1$

Implies that $0^{**} = 1^*$ Implies that $0 = 1^*$ $\therefore 1^* = 0$

x. If $a \in S(L)$ then, $a = b^*$ for some $b \in L$

But $a^* = a^{***}, \forall a \in L$

Now, $a^{**} = b^{***} = b^{*} = a$ Hence, $a^{**} = a$

Conversely, if $a = a^{**}$ then $a = b^{*}$, thus $a \in S(L)$

Hence $a \in S(L)$ iff $a = a^{**}$

xii. Let $a, b \in S(L)$ then $a = a^{**}, b = b^{**}$.

Since $a \wedge b \leq a$ implies that $(a \wedge b)^{**} \leq a^{**} = a$ $\quad \therefore a \geq (a \wedge b)^{**}$.

Again, since $a \wedge b \leq b$ implies that $(a \wedge b)^{**} \leq b^{**} = b$

$\therefore (a \wedge b)^{**} \leq b$ implies that $b \geq (a \wedge b)^{**}$. Implies that $a \wedge b = 0$

But $(a \wedge b) \leq (a \wedge b)^{**}$................................(B)

From (A) and (B) we get, $(a \wedge b) = (a \wedge b)^{**}$ implies that, $a \wedge b \in S(L)$

If $x \in S(L)$ such that $x \leq a$ and $x \leq b$ then $x \leq a \wedge b$ is a greatest lower bound of $S(L)$.

Therefore $a \wedge b = \inf S(L)$, $\{a, b\} \in S(L)$

xii. For $a, b \in S(L)$ $a^{*} \geq a^{*} \wedge b^{*}$

Implies that $a^{**} \leq (a^{*} \wedge b^{*})^{*}$ $[by(ii)]$

Implies that $a \leq (a^{*} \wedge b^{*})^{*}$ $[by(i)]$

Again, $b^{*} \geq a^{*} \wedge b^{*}$ implies that $b^{**} \leq (a^{*} \wedge b^{*})^{*}$ $[by(ii)]$

Implies that $b \leq (a^{*} \wedge b^{*})^{*}$ $[by(i)]$

$\therefore (a^{*} \wedge b^{*})^{*}$ is a upper bound of $\{a, b\}$ in $S(L)$

Let $x \in S(L)$ such that $a \leq x, b \leq x$ then $a^{*} \geq x$

$B^{*} \geq x^{*}$ $[by(ii)]$

$\therefore a^{*} \wedge b^{*} \geq x^{*}$ implies that $(a^{*} \wedge b^{*}) \leq x^{**} = x$. Implies that $(a^{*} \wedge b^{*}) \leq x$

$\therefore (a^{*} \wedge b^{*})^{*}$ is a least upper bound of $\{a, b\}$ in

$\sup \{a, b\} = (a^{*} \wedge b^{*})$

Again $(a \wedge b)^{**} = ((a \vee b)^{*}) = (a^{*} \wedge b^{*})^{*}$

Hence $\sup\{a,b\} = (a \vee b)^{**} = (a^* \wedge b^*)^*$

xiii. From (ix) we have $0^* = 1$, $1^* = 0$ then $0,1 \in S(L)$ and $1 \in D(L)$

Let $x \in S(L) \cap D(L)$ then $x \in S(L)$ and $x \in D(L)$,

Such that $x = x^{**}$, $x^* = 0$ then $x = (x^*)^* = 0^* = 1$

Hence $S(L) \cap D(L) = \{1\}$

xiv. Let $a, b \in D(L)$ then $a^* = 0$, $b^* = 0$. Implies that $a^{**} = b^{**} = 0^* = 1$

Now, $(a \wedge b)^{**} = a^{**} \wedge b^{**} = 1 \wedge 1 = 1$ $[by (ii)]$

$(a \wedge b)^* = (a \wedge b)^{***} = 1^* = 0$. Implies that $a \wedge b \in D(L)$

xv. If then $a^* = 0$ and $a \le b$

Implies that $a^* \ge b^*$. Implies that $b^* \le a^* = 0$

Implies that $b^* = 0$. Hence $b \in D(L)$

xvi. From (v) we have $(a \vee a^*)^* = a^* \wedge a^{**} = a^* \wedge (a^*)^* = 0$

Hence $a \vee a^* \in D(L)$

xvii. Let $\varphi : L \to S(L)$ defined by $\varphi(x) = x^{**}$

Then, $\varphi(x \wedge y) = (x \wedge y)^{**} = x^{**} \wedge y^{**} = \varphi(x) \wedge \varphi(y)$

$\therefore \varphi$ is a meet homomorphism. □

Theorem 4.1.2 Let $a,b \in L$ and L be pseudocomplemented meet semilattice and let $a,b \in L$, verify that $(a \wedge b)^* = (a^{**} \wedge b)^* = (a^{**} \wedge b^{**})^*$

Proof: We know that, $(a \wedge b)^* = (a \wedge b)^{**} = ((a \wedge b)^{**})^*$

i.e., $(a \wedge b)^* = (a^{**} \wedge b^{**})^*$ ....(i)

Again $(a^{**} \wedge b)^* = (a^{**} \wedge b)^{***} = ((a^{**} \wedge b)^{**})^* = (a^{****} \wedge b^{**})^*$

i.e., $(a^{**} \wedge b)^* = (a^{**} \wedge b^{**})^*$ ....(ii)

Now from (i) and (ii) we get,

$(a \wedge b)^* = (a^{**} \wedge b)^* = (a^{**} \wedge b^{**})$ □

Theorem 4.1.3 Let L be pseudocomplemented distributive lattice. Prove that for each $a,b \in L$, $[a)$ is a pseudocomplemented distributive lattice, infect, the pseudo-complement of $x \in [a)$ in $[a)$ is $x^* \vee a$.

Proof: Let $x \in [a)$ then $x \vee (x^* \vee a) = (x \vee x^*) \vee a = 1 \vee a = a$

Furthermore if $x \vee t = 1$ then $t \geq x^*$

$\Rightarrow t \vee a \geq x^* \vee a$

$\Rightarrow t \geq x^* \vee a$ [since $t \in [a) \Rightarrow t \geq a \Rightarrow t \vee a = t$]

From the above it follows that $x^* \vee a$ is the pseudocomplemented of x.

Therefore, $[a)$ is a pseudocomplemented distributive lattice. □

Theorem 4.1.4 A pseudocomplemented lattice L and $a,b \in L$ then prove that $a^{**} \vee b^{**} = (a \vee b)^{**}$.

Proof: If L is a pseudocomplemented meet semilattice then $a \vee b = (a^* \wedge b^*)^*$ where $a,b \in S(L)$

Now, for $a,b \in L$ and $a^{**}, b^{**} \in S(L)$

So, $a^{**} \vee b^{**} = (a^{***} \wedge b^{***})^* = (a^* \wedge b^*)^* = (a \vee b)^{**}$

Implies that $a^{**} \vee b^{**} = (a \vee b)^{**}$. □

Theorem 4.1.5 Let L be a pseudocomplemented meet semilattice and $a,b \in L$. Then prove that $(a \wedge b) = (a^{**} \wedge b)^* = (a^{**} \wedge b^{**})^*$.

Proof: Since L is a pseudocomplemented meet semilattice, then $a \leq a^{**}$

Implies that $a \wedge b \leq a^* \wedge b^*$

Implies that $(a \wedge b) \leq (a^{**} \wedge b^*)$................$(i)$

Again, $b \le b^{**}$ implies that $a^{**} \wedge b \le a^{**} \wedge b^{**}$.

Implies that $a^{**} \wedge b \le (a \wedge b)^{**}$

Implies that $(a^{**} \wedge b)^* \ge (a \wedge b)^{***}$.

Implies that $(a^{**} \wedge b)^* \ge (a \wedge b)^*$ ....(ii)

From (i) and (ii) we get,

$(a \wedge b)^* = (a^{**} \wedge b)^*$ ....(iii)

Again, $b \le b^{**}$ implies that $a^{**} \wedge b \le a^{**} \wedge b^{**}$.

Implies that $(a^{**} \wedge b) \ge (a^{**} \wedge b^{**})^*$ ....(iv)

Again, $a^{**} \le a^{***}$ implies that $a^{**} \wedge b^{**} \le a^{****} \wedge b^{**} = (a^{**} \wedge b)^{**}$.

Implies that $(a^{**} \wedge b^{**}) \ge (a^{**} \wedge b)^{***}$.

Implies that $(a^{**} \wedge b^{**})^* \ge (a^{**} \wedge b^{**})^*$ ....(v)

From (iv) and (v) we get,

$(a^{**} \wedge b)^* = (a^{**} \wedge b^{**})^*$ ....(vi)

From (iii) and (vi) we get, $(a \wedge b)^* = (a^{**} \wedge b)^* = (a^{**} \wedge b^{**})^*$. □

Theorem 4.1.6 Every distributive algebraic lattice is pseudocomplemented.

Or, Let L be a distributive algebraic lattice, then $L \cong I(S)$, where S is a join semi lattice with 0.

Proof: Let $I, I_k \in I(s)$ for $k \in K$ (index set)

Then $I \wedge I_r \subseteq I \wedge v(I_k / k \in K)$ for any $r \in K$.

Clearly, $v(I_k / k \in K) \subseteq I \wedge v(I_k / k \in K)$.

To prove the reverse inequality,

Let $(a \in I) \wedge v(I_k / k \in K)$.

Then $a \in I$ and $a \in v(I_k / k \in K)$.

Then there exist indices $\lambda_1, \lambda_2 ....\lambda_n$

Such that $a \le i_{\lambda 1} \vee i_{\lambda 2} \vee ....\vee i_{\lambda n}$

For some $i_{\lambda k} \in I_{\lambda k}$ for some $k \in 1,2,...........n$.

Thus $a \in I \wedge (I_{\lambda 1} \vee I_{\lambda 2} \vee ....\vee I_{\lambda n}$ and so on.

$a = I \wedge (I_{\lambda 1} \vee I_{\lambda 2} \vee ....\vee I_{\lambda n}$

$(I \wedge I_{\lambda 1}) \vee (I \wedge I_{\lambda 2}) \vee ....\vee (I \wedge I_{\lambda n})$ as $I(S)$ is distributive $\subseteq v(I_k / k \in K)$

$i.e., I \wedge v(L_k / k \in K \subseteq Iv(I_k / k \in K)$.

Therefore $I_k / I \wedge v(L_k / k \in K) = Iv(I_k / k \in K)$.

This shows that $I(S)$ has the join infinite distributive property.

Moreover as $0 \in S$, $I(S)$ is complemented.

Therefore $I(S)$ is pseudocomplemented and so L is pseudocomplemented. □

Theorem 4.1.7 Suppose L be a pseudocomplemented distributive lattice. Define the relation R by $x \equiv y(R)$ if and only if $x^* = y^*$ then R is a congruence on L and $L / R \cong S(L)$.

Proof: Here we have $x \equiv y(R)$

$\Leftrightarrow x^* = y^*$, then $x^* = x^*$.

Implies that $x = x(R)$. Implies that R is reflexive.

Also if $x \equiv y(R)$, then $x^* = y^*$. Implies that $y^* = x^*$

Implies that $y = x(R)$. Implies that R is symmetric.

Let $x \equiv y(R) y = z(R)$ then $x^* = y^*$ and

implies that $x^* = z^*$. Implies that $x \equiv z(R)$.

Implies that R is transitive.

Implies that R is an equivalence relation.

Now suppose $x \equiv y(R)$ and $t \in L$ then $x^* = y^*$

Implies that $x^{**} = y^{**}$

Now, $(x \wedge t)^{**} = x^{**} \wedge t^{**} = (y \wedge t)^{**}$

$$\Rightarrow (x \wedge t)^{**} = (y \wedge t)^{**}$$

$$\Rightarrow (x \wedge t)^{*} = (y \wedge t)^{*}$$

$$\Rightarrow x \wedge t = y \wedge t \equiv y \wedge t(R).$$

And $(x \vee t)^* = x^* \wedge t^* = y^* \wedge t^* = (y \vee t)^*$

Implies that $x \vee t = y \vee t(R)$. So R is congruence relation on L.

Define $\varphi : L/R \to S(L)$ by $\varphi((a]R) = a^{**}$

Then $\varphi([a] \wedge [b]) = \varphi([a \wedge b]) = (a \wedge b)^{**}$

$= a^{**} \wedge b^{**} = \varphi([a]) \wedge \varphi([b])$

and $\varphi([a]) \vee [b]) = \varphi([a \vee b]) = (a \vee b)^{**} = (a^* \wedge b^*)^*$

$= (a^{***} \vee b^{***})^* = a^{**} \vee b^{**}$

$= \varphi([a]) \vee \varphi([b])$ $\therefore \varphi$ is a homomorphism.

To show that φ is one-one.

Let $a^{**} = b^{**}$ implies that $a^* = b^*$

Implies that $a \equiv b(R)$. Implies that $[a] = [b]$ $\therefore \varphi$ is one-one.

Let $a \in S(L)$ then $a = a^{**} \Rightarrow a =$ $([a\})$ $\Rightarrow \varphi$ is onto.

Hence $\varphi : L/R \to S(L)$ is an isomorphism.

Therefore $L/R \equiv S(L)$. □

Following theorem gives a discription of semilattice which is due to [18]

Theorem 4.1.8 Suppose L be a pseudocomplemented meet semilattice and $S(L) = \{a^* / a \in L\}$. Then the partial ordering of L partially orders $S(L)$ and forms

$S(L)$ into a Boolean lattice. For $a,b \in S(L)$ we have $a \wedge b \in S(L)$ and then join in $S(L)$ is described by $a \vee b = (a^* \wedge b^*)^*$.

Proof: We derived with the following observations:

i. $\forall a \in L,\ a \le a^{**}$

ii. $a \le b \Rightarrow a^* \ge b^*$

iii. $a^* = a^{***}$

iv. $a \in S(L)$ iff $a = a^{**}$

v. For $a,b \in S(L), a \wedge b \in S(L)$

vi. For $a,b \in S(L), a \vee b = (a^* \wedge b^*)^*$

vii. Since $a^* \wedge a = a \wedge a^* = 0$

Also $a^* \wedge a^{**} = 0$ $a^* \wedge a^{**} = 0$

So $a \le a^{**}$ from the definition of pseudocomplement lattice.

viii. $a \le b$, *so* $a \wedge b^* \le b \wedge b^* = 0$ *i.e.,* $a \wedge b^* = 0 \Rightarrow b^* \le a^*$

From the definition of pseudocomplement lattice.

ix. By $(\mathrm{i})\mathrm{a}^* \le (\mathrm{a}^*)^{**} = \mathrm{a}^{***}$

Again $a \le a^{**}$ by (i)

So by $(\mathrm{ii})\mathrm{a}^{***} \le \mathrm{a}^*$. Hence $a^* = a^{***}$

x. Let $a \in S(L)$ then $a = b^*$ for some $b \in L$

Hence $a^{**} = b^{***} = b^* = a$

If $a = (a^*)^*$ then $a = (a^*)^*$. And so, $a \in S(L)$

xi. $a,b \in S(L)$ then $a = a^{**} b = b^{**}$

So, $a \ge (a \wedge b)^{**} \ge (a \wedge b)^{**}$ So, $(a \wedge b)^{**} \le a \wedge b$

Again, by (i) and $(ii) a \wedge b \le (a \wedge b)^{**}$. Hence $a \wedge b = (a \wedge b)^{**}$

So, $a \wedge b \in S(L), a \ge a \wedge b \Rightarrow a^{**} \ge (a \wedge b)^{**}$ by (ii)

$\Rightarrow a \geq (a \wedge b)^{**}$

For $a, b \in S(L)$. We have $a^* \geq a^* \wedge b^*$ So by (ii) and $a^* \leq (a^* \wedge b^*)^*$

Similarly, $b \leq (a^* \wedge b^*)^*$

Now if $a \leq x, b \leq x (x \in S(L))$ then $a^* \geq x^*, b^* \geq x^*$

So, $a^* \wedge b^* \geq x^*$. Hence, $x^{**} \geq (a^* \wedge b^*)^*$

i.e., $x \geq (a^* \wedge b^*)^*$ as $x \in S(L)$

Hence, $(a^* \wedge b^*)^* = Sup\ \{a,b\} = a \vee b \in S(L)$. Thus $S(L)$ is a lattice.

Moreover $0,1 \in S(L)$. Therefore $S(L)$ is a bounded lattice.

Now for any $a \in S(L), a \wedge a^* = 0$. And $a \vee a^* (a^* \wedge a^{**}) = 0^* = 1$

i.e., a^* is the complement of a in $S(L)$. Hence $S(L; \wedge, \vee)$ is a complemented lattice.

Then we only to show that $S(L)$ is distributive.

Let $x, y, z \in S(L)$

Then $x \wedge z \leq (x \vee (y \wedge z)$ and $y \wedge z \leq (x \vee (y \wedge z)$. Hence $x \wedge z \wedge (x \vee (y \wedge z)) = 0$

and $y \wedge z \wedge (x \vee (y \wedge z))^* \leq x^* = 0$. Thus, $z \wedge (x \vee (y \wedge z))^* \leq x^*$ and y^*

and so $z \wedge (x \vee (y \wedge z))^* \leq x^* \wedge y^*$. Consequently, $z \wedge (x \vee (y \wedge z)) \wedge (x^* \wedge y^*) = 0$

Which implies $z \wedge (x^* \wedge y^*)^* = (x \vee (y \wedge z))^{**} = x \vee (y \wedge z)$

So by (vi) and $(iv) \Rightarrow z \wedge (x \vee y) = x \vee (y \wedge z)$

Therefore $S(L)$ is distributive. □

4.2 Stone Lattice & Minimal prime Ideal

Boolean Lattice: A complemented distributive lattice is called Boolean lattice.

Stone Lattice: *A* distributive pseudocomplemented lattice L is called stone lattice if for $a \in L$, $a^* \vee a^{**} = 1$

Example 4.2.1 Every Boolean lattice is stone lattice but converse is not true.

Stone algebra: A complemented distributive lattice is called a stone algebra if for each $a^* \vee a^{**} = 1$

Generalized Stone Lattice: A lattice L with 0 is called generalized stone lattice if $(x]^* \vee (x]^{**} = L$ for each $x \in L$.

Theorem 4.2.1 Let L be a complemented distributive lattice, then show that the following conditions are equivalent:

(i) L is a stone algebra.

(ii) For $a,b \in L, (a \wedge b)^* = a^* \vee b^*$

(iii) $a,b \in S(L)$ implies $a \vee b \in S(L)$

iv. $S(L)$ is a sub algebra.

Proof: $(a) \Rightarrow (b)$

Suppose (a) holds, $i.e., L$ is a stone algebra.

We have, $(a \wedge b)^* = a^* \vee b^*$

Let $a,b \in L$, $(a \wedge b) \wedge (a^* \vee b^*) = (a \wedge b \wedge a^*) \vee (a \wedge b \wedge b^*)$ [Since L is a distributive lattice]

$= (a \wedge a^* \wedge b) \vee (a \wedge b \wedge b^*) = (0 \wedge b) \vee (a \wedge 0) = 0 \vee 0 = 0$

Now suppose $x \in L$ such that $(a \wedge b) \wedge x = 0$

$\Rightarrow (b \wedge x) \wedge a = 0 \Rightarrow b \wedge x \leq a^*$

Meeting both sides with a^{**} then we get, $a^{**} \wedge (b \wedge x) \leq a^{**} \wedge a^* = 0$

$\Rightarrow (x \wedge a^{**}) \wedge b = 0 \ \Rightarrow x \wedge a^{**} = b^*$

Since L is a stone algebra, then we have, $a^* \vee a^{**} = 1$

Now, $x = x \wedge 1 = x \wedge (a^* \vee a^{**}) = (x \wedge a^*) \vee (x \wedge a^{**}) \leq a^* \vee b^*$

Hence $a^* \vee b^*$ is the complement of $a \wedge b$

$i.e., (b)$ holds .

$(b) \Rightarrow (c)$

Suppose (b) holds,

Let $a, b \in S(L)$ we have, $a = a^*$ and $b = b^{**}$

$\therefore a \vee b = (a^{**} \vee b^{**}) = (a^* \vee b^*)^* = (a \vee b)^{**}$

$\Rightarrow a \vee b \in S(L)$

$(c) \Rightarrow (d)$

Suppose (c) holds,

$i.e., a, b \in S(L)$. Implies that $a \vee b \in S(L)$

As $a \vee b \in S(L)$. So we have, $a \vee b \in S(L)$

Hence (d) holds,

$i.e., S(L)$ is a subalgebra.

$(d) \Rightarrow (a)$

Suppose (d) holds,

$i.e., S(L)$ is a sub algebra of L.

Now for any $a \in L, a^* \in S(L), a^{**} \in S(L)$

Hence $a^* \vee a^{**} = (a^{**} a^{***})^*$ [since $a \vee b = (a^* \wedge b^*)^*$]

$= 0^* = 1$. Hence L is stone algebra. $i.e., (a)$ holds. □

Theorem 4.2.2 Show that a distributive pseudocomplemented lattice L is a stone Lattice iff $(a \vee b)^{**} = a^{**} \vee b^{**}$ $\forall a, b \in L$.

Proof: Let L be a pseudocomplemented distributive lattice. If L is a stone lattice,

then $\forall a, b \in L$. We have, $(a \vee b)^* = a^* \wedge b^*$

Hence $(a \vee b)^{**} = (a^* \wedge b^*)^* = a^{**} \vee b^{**}$

Conversely, let $(a \vee b)^{**} = a^{**} \vee b^{**}$ $\forall a, b \in L$

Now for $x \in L$, let $x^* \vee x^{**} = y$, then

$(x^*\vee X^{**})^{**}=y^{**} \Rightarrow y=y^{**} \Rightarrow (x^*\vee X^{**})^{**}=y^{**}$

Now, $y^*=(x^*\vee x^{**})^*=x^{**}\wedge x^{***}=x^{**}\wedge x^*=0$

$\therefore y^{**}=0^*=1 \Rightarrow y=1$. Hence $x^*\vee x^{**}=1$

Therefore L is stone lattice. □

Proposition 4.2.1 If p is a prime ideal of a lattice L, then $\dfrac{L}{R(p)}$ is a two element chain. The elements are $p, L-p$.

Proof: Let $x, y \in L-p$

If for some $1 \in L, x\wedge 1 \in p$,

Then $1\in p$ [since $x \notin p$ and p is prime]

Hence $y\wedge 1 \in p$

i.e., $\forall\ 1 \in L, x\wedge 1 \in p \Leftrightarrow y \wedge 1 \in p \Rightarrow x \equiv y \quad R(p)$ □

Proposition 4.2.2 Show that in a stone algebra every prime ideal contain exactly one minimal prime ideal.

Proof: Let P be a prime ideal and q_1 and q_2 be two minimal prime ideals contains in p with $q_1 \neq q_2$,

Let $x \in q_1 - q_2$, then $x\in q_1$ but $x \notin q_2$

Now $x\wedge x^*=0\in q_2 \Rightarrow x^*\in q_2 \Rightarrow x^* \in p$

Again since q_1 minimal, then $x\in q_1 \Rightarrow x^{**}\in q_1 \Rightarrow x^{**}\in p$

Hence $x^*\vee x^{**}\in p$ which contradict the fact that p is prime. Hence $q_1=q_2$

Hence in a stone algebra every prime ideal contains exactly one minimal prime ideal.

□

The following theorem is an extension of a fundamental result in lattice theory [13, Lemma 4, pp 169]. Though our proof is similar to their proof, we include the proof for the convenience for the reader.

Theorem 4.2.3 Let L be a distributive lattice with complemented. Then L is a stone algebra iff $P \vee Q = L$ for any two distinct minimal prime ideal.

Proof: First consider L is a stone algebra.

Let P and Q are two distinct minimal prime ideals.

Let $a \in Q - P$. Then $a \in P$

Now $a \wedge a^* = 0 \in p$. since p is prime and $a \in p$,

So $a^* \in p$,

Now let $L - Q$ is a minimal dual prime ideal.

Thus $(L - Q) \vee [a) = L$,

So $a = x \wedge a$ for some $x \in L - Q$

$\Rightarrow a^* \geq x \in L - Q \Rightarrow a^* \in L - Q \Rightarrow a^* \notin Q \Rightarrow a^* \in p - Q$

Similarly we have, $a^{**} \in p - Q$

Hence $a^* \vee a^{**} = 1 \Rightarrow 1 \in p \vee Q \Rightarrow L = p \vee Q$

Conversely, Suppose $p \vee Q = L$ for any two distinct minimal prime ideals.

We have to show that L is a stone algebra.

If L is not stone algebra, then there exists $a \in L$,

Such that $a^* \vee a^{**} \neq 1$

Then there exists a prime ideal R

Such that $a^* \vee a^{**} \in R$

We claim that, $(L - R) \vee [a^*) = L$ then $x \wedge a^* = 0$,

For some $x \in (L - R)$

$\Rightarrow a^{**} \geq x \in (L - R) \Rightarrow a^{**} \in (L - R) \Rightarrow a^{**} \notin R$

Which is contradiction.

Hence $(L - R) \vee [a^*) \neq L$

Let F be a minimal dual prime ideal containing $(L-R)\vee[a^*)$. and G be a maximal dual prime ideal containing $(L-R)\vee[a^{**})$.

Put $p = L-F$ and $Q = L-G$

Then p and Q are minimal prime ideal and $p \neq Q$

As $a^* \in Q$ but $a^* \notin p$ and $a^{**} \in p$ but $a^{**} \notin Q$

i.e., p and Q are distinct.

Also $p, Q \subseteq R$ and thus $P \vee Q \subseteq R \neq L$

Which is also contradiction.

Hence L is a stone algebra. □

Theorem 4.2.4 Let L be a complemented distributive lattice and p be a prime ideal of L. Then the following conditions are equivalent:

i. p is minimal.

ii. $x \in p, \Rightarrow x^* \notin p$

iii. $x \in p, \Rightarrow x^{**} \in p$

iv. $p \wedge D(L) = \varphi$

Proof: $(i) \Rightarrow (ii)$

Let p be minimal. Suppose, If (ii) fails there exists $x \in p$.

Such that $x^* \in p$. Let $D = (L-p)\vee[x)$ then $0 \in D$.

For otherwise $0 = q \wedge x$ for some $q \in L-p$.

Which implies that $q \leq x^* \in p$.

Therefore, $q \in p$, which is contradiction.

Hence $0 \notin p \cap D$,

Then by stone's representation theorem there exists a prime ideal Q, Such that $Q \cap (L-p) = \varphi$ This implies $Q \cap (L-p) = \varphi$

and so $Q \subseteq p$ But $Q \neq p$ as $x \in Q$

This contradict the minimally of p

Hence (ii) follows.

$(ii) \Rightarrow (iii)$ suppose (ii) holds and $x \in p$

Now $x^* \wedge x^{**} = 0 \in p$

But $x^* \notin p$ and p is prime. So $x^{**} \in p$ *i.e.*, (iii) holds.

$(iii) \Rightarrow (iv)$ suppose (iii) holds

Let $x \in p \cap D(L)$. Then $x \in p$ and $x \in D(L)$

$\therefore x \in D(L)$ implies $x^* = 0$. By (iii) $x^{**} \in p$

$\therefore x^{**} = (x^*)^* = 0^* = 1 \in p$

Which is impossible as p is prime.

So (iv) holds.

Suppose p is not minimal, then there exists a prime ideal $Q \subset p$

Let $x \in p - Q$

Now $x \wedge x^* = 0 \in Q$. Since $x \in Q$ and Q is prime.

So, $x^* \in Q \subset p$. Then $x, x^* \in p$

So, $x \vee x^* \in p$

Now, $(x \vee x^*)^* = x^* \wedge x^{**} = 0$

Implies that $x \vee x^* \in D(L)$ *i.e.*, $p \cap D(L) \neq \varphi$

and so (iv) does not hold. □

In [18, Lemma 8] has proved the following result lattices we generalize it to pseudocomplemented lattices.

Lemma 4.2.1 Let L be a distributive lattice with 0. Let $0 \leq x \in L$ and the interval $[0,x]$, is complemented. If y^* is the relative complemented of y in $[0,x]$, then $(y^*] = (y]^* \wedge (x](x]$ and $(y^{**}] = (y\} \wedge (x]$. □

Lemma 4.2.2 Let L be a distributive lattice with 0. For any $r \in L$

and any ideal $I, ((r] \wedge I]^* \wedge (I] = I^* \wedge (r]$. □

Theorem 4.2.5 A distributive lattice L with 0 is a generalized stone lattice if and only if each interval $[0,x]$, $0 < x \in L$, L is stone lattice.

Proof: Let L with 0 be a generalized stone and let $P \in [0,x]$.

Then $(P]^* \vee (P]^{**} = L$. So $x \in (P]^* \vee (P]^{**}$ implies $x = r \vee I$, for some

$r \in (P]^*, I \in (P]^{**}$

Now $r \in (P]^*$ implies $r \wedge P = 0$, also $0 \leq r \leq x$.

Suppose $t \in [0,x]$ such that $t \wedge P = 0$, then $t \in (P]^*$ implies $t \wedge I = 0$ Therefore, $t \wedge x = t \wedge (r \vee I) = (t \wedge r) \vee (t \wedge I) = (t \wedge r) \vee 0 = t \vee r$

Implies $t = t \wedge r$ implies $t \leq r$

So, r is the relative complement of P in $[0,x]$, *i.e.*, $r = P$.

Since $I \in (P]^{**}$. and $r \in (P]^*$, so $I \wedge r = 0$. Let $q \in [0,x]$ such that $q \wedge r = 0$ Then as $x = r \vee I$ so $q \wedge x = (q \wedge r) \vee (q \wedge I)$

Implies $q = q \leq I$ implies $q \leq I$

Hence, I is the relative complement of $r = P^*$ in $[0,x]$, *i.e.*, $I = P^{**}$ implies $x = r \vee I = P^* \vee P^{**}$. Thus $[0,x]$, is a stone lattice.

Conversely, suppose $[0,x]$, $0 < x \in L$ is a stone lattice. Let $P \in L$,

Then $P \wedge x \in [O,P]$ Since $[O,P]$ is a stone lattice, then

$(P \wedge x)^* \vee (p \wedge x)^{**} = P$ where $(P \wedge x)^*$ is the relative

complment of $(P \wedge x)$ in $[O,P]$

Therefore $P \in ((p] \cap (p \wedge x]) \vee ((p] \cap (P \wedge x]^{**})$

So, we can take $P = r \vee 1$, for $r \in (P \wedge x]^*$, $1 \in (p \wedge x]^{**}$

Now, $r \in (P \wedge x]^*$ implies $r \wedge P \wedge x = 0$ implies $r \wedge x = 0$ implies

$r \in (x]^{**}$ and $1 \in (P \wedge x]^{**}$

Now $P \wedge x \leq x$ implies $(P \wedge x]^{**} \subseteq (x]^{**}$

And so $1 \in (x]^{**}$

Therefore $P = r \vee 1 \in (x]^* \vee (x]^{**}$ and so $L \subseteq (x]^* \vee (x]^{**}$

But $(x]^* \vee (x]^{**} \subseteq L$ is obvious.

Hence $(x]^* \vee (x]^{**} = L$ and S on L is generalized stone. □

Following theorem is a generalization of [14, Proposition 5.5(b)]

Theorem 4.2.7 Suppose L be a distributive lattice with 0. If L is generalized stone, then it is normal.

Proof: Let P and Q be two minimal prime ideals of L. Then P, Q are unordered. Let $x \in P$,

Then $(x] \wedge (x]^* = (0] \subseteq Q$ implies $(x]^* \subseteq Q$. Since P is minimal,

so $(x]^{**} \subseteq P$. Again, as L is generalized stone,

so $(x]^* \vee (x]^{**} = L$. This implies $P \vee Q = L$ and so L is normal. □

Lemma 4.2.4 If L_1 is a sublattice of a distributive lattice L and P_1 is minimal prime ideal in L_1, then there exists a minimal prime ideal P in L such that $P_1 = L_1 \cap P$ □

Following theorem is generalization of [14, theorem 5 p.115]

Theorem 4.2.8 A sectionally pseudocomplementetd distributive lattice L is generalize stone if and only if any two minimal prime ideals are comaximal.

Proof: Suppose L is generalized Stone. So by Theorem 2.3.7 any two minimal prime ideals are comaximal. To prove the converse, let P,Q be two minimal prime ideals of L. We need to show that $[0,x]$ is stone, For each $x \in L$ Let P_1,Q_1 be two minimal prime ideals in $[0,x]$. Using Lemma 2.2.4. there exists minimal prime ideals P,Q in L such that $P_1 = P \cap [0,x], Q_1 = Q \cap [0,x]$.

Therefore $P_1 \vee Q_1 = (P \cap [0,x]) \vee (Q \cap [0,x]) = [P \vee Q] \cap [0,x] = L \cap [0,x] = [0,x]$.

Therefore $[0,x]$ is stone. So L is generalized stone. □

Corollary 4.2.1 A distributive lattice L is generalized stone if and only if it is sectionally complemented and normal. Figure 2.1 the lattice L is in fact a generalized stone lattice, as it is both sectionally complemented and normal.

Corollary 4.4.4: A distributive lattice L with 0 is generalized stone if and only if it is normal and sectionally complemented.

4.3 Sectionally pseudocomplemented Lattices

Sectionally pseudocomplemented Lattices: A lattice L with 0 is called sectionally pseudocomplemented if interval $[0,x]$ for each $x \in L$ is pseudocomplemented.

Note: Every finite distributive lattice is sectionally pseudocplemented.

Following figure 2.3 gives an example of a distributive lattice with 0 which is not sectionally pseudocomplemented.

In R^2 consider the set:

$E = \{(0,y)/0 \le y < 5\} \cup \{(2,y)/0 \le y < 5\} \cup \{(3,5),(4,5),(3,6)\}$

Define the partial ordering $\le$ on E by $(x,y) \le (x_1,y_1)$ if and only if $x \le x_1$ and $y \le y_1$ here E is clearly a distributive lattice. This is not a lattice as the supremum of (3,6) and (4,5) does not exist. Consider the interval $[0,p]$ observe that in t his interval (2,0) has no relative pseudocomplemented. So $(E,\leq)$ is not sectionally pseudocomplemented.

Normal Lattice: A distributive lattice L with 0 is called normal lattice if each prime ideal of L contains a unique minimal prime ideal. Equivalently, L is called normal if each prime filter of L is contained in a unique ultrafilter (maximal and proper of L).

Dense Lattice: A lattice L with 0 is called Dense lattice if $(x]^* = (0]$ for each $x \neq 0$ in L.

The following theorem is generalizasion of [8]

Theorem 4.3.1 If L is a distributive sectionally pseudocomplemented lattice, then L_F is a distributive pseudocomplemented lattice.

Proof: Suppose L is sectionally pseudocomplemented. Since L_F is a distributive lattice. Let $[x] \in L_F$, Then $[0] \subseteq [x] \subseteq F$. Now $0 \leq x \wedge f \leq f$, for all $f \in F$.

Let y be the pseudocomplemented of $x \wedge f$ in $[0, f]$ then $y \wedge x \wedge f = 0$ implies $[y \wedge f] \wedge [x] = [0]$, that is $[y] \wedge [x] = [0]$.

Suppose $[z] \wedge [x] = [0]$, for some $[z] \in L_F$ then $z \wedge x \equiv 0(\psi_F)$. This implies $z \wedge x \wedge f' = 0$...............(i)

For some $f'' \in F$. Since $z \equiv z \wedge f(\psi_F)$, so $z \wedge f'' = z \wedge f = \wedge f''$.........$(ii)$ for some $f'' \in F$. From (i) and (ii) we get $x \wedge x \wedge f' \wedge f'' = 0$ and $x \wedge f' \wedge f'' = z \wedge x \wedge f' \wedge f''$ Setting $g = f' \wedge f''$ we have $z \wedge g \wedge = z \wedge g \wedge f$, which implies $z \wedge g \leq f$ and $z \wedge g \wedge f = 0$ So $0 \leq z \wedge g \leq f$ and $z \wedge g \leq y$.

Hence, $[z \wedge g] \subseteq [y]$ But $[z] = [z \wedge g]$ as $g \in F$

Therefore, $[z] \subseteq [y]$, and so L_F is a pseudocomplemented distributive lattice. □

Theorem 4.3.2 Suppose L be a relatively pseudocomplemented lattice. Let $x \leq y \leq z$ in L and l be the relative pseudocomplement of y in $[x,z]$. Then for any $r \in L$, $l \wedge r$ is the relative pseudocomplement of $y \wedge r$ in $[x \wedge r, z \wedge r]$

Proof: Suppose $t \wedge r$ is the relative pseudocomplement of $y \wedge r$ in $[x \wedge r, z \wedge r]$ Since l is the relative pseudocomplement of y in $[x,z]$, so $l \wedge y = x$ Thus, $(l \wedge r) \wedge (y \wedge r) = x \wedge r$ This implies $l \wedge r \leq t \wedge r$

Again, $x \leq l \vee (t \wedge r) \leq z$ and $y \wedge (l \vee (t \wedge r)) = (y \wedge l) \vee ((y \wedge r) \wedge (t \wedge r)) = x \vee (x \wedge r)$

implies $l \vee (t \wedge r) \leq l$; $[\mathrm{x} \wedge \mathrm{r}, \mathrm{z} \wedge \mathrm{r}] \leq \mathrm{l}$ i.e, $\mathrm{l} = \mathrm{l} \vee (\mathrm{t} \wedge \mathrm{r})$

Hence $t \wedge r \leq l$, and so $t \wedge r \leq l \wedge r$. This implies $t \wedge r = l \wedge r$

Therefore $l \wedge r$ is the relative pseudocomplement of $y \wedge r$ in $[x \wedge r, z \wedge r]$. □

[17] extended the notion of pseudocomplementation for meet semilattices, following theorem generalises.

Theorem 4.3.3 If L is a distributive relatively pseudocomplemented lattice, then L_F is a distributive relatively pseudocomplemented lattice.

Proof: Since L_F is a distributive lattice. Let $[x],[y],[z] \in L_F$ with $[x] \subseteq [y] \subseteq [z]$. Then $[x] = [x \wedge y]$ and $[y] = [y \wedge z]$. Thus, $y \equiv x \wedge y(\psi_F)$ and $y \equiv x \wedge y(\psi_F)$ This implies $x \wedge f = x \wedge y \wedge f$ and $y \wedge g = y \wedge z \wedge g$ for some $f, g \in F$ then

$y \wedge f \wedge g = y \wedge z \wedge f \wedge g$, and $y \wedge f \wedge g = y \wedge z \wedge f \wedge g$, and so

$x \wedge f \wedge g \leq y \wedge f \wedge g \leq z \wedge f \wedge g$, that is $x \wedge h \leq y \wedge h \leq z \wedge g$

where $f \wedge g \in F$

Suppose t is the relative pseudocomplement of $y \wedge h$ in $[x \wedge h, z \wedge h]$. Then $t \wedge y \wedge h = x \wedge h$, and so $[t] \wedge [y \wedge h] = [x \wedge h]$. That is, $[t] \wedge [y] = [x]$ as $y \equiv y \wedge h(\psi_F)$ $y \equiv y \wedge h(\psi_F)$ and $x \equiv x \wedge h(\psi_F)$ Moreover, $[t] \wedge [z] = [t] \wedge [z \wedge h] = [t \wedge z \wedge h] = [t]$ implies $[x] \subseteq [t] \subseteq [z]$

We claim that $[t]$ is the relative pseudocomplement of $[y]$ in $[[x],[z]]$ in L_F

Suppose $[l]\wedge[y]=[x]$ for some $[l]\in[[x],[z]]$. Then $l\wedge y\equiv x(\psi_F)$ and so

$l\wedge y\wedge f'=x\wedge f'$ for some $f'\in F$ Again $[l]\subseteq[z]$ implies $l\equiv l\wedge z(\psi_F)$, and so

$l\wedge g'=l\wedge z\wedge g'$ for some $g'\in F$.Then $l\wedge y\wedge f'\wedge g'=x\wedge f'\wedge g'$ and

$l\wedge f'\wedge g'=l\wedge z\wedge f'\wedge g'$

Thus, $l\wedge k=l\wedge x\wedge k$ and $l\wedge k=l\wedge z\wedge k$ where $k=f'\wedge g'\in F$

This implies $x\wedge h\wedge k\leq l\wedge h\wedge k\leq z\wedge h\wedge k$

and $(l\wedge h\wedge k)\wedge(y\wedge h\wedge k\leq z\wedge h\wedge k$

Then $[l\wedge h\wedge k\leq t\wedge k]$. Hence $[l]=[l\wedge h\wedge k]\subseteq[t\wedge k]=[t]$

And so t is the relative pseudocomplement of $[y]$ in $[[x],[y]]$.

Therefore, L_F is relative pseudocomplemented. □

The following theorem is extension of [25, theorem 4.1]

Theorem 4.3.4 For a distributive sectionally pseudocomplemented lattice L, the following conditions are hold:

i. If L is generalized stone then L_F is stone for any filter F of L.

ii. L is generalized stone if and only if for each prime filter F of L, L_F is dense Lattice

Proof: (i) Let $\psi_F(x),\psi_F(y)\in L_F$ be such that $\psi_F(x)\wedge\psi_F(y)=0$ Then, $x\wedge y\equiv 0(\psi_F)$, which implies that $x\wedge y\wedge f=0$ for some $f\in F$. Since L is generalized stone, then L is normal, so $(x]^*\vee(y\wedge f]^*=L$

Hence $(\Psi_F(x)]^*\vee(\Psi_F(y)]^*=(\Psi_F(x)]^*\vee(\Psi_F(y\wedge f]^*$

$=\psi_F((x)]^*\vee(y\wedge f]^*)=\psi_F(L)=L_F$

Thus, L_F is normal.

Again, since L is sectionally pseudocomplemented, then L_F L_F is pseudocomplemented, Hence L_F is stone.

(iii) Suppose L is generalized stone. Let $\psi_F(x) \neq 0$ and $\psi_F(q) \in (\psi_F(x)]^*$. Then $\psi_F(q) \wedge \psi_F(x) = 0$. Then F is contained in a unique ultra filter of L. Thus L_F has a unique ultra filter; and so L_F has a unique minimal prime ideal.

But the zero ideal of L_F (as $0 \in L$) is the intersection of all the minimal prime ideals of L_F. Hence, by uniqueness, it is (minimal) prime ideal of L_F. Hence $\psi_F(q) = 0$ showing that L_F is dense.

Conversely, let L_F be dense for each prime filter F of L. Suppose $x, y \in L$ are such that $x \wedge y = 0$ Then $\psi_F(x \wedge y) = \psi_F(0) = 0$ That is $\psi_F(x) \wedge \psi_F(y) = 0$ which implies that $\psi_F(x) = 0$ or $\psi_F(y) = 0$ as L_F is dense. Hence, either $(\psi_F(x)]^* = L_F$ or $(\psi_F(y)]^* = L_F$. Thus $\psi_F((x)] \vee (y]^*) = L_F = \psi_F(L)$ and so $(x]^* \vee (y]^* = L)$ Therefore L is normal.

Again, since L is sectionally pseudocomplemented, so L is generalized stone. □

Chapter Five

Directoid equipped with sectionally switching mapping

Introduction: It is shown that every directoid equipped with sectionally switching mappings can be represented as a certain implication algebra. The concept of directoid was introduced by J. Jezek and R. Quackenbush [19] in the sake to axiomatize algebraic structures defined by on upward directed ordered set. In certain sense, directoids generalize semilattices.

5.1 Basic concepts

An ordered set $(B;\leq)$ is upward directed if $U(x,y)\neq\phi$ for every $x,y\in B$, where $U(x,y)=\{a\in B; x\leq a \text{ and } y\leq a\}$. Elements of $U(x,y)$ referred to be common upper bounds of x,y. Of course, if $(B,\leq)$ has a greatest element then it is upward directed.

Let $(B;\leq)$ be an upward directed set and $\vee$ denotes a binary operation on B. The pair $B=(B;\vee)$ is called directoid if the following axioms are hold:

1. $x\vee y\in U(x,y)$ for $x,y\in B$
2. If $x\leq y$ then $x\vee y=y$ and $y\vee x=y$........................(A)

Switching Algebra: The system $\{\{0,1\},\wedge,\vee,'\}$ is a two elements Boolean algebra which are called a switching algebra.

Proposition 5.1.1 A groupoid $B=(B;\leq)$ is a directoid if only if it satisfies the following axioms:

1. $x\vee x=x$..(1)
2. $(x\vee y)\vee x=x\vee y$...(2)
3. $y\vee(x\vee y)=x\vee y$...(3)
4. $x\vee((x\vee y)\vee z)=(x\vee y)\vee z$....(*skew asssociativity*).........(4)

Then the binary relation $\leq$ defined on B by the rule $x\leq y$ if and only if $x\vee y$ is an order and $x\vee y\in U(x,y)$ for each $x,y\in B$.

A directoid $B = (B;\vee)$ is called commutative if it satisfies the axiom

5. $x \vee y = y \vee x$..(5)

5.2 Switching Involutions

Let $(B;\vee)$ be an ordered set with a greatest element 1. For $p \in B$, the interval $[p,1]$ will be called a section. A mapping f of $[p,1]$ into itself will be called a sectional mapping. If f is a sectional mapping on $[p,1]$ and $x \in [p,1]$ then $f(x)$ will be denoted by x^p.

A sectional mapping on $[p,1]$ is called switching mapping if $p^p = 1$ and $1^p = p$ then it is called an involution if $x^{pp} = x$ for each $x \in [p,1]$. Hence any involution is a bijection and if a sectional mapping on $[p,1]$ is a switching involution then $p^p = 1$ iff $x = p$ and $x^p = p$ iff $x = 1$. $(B;\leq,1)$ will be called with sectional switching involutions if there is a sectional switching involution on the section $[p,1]$ for each $p \in B$.

Lemma 5.2.1 Let $B = (B;\circ,1)$ be an algebra of type $(2,0)$ satisfying the following axioms:

1. $x \circ x = 1,\ x \circ 1 = 1$..(6)
2. $x \circ y = 1$ implies $y = (y \circ x) \circ x$...................(7)
3. $x \circ ((((x \circ y) \circ y) \circ z) \circ z) = 1$.........................(8)

Define a binary relation $\leq$ on B by the setting $x \leq y$ if and only if $x \circ y = 1$......(*)

Then $(B;\leq)$ is an ordered set with a greatest element 1 where for each $p \in B$ the mapping $x \mapsto x^p = x \circ p$ is a sectional switching involution on $[p,1]$.

Proof: By (6) and (7) we infer immediately

$1 \circ x = (x \circ x) \circ x = x$.......................................(**)

Due to (6) the relation $\leq$ is reflexive and $x \leq 1$ for each $x \in B$. Suppose $x \leq y$ and $y \leq x$. Then $x \circ y = 1, y \circ 1 = 1$ and, by (7), $y = (y \circ x) \circ x = 1 \circ x = x$ thus $\leq$ is anti-symmetrical. Suppose $x \leq y$ and $y \leq z$. Then $\leq x \circ y = 1, y \circ z = 1$ and by (6) and (7) we have, $x \circ z = x \circ (1 \circ z) = x \circ ((y \circ z) \circ z) = x \circ (((1 \circ y) \circ z) \circ z) = x \circ ((((x \circ y) \circ y) \circ z) \circ z) = 1$ thus $x \leq z$ proving transitivity of $\leq$.

Now, let $p \in B$ and $x \in [p,1]$. Then $p \leq x$ an hence $p \circ x = 1$. Due to (7) we conclude $x^{pp} = (x \circ p) \circ p = x$ thus every sectional mapping $x \mapsto x^p = x \circ p$ is an involution on $[p,1]$. Applying (6) and (**) we infer that it is a switching mapping. □

Lemma 5.2.2 Let $B = (B; \circ, 1)$ satisfy (6), (7), (8) and

1. $y \circ (x \circ y) = 1$
2. $x \circ ((x \circ y) \circ y) = 1$

then $(x \circ y) \circ y \in U(x, y)$ for each $x, y \in B$.

Proof: By Lemma 5.2.1, $\leq$ defined by (*) is an order on B. Replace x by $x \circ y$ in (4) we obtain $y \circ ((x \circ y) \circ y) = 1$ thus $y \leq (x \circ y) \circ y$. By (5) we have $x \leq (x \circ y) \circ y$ thus $(x \circ y) \circ y \in U(x, y)$. □

5.3 d- implication algebras

The concept of implication algebra was introduced by J.C. Abbott [18]. It is a groupiod $B = (B\,;\,0)$ with a distinguished element 1 in which an order $\leq$ can be introduced by $x \leq y$ if and only if $x \circ y = 1$. It was shown [1] that $(B; \leq)$ is semi lattice $x \vee y = (x \circ y) \circ y$ and ,moreover, every section [p,1] is equipped by a sectional antitone involution $x^p = x \circ p$.

Let us note the name implication algebra express the fact that $x \circ y$ is interpreted as a connective implication $x \Rightarrow y$

Theorem 5.3.1 An algebra $B = (B;0;1)$ satisfying (1) – (5) will be called a weak d-implication algebra . We can state

Let $B = (B;0;1)$ be a weak d-implication algebra. Define a binary operation $\vee$ on B

By $x \vee y = (x \circ y) \circ y$

and for each $p \notin B$ define $x^p = x \circ p$. Then $D(B) = (B; \vee)$ is a directoid with the greatest element 1 with sectionally switching involutons whose induced order coincides with that of B.

Proof: Define $x \vee y = (x \circ y) \circ y$ and $x^p = x \circ p$, for $x \in [p,1]$.

(a) Let $x \circ y = 1$.Then $x \vee y = (x \circ y) \circ y = 1 \circ y = y$.

(b) Let $\leq$ be the induced order on B. By $(B4)$ we have $x \circ y \in [y,1]$. Suppose now $x \vee y = y$. Then, since the sectional mapping on $[y,1]$ is an involution, we infer

$x \circ y = (x \circ y)^{yy} = (x \circ y) \circ y) \circ y = (x \vee y)\circ = y \circ y = 1$

we have shown $x \circ y = 1$ if and only if $x \vee y = y$ thus order on B defined by (*) coincides with that of $(B;\vee)$ defined by B. The fact that $(B;\vee)$ is a directoid by Lemma 5.2.2 and the fact that $x \leq y$ gets $x \vee y = y = (x \circ y) \circ y = 1 \circ y = y$ and ,by $(B2)$, also $y \vee x = (y \circ x) \circ x = y$. By Lemma 5.2.1 sectional mappings $x \mapsto x^p$ for $x \in [p,1]$ are switching involutions. □

Theorem 5.3.2 Let $D = (D;\vee,1)$ be a directoid with a greatest element 1, $\leq$ its induced order. Let for each $p \in D$ there exists a sectional switching involution $x \mapsto x^p$ on [p,1]. Define $x \circ y = (x \vee y)^y$

Then $B(D) = (D;0;1)$ is a weak d- implication algebra.

Proof: Since $y \leq x \vee y$ in D, we have $x \vee y = [y,1]$ and hence the definition of the new operation "0" is sound. Moreover, $(x \circ y) \circ y == (x \vee y)^{yy} = x \vee y$.

We have to verify the conditions (1) – (5).

(1): $x \circ x = (x \vee 1)^x = 1^x = 1$ and $x \circ 1 = (x \vee 1)^1 = 1^1 = 1$.

(2): Suppose $x \circ y = 1$.Then $(x \vee y)^y = 1$ thus (since the sectional mapping is a switching bijection) also $x \vee y = y$. Conversely, if $x \vee y = y$ then $x \circ y = 1$, i.e., the order induced on D coincides with that given by (*) in Theorem 5.3.1 . Hence, if $x \circ y = 1$ then $x \leq y$ thus $y \in [x,1]$, i.e., $(y \circ x) \circ x = y^{xx} = y$.

(3): By (4) we have $x \leq (x \vee y) \vee z$ thus $(x \circ ((((x \circ y) \circ z) \circ z) = x \circ ((x \vee y) \vee z) = 1$

(4): Since $x \vee y \in [y,1]$, we have $x \circ y = (x \vee y)^y \in [y,1]$ thus $y \leq x \circ y$ whence $y \circ (x \circ y) = 1$

(5): Since $y \leq x \vee y$ we have $(x \circ y) \circ y = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y$.

Thus $x \leq x \vee y = (x \circ y) \circ y$ proving $x \circ ((x \circ y) \circ y) = 1$. □

Lemma 5.3.3 Let $B=(B;\circ,1)$ be a d-implication algebra. Define a binary relation $\leq$ on B by the setting $x \leq y$ if and only if $x \circ y = 1$. Then $\leq$ is an order on B and 1 is greatest element.

Proof: By (1), $\leq$ is reflexive. Suppose $x \leq y$ and $y \leq x$. Then $x \circ y = 1, y \circ x = 1$ and due to (1), also $x = 1 \circ x = (y \circ x) \circ x = (x \circ y) \circ y = 1 \circ y = y$, i.e., $\leq$ is anti-symmetrical. Transitivity of $\leq$ can be shown identically as in the proof of Lemma 5.2.1. By (1) $x \leq 1$ for each $x \in B$. □

Theorem 5.3.3 Let $B=(b,\circ,1)$ be a d-implication algebra. Define $x \vee y = (x \circ y) \circ y$ and for $x \in [y,1]$ let $x^y = x \circ y$. Then $C(B)=(B;\vee)$ is a commutative directoid with a greatest element 1 and with sectionally switching involutions.

Proof: By Lemma 5.3.3, $(B;\leq)$ is an ordered set where $x \leq y$ if and only if $x \circ y = 1)$ and 1 is a greatest element of $(B;\leq)$. Due to (1) we infer $x \vee y = y \vee x$.

By (1) and (3) we have, $x \circ (x \vee y) = x \circ ((x \circ y) \circ y) = x \circ ((((x \circ y) \circ y) \circ y) \circ y) = 1$ thus $x \leq x \vee y$. Analogously $y \leq x \vee y$ thus $x \vee y \in U(x,y)$. Further, if $x \leq y$ then $x \vee y = (x \circ y) \circ y = 1 \circ y = y$.

We have shown that $(B;\vee)$ is commutative directoid. Analogously as in the previous proofs, the induced order of $(B;\vee)$ coincides with $\leq$. Hence, 1 is a greatest element of $(B;\vee)$.

Now, let $y \in B$ and $x \in [y,1]$. Then $y \leq x$ and hence $x^{yy} = (x \circ y) \circ y = x \vee y = x$. Further, $y^y = y \circ y = 1$ thus for each $y \in B$ the mapping $x \mapsto x^y$ is a sectional switching involution on $[y,1]$. □

Theorem 5.3.4 Let $C=(C;\vee,1)$ be a commutative directoid with a greatest element 1. Let $\leq$ be its induced order and for each $p \in C$ there exists a sectional switching mapping involution $x \mapsto x^p$ on $[y,1]$. Define $x \circ y = (x \vee y)^y$. Then $B(C)=(C;\circ,1)$ is a d-implication algebra.

Proof: It was shown in Theorem 5.3.2 that "$\circ$" is correctly defined operation on C satisfying (1) and (3), and that $(x \circ y) \circ y = x \circ y$. Since $x \vee y = y \vee x$, (1) is evident. It remains to prove (2). Since $y \leq x \vee y$, we derive

$((x \circ y) \circ y) \circ y = (x \vee y) \circ y = (x \vee y)^{y} = x \circ y.$ □

Chapter Six

Boolean lattices with sectional switching mapping

Introduction: Sectional switching mapping were introduced by Chajda, and P. Emanovsky [3] and studied by several authors. Consider a Boolean lattice $L=(L,\vee,\wedge,1)$ with a greatest element 1. An interval $[a,1]$ for $a\in L$ is called a section . In each Section $[a,1]$ an antitone bijection is defined. We characterize these lattices by means of two induced binary operations providing that the resulting algebras from a variety. A mapping f, of $[a,1]$ on to itself is called a switching mapping if $f(a)=1, f(1)=a$ and for $x\in[a,1], a\neq x\neq 1$. We have $a\neq f(x)\neq 1$. If for $p,q\in L, p\leq q$ the mapping on the section $[a,1]$ is determined by that of $[1,1]$, We say that the compatibility condition is satisfied. We shall get conditions for antitone of switching mapping and a connections with complementation in sections will be shown.

6.1 Basic concepts

Let $L=(L,\vee,\wedge,1)$ be a lattice with the greatest element 1. for $a\in L$, the interval $[a,1]$ will be called a section.

A mapping $f: x\mapsto y$ is called an involution if $f(f(u))=x$ for each $x\in X$.

Let $(X,\leq)$ be an ordered set . A mapping $f: x\mapsto y$ is antitone if , $x\leq y$ implies $f(y)\leq f(x)$ for all $x,y\in X$.

A weakly switching mapping : $x\mapsto x^{l}$ will be called a switching mapping if $a\neq x^{a}\neq 1$ for each $x\in[a,1]$ with $a\neq x\neq 1$.

We induced lattices with 1. where for each $a\in L$ there is a mapping on the section $[a,1]$; such a structure will be called lattice with sectional mappings.

We study the following notation: for each $a \in L$ and $x \in [a,1]$ denote by x^a the image of x in this sectional mapping on $[a,1]$. Thus $: x \mapsto x^a$ is a symbol for the corresponding sectional mapping on the section [a,1].

Let $L = (L,\vee,\wedge,1)$ be a Lattice with sectional mapping. Define the so-called induced operation on L by the rule $x \vee y = (x \vee y)^y$ .Since $x \vee y \in [y,1]$ for any $x, y \in L$. Also, conversely, if "v" is induced on L, then for each $a \in L$ and $x \in [a,1]$. We have $x \vee a = (x \vee a)^a = x^a$.

6.2 Switching mapping

A mapping $: x \mapsto x^a$ on the section $[a,1]$ is weakly switching if $a^a = 1, 1^a = a$, in other words, a weakly switching mapping "switches" the bound element of the section.

Lemma 6.2.1 A lattice $L = (L,\vee,\wedge,1)$ with section involutions the following properties are equivalent for $a \in L$

(i) $: x \mapsto x^a$ is antitone,

(ii) The section $[a,1]$ is a lattice where $x \wedge_a y = (x^a \vee y^a)^a$ (De Morgan law).

Proof: (i) $\Rightarrow (ii)$: Since the sectional mapping on [a,1] is an antitone involution, it is a bijection and $x, y \le x \vee y$ implies $x^a, y^a \ge (x \vee y)^a$ and the existence of supremum for $x, y \in [a,1]$ yields existence of the infimum $x \wedge_a y$.

Hence $x^a \wedge_a y^a \ge (x \vee y)^a$.

However, $x^a, y^a \ge x^a \wedge_a y^a$.

thus, due to $x = x^{aa}, y = y^{aa}$,

we obtain $x, y \le (x^a \wedge_a y^a)^a$.

Whence $x \vee y \le (x^a \wedge_a y^a)^a$ *i.e.* $(x \vee y)^a \ge x^a \wedge_a y^a$

All together, we obtain (ii)

(ii) $\Rightarrow (i)$: Let $x, y \in [a,1]$

and suppose $x \le y$.

Then $x \vee y = y$ and, by (ii)

$y^a = (x \vee y)^a = x^a \wedge_a y^a$

Thus $y^a \leq x^a$,

i.e. the sectional mapping on $[a,1]$ is antitone. □

Lemma 6.2.2 A Lattice $L = (L, \vee, \wedge, 1)$ with sectional mappings.

(i) if the sectional mapping $: x \mapsto x^l$ is an involution for each $1 \in L$ then the induced operation satisfies the identity $(x \vee y) \vee y = (y \vee x) \vee x = x \vee y$ ....(A).

(ii) if the sectional mapping $: x \mapsto x^l$ is weakly switching for each $1 \in L$ and the induced operation an involution satisfies (A), then every sectional mapping is an involution.

Proof: (i) Since. $x \vee y \in [y,1]$

We have $x \vee y = (x \vee y)^y \geq y$.

Thus, if the sectional mapping is an involution we conclude,

$(x \vee y) \vee y = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y,$

Whence (i) is evident.

(ii) Let each sectional mapping be weakly switching, let $1 \in L$ and $x \in [1,1]$.

Then $x \vee I = x$ by(i)

and $x^{ll} = (x \vee l) \vee l = (l \vee x) \vee x = ((l \vee x)^x \vee x)^x = (x^x \vee x)^x = (l \vee x)^x = 1^x = x$

and thus $: x \to x^l$ is an involution. □

Lemma 6.2.3 A lattice $L = (L, \vee, \wedge, 1)$ with sectional mappings. Let $\leq$ be its induced order. Then $x \leq y$ if and only if $x \vee y = 1$.

Proof: If $x \leq y$, then $x \vee y = ((x \vee y)^y = y^y = 1$,

Conversely, if $x \vee y = 1$, *then* $(x \vee y)^y = 1$,

Since it is a switching mapping, $x \vee y = y$, *whence*, $x \leq y$. □

Lemma 6.2.4 A lattice $L = (L, \vee, \wedge, 1)$ with sectional weakly switching mappings.

Then L satisfies identities,

$x \vee x = 1, 1 \vee x = x, x \vee 1 = 1$.........................(B)

Proof: Since $x \vee y = ((x \vee y)^y$.

Thus $x \vee y = ((x \vee y))^{y} = y^{y} = 1,$

Again, since in a sectional switching mappings.

$x^{ll} = (x \vee l)^{\vee l}$
$= (l \vee x)^{\vee x}$
$= ((l \vee x)^{x} \vee x)^{x}$
$= (x^{x} \vee x)^{x}$
$= (1 \vee x)^{x} = 1^{x} = x$

further, $x^{ll} = (x \vee l)^{\vee l}$
$= x \vee l = (x \vee l)^{x} = x^{x} = 1$ □

Theorem 6.2.5 A lattice $L = (L, \vee, \wedge, 1)$ with sectional switching mappings.

(i) If L satisfies the identity $(((x \vee y) \vee y) \vee z) \vee (x \vee z) = 1$ ....(c)

then every switching mapping on L is antitone.

(ii) If every sectional switching mappings. on L is an involution then it antitone if and only if L satisfies (c)

Proof: (i) Let $z \in; x, y \in [z,1]$ and $x \leq y$.

By Lemma 6.2.3 We have $y \vee z = 1$, and by

Lemma 6.2.4 and (c) We conclude:

$(y \vee z) \vee (x \vee z) = ((1 \vee y) \vee z) \vee (x \vee z)$
$= (((x \vee y) \vee y) \vee z) \vee (x \vee z) = 1.$

By Lemma 6.2.3 we have $\vee z \leq x \vee z$

and thus $y^{z} = y \vee z \leq x \vee z = x^{z}$

(ii) Let the sectional switching mappings. on L are antitone involutions.

By Lemma 6.2.1 we have $(x \vee y) \vee y = x \vee y$.

Since $x \vee y \vee z \geq x \vee z$ and $x \vee y \vee z, x \vee z \in [z,1]$

We obtain , $((x \vee y) \vee y) \vee z = (x \vee y \vee z)^{z} \leq (x \vee z)^{z} = x \vee z$.

By Lemma 6.2.3 we conclude $(((x \vee y) \vee y) \vee z) \vee (x \vee z) = 1$. □

6.3 The compatibility condition

Consider a lattice with sectional mappings where the mapping in a smaller section is determined by that of a greater one.

We say that $L=(L,\vee,\wedge,1)$ satisfies the compatibility condition if $p \le q \le x$ implies that $x^q = x^p \vee q$ ....(D)

It is easy to verify that (D) can be equivalently expressed as the following identity,

$(y \vee z) \vee (x \vee y) = ((y \vee z) \vee x) \vee (x \vee y)$................................(E)

Since $x \le x \vee y \le x \vee y \vee z$ and $(y \vee z) \vee (x \vee y) = (x \vee y \vee z)^{(x \vee y)}$

$(y \vee z) \vee x = (x \vee y \vee z)^x$

Lemma 6.3.1 A lattice $L=(L,\vee,\wedge,1)$ with sectional switching mappings satisfying the compatibility condition . Then

(i) $x \vee x^l = 1$ for each $l \in L$ and each $x \in [l,1]$

(ii) If $z \mapsto z^l$ is a switching mappings for $l \ne 1$ then $x^l \ne x$ and if $x < y$ then $x^l \ne y^l$ for $x, y \in [l,1]$

(iii) If all the sectional mappings are switching, then no section of L can be a chain with more then two elements.

Proof: (i) Since $l \le x$, we conclude directly by (E) $1 = x^x = x^l \vee x$

(ii) If $z \mapsto z^l$ is a switching mapping on [1,1] and $x, y \in [l,1]$, then if $x^l = x$, by (i) ,

We obtain $1 = x^l \vee x = x$ and , hence,

$1 = x^l = 1^l = l$, a contradiction.

If $x < y$ and $x^l = y^l$, then by (E) and (i) , $y^x = y^l \vee x = x^l \vee x = 1$

Since the sectional mapping is switching, it yields $y = x$, a contradiction.

(iii) Suppose that [1,1] is a chain with more then two elements.

Then there exists, $x, \in [l,1], l \ne x \ne 1$

We have, $x^l \ne l, x^l \ne 1$ and by (i) ,

$1 = x^l \vee x = \max(x, x^l)$, a contradiction. □

Theorem 6.3.2 A Lattice $L = (L,\vee,\wedge,1)$ with sectional switching mapping satisfying the compatibility condition. If $x \mapsto x^l$ is antitone on [1,1], then x^1 is a complement of x for each $x,\in [l,1]$.

Proof: Considers the sectional switching mapping on $[l,1]$ is antitone.

By Lemma 6.3.5 we have $x \vee x^l = 1$ and $x^l \vee x^{ll} = 1$ for each $x,\in [l,1]$.

Take $z = x \wedge_l x^l$. Then $z \le x,\ z \le x^l$ and, due to the antitone property of mapping, also $z^l \ge x^l,\ z^l \ge x^{ll}$. Thus, $z^l \ge x^l \vee x^{ll} = 1$

Therefore, it follows that, $z^l = 1$, *i.e.*,$z = l$ and x^l is complement of x in the lattice $([l,1]\vee,\wedge,1)$. □

Theorem 6.3.3 A lattice $L = (L,\vee,\wedge,1)$ with sectionally antitone involutions satisfying the compatibility condition. Then for each $l \in L$ the section [1,1] is an orthomodular lattice where x^l is an orthocomplement of $x,\in [l,1]$.

Proof: Since sectionally antitone involutions are switching mappings, thus by Lemma 6.1.1 and Theorem 6.3.7, $[l,1]$ is a lattice and x^l is a complement of $x,\in [l,1]$.

Since this sectional mapping is an involution, we have $x^{ll} = x$ and due to antitony, $x \le y$ implies $y^l \le x^l$ for $x, y \in [l,1]$, thus x^l is an orthocomplement of x in $[l, 1]$, using the compatibility condition $l \le x \le y$ implies $y^x = y^l \vee x$

and hence $y \wedge_l (x \vee y^l) = y \wedge_l y^x = y \wedge_x y^x = x$

which is the orthomodular condition in the lattice $([l, 1],\vee,\wedge_l)$ □

Theorem 6. 3.4 A Lattice $L = (L,\vee,\wedge,1)$ with sectionally antitone involutions. If for $l \in L$ and each $x, y \in [l, 1]$, the relation

$(x^l \vee y) \vee x^l = (y^l \vee x)^l \vee y^l$ ....(F)

holds, then $([l,1],\vee,\wedge_l)$ is a Boolean algebra.

Proof: Due to Lemma 6.1.1 $([l,1],\vee,\wedge_l)$ is a lattice and we can use De Morgan's law for each section. Let $a \in [l,1]$.

Using of the identity (F), we obtain

$$a \vee a^l = a^{ll} \vee a^l = (a^l \vee l)^l \vee a^l$$

$$= (l^l \vee a)^l \vee l^l = (1 \vee a)^l \vee 1 = 1$$

Due to the De Morgan's law, we have,

$$a \vee_l a^l = a^{ll} \vee_l a^l = (a^l \vee a)^l = 1^l = l.$$

Hence, a^l is a complement of a in [1, 1].

Let $u \in [l, 1]$ is a complement of a in [1, 1],

i.e. $a \vee u = 1$ and $a \wedge_l u = l$.

Using the identity (F) and the De Morgan's law again, we derive,

$$a = l \vee a = (a \vee u)^l \vee a$$

$$= (a^{ll} \vee u)^l \vee a^{ll}$$

$$= (u^l \vee a^l)^l \vee u^l$$

$$= (u \wedge_l a) \vee u^l$$

$$= l \vee u^l = u^l$$

Thus, $a^l = u^{ll} = u$, and the complement is unique.

Since the involution is an antitone unique complementation, then, according to $([l,1],\vee,\wedge_l)$ is distributive. □

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