A STUDY ON COMPLEMENTED LATTICES AND BOOLEAN FUNCTION.

By

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To my parents,
who have profoundly
influenced my life

Declaration

We hereby declare that the thesis entitled "A Study on Complemented Lattices and Boolean Function," submitted for the partial fulfillment for the Master of Philosophy degree is done by the student (Roll No.M.Phil 0651503. Session-January 2006 himself and is not submitted elsewhere for any other degree or diploma.

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Abstract

In this thesis the nature of Complemented lattice and Boolean function is studied. Lattice theory is a part of Mathematics. In Modern algebra, Abstract algebra and Boolean function are Lattice theory play important role. A non empty set P together with a binary relation R is said to form a partially ordered set or a *poset* if the following conditions hold:

i) Reflexivity

ii) Anti-symmetry

iii) Transitivity

A poset (L, $\leq$) is said to form a lattice if for every a, b $\in$ L if Sup {a,b} and Inf {a,b}exist in L. A lattice is said to be complemented lattice if every element has complement. In this thesis we give several results on complemented lattice, Boolean function and Boolean algebra which will certainly extend and generalize many results in lattice theory. The thesis contains four chapters.

Chapter one: We have discussed the basic definition of set, partially order set, relations, functions etc.

Chapter two: We have discussed lattice, sublattice, convex sublattice, complemented lattice, ideal, Filter, Prime ideal, Principle ideal and Principle Dual ideal. We have proved that two bounded lattices A and B are complemented if and only if $A \times B$ is complemented. In this chapter we have also discussed the definition of upper bound, least upper bound, lower bound, greatest lower bound and relatively complemented lattice, and established relation among them. We also studied some other properties of these concepts and we have showed that two lattices A and B are .relatively complemented if and only if the cross product of two lattices A and B is relatively complemented.

Chapter three: We have discussed Boolean algebra, Boolean lattice and Boolean function.

Let $(A, \wedge, \vee, ', 0, 1)$ be a Boolean algebra. Expressions involving members of A and the operations $\wedge, \vee$ and complementation are called Boolean expressions or Boolean polynomials. For example, $x \vee y', x, x \wedge 0$ are etc. all Boolean expressions. Any function

specifying these Boolean expressions is called a Boolean function. Thus if $f(x,y) = x \wedge y$ then f is the Boolean function and $x \wedge y$ is the Boolean expressions (or value of the function f). Since it is normally the functional value (and not the function) that we are interested in, we call these expressions the Boolean function.

We will denote least and greatest elements of a Boolean algebra by 0 and 1 respectively. In fact, most of the times we will confine ourselves to Boolean algebras that contain only these two elements. We also discus in this chapter Disjunctive Normal form (DN form), Conjunctive Normal form (CN form), Length and Cover. A Boolean function is said to be in DN form in n variables $x_1, x_2 \ldots\ldots\ldots\ldots, x_n$ if it can be written as join of terms of the type $f_1(x_1) \wedge f_2(x_2) \wedge \ldots\ldots\ldots \wedge f_n(x_n)$ where $f_i(x_i) = x_i$ or x_i' for all i =1,2,3,------,n and no two terms are same. Also 1 and 0 are said to be in DN form. We also prove the theorem: 'Every Boolean function can be put in DN form'. Here we give several results on DN form, CN form, Homomorphisom, Iso-morphisom and Indomorphisom.

Chapter four : In this chapter we have studied series combination, parallel combination, don't care condition and Bridge circuits. By a switch we mean a contact or a device in an electric circuit which lets the current to flow through the circuit. The switch can assume two states 'closed ' or 'open ' (ON or OFF). In the first case the current flows and in the second the current does not flow. We will use a,b,c, ...x,y,z........etc. to denote switches in a circuit. Two switches a,b are said to be connected 'in series' if the current can pass only when both are in closed state and current does not flow if any one or both are open. Two switches a, b, are said to be connected 'in parallel' if current flows when any one or both are closed and current does not pass when both are open. In this chapter we also solve some circuit problems.

Contents

CHAPTER 1

Set, Relation and Function.

Introduction : In this chapter we have discussed the basic definition of sets. We also discussed the definition of Relations and Functions.

1.1: The basic definition of sets:

Definition (Set): Any collection well defined of objects or list is called a set and its objects are called elements or members [1].

Note: The terms well defined means whether any given object belongs to or does not belong to the set.

Notation of sets : Sets are usually denoted by capital letters A, B, C etc. and the elements in sets are usually denoted by small letters a, b, c etc.

Example 1.1.1: Let A = {a, e, i, o, u}

Here the set is read "A is the set whose elements are a, e, i, o, u". This type denoting is termed as Roaster method.

Example 1.1.2 : Let A = { x : x is odd}

Here the set is read "A is the set of numbers x such that x is odd". This type denoting is termed as Set Builder method.

Definition (Subset) : A set 'A' is said to be a subset of a set B if each element of A is also an element of B and is written $A \subset B$ (to be read *A is a subset of B* or *A is contained in B*)

Example 1.1.3 : Let A = { 1,2,3,5,6} , B= {1,2,3}, C={1,5,6} and D= {1,6,7}

Here $B \subset A$, $C \subset A$ but $D \not\subset A$

Definition (Null set or void set or empty set) : A set that contains no elements is called the empty set or void set or null set and it is denoted by the symbol ϕ.

Example 1.1.4 : Let A ={x: x^2=5, x is even}

Here A = ϕ = {}.

Note: The empty set ϕ is a subset of every set, $\phi \subset A$, $\phi \subset B$, $\phi \subset C$, $\phi \subset D$.

Definition (Union of two sets) : Let A and B be two sets. Then the union of A and B is the set of all elements which belong to A or to B or to both A and B and it is denoted by $A \cup B$ which is read " A union B" or as A cup B.

Thus $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Example 1.1.5: Let A={1,2,3}, B={1,3,5,7} Then $A \cup B$={1,2,3,5,7}.

Definition (Intersection of two sets) : Let A and B be two sets. Then the intersection of A and B is the set of all elements which belong to both A and B and it is denoted by $A \cap B$ which is read " A intersection B" or A cap B"

Thus $A \cap B = \{x: x \in A \text{ and } x \in B\}$

Example 1.1.6 : Let A = {1,2,5}, B= {1,7} and C = {2,3,5}

Thus $A \cap B$ ={1}, $A \cap C$ ={2,5} and $B \cap C = \phi$.

Definition (Difference of sets): Let A and B be two sets. Then the difference of A and B is the set of all elements which belong to A but which do not belong to B and it is denoted by A-B. Here A-B is read "A minus B" or "A difference B".

Example 1.1.7 : Let A={1,6,7,8} and B={1,2,6,9,10} then A-B={7,8} and B-A={2,9,10}.

Definition (Universal set): For many purposes all sets under investigation are considered subsets of a fixed set and we call this fixed set the universal set or universe of discourse. For convenient sometimes we will denote this set by U.

Example 1.1.8 : In space geometry, the universal set consists of all the points in space.

Definition (Complement) : Let A be a set. Then the complement of A is the set of elements which do not belong to A, i.e. U-A where U is the universal set and it is denoted by A' or A^c. Thus $A' = \{x : x \in U, x \notin A\} = U - A$

Simply we can write $A' = \{x: x \notin A\}$

Example 1.1.9 : Let U={1,2,3,4,5,6,7}, A={2,4,6}, Then A'=U-A={1,3,5,7}

Definition (Equal set) : Two sets A and B are said to be equal if and only if they have the same elements and is written A=B

Example1.1.10: Let A={1,2,3} , B={1,2,1,3}, C={1,1,2,2,3,3} and D={1,2,3,1,2,3}
Then A=B=C=D.

Note: A set is not changed by repeating its elements or if its elements are rearranged.

Problem 1.1.11: Which of these set are equal : {r,s,t}, {r,t,s}, {s,t,r,s}, {t,s,t,r}, {s,r,s,t}.

Ans : These sets are equal as reordering and repetition does not change a set.

Definition (Ordered pair) : An ordered pair consists of any two elements a and b is denoted by (a,b) where a is designated as the first element and b is designated as the second element.

Two ordered pairs (p,q) and (r,s) are equal if and only if p=r and q=s .

Example 1.1.12: Here (1,2),(3,11),(1,1),(2,2) etc. are all ordered pairs.

Definition (Product sets): The product set of two sets A and B, written A x B, consists of all ordered pairs (a,b) where $a \in A$ and $b \in B$.

Thus A x B=$\{(a,b): a \in A, b \in B.\}$

A x B is also called the Cartesian product of A and B.

Example 1.1.13: Let A=$\{a,b\}$ and B=$\{1,2\}$, then

A x B=$\{(a,1),(a,2),(b,1),(b,2)\}$ and

B x A=$\{(1,a),(1,b),(2,a),(2,b)\}$.

Definition (Proper subset) : If A is a subset of B and $A \neq B$, then A is called a proper subset of B.

1.2: Relations:

Definition (Relation) : Let A and B be sets. A binary relation or, simply, a relation from A to B is a subset of $A \times B$.

Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair $a \in A$ and $b \in B$, exactly one of the following is true;

i) $(a,b) \in R$; we then say " a is R-related to b", written aRb.

ii) $(a,b) \notin R$; we then say " a is not R -related to b", written $a \not R b$.

The domain of a relation R from A to B is the set of all first elements of the ordered pairs which belong to R, and so it is a subset of A; and the range of R is the set of all second elements, and so it is a subset of B.

Sometimes R is a relation from a set A to itself, that is R is a subset of $A^2 = A \times A$

In such a case, we say that R is a relation on A.

Example 1.2.1 : Let A={1,2,3},and B={x,y,z}, and let R= {(1,y), (1,z), (3,y)}. Then R is a relation from A to B. Science R is a subset of A $\times$B. with respect to this relation 1Ry, 1Rz, 3Ry,but 1 $\not R$ x, 2 $\not R$ x, 2 $\not R$ y, 2 $\not R$ z, 3 $\not R$ x, 3 $\not R$ z.

The domain of R is {1,3}, and the range is {y,z}.

Definition (Reflexive Relation): A relation R in a set A is called a reflexive relation if , $\forall$ a $\in$ A, (a,a) $\in$ R.

Example 1.2.2: Let A={1,2,3} then R={(1,1),(1,2),(2,2),(3,3)} is a reflexive relation.

Definition (Symmetric Relation): Let R be a subset of A $\times$ A i.e., let R be a relation of in A. Then R is called a symmetric relation if (a,b) $\in$ R. implies (b,a) $\in$ R

Example1.2.3: Let S={1,2,3}and let R={(1,2),(1,3),(2,3),(2,1), (3,1),(3,2)}. Then R is a symmetric relation.

Definition (Anti-Symmetric Relation): A relation R in a set A i.e. a subset of A $\times$ A is called an anti- symmetric relation if (a,b) $\in$ R and (b,a) $\in$ R implies a=b. In other words, if a $\neq$ b then possibly a is related to b or possibly b is related to a but never both.

Example 1.2.4: Let A be a family of sets, and R be the relation in A defined by "x is a subset of y". Then R is anti- symmetric since $C \subseteq D$ and $D \subseteq C$ implies $C=D$.

Definition (Transitive Relation): A relation R in a set A is called a transitive relation if (a,b) $\in$ R and (b,c) $\in$ R implies (a,c) $\in$ R . In other words, if a is related to b and b is related to c, then a is related to c.

Example 1.2.5: Let B={a,b,c} and let R={(a,b),(c,b),(b,a),(a,c)}. Then R is not a transitive relation since (c,b) $\in$ R and (b,a) $\in$ R but (c,a) $\notin$ R.

Definition (Equivalence Relation): A relation R in a set A is an equivalence relation if

1) R is reflexive, that is, for every $a \in A$, $(a,a) \in R$.
2) R is symmetric, that is, $(a,b) \in R$ implies $(b,a) \in R$.
3) R is transitive, that is, $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$.

Example 1.2.6: Let X= {a,b,c} be a set and let

R={(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b)(c,c)} be a relation of $A \times A$ then the relation R is an equivalence relation, since

i) R is reflexive, as (a,a),(b,b),(c,c) $\in$ R
ii) R is symmetric, as (a,b),(a,c),(b,c) $\in$ R also (b,a), (c,a), (c,b) $\in$ R and
iii) R is transitive, as (a,c),(c,b) $\in$ R also (a,b) $\in$ R.

1.3: Functions:

Definition (Function): Suppose that to each element of a set A we assign a unique element of a set B; the collection of such assignments is called a function from A in to B. The set A is called the domain of the function, and the set B is called the target set. Functions are ordinarily denoted by symbols. For example, let f denote a function from A in to B. Then we write $f: A \rightarrow B$.

Which is read ; " f is a function from A into B." or "f takes A into B." or "f maps A into B.

Example 1.3.1 : consider the function $f(x) = x^3$ i.e. f assigns to each real number its cube. Then the image of 2 is 8 and so we may write f(2)=8. Similarly, f(-3)= -27, and f(0) = 0.

Example 1.3.2 : Figure 1.1 defines a function f from A ={a,b,c,d}, into B= {r,s,t,u} in the obvious way. That is f(a)=s, f(b)=u, f(c)=r, f(d)=s. The image of f is the set {r,s,u}. Note that t does not belong to the image of f because t is not the image of any element of A under f.

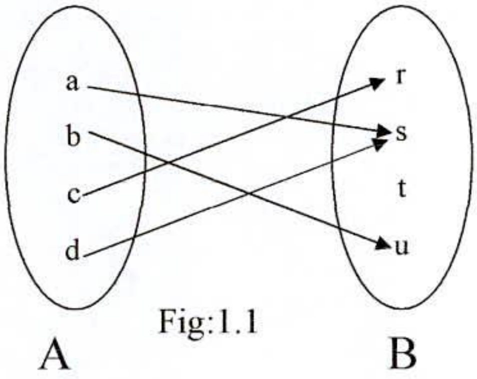


Fig:1.1

Definition (Identity Function): Consider any set A. Then there is a function from A into A which sends each element into itself. It is called the identity function on A and it is usually denoted by I_A or simply I . In other words, the identity function $I_A : A \to A$ is defined by $I_A(a) = a$ for every element $a \in A$

Definition (Composition of functions): Consider, the functions f:$A \to B$ and g:$B \to C$ that is, where the target set B of f is the domain of g. This relationship can be pictured by the following diagram:

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Let $a \in A$; then its image f(a) under f is in B which is the domain of g. Accordingly, we can find the image of f(a) under the function g, that is, we can find $g(f(a))$. Thus we have a rule which assigns to each element a in A an element $g(f(a))$ in C or, in other words, f and g give rise to a well defined function from A to C. This new function is called the composition of f and g, and it is denoted by gof.

More briefly, if f:$A \to B$ and g:$B \to C$, then we define a new function gof:$A \to C$ by $(gof)(a) \equiv g(f(a))$.

Here $\equiv$ is used to mean equal by definition. Note that we can now add the function *gof* to the above diagram of f and g as follows;

$$A \xrightarrow{f} B \xrightarrow{g} C$$
$$\searrow gof \nearrow$$

We emphasize that the composition of f and g is written gof, and not fog; that is, the composition of functions is read from right to left, and not from left to right.

Definition (One-to-one function) : A function $f : A \rightarrow B$ is said to be One-to-one if different elements in the domain A have distinct images. Another way of saying the same thing follows:

f is one-to-one if $f(a)=f(a')$ implies $a = a'$

Definition (Onto function) : A function $f : A \rightarrow B$ is said to be an onto function if every element of B is the image of some element in A or in other words, if the image of f is the entire target set B. In such a case we say that f is a function of A onto B or that f maps A onto B. That is f maps A onto B if $\forall b \in B, \exists a \in A$ such that $f(a)=b$.

Definition (Inverse Function) : A function $f : A \rightarrow B$ is said to be invertible if its inverse relation f^{-1} is a function from B to A. Equivalently $f : A \rightarrow B$ is invertible if there exists a function

$f^{-1} : B \rightarrow A$, called the inverse of f, such that $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$.

Example 1.3.3 : Consider functions $f_1:A \rightarrow B$, $f_2:B \rightarrow C$, $f_3:C \rightarrow D$ and $f_4:D \rightarrow E$ defined by fig-1.2. Now f_1 is one to one since no element of B is the image of more than one element of A. Similarly, f_2 is one-to-one. However, neither f_3 nor f_4 is one-to-one since $f_3(r)=f_3(u)$ and $f_4(v) = f_4(w)$.

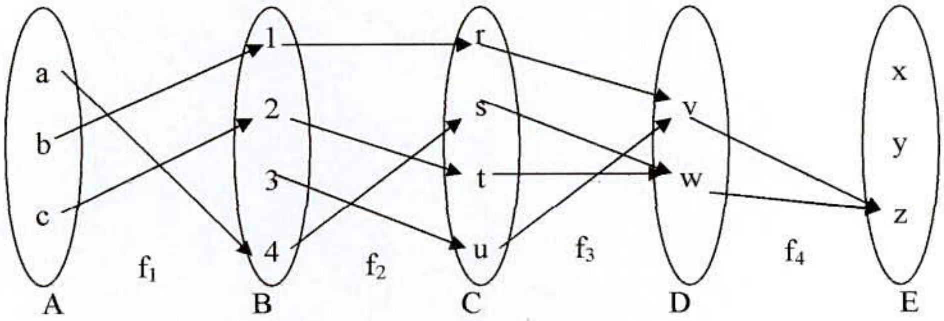


Fig : 1.2

As far as being on to is concerned, f_2 and f_3 are both onto functions since every element of C is the image under f_2 of some element of B and every element of D is the image under f_3 of some element of C. i.e.$f_2(B)=C$ and $f_3(C) = D$. On the other hand f_1 is not onto since $3 \in B$ but 3 is not the image under f_1 of any element of A, and f_4 is not onto since, for example, $x \in E$ but x is not the image under f_4 of any element of D.

Thus f_1 is one-to-one but not onto, f_3 is on to but not one to one, and f_4 is neither one-to-one nor onto. However, $f_2$is both one to one and onto i.e.- f_2 is a one to one correspondence between A and B. Hence f_2 is invertible and f_2^{-1} is a function from C to B.

Example 1.3.4 : Let $f{:}A \rightarrow B$ and $g{:}B \rightarrow C$ be the function defined by fig-1.3 .

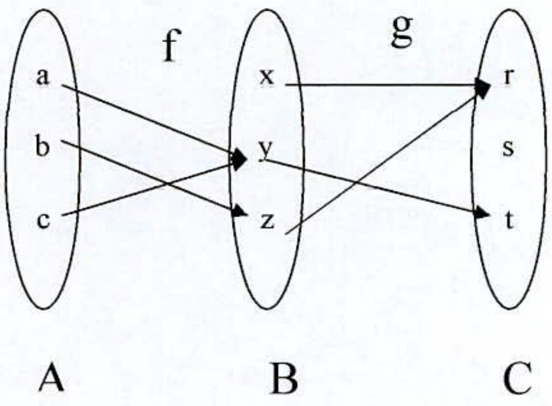


Fig : 1.3

We compute $gof{:}A \rightarrow C$ by its definition: $(gof)(a) \equiv g(f(a))=g(y)=t$, $(gof)(b) \equiv g(f(b))=g(z)=r$, $(gof)(c) \equiv g(f(c))=g(y)=t$ observe that the composition gof is equivalent to "following the arrows" from A to C in the diagrams of the functions f and g.

Example 1.3.5: Let $f{:}R \rightarrow R$ and $g{:}R \rightarrow R$ be defined by $f(x)=x^2$ and $g(x)=x+3$. Then $(gof)(2) \equiv g(f(2))=g(4)=7$: $(fog)(2) \equiv f(g(2))=f(5)=25$

Thus the composition functions gof and fog are not the same function. We compute a general formula for these functions:

$(gof)(x) \equiv g(f(x))=g(x^2)=x^2+3$

$(fog)(x) \equiv f(g(x))=f(x+3)=(x+3)^2=x^2+6x+9.$

CHAPTER 2

Lattice, Sublattice and Complemented Lattice

Introduction:

In this chapter we discuss Ideals, Complete lattices and Relatively Complemented lattices along with the Lattice, Sublattice and Complimented Lattice. Complete lattices and semi lattices have been studied by several authors e.g. Papert [12], Rozen [14], Varlet [15]. A lattice L is called complete lattice if for every non empty subset of L has its Sup and Inf in L. In this chapter we also proved "Two bounded lattices A and B are complemented iff $A \times B$ is complemented".

2.1: Poset, Greatest element, Least element, Bounded Poset, Upper bound, Least upper bound, Lower bound, Greatest lower bound, Chain.

Definition(Poset): A non empty set P, together with a binary relation R is said to form a partially ordered set or a poset if the following conditions hold:

1. Reflexivity: aRa, for all $a \in P$

2. Anti-symmetry : If aRb, bRa then a=b $(a, b \in P)$

3. Transitivity : If aRb, bRc then aRc

Example 2.1.1: The set N of natural numbers under divisibility forms a poset. Thus here, $a \le b$ means $a | b$ (a divides b).

Definition (Greatest element): Let P be a poset. If $\exists$ an element $a \in P$ then a is called greatest or unity element of P. Greatest element if it exists, will be unique.

Definition (Least element): An element $b \in P$ will be called least or zero element of P if $b \le x \ \forall \ x \in P$. It is denoted by 0. Least element if it exists, will be unique.

Example 2.1.2 : Let A={1,2,3}. Then $(P(A), \subseteq)$ is a poset.

Let B={ φ, {1,2}, {2}, {3}}

Then $(B, \subseteq)$ is a poset with φ as least element. B has no greatest element.

Let C={{1,2}, {2}, {3}, {1,2,3}}

Then C has greatest element {1,2,3} but no least element.

If D={ φ, {1}, {2}, {1,2} }.Then D has both least and greatest elements namely φ and {1, 2}.

Again E={{1},{2},{1,3}} has neither least nor greatest element.

Definition (Bounded Poset): If a poset has least and greatest element we called it

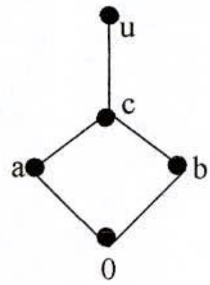


Fig: 2.1

bounded poset. Ex. P={x: $0 \le x \le u$ }

Definition (Upper bound): Let S be a non- empty subset of a poset P. An element $a \in P$ is called an upper bound of S if $x \le a \ \forall \ x \in S$.

Definition (Least upper bound) : If a is an upper bound of S such that $a \le b$ for all upper bounds b of S then a is called least upper bound (l.u.b.) or supremum of S. We write Sup S for Supremum S.

It is clear that there can be more than one upper bound of a set. But Sup, if it exists, will be unique.

Definition (Lower bound): An element $a \in P$ is called a lower bound of S if $a \le x$. $\forall x \in S$.

Definition (Greatest Lower bound): If a is a lower bound of S then a will called greatest lower bound (g.l.b) or infimum of S (inf S) if $b \le a$ for all lower bounds b of S.

Example 2.1.3 : Let $(z,\leq)$be the poset of integers . Let S= {-- -2, -1, 0,1,2}then 2=Sup S

Again in the poset (R, $\leq$) of real numbers if S=$\{x \in R : x < 0\}$ then

Sup S= 0 (and it does not belong to S).

Definition (Chain): If P is a poset in which every two members are comparable then P is called a totally ordered set or a toset or a chain.

The greatest element is comparable with all elements of the poset.

2.2 Lattice, Sublattice, Convex Sublattice.

Definition (Lattice): A poset (L, $\leq$) is said to form a lattice if for every a,b$\in$L, Sup{a,b} and Inf {a,b}exist in L

In that case, we write

Sup{a,b}=$a \vee b$ (read a join b)

Inf{a,b}=$a \wedge b$ (read a meet b)

Other notations like a+b and a.b or $a \cup b$ and $a \cap b$ are also used for Sup{a,b} and Inf{a,b}([3],[7],[10],[13]).

Example 2.2.1: Let X be a non empty set, then the poset (P(X),$\subseteq$) of all subset of X is a lattice. Here for A,B$\in$P(X)

we will have $\mathbf{A} \wedge B = A \cap B$ and $\mathbf{A} \vee B = A \cup B$

As a particular case, when X={1,2,3}.

P(X)=$\{\varphi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.It is represented by the following figure (Fig-2.2).

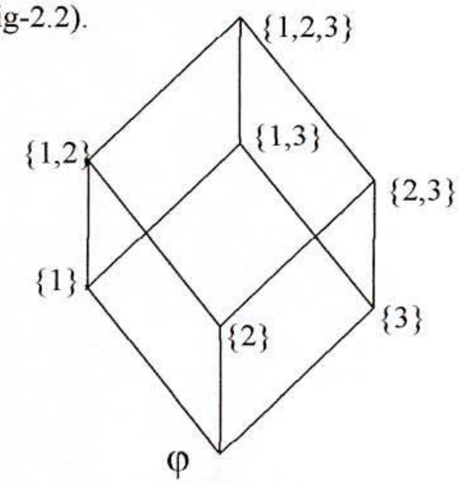


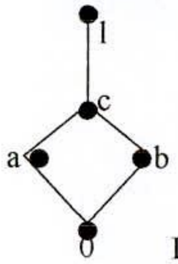
Fig: 2.2

Definition (Sublattice): A non empty subset S of a lattice L is called a sublattice of L if $a,b \in S$, Sup{a,b}and Inf {a,b}exist in S.

(It is understood that $\wedge$ and $\vee$ are taken in L)

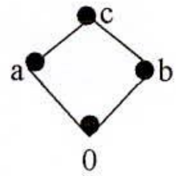
Example2.2.2: Let L={0, a, b, c,1}, S= {0, a, b, c} S_1={0, a, b, 1}

S is sublattice. (Fig-2.4)



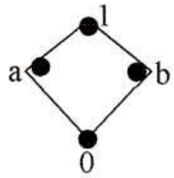
L

Fig: 2.3



S

Fig: 2.4



S_1

Fig: 2.5

S_1(in Fig: 2.5) is not sublattice.

Definition (Convex Sublattice): A sublattice S of a lattice L is called a convex sublattice if for all a, b $\in$ S

$[a \wedge b, a \vee b] \subseteq S$

Example 2.2.3: In the lattice {1,2,3,4,6,12} under divisibility {1,6} is a sublattice which is not convex as $2,3 \in [1,6]$ but $2,3 \notin \{1,6\}$.

Thus $[1,6] \not\subset \{1,6\}$.

2.3: Ideal, Filter or Dual ideal, Prime ideal, Principle ideal, Principle dual ideal.

Definition (Ideal): A non empty subset I of a lattice L is called an ideal of L if(i) $a, b \in I \Rightarrow a \vee b \in I$

(ii) $a \in I$ and $x \in L \Rightarrow x \wedge a \in I$ ([11]).

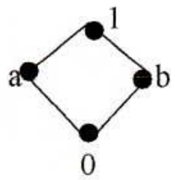


Fig: 2.6

Here in Fig:2.6, Let I={0,a}, $0 \vee a = a \in I$, Again $0 \in I$ and $b \in L$ $\Rightarrow 0 \wedge b = 0 \in I$

So, I={0,a} is an ideal.

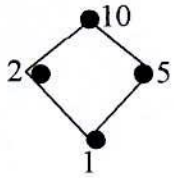


Fig: 2.7

In fig: 2.7: Let I= {2,10}

$2 \vee 10 = 10 \in I$

But $2 \in I$ and $5 \in L$

$2 \wedge 5 = 1 \notin I$

Here I={2,10} is not an ideal.

But I_1={1,2}and I_2={1,5} are ideal.

Definition (Filter or Dual ideal): A subset D of a lattice L is said to be filter or dual ideal.

If (i) $a, b \in D \Rightarrow a \wedge b \in D$

(ii) $a \in D$ and $x \in L$ then $a \vee x \in D$

In the following figure (Fig: 2.8), {a,1} and {b, 1} are filter or dual ideal.

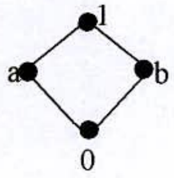


Fig: 2.8

Definition (Prime ideal): An ideal P is said to be Prime ideal. If $P \subset L$ and

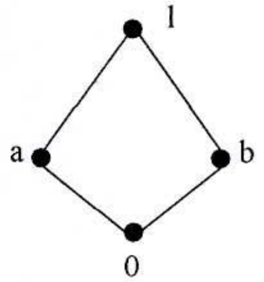


Fig: 2.9

i) $a \wedge b \in P$

ii) *either* $a \in P$ or $b \in P$ ([9]).

In Fig:2.9, we get I={0} is not prime ideal.

$a \wedge b = 0 \in I$

I={0,a} is a prime ideal.

{0, a} and {0,b} are prime ideal.

I={a,1} is not prime ideal.

Definition:(Principal ideal): Let L be a lattice and $a \in L$ be any element.

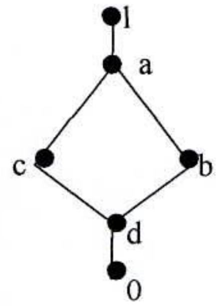


Fig: 2.10

Let $(a] = \{x \in L | x \le a\}$ then $(a]$ forms an ideal of L. It is called principal ideal generated by a ([8]).

In Fig:2.10: $(a] = \{0, a, b, c, d\}$ and $(d] = \{0, d\}$ be two principal ideal.

Definition(Principal Dual ideal): Let L be a lattice and $a \in L$ be any element. The set $[a) = \{x \in L, x \geq a\}$ forms a dual ideal of L, called the principle dual ideal generated by a.

Problem 2.3.1: Show that the union of two sublattices may not be a sublattice.

Proof.: Consider the lattice L ={1,2,3,4,6,12} of factor S of 12 under divisibility.

Let S= {1,2}, S_1={1,3} are sublattices of L. But $S \cup S_1$ ={1,2,3} is not sublattice.

Because {2,3} $\in S \cup S_1$

But $2 \vee 3 = 6 \notin S \cup S_1$

Hence the union of two sublattices may not be a sublattice.

Problem 2.3.2 : If A is an ideal and A' is a dual ideal of a lattice L such that $A \cap A' \neq \phi$, then show that $A \cap A'$ is a convex sublattice of L.

Proof.: Since A and A' are sublattices

$\therefore A \cap A'$ is a sublattice.

Let $a, b \in A \cap A'$ where $a \leq b$

$\Rightarrow a, b \in A$ and $a, b \in A'$

$\Rightarrow [a, b] \subseteq A$ and $[a, b] \subseteq A'$

As A and A' will be convex sublattices.

Thus $[a, b] \subseteq A \cap A'$, so $A \cap A'$ is a convex sublattice of L.

Problem 2.3.3 : Prove that intersection of two ideal is an ideal.

Proof: Let I_1={0,a} and I_2= {0,b} be two ideal.

$I_1 \cap I_2 = \{0\} = I_3$

Here $1 \in L, 0 \in I_3$

$1 \wedge 0 = 0 \in I_3$

Hence intersection of two ideal is an ideal.

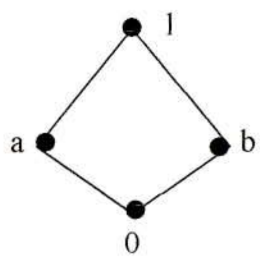


Fig: 2.11

2.4: Complement and Relatively Complemented Lattice.

Definition (complements): Let [a,b] be an interval in a lattice L.

Let $x \in [a,b]$ be any element. If $\exists y \in L$ s.t.,

$x \wedge y = a, \quad x \vee y = b.$

then y is a complement of x relative to [a,b], or y is a relative complement of x in [a,b].

Observations:

i) If such a y exists then y lies in [a,b] as $a = x \wedge y \le y \le x \vee y = b$.

ii) If y is relative complement of x, x will be relative complement of y.

iii) An element x may or may not have a relative complement. A relative complement may or may not be unique.

Let us consider the pentagonal lattice as in the Fig- 2.12

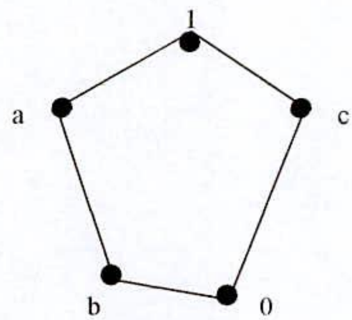


Fig-2.12

b has no complement relative to [0,a] where as a and b are both complement of c relative to [0,1] and b has only one complement c relative to [0,1]

iv) If a, b are unique complements of each other relative to [a,b] then $a \wedge b = a$, $a \vee b = b$. Thus a, b are each others complement.

Let x be any other complement of a relative to [a, b]

Then $a \wedge b = a = a \wedge x$

$a \vee b = b = a \vee x$

Now $b = a \vee x = (a \wedge x) \vee x = x$

Definition (Complemented interval): If every element x of an interval [a, b] has at least one complement relative to [a, b], the interval [a, b] is said to be complemented.

Definition (Relatively Complemented Lattice): If every interval in a lattice is complemented the lattice is said to be relatively complemented. Suppose now L is a bounded lattice. If for any $x \in L$, $\exists y \in L$ s.t., $x \wedge y=0$, $x \vee y=1$, y is called complement of x (we need not say relative to [0,1]). Further, if every element of L has a complement, we say L as complemented. Thus a bounded lattice is complemented if the interval $[0,1] = L$ is complemented. If L is bounded lattice and is relatively complemented then L is complemented but not conversely.

Let us again consider the pentagonal lattice as shown below,

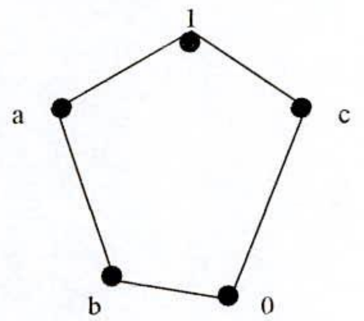


Fig-2.13

[0,1] is complemented as a and c are each others complement; b and c are each others complement and of course, 0 and 1 are each others complement. This lattice is not relatively complemented as b has no complement relative to [0,a] and so [0,a] is not complemented. The lattice given by the following diagram (Fig.-2.14) is not complemented as a has no complement (relative to [0,1]).

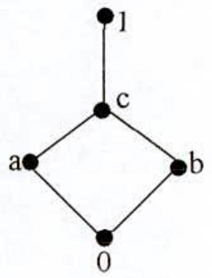


Fig :2.14

The lattice given by the figure below is both relatively complemented as well as complemented.

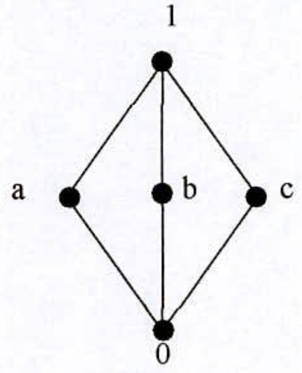


Fig :2.15

Definition (Complemented lattice) : A lattice is said to be complemented lattice if every element has complement.

In Fig:2.16, we have

0 is the complement of 1

a is the complement of b

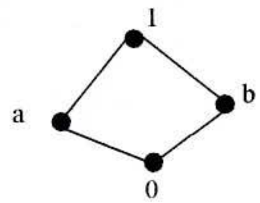


Fig: 2.16

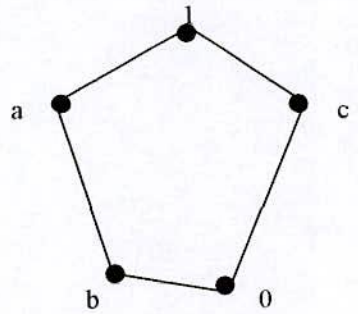


Fig-2.17

This lattice (Fig:2.17)is not relatively complemented as b has no complement in [0,a] and so [0,a] is not complemented.

Problem 2.4.1 : Let S = {a,b,c,d,e,f,g}be ordered as in fig. 2.18 and let X = {c,d,e}.

(a) Find the upper and lower bounds of X.

(b) Identify sup (X) (the supremum of X) and inf(X) (the infimum of X), if either exists.

Solution:

(a) The elements e, f and g succeed every element of X; hence e, f and g are the upper bounds of X. The element 'a' precedes every elements of X; hence it is the lower bound of X. Note that b is not a lower bound since b does not precede c; in fact, b and c are not comparable.

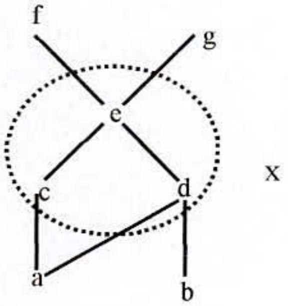


Fig: 2.18

(b) Since e precedes both f and g, we have e = sup (X). Likewise, since 'a' precedes (trivially) every lower bound of X, we have a = inf (X). It may be noted that sup (X) belongs to X but inf (X) does not belong to X.

Problem 2.4.2 : Let S = {1,2,3,....,8} be ordered as in Fig. 2.19 and let A = {2,3,6}

(a) Find the upper and lower bounds of A.

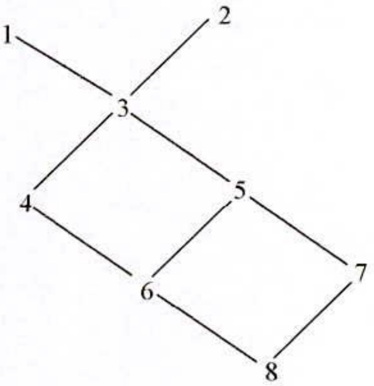


Fig: 2.19

(b) Identify sup(A) and inf(A) if either exists.

Solution:

(a) The upper bound is 2, and the lower bounds are 6 and 8

(b) Here sup(A)= 2 and inf(A) = 6

Problem 2. 4. 3 : Repeat problem 2.4.2 for the subset B= {1,2,5} of S

(a) There is no upper bound for B since no element succeeds both 1 and 2. The lower bounds are 6, 7, 8.

(b) Trivially, sup(A) does not exist since there are no upper bounds. Although A has three lower bounds, inf(A) does not exist since no lower bound succeeds both 6 and 7.

Problem 2. 4. 4 : Consider the ordered set A = {a,b,c,d,e} in fig. 2.20(a). Find the Hasse diagram of the collection p(A) of predecessor sets of the elements of A ordered by set inclusion.

The elements of p(A) follow:

p(a) = {a, c, d, e}, p(b) = {b, c, d, e}, p(c) = {c, d, e}, p(d) = {d}, p(e)={e}

Fig. 2.20(b) gives the diagram of p(A) ordered by set inclusion. Observe that the two diagrams in fig. 2.20 are identical except for the labeling of the vertices.

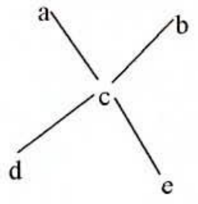


Fig: 2.20(a)

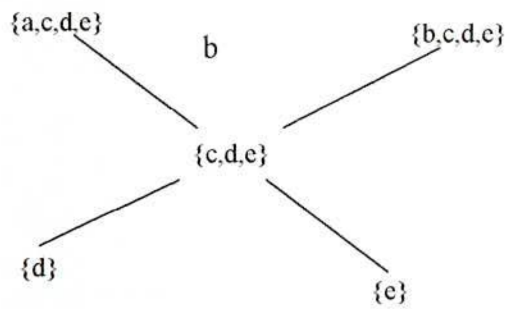


Fig: 2.20(b)

Problem 2.4.5 : Suppose the set P={1,2,3,.....}of positive integers is ordered by divisibility. Insert the correct symbol. $\prec$, $\succ$ or || (not comparable). between each pair of numbers:

(a) 2___8, (b) 18___24, (c) 9___3, (d) 5___15

Solution:

(a) Since 2 divides 8, 2 precedes 8; hence $2 \prec 8$.

(b) 18 does not divide 24, and 24 does not divide 18; hence 18||24

(c) Since 9 is divisible by 3, $9 \succ 3$.

(d) Since 5 divides 15, $5 \prec 15$

Problem 2.4.6 : Let P ={1,2,3,.....} be ordered by divisibility. State whether each of the following is a chain (linearly ordered subset) in P.

(a) A = {24, 2,6} (c) C = {2,8,32,4} (e) E = {15, 5, 30}

(b) B = {3,15,5} (d) D = {7} (f) P = {1,2,3,......}

Solution:

(a) Since 2 divides 6 which divides 24, A is a chain in P.

(b) Since 3 and 5 are non comparable, B is not a chain in P.

(c) C is a chain in P since $2 \prec 4 \prec 8 \prec 32$, that is 2|4|8|32 where | means divides.

(d) Any set consisting of one element is linearly ordered; hence D is a chain in P.

(e) Here $5 \prec 15 \prec 30$; hence E is a chain in P.

(f) P is not linearly ordered e.g. 2 and 3 are non comparable; hence P itself is not a chain in P.

Problem 2.4.7 : Let A = {1,2,3,4,5} be ordered by the diagram in Fig. 2.21. Insert the correct symbol, $\prec$, $\succ$ or || (not comparable). between each pair of element:

(a)1___5 (b) 2____3 (c) 4_____1 (d) 3_____4

Solution:

(a) Since there is a "path"(edges slanting upward) from 5 to 3 to 1, 5 precedes 1; hence $1 \succ 5$.

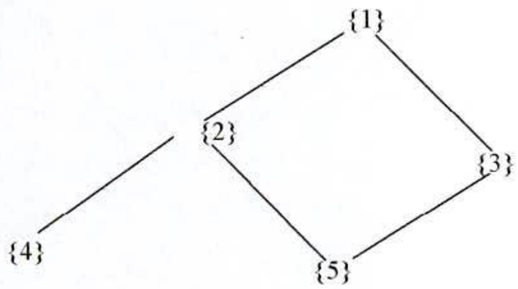


Fig : 2.21

(b) There is no path from 2 to 3, or vice versa; hence 2||3.

(c) There is a path from 4 to 2 to 1; hence $4 \prec 1$.

(d) Neither $3 \prec 4$ nor $4 \prec 3$; hence 3||4.

Problem 2.4.8 : Let A={1,2,3,4,5}be ordered as follows in Fig. 2.21

(a) Find all minimal and maximal element of A.

(b) Does A have a first element or a last element?

Solution:

(a) No element strictly precedes 4 or 5, so 4 and 5 are minimal elements of A. No element strictly succeeds 1, so 1 is a maximal element of A.

(b) A has no first element. Although 4 and 5 are minimal elements of A, neither precedes the other. However, 1 is a last element of A since 1 succeeds every element of A.

Problem 2.4.9 : Consider the ordered set A in Fig. 2.21. For each $a \in A$. Let p(a) denote the set of predecessors of a, that is , p(a) = $\{x : x \leq a\}$

Solution:

Let p(A) denote the collection of all predecessor sets of A, and let p(A) be ordered by set inclusion. Draw the Hasse diagram of p(A).

The elements of p(A) follow:

p(1) = {1,2,3,4,5}, p(2) = {2,4,5}, p(3) = {3, 5} , p(4) = {4}, p(5) = {5}

Fig. 2.22 gives the Hasse diagram of p(A) ordered by set inclusion. [Observe that the diagrams of A and p(A) are identical except for the labeling of the vertices.

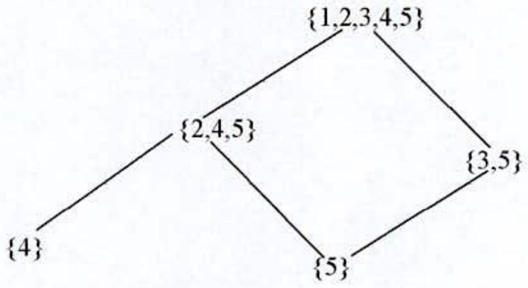


Fig : 2.22

2.5 : Modular Lattice, Jordan-Dedekind condition, Atoms and covers.

Definition (Modular Lattice): A lattice L is called a modular lattice if $\forall a, b, c \in L$, with $a \geq b$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad [= b \vee (a \wedge c)]$$

Example 2.5.1 : The lattice given by the following diagrams are modular

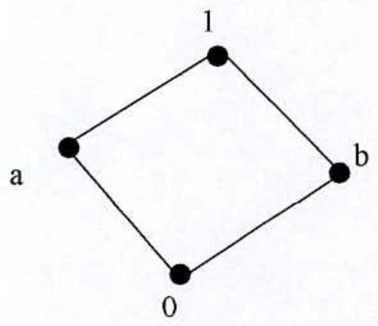


Fig: 2.23

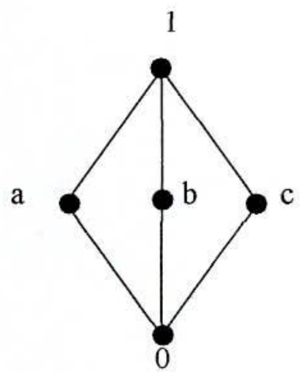


Fig: 2.24

In the first we cannot find any triplet a, b, c s.t., a>b and c is not comparable with a or b. Hence it is modular.

By similar argument the second lattice is also seen to be modular.

The following theorems are generalization of some theorem of [13].

Theorem 2.5.2 : Homomorphic image of a modular lattice is modular.

Proof : Let $\theta : L \to M$ be an onto homomorphism and suppose L is modular.

Let x, y, z$\in$M be three elements with x > y.

Since θ is onto homomorphism $\exists$ a, b, c $\in$L s.t., θ(a) = x, θ(b) = y, θ(c) = z where a > b

Now L is modular a, b, c $\in$ L, a > b thus we get $a \wedge (b \vee c) = b \vee (a \wedge c)$

Now

$$\begin{aligned} x \wedge (y \vee z) &= \theta(a) \wedge (\theta(b) \vee \theta(c)) \\ &= \theta(a) \wedge (\theta(b \vee c)) = \theta(a \wedge (b \vee c)) \\ &= \theta(b \vee (a \wedge c)) = \theta(b) \vee \theta(a \wedge c) \\ &= \theta(b) \vee [\theta(a) \wedge \theta(c)] = y \vee (x \wedge z) \end{aligned}$$

Hence M is modular.

Theorem 2. 5. 3: Two lattices L and M are modular if and only if L x M is modular.

Proof. : Let L and M be modular

Let (a_1, b_1) , (a_2, b_2), (a_3, b_3) $\in$L x M be three elements with $(a_1, b_1) \geq (a_2, b_2)$

Then $a_1, a_2, a_3 \in L$, $a_1 \geq a_2$

$b_1, b_2, b_3 \in M$, $b_1 \geq b_2$

and since L and M are modular, we get

$$a_1 \wedge (a_2 \vee a_3) = a_2 \vee (a_1 \wedge a_3)$$
$$b_1 \wedge (b_2 \vee b_3) = b_2 \vee (b_1 \wedge b_3)$$

Thus

$$\begin{aligned} &(a_1, b_1) \wedge [(a_2, b_2) \vee (a_3, b_3)] \\ &= (a_1, b_1) \wedge (a_2 \vee a_3, b_2 \vee b_3) \\ &= (a_1 \wedge (a_2 \vee a_3), b_1 \wedge (b_2 \vee b_3)) \\ &= (a_2 \vee (a_1 \wedge a_3), b_2 \vee (b_1 \wedge b_3)) \\ &= (a_2, b_2) \vee (a_1 \wedge a_3, b_1 \wedge b_3) \\ &= (a_2, b_2) \vee [(a_1, b_1) \wedge (a_3, b_3)] \end{aligned}$$

Hence L x M is modular.

Conversely , Let L x M be modular.

Let $a_1, a_2, a_3 \in L$, $a_1 \geq a_2$

$b_1, b_2, b_3 \in M$, $b_1 \geq b_2$

Then (a_1, b_1) , (a_2, b_2), $(a_3, b_3) \in L \times M$ and $(a_1, b_1) \geq (a_2, b_2)$.

Since L x M is Modular, we find

$$(a_1, b_1) \wedge [(a_2, b_2) \vee (a_3, b_3)] = (a_2, b_2) \vee [(a_1, b_1) \wedge (a_3, b_3)]$$

$$\text{Or, } (a_1, b_1) \wedge (a_2 \vee a_3, b_2 \vee b_3) = (a_2, b_2) \vee (a_1 \wedge a_3, b_1 \wedge b_3)$$

$$\text{Or, } (a_1 \wedge (a_2 \vee a_3), b_1 \wedge (b_2 \vee b_3)) = (a_2 \vee (a_1 \wedge a_3), b_2 \vee (b_1 \wedge b_3))$$

$$\Rightarrow \quad a_1 \wedge (a_2 \vee a_3) = a_2 \vee (a_1 \wedge a_3)$$

$$b_1 \wedge (b_2 \vee b_3) = b_2 \vee (b_1 \wedge b_3)$$

$\Rightarrow$ L and M are Modular.

Remark : It is important to point out here that in the above theorem, the assertion is that a non modular lattice contains a pentagonal sublattice and not only a pentagonal subset. In other words, it is possible that we may have a modular lattice which contains pentagonal subset. Consider, for instance, the lattice L of factors of120. The lattice is given by the diagram.

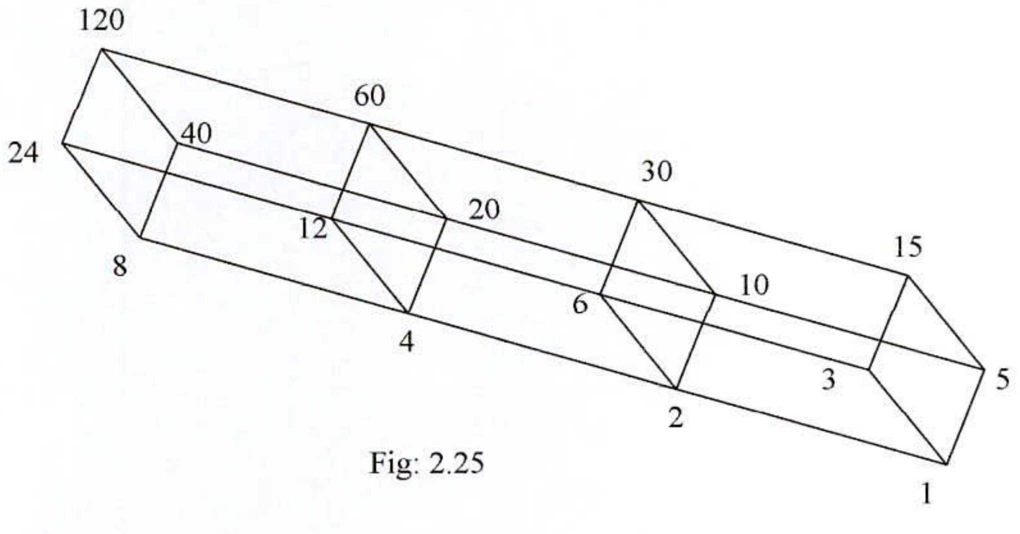


Fig: 2.25

We notice S = { 2, 6, 10, 12, 60} is a pentagonal subset of L but not a sublattice. For, in L, $10 \vee 6 = 30 \notin S$. Again L is modular, as it is cardinal product of three chains A ={0<1<2<3}, B= {0<1}, C ={0<1} and a chain being modular gives product of chains to be modular.

Definition (Jordan-Dedekind condition) : Let L be a lattice of finite length. We say L satisfies the Jordan-Dedekind condition if all maximal chains between same end points have same (finite) length.

Definition (Atoms and Covers) : Let L be a lattice of finite length with least element o.

An element $x \in L$ is said to have height or dimension n if $l\ [o,x]=n$ and, in that case, we write $h(x)=n$.

An element a in a lattice L is called an atom if it covers o. In other words, a is an atom iff $a \neq 0$ and $x \wedge a=a$ or $x \wedge a= 0$ $\forall x \in L$

Clearly then $h(a)= 1$.

An element b is called dual atom, if 1, the greatest element of the lattice covers b.

The following theorem is a generalizations of a well-known results.

Theorem 2.5.4: Let L be a lattice of finite length. Suppose in L, whenever, x,y cover $x \wedge y$ implies $x \vee y$ covers x and y, then L satisfies the Jordan- Dedekind condition.

Proof : Let a, b be any two comparable points $(a \leq b)$.We show all maximal chains from a to b have same length $l\ [a,b]$.

Since all chains from a to b are finite, at least one maximal chain exists of finite length from a to b . We show all maximal chains are of the same length.

We prove the result by induction on n, the length $l\ [a,b]$, if $l[a,b]=1$.then b covers a and thus there is only one maximal chain from a to b with length 1 and hence the result holds for n=1.

Let the result be true for n=m-1

Let $a<x_1<x_2< \ldots. < x_m=b$

$a<y_1<y_2< \ldots\ldots < y_k=b$

be two maximal chains from a to b of lengths m and k. We show k=m.

Case (i) If $x_1=y_1$

Then $x_1<x_2<\ldots.<x_m =b$

$x_1=y_1<y_2<\ldots.<y_k=b$

are two maximal chains from $x_1$to b with lengths m-1, k-1 and as the result holds for m-1,

$k-1=m-1 \Rightarrow k=m$

Case (ii) $x_1 \neq y_1$

Here $x_1$and $y_1$cover $a=x_1 \wedge y_1$

Thus by given condition $x_1 \vee y_1$ covers x_1 and y_1

Let $x_1 \vee y_1=t$

Since $x_1<b$, $y_1<b$

$t=x_1 \vee y_1 \leq b$

and we find t and b are comparable.

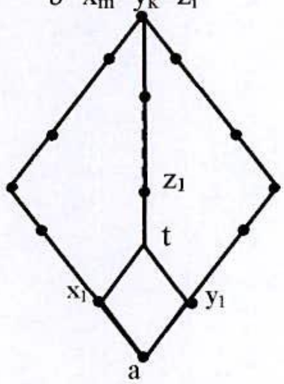


Fig: 2.26

Let $t<z_1<z_2<\ldots\ldots<z_i=b$.be a maximal chain from t to b with length i

Now

$x_1<x_2<\ldots\ldots\ldots<x_m=b$

$x_1<t<z_1<\ldots\ldots\ldots<z_i=b$

are two maximal chains from x_1 to b of lengths m-1 and i+1 (Note t covers x_1).

But the result holds for m-1 and thus i+1=m-1.

Again, the chains

$y_1<y_2<\ldots\ldots\ldots<y_k=b$

$y_1<t<z_1<\ldots\ldots\ldots<z_i=b$

are maximal chains from y_1 to b with lengths k-1 and i+1 i.e., are maximal chains from y_1 to b with lengths k-1 and m-1.

But result holds for m-1, and so k-1=m-1

$\Rightarrow k=m$

i.e., the result holds for n=m.

Hence by induction hypothesis, the result holds for all n and our assertion is proved.

Example 2.5.5 : Let $X \neq \varphi$ finite set and $\rho(X)$ be the power set of X. We know $(\rho(X), \subseteq)$ forms a lattice with least element φ and greatest element x. Also for any A, B $\in \rho(X)$

Solution:

$A \wedge B = A \cap B$ and $A \vee B = A \cup B$

Since

$$A \wedge (X - A) = A \cap (X - A) = \phi$$
$$A \vee (X - A) = A \cup (X - A) = X$$

we find X-A is complement of A (relative to $[\varphi, X]$)

Thus $\rho(X)$ is complemented.

Suppose B is any complement of A, then

$$A \wedge B = \phi = A \cap B$$
i.e.,
$$A \vee B = X = A \cup B$$
$$A \cap B = A \cap (X - A)$$
$$A \cup B = A \cup (X - A)$$
$$B = X - A$$

Or that X-A is unique complement of A. Thus this is an example of a uniquely complemented lattice.

We show further that $\rho(X)$ is also relatively complemented. Consider any interval [A,B] of $\rho(X)$.

Let, $C \in [A, B]$ be any member. Then

$$C \cap (A \cup (B - C)) = (C \cap A) \cup (C \cap (B - C)) = A \cup \phi = A$$
$$C \cup (A \cup (B - C)) = (C \cup A) \cup (B - C) = C \cup (B - C) = B$$

Showing that $A \cup (B - C)$ is complement of C relative to [A,B].

Since C was any element of any interval of $\rho(X)$. we find $\rho(X)$ is relatively complemented.

Theorem 2.5.6 : Two bounded lattices A and B are complemented if and only if A x B is complemented.

Proof : - Let A and B be complemented and suppose o, u and o', u' are the universal bounds of A and B respectively.

Then (o,o') and (u, u') will be least and greatest elements of AxB.

Let $(a,b) \in AxB$ be any element.

Then $a \in A$, $b \in B$ and as A, B are complemented $\exists a' \in A, b' \in B$ s.t.,

$a \wedge a'=o$, $a \vee a'=u$, $b \wedge b'=o'$, $b \vee b'=u'$

Now $(a,b) \wedge (a', b')=(a \wedge a', b \wedge b')=(o,o')$

$(a,b) \vee (a',b')= (a \vee a', b \vee b')=(u,u')$

Shows that (a',b') is complement of (a,b) in AxB.

Hence AxB is complemented

Conversely, let AxB be complemented.

Let $a \in A$, $b \in B$ be any elements.

Then $(a,b) \in AxB$ and thus has a complement, say (a',b')

Then $(a,b) \wedge (a',b')=(o,o')$, $(a,b) \vee (a',b')=(u,u')$

$\Rightarrow (a \wedge a',b \wedge b')=(o,o')$, $(a \vee a', b \vee b')=(u,u')$

$\Rightarrow a \wedge a'=o$, $a \vee a'=u$

$b \wedge b'=o'$, $b \vee b'=u'$

i.e. a' and b' are complements of a and b respectively. Hence A and Bare complemented.

Theorem 2.5.7 : Two lattices A and B are relatively complemented if and only if A x B is relatively complemented.

Proof : - Let A, B be relatively complemented.

Let $[(a_1, b_1), (a_2, b_2)]$ be any interval of Ax B and suppose (x, y) is any element of this interval.

Then $(a_1, b_1) \leq (x, y) \leq (a_2, b_2)$

$a_1, a_2, x \in A$

$b_1, b_2, y \in B$

$\Rightarrow \quad a_1 \leq x \leq a_2$

$\quad b_1 \leq y \leq b_2$

$\Rightarrow x \in [a_1, a_2]$ an interval in A

$y \in [b_1, b_2]$ an interval in B

Since A, B are relatively complemented so x, y have complements relative to $[a_1, a_2]$ and $[b_1, b_2]$ respectively.

Let x' and y' be these complements, Then

$x \wedge x' = a_1,\ y \wedge y' = b_1$

$x \vee x' = a_2,\ y \vee y' = b_2$

Now $(x,y) \wedge (x', y') = (x \wedge x', y \wedge y') = (a_1, b_1)$

$(x,y) \vee (x', y') = (x \vee x', y \vee y') = (a_2, b_2)$

$\Rightarrow (x', y')$ is complement of (x, y) relative to $[(a_1, b_1), (a_2, b_2)]$

Thus any interval in A x B is complemented .

Hence A x B is relatively complemented

Conversely, let A x B be relatively complemented.

Let $[a_1, a_2]$ and $[b_1, b_2]$ be any intervals in A & B.

Let $x \in [a_1, a_2]$, $y \in [b_1, b_2]$ be any elements.

Then $a_1 \leq x \leq a_2$, $b_1 \leq y \leq b_2$

$\Rightarrow (a_1, b_1) \leq (x, y) \leq (a_2, b_2)$

$\Rightarrow (x, y) \in [(a_1, b_1), (a_2, b_2)]$, an interval in A x B

$\Rightarrow$ (x,y) has a complement, say (x', y') relative to this interval.

Thus $(x,y) \wedge (x', y') = (a_1, b_1)$

$\quad (x,y) \vee (x', y') = (a_2, b_2)$

or, $\quad (x \wedge x', y \wedge y') = (a_1, b_1)$

$\quad (x \vee x', y \vee y') = (a_2, b_2)$

$\Rightarrow \quad x \wedge x' = a_1,\ x \vee x' = a_2$

$y \wedge y' = b_1,\ y \vee y' = b_2$

$\Rightarrow$ x' is complement of x relative to $[a_1,a_2]$

y' is complement of y relative to $[b_1, b_2]$

which in turn imply that A, B are relatively complemented.

Theorem 2.5.8: Dual of a complemented lattice is complemented.

Proof: Let (L,p) be a complemented lattice with o, 1 as least and greatest elements.

Let $(\overline{L}, \overline{P})$ be the dual of (L,p) Then 1,o are least and greatest elements of $\overline{L}$.

Let $a \in \overline{L} = L$ be any element.

Since $a \in L$, L is complemented, $\exists$ $a' \in L$ s.t.

$a \wedge a' = 0, a \vee a' = 1$ in L.

i.e., 0=Inf{a,a'} in L.

$\Rightarrow 0 \ \rho \ a, o \rho \ a'$

$\Rightarrow a \ \overline{\rho} \ 0, a' \overline{\rho} \ 0$ in $\overline{L}$

$\Rightarrow$ o is an upper bound of {a,a'} in $\overline{L}$

If k is any upper bound of {a,a'} in $\overline{L}$ then $a \ \overline{\rho} \ k, a' \overline{\rho} \ k$

$\Rightarrow k \ \rho \ a, \ k \ \rho \ a' \Rightarrow k \ \rho \ o$ as o is Inf.

$\Rightarrow 0 \ \overline{\rho} \ k$

i.e., o is l.u.b. $\{a, a'\}$ in $\overline{L}$

i.e., $a \vee a' = 0$ in $\overline{L}$

Similarly, $a \wedge a' = 1 \, in \, \overline{L}$

or that a' is complement of a in $\overline{L}$

Hence $\overline{L}$ is complemented.

Problem 2.5.9 : Show that no ideal of a complemented lattice which is a proper sublattice can contain both an element and its complement.

Solution: Let L be a complemented lattice. Then o, $1 \in$ L. Let I be an ideal of L such that I is a proper sublattice of L. Suppose $\exists$ an element x in I such that its complement x' is also in I.

Then $x \wedge x' = o$, $x \vee x' = 1$

since I is a sublattice, $x \wedge x'$, $x \vee x'$ are in I. i.e., o, $1 \in I$.

Now if $l \in$ L be any element then as $1 \in I$

$l \wedge 1 \in I$

$\Rightarrow$ $l \in I \Rightarrow L \subseteq I \Rightarrow I = L$, a contradiction.

Problem 2.5.10 : Let L be a uniquely complemented lattice and let a be an atom in L. Show that a' the complement of a is a dual atom of L.

Solution: Since L is uniquely complemented lattice, every element has a unique complement.

Suppose a' is not a dual atom, then $\exists$ at least one x s.t., $a' < x < 1$

$\Rightarrow$ $a' \vee a \leq x \vee a$

$\Rightarrow$ $1 \leq x \vee a \leq 1$

$\Rightarrow$ $1 = x \vee a$.

Now if $a \leq x$ then $x \vee a. = x \Rightarrow$ x=1, not true. Again if $a \leq$ x, then $a \not\wedge x = 0$ (note a is an atom)

thus $a \wedge x = 0$, $a \vee x = 1$ $\Rightarrow$ $x = a'$,again a contradiction.

Hence a' is a dual atom.

CHAPTER 3

Boolean Algebra and Boolean Function

Introduction:

A complemented distributive lattice is called a Boolean Lattice. Let $(A,\wedge,\vee,',0,1)$ be a Boolean Algebra. Expressions involving members of A and the operations $\wedge,\vee$ and complementation are called Boolean expression. Any function specifying these Boolean expressions is called a Boolean function. A Boolean function is said to be in Disjunctive normal form (DN form) in n variables $x_1, x_2, x_3,............, x_n$ if it can be written as join of terms of the type $f_1(x_1)\wedge f_2(x_2)\wedge f_3(x_3)\wedge...........\wedge f_n(x_n)$. where $f_i(x_i)=x_i$ for all i = 1,2,3,.........,n and no two terms are same. A Boolean function f is said to be in Conjunctive Normal Form (CN form) in n variables $x_1, x_2, x_3,.........\ ..., x_n$ if f is meet of terms of the type $f_1(x_1)\vee f_2(x_2)\vee.........\ ....\ .....\vee f_n(x_n)$ where $f_i(x_i)=x_i$ or $x_{i'}$ for all i=1,2,3,............, n and no two terms are same.

3.1: Boolean Algebra and Boolean Lattice:

Definition (Boolean Algebra and Boolean Lattice): A complemented distributive lattice is called a Boolean Lattice. Since complements are unique in a Boolean Lattice we can regard a Boolean Lattice as an algebra with two binary operations $\wedge$ and $\vee$ and one unary operation $'$. Boolean Lattices so considered are called Boolean algebras. In other words, by a Boolean Algebra, we mean a system consisting of a non empty set L together with two binary operations $\wedge$ and $\vee$ and unary operation $'$,0 and 1 satisfying ($\forall a, b, c \in L$)

(i) $a \wedge a = a, a \vee a = a$

(ii) $a \wedge b = b \wedge a, a \vee b = b \vee a$

(iii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c, a \vee (b \vee c) = (a \vee b) \vee c$

(iv) $a \wedge (a \vee b) = a, a \vee (a \wedge b) = a$

(v) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(vi) $\forall a \in L, \exists a' \in L$ s.t., $a \wedge a' = 0. a \vee a' = 1$

where 0, 1 are elements of L satisfying $0 \le x \le 1 \quad \forall x \in L$. ($a'$ will be unique and is the complement of a)([1],[2],[4],[5],[6],[10]).

Since we have mentioned above that, $\wedge$, $\vee$ and $'$ are operations on L. It is clear that closure properties hold in L i.e.- for all $a,b \in L$, $a \wedge b$, $a \vee b$, $a' \in L$.

Example 3.1.1 : Let B={0,a,b,1}. If we define $\wedge$, $\vee$ and complementation $'$ by

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$'$	
0	1
a	b
b	a
1	0

Then B forms a Boolean algebra under these operations. Since a Boolean Algebra is distributive (and thus, modular) and complemented, all properties of modular, distributive and complemented lattices hold in a Boolean algebra.

Theorem 3.1.2 : Prove that Boolean sublattice may not be a Boolean subalgebra.

Proof: A subalgebra (or a Boolean subalgebra) is a non empty subset S of a Boolean algebra L. Such that , $a, b \in S \Rightarrow a \wedge b, a \vee b, a' \in S$

We thus realize that a subalgebra differs from a sublattice in as much as it is closed under complementation also. Notice that if [a, b] be an interval in a Boolean algebra L, where a>0, then [a, b] is a sublattice of L, but is not subalgebra as

$$\begin{aligned} a \in [a,b] &\Rightarrow a' \in [a,b] \\ &\Rightarrow a \wedge a' \in [a,b] \\ &\Rightarrow 0 \in [a,b] \end{aligned}$$

which is not possible as a>0. Hence a Boolean sublattice may not be a Boolean subalgebra.

Theorem 3.1.3 : In a Boolean algebra, the following results hold.

i) $(a')' = a$

ii) $(a \wedge b)' = a' \vee b'$ [De Morgan's Law]

iii) $(a \vee b)' = a' \wedge b'$ [De Morgan's Law]

iv) $a \leq b \Leftrightarrow a' \geq b'$

v) $a \leq b \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a' \vee b = 1$

Prof: (i) Let $(a')' = a''$ then

$$a \wedge a' = 0, \quad a \vee a' = 1$$
$$a' \wedge a'' = 0, \quad a' \vee a'' = 1$$
$$\Rightarrow a \wedge a' = a'' \wedge a', \quad a \vee a' = a'' \vee a'$$
$$\Rightarrow a'' = a \qquad \therefore (a')' = a.$$

(ii) We have

$$\begin{aligned} (a \wedge b) \wedge (a' \vee b') &= [(a \wedge b) \wedge a'] \vee [(a \wedge b) \wedge b'] \\ &= [(a \wedge a') \wedge b] \vee [a \wedge (b \wedge b')] \\ &= [0 \wedge b] \vee [a \wedge 0] = 0 \vee 0 = 0 \end{aligned}$$

$$\begin{aligned} (a \wedge b) \vee (a' \vee b') &= (a' \vee b') \vee (a \wedge b) \\ &= [(a' \vee b') \vee a] \wedge [(a' \vee b') \vee b] \\ &= [(a' \vee a) \vee b'] \wedge [a' \vee (b' \vee b)] \\ &= (1 \vee b') \wedge (a' \vee 1) = 1 \wedge 1 = 1 \end{aligned}$$

Hence $(a \wedge b)' = a' \vee b'$

(iii) This can be proved in the similar way used to prove (ii).

(iv) $a \le b \Rightarrow a = a \wedge b$

$$\Rightarrow a' = (a \wedge b)' = a' \vee b'$$

$$\Rightarrow b' \le a'$$

$$b' \le a' \Rightarrow b'' \ge a'' \Rightarrow b \ge a$$

(v)$a \le b \Rightarrow a \wedge b' \le b \wedge b' \Rightarrow 0 \le a \wedge b' \le 0$

$$\Rightarrow a \wedge b' = 0$$

Again,Let,$a \wedge b' = 0$

Then $a = a \wedge 1 = a \wedge (b \vee b') = (a \wedge b) \vee (a \wedge b')$

$$= (a \wedge b) \vee 0 = (a \wedge b)$$

$$\Rightarrow a \le a \wedge b.$$

Second result follows similarly.

3.2: Boolean Functions:

Definition (Boolean Functions): Let *(A, ∧, v, ′ ,0,1)* be a Boolean algebra. Expressions involving members of A and the operations $\wedge, \vee$ and complementation are called Boolean expressions or Boolean polynomials. For example, $x \vee y', x, x \wedge 0$ etc. are all Boolean expressions. Any function specifying these Boolean expressions is called a Boolean function. Thus if f(x,y)=x $\wedge$y then f is the Boolean function and x$\wedge$ y is the Boolean expression (or value of the function f). Since it is normally the functional value (and not the function) that we are interested in, we call these expressions the Boolean functions.

In what follows, we'll be denoting least and greatest elements of a Boolean algebra by 0 and 1 respectively. In fact, most of the times we'll confine ourselves to Boolean algebras that contain only these two elements.

Disjunctive Normal form (DN form)

A Boolean function (expression) is said to be in disjunctive normal form(DN form) in n variables x_1, x_2, -----, x_n. If it can be written as join of terms of the type

$f_1(x_1) \wedge f_2(x_2) \wedge .... \wedge f_n(x_n)$ Where $f_i(x_i)=x_i$ or x'_i for all i=1,2,3,----, n. and no two terms are same. Also 1 and 0 are said to be in disjunctive normal form.

Again, in that case, terms of the type $f_1(x_1) \wedge f_2(x_2) \wedge .... \wedge f_n(x_n)$ are called minterms or minimal polynomials. (A normal form is also called a canonical form).

For instance, $(x \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z)$ is in disjunctive normal form (in 3 variables) and each of the brackets is a minterm.

Thus each minturm is a meet of all the n variables with or without a prime (i.e. x_i or x_i'). If we have three variables x, y, z then any function in the DN form will be join of some or all the minterms.

$$
\begin{array}{llll}
x \wedge y \wedge z & x' \wedge y \wedge z & x \wedge y' \wedge z & x \wedge y \wedge z' \\
x \wedge y' \wedge z' & x' \wedge y \wedge z' & x' \wedge y' \wedge z & x' \wedge y' \wedge z'
\end{array}
$$

Which will be 2^n (n=3) in number.

Consider the function.

$$
\begin{aligned}
f(x,y) &= x \vee (x' \wedge y) = x \wedge (y \vee y') \vee (x' \wedge y) \\
&= (x \wedge y) \vee (x \wedge y') \vee (x' \wedge y)
\end{aligned}
$$

Which is in DN form and contains three minterms out of four (possible) minterms $x \wedge y, x' \wedge y, x \wedge y', x' \wedge y'$ in 2 variables.

Example 3.2.1: A complete DN form in 3 variables is

$$
f(x,y,z) = (x\wedge y\wedge z)\vee(x'\wedge y\wedge z)\vee(x\wedge y'\wedge z)\vee(x\wedge y\wedge z')\vee(x\wedge y'\wedge z')\vee(x'\wedge y\wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z')
$$

Theorem 3.2.2 : Every Boolean function can be put in disjunctive normal form.

Proof.: We prove the result by taking the following steps.

(1) If primes occur outside brakets, then open the brakets by using De Morgan's laws. $(a \wedge b)' = a' \vee b'$, $(a \vee b)' = a' \wedge b'$

(2) Open all brakets by using distributivity and simplify using any of the definition conditions like indempotency , absorption etc.

(3) If any of the terms does not contain a certain variable x_i(or x_i') then take meet of that term with $x_i \vee x_i'$. Do this with each such term. (it will not affect the function as $x_i \vee x_i' = 1$ and $1 \wedge a = a$)

Now ,open brakets and drop all terms of the type $a \wedge a'(=0)$. Again, if any of the terms occur more than once. These can be omitted because of idempotency. The resulting expression will be in DN form.

Hence every function in a Boolean algebra is equal to a function in DN form.

Problem 3.2.3 : Put the function $f = \left[(x \wedge y')' \vee z'\right]' \wedge (x' \vee z)'$ in the DN form.

Solution : We have $f = [(x' \vee y'') \vee z'] \wedge (z' \wedge x'')$.

$$
\begin{aligned}
&= (x' \vee y \vee z') \wedge (z' \wedge x) \\
&= (x' \wedge z' \wedge x) \vee (y \wedge z' \wedge x) \vee (z' \wedge z' \wedge x) \\
&= 0 \vee (x \wedge y \wedge z') \vee (x \wedge z') \\
&= (x \wedge y \wedge z') \vee [(x \wedge z') \wedge (y \vee y')] \\
&= (x \wedge y \wedge z') \vee [(x \wedge z' \wedge y) \vee (x \wedge z' \wedge y')] \\
&= (x \wedge y \wedge z') \vee (x \wedge y' \wedge z')
\end{aligned}
$$

Problem 3.2.4: Put the function:

$f = [(x' \wedge y) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x' \wedge y' \wedge z \wedge t) \vee t']'$ in the DN form.

Solution : We have

$$
\begin{aligned}
f &= (x' \wedge y)' \wedge (x \wedge y \wedge z')' \wedge (x \wedge y' \wedge z)' \wedge (x \wedge y \wedge z' \wedge t)' \wedge t \\
&= \left[(x \vee y') \wedge (x' \vee y' \vee z)\right] \wedge (x' \vee y \vee z') \wedge [(x \vee y \vee z \vee t') \wedge t] \\
&= \left[(x \wedge x') \vee (x \wedge y') \vee (x \wedge z) \vee (y' \wedge x') \vee (y' \wedge y') \vee (y' \wedge z)\right] \wedge (x' \vee y \vee z') \\
&\quad \wedge \left[(x \wedge t) \vee (y \wedge t) \vee (z \wedge t) \vee (t \wedge t')\right] \\
&= \left[(x \wedge y') \vee (x \wedge z) \vee (y' \wedge x') \vee y' \vee (y' \wedge z)\right] \wedge \\
&\quad \left[(x' \wedge y \wedge t) \vee (x' \wedge z \wedge t) \vee (y \wedge x \wedge t) \vee (y \wedge t) \vee (y \wedge z \wedge t) \vee (z' \wedge x \wedge t) \vee (z' \wedge y \wedge t)\right]
\end{aligned}
$$

$$= (x \wedge y' \wedge z' \wedge t) \vee (x \wedge z \wedge y \wedge t) \vee (y' \wedge x' \wedge z \wedge t) \vee (y' \wedge z \wedge t \wedge x')$$
$$\vee (y' \wedge z' \wedge x \wedge t) \vee (y' \wedge z \wedge x' \wedge t)$$
$$= (x \wedge y' \wedge z' \wedge t) \vee (x \wedge y \wedge z \wedge t) \vee (x' \wedge y' \wedge z \wedge t)$$

Example 3.2.5 :

Let A= {0,1} and $f : A^2 \to A$ be defined by

$f(x,y) = (x \wedge y) \vee (x' \wedge y) \vee (x \wedge y') \vee (x' \wedge y')$

i.e.- f is incomplete DN from. We calculate all values of f(x,y), $x,y \in A$

$$f(0,0) = (0 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) = 1$$
$$f(1,0) = (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) \vee (1 \wedge 0) = 1$$
$$f(0,1) = (0 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) = 1$$
$$f(1,1) = (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0) = 1$$

(Note $x = 0 \Leftrightarrow x' = 1$)

We thus notice that in each case, one minterm is $1 \wedge 1 = 1$ and all others are zero. Also the resulting value of f(x,y) is always I.

If we go through similar process, with a function f which is in complete DN from in 3 variables x, y, z we'll get the same result.

Problem 3.2.6 : Write the function $x \vee y'$ in the disjunctive normal form in three variables x,y,z.

Solution: We have,

$$x \vee y' = [x \wedge (y \vee y') \wedge (z \vee z')] \vee [y' \wedge (x \vee x') \wedge (z \vee z')]$$
$$= [\{(x \wedge y) \vee (x \wedge y')\} \wedge (z \vee z')] \vee [\{(y' \wedge x) \vee (y' \wedge x')\} \wedge (z \vee z')]$$
$$= (x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (y' \wedge x \wedge z) \vee (y' \wedge x \wedge z') \vee$$
$$(y' \wedge x' \wedge z) \vee (y' \wedge x' \wedge z')$$
$$= (x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee$$
$$(x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z').$$

Conjunctive Normal Form :

Definition: (Conjunctive Normal Form): A Boolean function f is said to be in Conjunctive Normal Form (CN form)in n variables $x_1, x_2, x_3, \ldots\ldots\ldots \ldots, x_n$ if *f* is meet of terms of the type $f_1(x_1) \vee f_2(x_2) \vee \ldots\ldots\ldots \ldots\ldots\ldots \ldots \vee f_n(x_n)$

where $f_i(x_i) = x_i$ or $x_{i'}$ for all i=1,2,3,............, n

and no two terms are same. Also 0 and 1 are said to be in CN form.

Problem 3.2.7: Put the function

$f = [(x \wedge y') \vee z'] \wedge (x' \vee z)'$ in the C.N form.

Solution: We have

$$
\begin{aligned}
f &= [(x' \vee y) \vee z'] \wedge (x \wedge z') \\
&= (x' \vee y \vee z') \wedge [(x \wedge z') \vee (y \wedge y')] \\
&= (x' \vee y \vee z') \wedge \{[(x \wedge z') \vee y] \wedge [(x \wedge z') \vee y']\} \\
&= (x' \vee y \vee z') \wedge [(x \vee y) \wedge (z' \vee y) \wedge (x \vee y') \wedge (z' \vee y')] \\
&= (x' \vee y \vee z') \wedge [\{x \vee y \vee (z \wedge z')\} \wedge \{(z' \vee y) \vee (x \wedge x')\} \wedge \{(x \vee y') \vee (z \wedge z')\} \wedge \{(z' \vee y') \vee (x \wedge x')\}] \\
&= (x' \vee y \vee z') \wedge (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (z' \vee y \vee x) \wedge (z' \vee y \vee x') \wedge \\
&\quad (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (z' \vee y' \vee x) \wedge (z' \vee y' \vee x') \\
&= (x \vee y \vee z) \wedge (x' \vee y \vee z') \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (x' \vee y' \vee z')
\end{aligned}
$$

Problem 3.2.8 : Put the function $x \wedge (y \vee z)$ in the C.N form.

Solution:

$$
\begin{aligned}
x \wedge (y \vee z) &= [x \vee (y \wedge y')] \wedge [(y \vee z) \vee (x \wedge x')] \\
&= (x \vee y) \wedge (x \vee y') \wedge (y \vee z \vee x) \wedge (y \vee z \vee x') \\
&= (x \vee y) \vee (z \wedge z') \wedge (x \vee y') \vee (z \wedge z') \wedge (y \vee z \vee x) \wedge (x' \vee y \vee z) \\
&= (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (x \vee y \vee z) \wedge (x' \vee y \vee z) \\
&= (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (x' \vee y \vee z)
\end{aligned}
$$

Problem 3.2.9: Find the D.N form of the function whose CN form is

$f=(x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (x' \vee y \vee z)$

Solution: We know $f=(f')'$. Thus

$f=[\{(x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (x' \vee y \vee z)\}']'$

$=[(x \vee y \vee z)' \vee (x \vee y \vee z')' \vee (x \vee y' \vee z)' \vee$

$(x \vee y' \vee z')' \vee (x' \vee y \vee z)']'$ (by De Morgan's Law)

$=[(x' \wedge y' \wedge z') \vee (x \wedge y' \wedge z) \vee (x' \wedge y \wedge z')$

$\vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z')]'$ (by De Morgan's Law)

$= (x \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z')$

3.3: Length and Cover :

Definition(Length and Cover) : A finite chain with n elements is said to have length n-l. (i.e..length is the number of links that the chain has).We say a covers b if $b < a$ and there exists no c.s.t., $b < c < a$. A chain $x_1 < x_2 < \ldots < x_n$ is called a maximal chain if each X_{i+1} covers X_i. Suppose now [a,b] is an interval in a lattice and if amongst all chains from a to b, there is one of maximum length n , we say [a, b] has length n. Thus it is the Sup of lengths of chains from a to b. We denote it by $l[a, b] = n$. In case some chains from a to b have infinite length we say [a, b] has infinite length.

If L be a lattice with least 0 and greatest element 1 then as $L = [0, 1]$, length of L is defined to be length of the interval $[0, 1]$.

Example 3.3.1: Consider the pentagonal lattice

It has five chains

$0 < 1, \quad 0 < a < 1, \quad 0 < b < 1.$

$0 < c < 1, \quad 0 < b < a < 1$

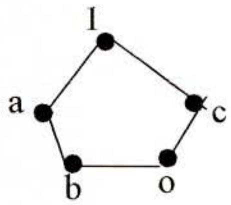


Fig :3.1

from 0 to 1. The last two being maximal chains. (Thus there can be more than one maximal chain from x to y ($x \leq y$) in a lattice).

The chains have lengths 1, 2, 2, 2, 3,

$l\,[0, 1\,] = 3$, and hence length of the pentagonal lattice is 3.

3.4: Homomorphism, Isomorphism and Endomorphism.

Definitions:(Homomorphisms, Isomorphism and Endomorphism)

Let L and M be lattices. A mapping $\theta : L \rightarrow M$ is called a meet homomorphism if $\theta (a \wedge b) = \theta (a) \wedge \theta (b)$

It is called a join homomorphism if $\theta(a \vee b)= \theta (a) \vee \theta (b)$

If θ is both meet as well as join homomorphisim , it is called a homomorphism (it is ,of course , clear that the operation $\wedge$ and $\vee$ on the left are those of L and on the right are of M). A homomorphism is also sometimes called a morphism.

If in addition, the map θ is also 1-1 onto we call θ to be an ismorphism. If θ is an ismorphism from L to L we call it an automorphism.

A homomorphism from L to L is called endomorphism. If $\theta : L\rightarrow M$ is onto homomorphism, we say M homomorphic image of L

Example 3.4.1: Let L and M be the lattices

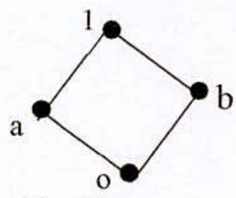


Fig :3.2

Fig :3.3

Define θ: $L \rightarrow M$, s.t,

$\theta (o) = p$, $\theta (a) = q$, $\theta (b) = p$, $\theta (1) =q$

Then θ is a homomorphism.

$\theta (a \wedge b)= \theta (o) = p$, $\theta (a) \wedge \theta (b) = q \wedge p =p$

$\theta (o \vee a)= \theta (a) = q$, $\theta (o) \vee \theta (a) = p \vee q = q$ etc.

The map $\varphi : L \rightarrow M$, S.t., φ (o) =p, φ (a) =x, φ (b) =x, φ (1) = q is neither a meet, nor a join homomorphism as

$\varphi (a \wedge b) = \varphi (0) = p, \varphi(a) \wedge \varphi (b) = x \wedge x = x$

$\varphi (a \vee b) = \varphi (1) = q, \varphi (a) \vee \varphi (b) = x \vee x = x$

The map Ψ: $L \rightarrow M$, s.t, Ψ (0) = P, Ψ (a) = p, Ψ (b) = p, Ψ (1) = q is meet homomorphism, but not a join homomorphism as

$\Psi (a \vee b) = \Psi (1) = q, \Psi (a) \vee \Psi (b) = p \vee p = p$

$\Psi (a \wedge b) = \Psi (0) = p = p \wedge p = \Psi (a) \wedge \Psi (b)$

$\Psi (a \wedge 0) = \Psi (0) = p = p \wedge p = \Psi (a) \wedge \Psi (0)$

$\Psi (a \wedge 1) = \Psi (a) = p = p \wedge q = \Psi (a) \wedge \Psi (1)$ etc.

Finally, the map $\sigma : L \rightarrow M$, s.t., σ (0) = p, σ (a) = σ (b) q= σ (1) is a join homomorphism but not a meet homomorphism.

Theorem 3.4.2 : Any meet (join) homomorphism preserves order.

Proof : Let $\theta : L \rightarrow M$ be a meet homomorphism,.

Let $a \leq b$ in L.

Then $a = a \wedge b$

$\Rightarrow \theta (a) = \theta (a \wedge b) = \theta (a) \wedge \theta (b)$

$\Rightarrow \theta (a) \leq \theta (b)$

Dually, the result follows for join homomorphisms.

Hence, a homomorphisms preserves order. The converse, however, is not true. Consider the map φ in the above example, φ preserves order but is neither a meet nor a join homomorphism.

Problem: 3.4.3 : Let L, M be lattices. If $\theta : L \rightarrow M$ is onto homomorphism and L has least element then so has M.

Solution: Let 0 be least element of L. Then $(0) \leq (x)\ \forall\ x \in L$. $\theta(0) \leq \theta(x)$ (θ preserves order). Since θ is onto, any element $y \in M$ is of the for $\theta(a)$, $a \in L$. But $\theta(0) \leq \theta(x)$ $\forall\ x \in L$. i.e., $\theta(0)$ is least element of M.

Similarly, we can show that if L has greatest element, M would also have greatest element. Hence if L is a bounded lattice then so would be any of its homomorphic image. Having defined isomorphism in two ways, it should be our endevour to first establish the equivalence of the two definitions which we achieve through.

Theorem 3.4.4 : Lattices isomorphic as posets are isomorphic as algebras and conversely.

Proof: Let L and M be two lattices isomorphic as posets i.e., $\exists$ a 1-1 onto map $\theta: L \rightarrow M$ s.t., $a \leq b$ in $L \leftrightarrow \theta(a) \leq \theta(b)$ in M. To show that L, M are isomorphic as algebras, we need prove that θ is a homomorphism.

Let $a, b \in L$ be any elements.

Then $a \wedge b \leq a$, $a \wedge b \leq b$

$\Rightarrow \theta(a \wedge b) \leq \theta(a)$, $\theta(a \wedge b) \leq \theta(b)$

$\Rightarrow \theta(a \wedge b)$ is a lower bound of $\{\theta(a), \theta(b)\}$.

Suppose $y \in M$ is any lower bound of $\{\theta(a), \theta(b)\}$.

Since $y \in M$, θ is onto, $\exists\ x \in L$, s.t., $\theta(x) = y$.

So $\theta(x)$ is a lower bound of $\{\theta(a), \theta(b)\}$.

$\Rightarrow \theta(x) \leq \theta(a)$, $\theta(x) \leq \theta(b)$

$\Rightarrow x \leq a$, $x \leq b$

$\Rightarrow$ x is a lower bound of $\{a, b\}$

$\Rightarrow x \leq a \wedge b$

$\theta(x) \leq \theta(a \wedge b)$ in M

or that $\theta(a \wedge b)$ is g.l.b. $\{\theta(a), \theta(b)\}$

i.e. $\theta(a \wedge b) = \theta(a) \wedge \theta(b)$

Dually, we can show $\theta(a \vee b) = \theta(a) \vee \theta(b)$

Hence θ is a homomorphism.

Conversely, let L, M be isomorphic as algebras i.e. $\exists$ a 1-1 onto homomorphism $\theta: L \to M$. We need show $a \le b$ in L θ or θ $(a) \le \theta(b)$ in M.

Now $a \le b \Rightarrow = a \wedge b$

$\Rightarrow \theta(a) = \theta(a \wedge b) = \theta(a) \wedge \theta(b)$

$\Rightarrow \theta(a) \le \theta(b)$

Also

$$\theta(a) \le \theta(b) \Rightarrow \theta(a) = \theta(a) \wedge \theta(b)$$
$$\Rightarrow \theta(a) = \theta(a \wedge b)$$
$$\Rightarrow a \quad a \wedge b$$
$$\Rightarrow a \le b$$

Which proves our assertion.

Problem 3.4.5 : If L_1, L_2, M_1, M_2 are lattices such that $L_1 = M_1$ and $L_2 = M_2$ then show that $L_1 \times L_2 \cong M_1 \times M_2 \cong M_2 \times M_1$.

Solution : Let $f: L_1 \to M_1$ and $g: L_2 \to M_2$ be the given isomorphisms.

Define $\theta: L_1 \times L_2 \to M_1 \times M_2$ s.t.,

$$\theta((a, b)) = (f(a), g(b))$$

Then $\theta((a,b)) = \theta((c,d))$

$\Leftrightarrow (f(a), g(b)) = (f(c); g(d))$

$\Leftrightarrow f(a)=f(c),\ g(b)=g(d)$

$\Leftrightarrow \quad a=c,\ b=d$

$\Leftrightarrow \quad (a, b) = (c, d)$

shows that θ is well defined 1-1 map

Again, $\theta((a,b) \wedge (c, d) = \theta((a \wedge c, b \wedge d))$

$$= (f(a \wedge c), g(b \wedge d))$$
$$= (f(a) \wedge f(c), g(b) \wedge g(d))$$
$$= (f(a), g(b)) \wedge ((f(c), g(d))$$
$$= \theta((a,b)) \wedge \theta((c,d))$$

Similarly, $\theta((a, b) \vee (c, d)) = \theta((a, b)) \vee ((c, d))$

Showing thereby that θ is a homomorphism.

Finally, for any $(m_1, m_2) \in$

$M_1 \times M_2$, since $m_1 \in M_1$ & $m_2 \in M_2$ and f, g are onto,

$\exists l_1 \in L_1$, $L_2 \in L_2$ s.t., $f(l_1) = m_1$, $g(l_2) = m_2$

and $\theta((l_1, l_2)) = (f(l_1), g(l_2)) = (m_1, m_2)$

Shows that θ is onto and hence an isomorphism.

To show $M_1 \times M_2 \cong M_2 \times M_1$, we can define

$\varphi: M_1 \times M_2 \to M_2 \times M_1$ s.t.,

$\varphi: ((m_1, m_2)) = (m_2, m_1)$

It is now easy to verify that φ is an isomorphism.

We conclude this chapter with the following two theorem which are nice generalization of some results of [13].

Theorem 3.4.6 : Homomorphic image of a relatively complemented lattice is relatively complemented.

Proof : Let $\theta : L \to M$ be an onto homomorphism and suppose L is relatively complemented. Let $[a', b']$ be any interval in M, since θ is onto homomorphism, $\exists$ pre images a and b for a', b' respectively such that $\theta(a) = a'$, $\theta(b) = b'$ and $a \angle b$ (as $a' < b'$),

Thus [a, b] is an interval in L.

Let $y \in [a', b'] = [\theta(a), \theta(b)]$ be any element then as before $\exists$ a pre image x of y s. t., $\theta(x) = y$ and $a \leq x \leq b$.

Now L relatively complemented implies that X has a complement x' relative to [a, b],

i. e., $x \wedge x' = a$, $x \vee x' = b$

$\Rightarrow \quad \theta(x) \wedge \theta(x') = \theta(a)$, $\theta(x) \vee \theta(x') = \theta(b)$

$\Rightarrow \quad y \wedge \theta(x') = a'$, $y \vee \theta(x')\, b' = b'$

$\Rightarrow$ θ (x') is complement of y relative to [a', b'],

Thus each element in any interval in M has a complement, giving us the required result.

Definition(kernel of θ): Let θ :L $\rightarrow$M be an onto homomorphism. the set{x$\in$L: θ (x)=o'} where o' is least element of M is called kernel of θ and is denoted by kerθ. If M does not have the zero element, kerθ does not exist.

Theorem 3.4.7 : If $\theta : L \rightarrow M$ is an onto homomorphism, where L, M are lattices and 0' is least element of M, then Ker θ is an ideal of L.

Proof: Since θ is onto, 0' $\in$ M, thus ker θ $\neq \varphi$ as pre image of 0' exists in L.

Now x, y $\in$ ker $\theta \Rightarrow \theta(\chi)=0'=\theta(y)$

$\theta(\chi \vee y)=\theta(\chi) \vee \theta(y)=0' \vee 0' \Rightarrow \chi \vee y \in Ker\,\theta.$

Again $\chi \in Ker\,\theta$, $I \in L$, *gives* $\theta(\chi)=0'$

Also $\theta(x \wedge I)=\theta(x) \wedge \theta(I)=0' \wedge I=0'$

$\Rightarrow$ $x \wedge I \in Ker\ \theta.$

Hence Ker θ. is an ideal of L.

Theorem: 3.4.8. If: θ:L $\rightarrow$L be a homomorphism where L is a complete lattice then $\exists$ some a$\in$L, s.t., θ(a)=a.

Proof: Let S = $\{x \in L \,/\, x \leq \theta\ (x)\}$

Then $S \neq \varphi$ *as* $0 \in S$ *as* $0 \leq \theta\ (0)$ *(Note* $\theta\ (0) \in L$*)*.

Thus S is a non empty subset of a complete lattice and therefore Sup S exists. Let Sup S = a.

Now $x \leq a \quad \forall\ x \in S$

$\Rightarrow$ $\theta\ (x) \leq \theta\ (a) \quad \forall\ x \in S$

$\Rightarrow$ $x \leq \theta\ (x) \leq \theta\ (a) \quad \forall\ x \in S$

$\Rightarrow$ θ (a) is an upper bound of S

$\Rightarrow \quad a \leq \theta\,(a) \quad (Def.\ of\ Sup)$

$\Rightarrow$ $a \in S$ by def. of S and hence a is greatest element os S.

Also $a \leq \theta\,(a) \Rightarrow \theta\,(a) \leq \theta\,(\theta\,(a)\}$

$\Rightarrow \theta\,(a) \in S \ (Def.of\ S)$

a being greatest element of S then gives $\theta\,(a) \leq a$

i.e., $a \leq \theta\,(a) \leq a.$

Hence $\theta\,(a) = a,$ which proves our assertion.

Definition (Dual join homomorphism, Dual isomorphism) : A mapping $\theta : L \rightarrow M$ is called dual meet hom omorphism if $\theta\,(a \wedge b) = \theta\,(a) \vee \theta\,(b)$ and is called a dual join homomorphism if $\theta\,(a \vee b) = \theta\,(a) \wedge \theta\,(b)$.

It is called a dual homomorphism if it satisfies both the above conditions.

A 1-1 onto dual homomorphism is called a dual isomorphism.

CHAPTER 4

SWITCHING CIRCUIT DESIGNS

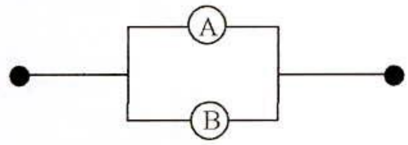
Introduction:

In this chapter we study series combination, Parallel combination, Don't care condition and Bridge circuits. In this chapter using the concepts of Boolean algebra and Boolean polynomials different circuits has been designed and analyzed.

4.1: Series combination, Parallel combination:

Let A, B,.....denote electrical switches, and let A and A' denote switches with the property that if one is on then the other is off, and vice versa. Two switches, A and B, can be connected by wire in a series or parallel combination as follows:

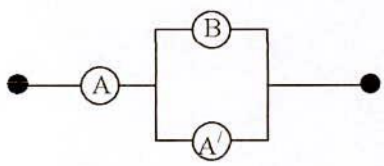
Series Combination , $A \wedge B$



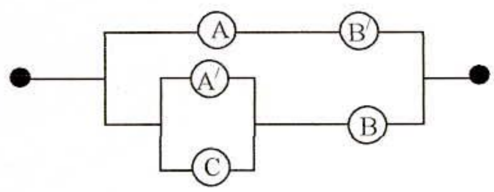
Parallel Combination , $A \vee B$

Let $A \wedge B$ and $A \vee B$, read as A meet B and A join B, denote respectively that A and B are connected in series and A and B are connected in parallel. A Boolean switching circuit design means an arrangement of wires and switches that can be constructed by repeated use of series and parallel combinations; hence it can be described by the use of the connectives $\wedge$ and $\vee$.

Example 4.1.1:



Circuit (1)



Circuit (2)

(1) $A \wedge (B \vee A')$

(2) $(A \wedge B') \vee [(A' \vee C) \wedge B]$

Circuit (1) can be described by $A \wedge (B \vee A')$, and circuit (2) can be described by $(A \wedge B') \vee [(A' \vee C) \wedge B]$.

Now let 1 and 0 denote, respectively, that a switch or circuit is on and that a switch or circuit is off. The next two tables describe the behavior of a series circuit $A \wedge B$ and parallel circuit $A \vee B$

A	B	$A \wedge B$
1	1	1
1	0	0
0	1	0
0	0	0

A	B	$A \vee B$
1	1	1
1	0	1
0	1	1
0	0	0

The next table shows the relationship between a switch A and a switch A'.

A	A'
1	0
0	1

Notice that the above three table are identical with the tables of conjunction, disjunction and negation for statements (and propositions). The only difference is that 1 and 0 are used here instead of T and F .

Theorem 4.1.2 : The algebra of Boolean switching circuits is a Boolean algebra. In order to find the behavior of a Boolean switching circuit, a table is constructed which is analogous to the truth tables for propositions.

Example 4.1.3: Consider circuit(1) in example 4.1.1. What is the behavior of the circuit, that is, when will the circuit be on (i.e.- when will current flow) and when will the circuit be off ? A "Truth" table is constructed for $A \wedge (B \vee A')$ as follows:

A	B	A'	$B \vee A'$	$A \wedge (B \vee A')$
1	1	0	1	1
1	0	0	0	0
0	1	1	1	0
0	0	1	1	0

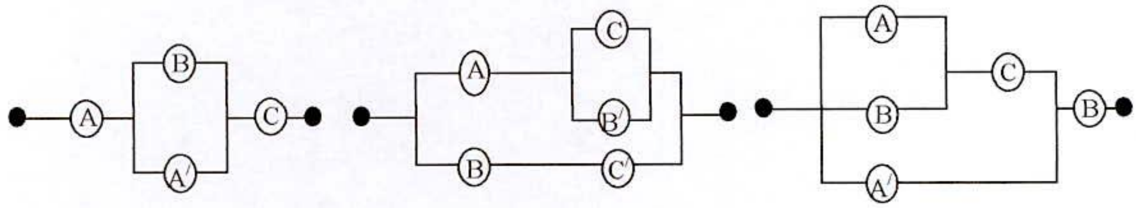
Thus current will flow only if both A and B are on.

Example 4.1.4 : The behavior of circuit (2) in example 4.2.1 is indicated by the following truth table for $(A \wedge B') \vee [(A' \vee C) \wedge B]$:

A	B	C	A'	B'	$A \wedge B'$	$A' \vee C$	$(A' \vee C) \wedge B$	$(A \wedge B') \vee ((A' \vee C) \wedge B)$
1	1	1	0	0	0	1	1	1
1	1	0	0	0	0	0	0	0
1	0	1	0	1	1	1	0	1
1	0	0	0	1	1	0	0	1
0	1	1	1	0	0	1	1	1
0	1	0	1	0	0	1	1	1
0	0	1	1	1	0	1	0	0
0	0	0	1	1	0	1	0	0

Remark 4.1.5 : Any combination of switches using the connectives $\wedge$ and $\vee$ such as $(A \wedge B') \vee [(A' \vee C) \wedge B]$, will also be called a Boolean polynomial.

Problem 4.1.6 : Determine the Boolean polynomial for each of the given three circuits



Circuit (1) Circuit (2) Circuit (3)

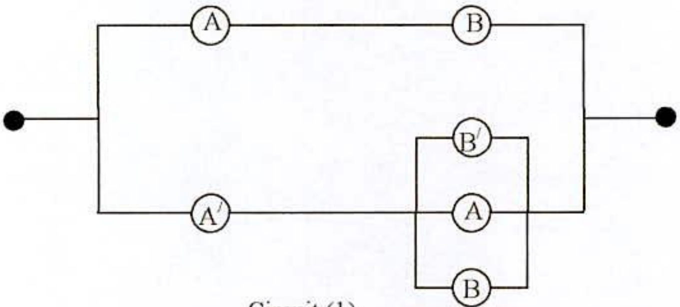
Solution:

(*i*) $A \wedge (B \vee A') \wedge C$ (ii) $[A \wedge (C \vee B')] \vee (B \wedge C')$ (iii) $\{[(A \vee B) \wedge C] \vee A'\} \wedge B$

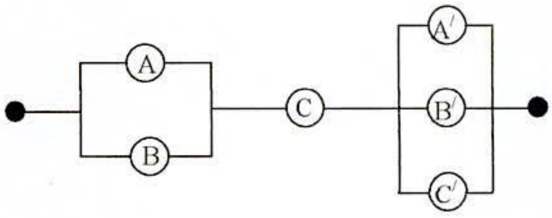
Problem 4.1.7 : Construct a circuit for each of the following Boolean polynomials:

$(1)(A \wedge B) \vee [A' \wedge (B' \vee A \vee B)]$ (2) $(A \vee B) \wedge C \wedge (A' \vee B' \vee C')$

Solution:



Circuit (1)



Circuit (2)

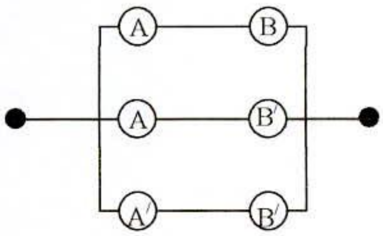
(1) Note that the series circuit $(A \wedge B)$ is in parallel with $(A' \wedge (B' \vee A \vee B))$ which is A' in series with the parallel combination $(B' \vee A \vee B)$.

(2) Note that the parallel circuit $(A \vee B)$ is in series with C and in series with the parallel circuit $(A' \vee B' \vee C')$.

Problem 4.1.8 : Construct an equivalent simpler circuit of the adjacent diagram.

Solution: We first write the Boolean polynomial which represents the circuit.

$(A \wedge B) \vee (A \wedge B') \vee (A' \wedge B')$



Now

$(A \wedge B) \vee (A \wedge B') \vee (A' \wedge B')$

$\equiv [A \wedge (B \vee B')] \vee (A' \wedge B')$

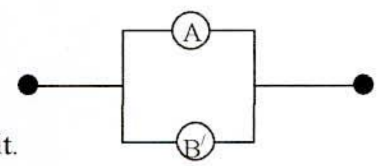
$\equiv [A \wedge U] \vee (A' \wedge B')$

$\equiv A \vee (A' \wedge B')$

$\equiv (A \vee A') \wedge (A \vee B')$

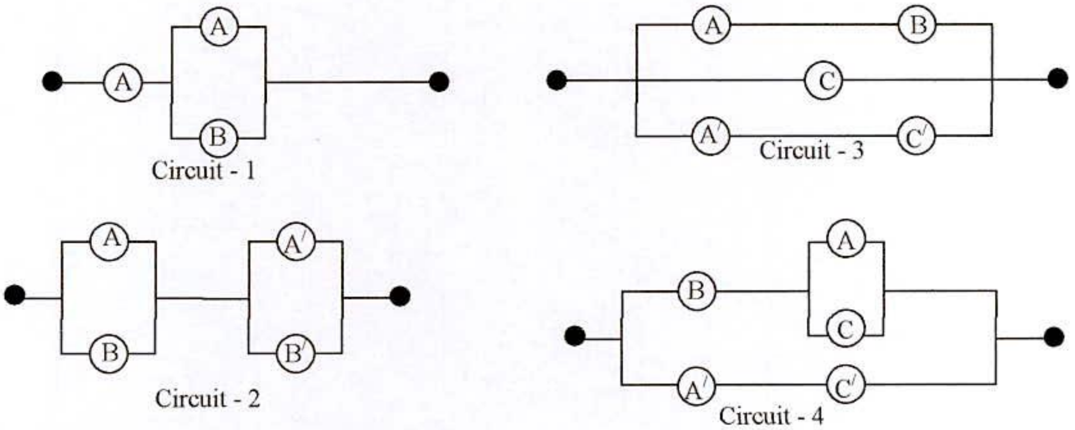
$\equiv U \wedge (A \vee B')$

$\equiv A \vee B'$



Hence the adjacent figure is an equivalent circuit.

Problem 4.1.9 : Determine the Boolean polynomial for each of the given circuits.



Solution: (i) $A \wedge (A \vee B)$ (ii) $(A \vee B) \wedge (A' \vee B')$

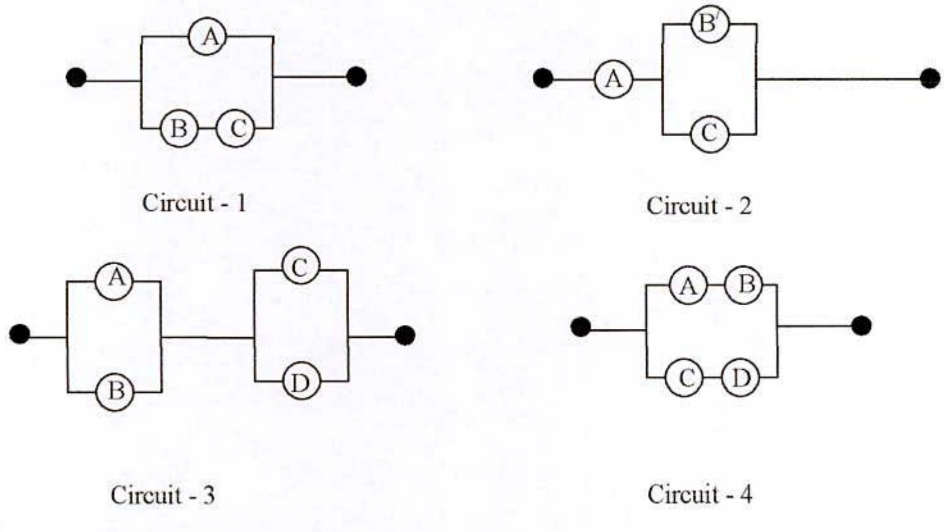
(iii) $(A \wedge B) \vee C \vee (A' \wedge C')$ (iv) $[B \wedge (A \vee C)] \vee (A' \wedge C')$

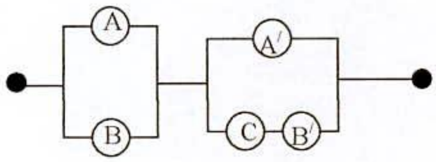
Problem 4.1.10 : Construct a circuit for each Boolean polynomial :

(1) $A \vee (B \wedge C)$ (2) $A \wedge (B \vee C)$ (3) $(A \vee B) \wedge (C \vee D)$ (4) $(A \wedge B) \vee (C \wedge D)$

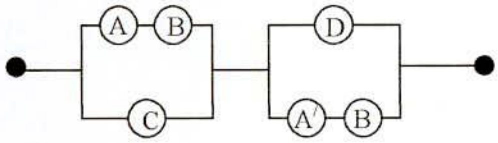
(5) $(A \vee B) \wedge [A' \vee (C \wedge B')]$ (6) $[(A \wedge B) \vee C] \wedge [D \vee (A' \wedge B)]$

Solution:





Circuit - 5



Circuit - 6

4.2: Don't Care Conditions:

suppose we have a circuit specified by a certain function. Suppose further that there is a condition which is impossible to happen. Then there is no harm if we include this condition as part of our circuit (function) as in any case this condition is not to occur. The advantage lies in the fact that addition of an extra condition sometimes leads to a simplification of the given circuit. Such conditions are called don't care conditions.

Example 4.2.1 : Suppose an interview board has three members A, B, C and a candidate is selected only when all the three members say yes. When a member has to say yes, he presses a button provided to him. When all the three press the respective buttons provided to them, a light goes on. If a,b,c stand for A,B,C saying yes, then the light shines only when a & b & c occur. The circuit would be constructed as follows:

Suppose now, member C would say yes if both A and B say yes. Then the condition $a \wedge b \wedge c'$ can never occur. Thus our original circuit can be put equal to $(a \wedge b \wedge c) \vee (a \wedge b \wedge c')$ which simplifies to $(a \wedge b) \vee (c \wedge c') = a \wedge b$ which is simpler than $a \wedge b \wedge c$. Thus $a \wedge b \wedge c'$ is a Don't Care Condition.

Translated in simple language, since C will say yes when A and B say yes, we need not provide C with a button to press. Indeed, if C presses the button when one of A or B have said no, the result is still no. So pressing of the button by C is immaterial.

Problem 4.2.2 : Suppose a circuit is defined by the following table where X denotes don't care condition. Draw the circuits with X=0 and 1

x	y	z	f(x,y,z)
0	0	0	1
1	0	0	1
0	1	0	0
0	0	1	X
1	1	0	0
1	0	1	1
0	1	1	0
1	1	1	0

Solution: The function (with X= 0) is

$$\begin{aligned}
f &= (x'\wedge y'\wedge z')\vee(x\wedge y'\wedge z')\vee(x\wedge y'\wedge z)\\
&= [(x'\vee x)\wedge(y'\wedge z')]\vee(x\wedge y'\wedge z)\\
&= (y'\wedge z')\vee(x\wedge y'\wedge z)\\
&= y'\wedge(z'\vee(x\wedge z))\\
&= y'\wedge((z'\vee x)\wedge(z'\vee z))\\
&= y'\wedge(z'\vee x)
\end{aligned}$$

If X=1, we get

$$\begin{aligned}
f &= y'\wedge(z'\vee x)\vee(x'\wedge y'\wedge z)\\
&= (y'\wedge z')\vee(y'\wedge x)\vee(x'\wedge y'\wedge z)\\
&= y'\wedge[z'\vee x\vee(x'\wedge z)]\\
&= y'\wedge[z'\vee\{(x\vee x')\wedge(x\vee z)\}]\\
&= y'\wedge[z'\vee(x\vee z)]\\
&= y'
\end{aligned}$$

The two circuits are given by

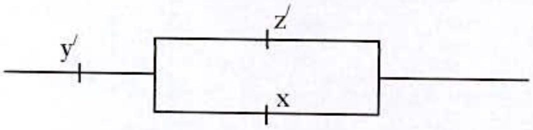


Fig : 4.1

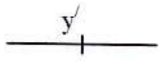


Fig : 4.2

Thus, if X is a don't care condition, the circuit can be simplified by applying the condition.

Example 4.2.3: Suppose we have a circuit defined by the function $f = a \wedge (b' \vee c) \vee (a' \wedge b' \wedge c')$. Suppose further that the conditions $a \wedge b \wedge c'$ and $a' \wedge b' \wedge c$ are impossible to occur.

Then our function can be replaced by

$$\begin{aligned}
f &= [a \wedge (b' \vee c) \vee (a' \wedge b' \wedge c')] \vee (a \wedge b \wedge c') \vee (a' \wedge b' \wedge c) \\
&= a \wedge (b' \vee c) \vee (a' \wedge b' \wedge c') \vee (a' \wedge b' \wedge c) \vee (a \wedge b \wedge c') \\
&= a \wedge (b' \vee c) \vee [(a' \wedge b') \wedge (c' \vee c)] \vee (a \wedge b \wedge c') \\
&= a \wedge (b' \vee c) \vee (a' \wedge b') \vee (a \wedge b \wedge c') \\
&= a \wedge [(b' \vee c) \vee (b \wedge c')] \vee (a' \wedge b') \\
&= a \wedge [(b' \vee c \vee b) \wedge (b' \vee c \vee c')] \vee (a' \wedge b') \\
&= (a \wedge 1) \vee (a' \wedge b') \\
&= (a \vee a') \wedge (a \vee b') \\
&= a \vee b'
\end{aligned}$$

Hence the circuit giver by the original function could be replaced by $a \vee b'$. The two circuits are not equal, but they differ only by cases which would never arise. Hence any one will give the desired result. The two circuits are given by

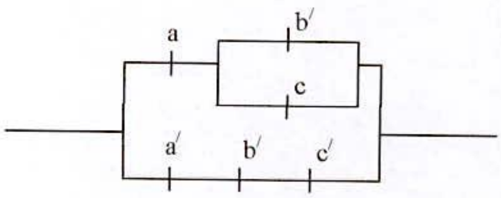


Fig : 4.3

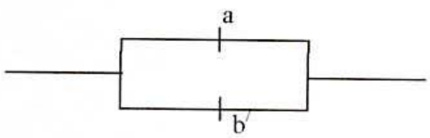


Fig : 4.4

4.3: Bridge Circuits:

Consider the following series-parallel circuit which is represented by the function

$$[a\wedge(d\vee(c\wedge e))]\vee[b\wedge(e\vee(c\wedge d))]$$
$$=(a\wedge d)\vee(a\wedge c\wedge e)\vee(b\wedge e)\vee(b\wedge c\wedge a)$$

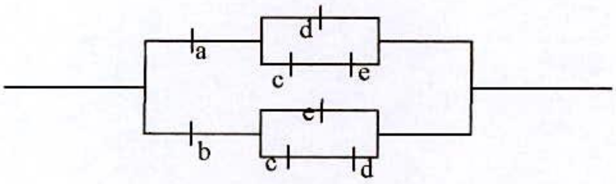


Fig : 4.5

We notice it can also be represented by the following circuit, which is not a series-parallel

circuit

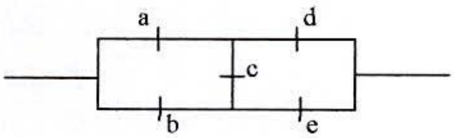


Fig : 4.6

Or

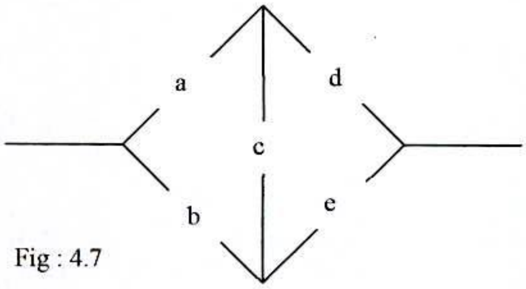


Fig : 4.7

This is called a bridge circuit. Expression (1) explains the routes through which the current could flow.

Consider another function

$$
\begin{aligned}
f &= (a \wedge b) \vee (a \wedge c \wedge z) \vee [\{a \vee (x \wedge y)\} \wedge \{c \vee (z \wedge b)\}] \\
&= (a \wedge b) \vee (a \wedge c \wedge z) \vee [(a \wedge c) \vee (a \wedge (z \wedge b)) \vee (x \wedge y) \wedge c \vee (x \wedge y) \wedge b] \\
&= (a \wedge b) \vee (a \wedge z \wedge c) \vee [(a \vee (x \wedge y)) \wedge c] \vee [(a \vee (x \wedge y)) \wedge (z \wedge b)]
\end{aligned}
$$

It is given by the following circuit

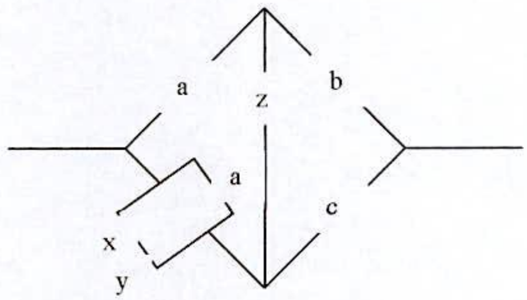


Fig : 4.8

The corresponding series-parallel circuit being

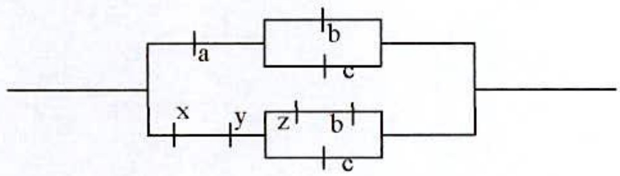


Fig : 4.9

Consider now the bridge circuit

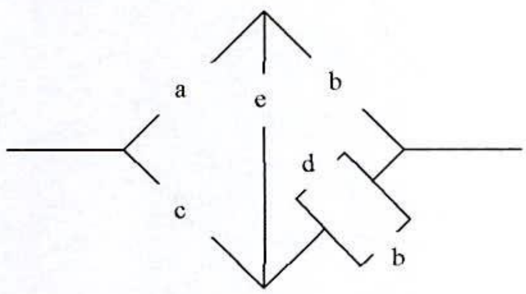


Fig : 4.10

The different paths through which the current can flow are given by

$a \wedge b, a \wedge e \wedge (d \vee b'), c \wedge (d \vee b')$ and $c \wedge e \wedge b$

Thus the function representing this circuit is

$(a \wedge b) \vee (a \vee e \wedge (d \vee b')) \vee [c \wedge (d \vee b')] \vee (c \wedge e \wedge b)$
$= \{a \wedge [b \vee (e \wedge d) \vee (e \wedge b')]\} \vee \{c \wedge [d \vee b' \vee e]\}$
$= a \wedge [b \vee e \vee (e \wedge d)] \vee c \wedge (d \vee b' \vee e)$
$= a \wedge (b \vee e) \vee c \wedge (d \vee b' \vee e)$ [as $b \vee (e \wedge b') = (b \vee e)$]

Now this is represented by the series-parallel circuit

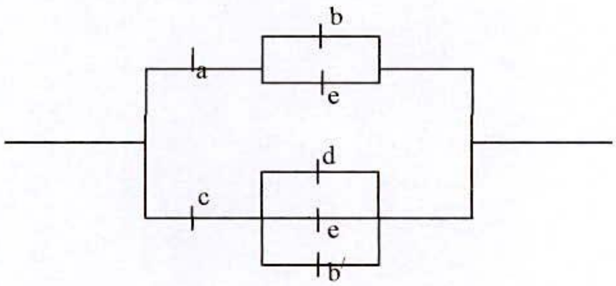


Fig : 4.11

One may remark here that it is not essential that the bridge circuit corresponding to a series- parallel.

Circuit would always have lesser number of switches.

For, instance, the bridge circuit

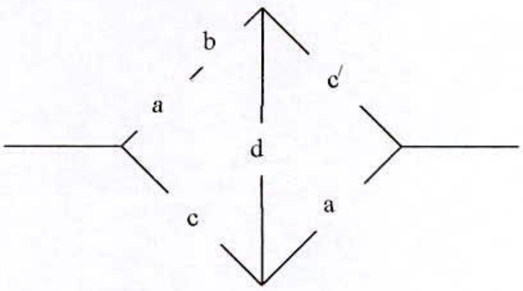


Fig : 4.12

which is represented by

$$[(a \wedge b) \wedge c'] \vee [a \wedge b \wedge d \wedge a] \vee (c \wedge a) \wedge (c \wedge d \wedge c')$$
$$= (a \wedge b \wedge c') \vee (a \wedge b \wedge d) \vee (c \wedge a)$$
$$= a \wedge [b \wedge (c' \vee d) \vee c]$$
$$= a \wedge [(b \wedge c') \vee c \vee (b \wedge d)]$$
$$= a \wedge [(b \vee c) \vee (b \wedge d)]$$
$$= a \wedge [(c \vee b) \vee (b \wedge d)]$$
$$= a \wedge [c \vee (b \vee (b \wedge d))]$$
$$= a \wedge (c \vee b)$$

Which has series parallel circuit

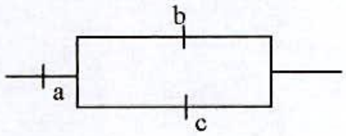


Fig : 4.13

having lesser number of switches.

Reader with a sharp eye would have noticed that the above conversions of bridge circuits to series parallel circuits actually involve the Wye to delta transformations.

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