Steady MHD Free Convection Heat and Mass Transfer Flow about a Vertical Porous Surface with Thermal Diffusion and Induced Magnetic Field

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A thesis submitted in partial fulfillment of the requirements for the degree of
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DECLARATION

This is to certify that the thesis work entitled **"Steady MHD Free Convection Heat and Mass Transfer Flow About a Vertical Porous Surface with Thermal Diffusion and Induced Magnetic Field"** has been carried out by **Md. Afruz-Zaman** in the department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

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ABSTRACT

In this study the thermal diffusion effect on the steady laminar free convection flow and heat transfer of viscous incompressible MHD electrically conducting fluid above a vertical porous surface is considered under the influence of an induced magnetic field. The governing non-dimensional equations relevant to the problem, containing the partial differential equations, are transformed by usual similarity transformations into a system of coupled non-linear ordinary differential equations and will be solved analytically by using the perturbation technique. On introducing the non-dimensional concept and applying Boussinesq's approximation, the solutions for velocity field, temperature distribution, mass concentration and induced magnetic field to the second order approximations are obtained for different selected values of the established dimensionless parameters. The influences of these various establish parameters on the velocity and temperature fields, mass concentration and the induced magnetic fields are exhibited under certain assumptions and are studied graphically. The effects of these dimensionless parameters on the coefficients of skin friction and heat transfer are also studied in tabular form in the present analysis. It is observed that the effects of thermal-diffusion and suction have great importance on the velocity, temperature, induced magnetic fields and mass concentration for several fluids considered, so that their effects should be taken into account with other useful parameters associated. It is also found that the dimensionless Prandtl number, Grashof number, Modified Grashof number and magnetic parameter have an appreciable influence on the concerned independent variables. Further, for more accuracy of the analytical approximate results, a numerical solution have been obtained by using standard initial value solver numerical procedure based on the sixth order Runge-Kutta integration scheme along with Nachtsheim-Swigert iteration technique. Finally, a comparison has been made between the numerical results and analytical approximate results and a very good agreement is found between the results.

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Nomenclature

x, y	Cartesian coordinates
t	time
f_w	Suction parameter
u	Fluid velocity in the x direction
v	Fluid velocity in the y direction
k	Thermal conductivity
M	Mach number
μ	Kinematic viscosity
η	Similarity variable
υ	Coefficient of kinematics viscosity
ψ	Stream function
ρ	Fluid density
θ	Dimensionless temperature
γ	Boundary layer thickness
τ_w	Skin friction coefficient
φ	Velocity potential
p	Pressure
$f(\eta)$	Similarity function
β	Coefficient of thermal expansion
q_w	Heat transfer coefficient
u_e	External velocity
L_c	Characteristic length
T	Fluid temperature

C	concentration of species
μ_e	Magnetic permeability
H_0	Applied constant magnetic field
H_x	Induced magnetic field
D_m	Chemical molecular
D_T	Thermal diffusivity
C_w	Concentration of species at the wall
k_T	Thermal diffusion ratio
U_0	Uniform velocity
V_0	Non-zero suction velocity
H_w	Induced magnetic field at the wall
g	Acceleration due to gravity
g_x	x component body force
R_e	Reynolds number, $R_e = \dfrac{UL}{\upsilon_0}$
C_p	Specific heat
G_r	Grashop number
F_r	Froude number, $F_r = \dfrac{U^2}{gL}$
P_r	Prandtl number, $P_r = \dfrac{\mu_0 C_{p0}}{k_0}$
E	Eckert number, $E = \dfrac{U^2}{C_{p0}\Delta T}$
S_c	Schmidt number, $S_c = \dfrac{\upsilon}{D_m}$
S_0	Soret number, $S_0 = \dfrac{(T_w - T_\infty)D_T}{(C_0 - C_\infty)U_0 x}$

INTRODUCTION

The convective heat and mass transfer process takes place due to the buoyancy effects owing to the differences of temperature and concentration, respectively. In dealing with the transport phenomena, the thermal and mass diffusions occurring by the simultaneous action of buoyancy forces are of considerable interest in practice. Further, heat and mass transfer in the presence of magnetic field, which is the subject matter of MHD, has different applications in natural phenomena and in many engineering problems. In recent times, the problems of natural convective heat and mass transfer flows through a porous medium under the influence of a magnetic field have been paid attention of a number of researchers because of their possible applications in many branches of science, engineering and geophysical process. Considering these numerous applications, MHD free convective heat and mass transfer flow in a porous medium have been studied by among others [34], [40] etc. Using Runge-Kutta fourth order technique along with shooting method, Sharma and Shing [42] investigated the unsteady MHD free convection boundary layer flow and heat transfer of viscous incompressible electrically conducting fluid along an isothermal vertical non-conducting porous plate. Choudhary and Sharma [10] studied the laminar mixed convection flow of an incompressible electrically conducting viscous fluid over a continuously moving porous vertical plate with combined buoyancy effects of thermal and mass diffusion under the action of a uniform transverse magnetic field, subject to constant heat and mass flux with induced magnetic field. However, Pantokratoras [30] showed that a moving electrically conducting fluid induced a new magnetic field, which interacts with the applied external magnetic fields and the relative importance of this induced magnetic field depends on the relative value of the magnetic Reynolds number ($R_m >> 1$). Alam [3] studied the steady two-dimensional problem of MHD free convection and mass transfer flow past an infinite vertical porous plate taking into account the effects of thermal diffusion and large suction.

In recent times, the problems of natural convective heat and mass transfer flows through a porous medium under the influence of a magnetic field have been paid attention of a number of researchers because of their possible applications in many branches of science, Engineering and geophysical process. Considering these numerous applications, MHD free convective heat and mass transfer flow in a porous medium have been studied by among others Rapits and Kafoussias [35], Sattar [39], Sattar and Hossain [40] etc. Besides, Kim [25] has been studied the effect of MHD of a micropolar fluid on coupled heat and mass transfer, flowing on a vertical porous plate moving in a porous medium.

Nevertheless, more complicate phenomenon arises between the fluxes and the driving potentials when heat and mass transfer occur simultaneously in case of a moving fluid. It has been observed that an energy flux can be generated not only due to the temperature gradients but also by composition gradients.

Moreover, it has been observed that an energy flux can be generated not only due to the temperature gradients but also by composition gradients. The energy flux produced by a composition gradient is referred to as the diffusion-thermo (Dufour) effect, whereas, mass flux caused by temperature gradients is known as the thermal-diffusion (Soret) effect. In some exceptional cases, for instance, in mixture between gases with very light molecular weight (H_2, He) and of medium molecular weight (N_2, air) the diffusion-thermo (Dufour) effect and in isotope separation the thermal-diffusion (Soret) effect was found to be of a considerable magnitude such that these effects cannot be ignored.

In view of the relative importance of these above mentioned effects many researchers have studied and reported results for these flows of whom the names are [14], [24], [7], [4] and [36] etc. Combined chemical reaction and Soret/Dufour effects on free convection heat and mass transfer in Darcian porous media were studied also very recently by [33]. In these studies it has been identified that Soret and Dufour effects are important for intermediate molecular weight gases in coupled heat and mass transfer in chemical process systems. Alam et al. [5] extensively investigated the Dufour and Soret effects on steady MHD free-forced convective and mass transfer flow past a semi-infinite vertical plate. A numerical study of the natural convection heat and mass transfer about a vertical surface embedded in a saturated porous medium under the influence of a magnetic field has been done by Postelnicu [32], taking into account the diffusion-thermo and thermal-diffusion effects.

Following the study to those of [10], [30] and [32], Hossain and Khatun [20] investigated the Dufour effect on combined heat and mass transfer of a steady laminar mixed free-forced convective flow of viscous incompressible electrically conducting fluid above a semi-infinite vertical porous surface under the influence of an induced magnetic field. They have used the perturbation technique to solve the problem.

Using the Galerkin finite element method, an analysis was performed by Reddy and Rao [37] to study the effect of thermal diffusion on an unsteady MHD free convective mass transfer flow of incompressible electrically conducting fluid past an infinite vertical porous plate with Ohmic dissipation. It is considered that the plate temperature oscillates with the same frequency as that of variable suction velocity and influence of uniform magnetic field is applied normal to the flow.

Based on the assumptions that the magnetic Reynolds number of the flow is of insignificant magnitude, the present study deals with the study of steady two-dimensional MHD free convection heat and mass transfer flow past an infinite vertical porous plate, taking into account the effects of thermal diffusion and large suction with induced magnetic field.

Considering various aspects of an MHD heat and mass transfer flow, the analyses presented here, as mentioned above, are classified mainly into two different methods, one is analytical approximate method and the other is numerical method. The analytical solutions are obtained by the method of perturbation technique. Numerical solution is obtained by the sixth order Runge-Kutta integration scheme along with Nachtsheim-Swigert iteration technique.

Therefore, this thesis is composed of six chapters. In **CHAPTER I**, available information regarding MHD heat and mass transfer flows along with various effects are summarized and discussed from both analytical and numerical point of view. In **CHAPTER II**, the basic governing equations related to the problem considered thereafter are shown in standard vector form and the detailed calculation techniques for the problem are given. **CHAPTER III** is concerned with the analytical solution of the problem based on perturbation technique has been discussed. In **CHAPTER IV**, the perturbation solutions and results discussions are presented. **CHAPTER V** deals with the numerical procedure based on the sixth order Runge-Kutta integration scheme along with Nachtsheim-Swigert iteration technique is discussed. A comparison between the numerical results and analytical approximate results are also given here. In **CHAPTER VI**, the conclusions gained from this work and brief descriptions for further works related to our present rehearse are discussed.

CHAPTER I

Available Information on MHD Heat and Mass Transfer Flows

1.1. Magnetohydrodynamics (MHD)

Magnetohydrodynamics (MHD) is the branch of magneto fluid dynamics, which deals with the flow of electrically conducting fluid in electric and magnetic field. Probably, the largest advancement towards an understanding of such phenomena comes from the field of astrophysics. It has long been suspected that most of the matter in the universe is in the form of plasma or highly ionized gaseous state and much of the basic knowledge in the area of electromagnetic fluid dynamics involved from these studies.

The field of MHD consists of the study of a continuous, electrically conducting fluid under the influence electromagnetic fields. Originally, MHD included only the study of partially ionized gases as well as the other names have been suggested, such as magneto fluid mechanics or magneto aerodynamics, but the original nomenclature has persisted. The essential requirement for problem to be analyzed under the law of MHD is that the continuum approach be applicable.

There are many natural phenomena and engineering problems susceptible to MHD analysis. It is the useful in astrophysics because much of the universe is filled with widely spaced charged particles and permeated by magnetic fields and so the continuum assumption becomes applicable. Engineers employ MHD principles in the design of heat exchangers, pumps and flow meters; in solving space vehicle propulsion, control and reentry problem; in designing communications and radar system; in creating novel power generating systems, and in developing confinement schemes for controlled fusion.

The MHD in the generation of electrical power with the flow of electrically conducting fluid through a right-hand transverse magnetic field is one of the most important applications. Recently, theses experiments with ionized gases have been performed with the hope of producing power on large scale in stationary plants with large magnetic fields. Generation of MHD power on a smaller scale is of interest of space applications.

Generally we know that, to convert the heat energy in to the electricity, several intermediate transformations are necessary. Each of these steps means a loss of energy. This naturally limits the over all efficiency, reliability and compactness of the conversion process. Method for the direct conversion to energy is now increasingly receiving attention. Of these, the fuel converts the chemical of fuel directly into electrical energy; fusion energy utilizes the energy released when two hydrogen molecules fuse into a heavier one, and thermoelectrically power generation uses a thermocouple. MHD power generation is another new process that has received worldwide attention.

The principal MHD effects were first demonstrated in the experiments of Faraday and Ritchie. Faraday [15] find out experiments with flow of mercury in glass tubes placed between poles of a magnet and discovered that a voltage was induced across the tube by the motion of the mercury across the magnetic field, perpendicular to the direction of flow and to the magnetic field. Faraday observed that the current generated by this induced voltage interacted with the magnetic field to slow down the motion of the fluid, and he was aware of the fact that the current produced its own magnetic fluid that obeyed Ampere right-hand rule and thus, in turn distorted the field of magnet.

Ritchie contemporary of Faraday [15] discovered in 1832 that when an electric field was applied to a conducting fluid perpendicularly to a magnetic field, it pumped the fluid in a direction perpendicular to both fields. Faraday also suggested that electrical power could be generated in a load circuit by the interaction of a flowing conducting fluid and a magnetic field.

The first astronomical application of the MHD theory occurred in 1899, when Bigalow suggested that the sun as a gigantic magnetic system. It remained, however, for Alfven [6] to make a most significant contribution by discovering MHD waves in the sun. These waves are produced by disturbances which propagate simultaneously in the conducting fluid and the magnetic field. The analogy that explains the generation of an Alfven wave is that of a harp string plucked while submerged in a fluid. The string provides the elastic force and the fluid provides the inertia force, and they combine to propagate a perturbing wave through the fluid and the string.

In summary, MHD phenomena result from the mutual effect of a magnetic field and conducting fluid flowing across it. Thus, an electromagnetic force is produced in a fluid flowing across a transverse magnetic field, and the resulting current and magnetic field combine to produce a force that resists the fluid's motion. The current also generates its own magnetic field which distorts the original magnetic field. An opposing or pumping force on the fluid can be produced by applying an electric field perpendicularly to the magnetic field. Disturbance in either the magnetic field or the fluid can propagate in both to produce MHD waves, as well as upstream and downstream-wake phenomena. The science of MHD is the detailed study of these phenomena, which occur in nature and are produced in engineering devices.

1.2. Electromagnetic Equations

MHD equations are the ordinary electromagnetic and hydrodynamic equations which have been modified to take account of the interaction between the motion of the fluid and electromagnetic field. The basic laws of electromagnetic theory are all contained in special theory of relativity. But it is always assumed that all velocities are small in comparison to the speed of light.

Before writing down the MHD equations we will first of all notice the ordinary electromagnetic equations (Cramer and Pai [12]). The mathematical formulation of the electromagnetic theory is known as Maxwell's equations which explore the relation of basic field quantities and their production. The Maxwell's electromagnetic equations are given by

Charge continuity $$\nabla.\mathbf{D} = \rho_c \tag{1.1}$$

Current continuity $$\nabla.\mathbf{J} = -\frac{\partial \rho_c}{\partial t} \tag{1.2}$$

Magnetic field continuity $$\nabla.\mathbf{B} = 0 \tag{1.3}$$

Ampere's law $$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{1.4}$$

Faraday's law $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.5}$$

Constitutive equations for D and B $$\mathbf{D} = \in' \mathbf{E} \tag{1.6}$$

$$\mathbf{B} = \mu_c \mathbf{H} \tag{1.7}$$

Lorentz force on a change $$\mathbf{F}_\mathrm{p} = q'(\mathbf{E} + \mathbf{q}_\mathrm{p} \times \mathbf{B}) \tag{1.8}$$

Total current density flow $$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{q} \times \mathbf{B}) + \rho_c \mathbf{q} \tag{1.9}$$

In equations (1.1) – (1.9), **D** is the displacement current, ρ_c is the charge density, **J** is the current density, **B** is the magnetic induction, **H** is the induced magnetic field, **E** is the electric field, ε' is the electrical permeability of the medium, μ_c is the magnetic permeability of the medium, $\mathbf{q_p}$ is the velocity of the charge, σ is the electrical conductivity, **q** is the velocity of the fluid and $\rho_c\,\mathbf{q}$ is the convection current due to charges moving with the fluid.

1.3. Fundamental Equations of fluid Dynamics of Viscous Fluids

In the study of fluid flow one determines the velocity distribution as well as the states of the fluid over the whole space for all time. There are six unknowns namely, the three components (u,v,w) of velocity **q**, the temperature T, the pressure p and the density ρ of the fluid, which are function of special co-ordinates and time. In order to determine these unknown we have the flowing equations:

(a) Equation of state, which connects the temperature, the pressure and the density of the fluid.

$$p = \rho RT \tag{1.10}$$

For an incompressible fluid the equation of state simply

$$\rho = \text{constant} \tag{1.11}$$

(b) Equation of continuity, which gives relation of conservation of mass of the fluid. The equation of continuity for a viscous incompressible fluid is

$$\nabla.\mathbf{q} = 0 \tag{1.12}$$

(c) Equation of motion, also known as the Navier-Stokes equations, which give the relations of the conservation of momentum of the fluid. For viscous incompressible fluid the equation of motion is

$$\rho\frac{D\mathbf{q}}{Dt} = \mathbf{F} - \nabla p + \mu\nabla^2\mathbf{q} \tag{1.13}$$

where F is the body force per unit volume and the last term on the right hand side represents the force per unit volume due to viscous stresses and p is the pressure. The operator,

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

is known as the material derivative or total derivative with respect to time which gives the variation of a certain quantity of the fluid particle with respect to time. Also ∇^2 represents the Laplacian operator.

(d) The equation of energy, which gives the relation of conservation of energy of the fluid. For an incompressible fluid with constant viscosity and heat conductivity the energy equation is

$$\rho C_p \frac{DT}{Dt} = \frac{\partial Q}{\partial t} + k\nabla^2 T + \phi \tag{1.14}$$

C_p is the specific heat at constant pressure, $\frac{\partial Q}{\partial t}$ is the rate of heat produced per unit volume by external agencies, k is the thermal conductivity of the fluid, ϕ is the viscous dissipation function for an incompressible fluid

$$\phi = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{1}{2}\left(Y_{xy}^2 + Y_{yz}^2 + Y_{zx}^2\right)\right]$$

where

$$Y_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$Y_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$Y_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

(e) The concentration equation for viscous incompressible fluid is

$$\frac{Dc}{Dt} = D_M \nabla^2 C \tag{1.15}$$

C is the concentration and D_M is the chemical molecular diffusivity.

1.4. MHD Approximations

The electromagnetic equation as given in (1.1) – (1.9) are not usually applied in their present form and requires interpretation and several assumptions to provide the set to be used in MHD. In MHD we consider a fluid that is grossly neutral. The charge density ρ_c in Maxwell's equations must then be interpreted as an excess charge density which is in general not large. If we disregard the excess charge density, we must disregard the displacement current. In most problems due to convection of the excess charge are small (Cramer and Pai [12]).

The electromagnetic equations to be used are then as follows:

$$\nabla.\mathbf{D} = \mathbf{0} \tag{1.16}$$

$$\nabla.\mathbf{J} = \mathbf{0} \tag{1.17}$$

$$\nabla.\mathbf{B} = \mathbf{0} \tag{1.18}$$

$$\nabla\times\mathbf{H} = \mathbf{J} \tag{1.19}$$

$$\nabla\times\mathbf{E} = \mathbf{0} \tag{1.20}$$

$$\mathbf{D} = \varepsilon^{/}\,\mathbf{E} \tag{1.21}$$

$$\mathbf{B} = \mu_c\,\mathbf{H} \tag{1.22}$$

$$\mathbf{J} = \sigma\,(\mathbf{E}+\mathbf{q}\times\mathbf{B}) \tag{1.23}$$

1.5. MHD Equations

We will now modify the equations of fluid dynamics suitably to take account of the electromagnetic phenomena.

(a) The MHD equation of continuity for viscous incompressible electrically conducting fluid remains the same

$$\nabla.\mathbf{q} = 0 \tag{1.24}$$

(b) The MHD momentum equation for a viscous incompressible and electrically conducting fluid is

$$\rho\frac{D\mathbf{q}}{Dt} = \mathbf{F} - \nabla p + \mu\nabla^2\mathbf{q} + \mathbf{J}\times\mathbf{B} \tag{1.25}$$

where F is the body force term per unit volume corresponding to the usual viscous fluid dynamic equations and the new term $\mathbf{J} \times \mathbf{B}$ is the force on the fluid per unit volume produced by the interaction of the current and magnetic field (called a Lorentz force).

(c) The MHD energy equation for a viscous incompressible electrically conducting fluid is

$$\rho C_p\frac{DT}{Dt} = \frac{\partial Q}{\partial t} + k\nabla 2T + \phi + \frac{J^2}{\sigma} \tag{1.26}$$

The new term is the Joule heating term and is due to the resistance of the fluid to the flow of current.

(d) The MHD equation of concentration for viscous incompressible electrically conducting fluid remains the same as

$$\frac{DC}{Dt} = D_M\nabla^2 C \tag{1.27}$$

1.6. Some Useful Dimensionless Parameters

(i) Reynolds number (R_e)

The Reynolds number R_e is the most important parameter of the fluid dynamics of a viscous fluid, which is defined by the following ratio

$$R_e = \frac{inertia\ \ force}{viscous\ \ force} = \frac{mass \times acceleration}{shear\, stress \times cross\ sectional\ area} = \frac{\rho L^3 \times \frac{U}{T}}{\mu \times \frac{U}{T} \times L^2} = \frac{\rho L U}{\mu} = \frac{LU}{\upsilon}$$

where, L and U denotes the Characteristic length and velocity respectively and $\upsilon = \frac{\mu}{\rho}$ is the kinematic viscosity (μ is the viscosity and ρ is the density).

For if R_e is small, the viscous force will be predominant and the effect of viscosity will be felt in the whole flow field. On the other hand if R_e is large the inertia force will be predominant and in such case the effect of viscosity to be confined in a thin layer, near to the solid wall or other restricted region, which is known as boundary layer. However if R_e is very large, the flow ceases to be laminar and becomes turbulent. The Reynolds number at which translation from laminar to turbulent occurs is known as critical Reynolds number.

Reynolds in 1883 found that for flow in a circular pipe becomes turbulent when R_e exceeds the critical value 2300,

$$\text{i.e. } R_e = \left[\frac{\overline{U}\, d}{\upsilon}\right]_{crit} = 2300$$

where $\overline{U}$ is the mean velocity and 'd' is the diameter of the pipe.

When the viscous force is pre-dominating force, Reynolds number must be similar for dynamic similarity of two flows.

(ii) Prandtl number (P_r)

The Prandtl number P_r is the ratio of the kinematic viscosity to the thermal diffusivity and is defined by

$$P_r = \frac{\upsilon}{a} = \frac{\frac{\mu}{\rho}}{\frac{k}{\rho c_p}} = \frac{\mu c_p}{k}$$

where c_p is the specific heat at constant pressure and k is the thermal conductivity. The value of

$\frac{k}{\rho c_p}$ is the thermal diffusivity due to the heat condition. The smaller value of $\frac{k}{\rho c_p}$ is, the narrower is the region which affected by the heat condition and it is known as the thermal boundary layer. the value of $\upsilon = \frac{\mu}{\rho}$ show the effect of viscosity of the fluid. Thus the Prandtl number shows that the relative importance of heat conduction and viscosity of a fluid. Evidently P_r varies from fluid to fluid. For air $P_r = 0.72$ (approx), for water at 15.5^0c, $P_r = 7.00$ (approx), for mercury Pr = 0.044, but for high viscous fluid it may be very large, e.g. for glycerin $P_r = 7250$.

(iii) Magnetic Force number (*M*)

The magnetic force number is the ratio of the magnetic force to the inertia force and is defined by

$$M = \frac{magnetic\ \ force}{inertia\ \ force} = \frac{\mu_3^e H_0^2 \sigma' L}{\rho U}$$

(iv) Schmidt number (S_c)

The Schmidt number is the ratio of the viscous diffusivity to the chemical molecular diffusivity and is defined by $S_c = \frac{viscous\ \ diffusivity}{chamical\ \ molecular\ \ diffusivity} = \frac{\upsilon}{D_m}$

(v) Grashof number (G_r)

The Grashof number is defined by $G_r = \frac{g\beta L^3 \nabla T}{\upsilon^2 T}$ and is measure of the relative importance of the buoyancy and viscous forces. The larger it is, stronger is the convective current.

(vi) Modified Grashof number (G_m)

The modified Grashof number is defined by $G_m = \frac{g\beta^* L^3 \nabla C}{\upsilon^2}$

(vii) Soret number (S_0)

The Soret number is defined by $S_0 = \frac{D_T (T_w - T_\infty)}{\upsilon (C_w - C_\infty)}$

(viii) Magnetic diffusivity (P_m)

The magnetic diffusivity is defined by $P_m = \mu_e \sigma' \upsilon$

(ix) Eckert number (E_c)

The Eckert number is defined by $E_c = \dfrac{U^2}{c_p(T_w - T_\infty)}$

1.7. Suction and Injection

For ordinary boundary layer flows of adverse pressure gradients, the boundary layer flow will eventually separate from the surface. Separation of the flow causes many undesirable features over the whole field; for instance if separation occurs on the surface of an airfoil, the lift of the airfoil will decrease and the drag will enormously increase. In some problems we wish to maintain laminar flow without separation. Various means have been proposed to prevent the separation of boundary layer flows; suction and injection are two of them.

The stabilizing effect of the boundary layer development has been well known for several years and till to date it is still the most of efficient, simple and common method of boundary layer control. Hence, the effect of suction on hydromagnetic boundary layer is of great interest in astrophysics. It is often necessary to prevent separation of the boundary layer to reduce the drag and attain high lift values.

Many authors have made mathematical studies on these problems, especially in the case of steady flow. Among them the name of Cobble [11] may be cited who obtained the conditions under which similarity solutions exist for hydromagnetic boundary layer flow past a semi-infinite flat plate with or without suction. Following this, Soundalgekar and Ramanamurthy [46] analyzed the thermal boundary layer. Then Singh [43] studied this problem for large values of suction velocity employing asymptotic analysis in the spirit of Nanbu [27]. Sing and Djukic [44] have again adopted the asymptotic method to study the hydromagnetic effect on the boundary-layer development over a continuously moving plate. In a similar way Bestman [8] studied the boundary layer flow past a semi-infinite heated porous plate for two component plasma.

On the other hand, one of the important problems facing the engineers engaged in high speed flow is the cooling of the surface to avoid the structural failures as a result of frictional heating and other factors. In these respect the possibility of using injection at the surface is a measure to cool the body in the high temperature fluid. Injection of secondary fluid through porous walls is of practical importance in film cooling of turbine blades combustion chambers. In such applications injection usually occurs normal to the surface and the injected fluid may be similar to or different from the primary fluid. In some recent applications, however, it has been recognized that the cooling efficiency can be enhanced by vectored injection at an angle other than 90^0 to the surface. A few workers including Ingar and Swearn [22] have theoretically proved this feature for a linear boundary layer. In addition, most previous calculations have been limited to injection rates ranging from small to moderate. Raptis et al. [35] studied the free convection effects on the flow field of an incompressible, viscous dissipative fluid, past an infinite vertical porous plate which is accelerated in its won plane. The fluid is subjected to a normal velocity of suction/injection proportional to $t^{-1/2}$ and the plate is perfectly insulated, i.e., there is no heat transfer between the fluid and the plate. Hasimoto [19] studied the boundary layer growth on an infinite flat plate started at time t = 0, with uniform suction or injection. Exact solutions of the Navier stokes equations of motion were derived for the case of uniform suction and injection whish was taken to be steady or proportional to $t^{-1/2}$ and the plate is perfectly insulated, i.e., there is no heat transfer between the fluid and the plate. Numerical calculations are also made for the case of impulsive motion of the plate. In the case of injection, velocity profiles have injection points. The qualitative natures of the flow in both the suction and injection cases are the same which are obtained from the results of the corresponding studies on steady boundary layer, so far obtained.

1.8. MHD Boundary Layer and Related Transfer Phenomena

Boundary layer phenomena occurs when the viscous effect may be considered to be confined in a very thin layer near to the boundaries and the non-dimensional diffusion parameter such as the Reynolds number, the Peclet number and the magnetic Reynolds number are very large. The boundary layers are then the velocity and thermal (or magnetic) boundary layers and each of its thickness is inversely proportional to the square root of the associated diffusion number. Prandtl observed from experimental flows that, in classical fluid dynamics boundary layer theory, for large Reynolds number, the viscosity and the thermal conductivity appreciably influences the flow only near a wall. When distance measurements in the flow direction are compared with a characteristic dimension in that direction, transverse measurement compared with the boundary layer thickness and velocities compared with the free stream velocity, the Navier-Stokes and energy equations can be considerably simplified neglecting small quantities. The flow directional component equations only remain and pressure is then only a function of the flow direction and can be determined from the non-viscous flow solution. Also the number of viscous term is reduced to the dominant term and the heat condition flow direction is negligible.

There are two types of MHD boundary layer flows, by considering the limiting cases of a very large and a negligible small magnetic Reynolds number. When the magnetic Reynolds number is large; the magnetic boundary layer thickness is small and is of nearly the same size of the viscous and thermal boundary layers and then the equations of the MHD boundary layer must be solved simultaneously. On the other hand, when the magnetic Reynolds number is very small and the magnetic field is oriented in an arbitrary direction relative to a confining surface; the flow direction component of the magnetic interaction and the corresponding joule heating is only a function of the transverse magnetic field component and the local velocity in the flow direction. Changes in the transverse magnetic boundary layer are negligible. The thickness of the magnetic boundary layer is very large and the induced magnetic field is negligible. In this case the magnetic field moves with the flow and is called frozen mass.

1.8.1. MHD and Heat Transfer

With the advent hypersonic flight, the field of MHD, as defined above, which has been associated largely with liquid metal pumping, has attracted the interest of aerodynamics [13]. It is possible to alter the flow and the heat transfer around high-velocity vehicles provided that the air is sufficiently ionized. Further more, the invention of high temperature facilities such as the shock tube and plasma jet has provided laboratory sources of flowing ionized gas, which provide an incentive for the study of plasma accelerators and generators.

As a result of this, many of the classical problems of fluid mechanics have been reinvestigated. Some of these analyses arose out of the natural tendency of scientists to investigate a new subject. In this case it was the academic problem of solving the equations if fluid mechanics with a new body force and another source of dissipation in the energy equation were presented. Some times there were no practical applications for these results. For Example, natural convection MHD flows have been of interest to the engineering community only since the introduction of liquid metal heat exchangers, whereas the thermal instability investigations described later in section 1.1 are directly applicable to the problems in geophysics and astrophysics.

1.8.2. Free Convection

In the Studies related to heat transfer, considerable effort has been directed towards the convective mode, in which the relative motion of the fluid provides an additional mechanism for the transfer of energy and material, the later being a more important consideration in cases where mass transfer, due to a concentration difference, occurs. Convection is inevitably coupled with the conductive mechanisms, since, although the fluid motion modifies the transport process, the eventual transfer of energy from one fluid element to another in its neighborhood is thorough conduction. Also, at the surface the process is predominantly that of conduction because the relative fluid motion is brought to zero at the surface. A study of the convective heat transfer therefore involves the mechanisms of conduction and sometimes those of radiative processes as well, coupled with that fluid flow. These make the study of this mode of heat or mass transfer very complex, although its importance in technology and in nature can hardly be exaggerated.

The heat transfer in convective mode is divided into two basic processes. If no externally induced flow is provided and flow arises naturally simply owing to the effect of a density difference, resulting from a temperature or concentration difference in a body force field, such as the gravitational field, the process is referred to the natural convection. On the other hand if the motion of the fluid is caused by an external agent such as the externally imposed flow of a fluid stream over a heated object, the process is termed as force convection. In the force convection, the fluid flow may be the result of, for instance, a fan, a blower, the wind or the motion of the heated object itself. Such problems are very frequently encountered in technology where the heat transfers to or from a body is often due to an imposed flow of a fluid at a different temperature from that of a body. On the other side, in the natural convection, the density difference gives rise to buoyancy effects, owing to which the flow is generated. A heated body cooling in ambient air generates such a flow in the region surrounding it. Similarly, the buoyant flow arising from heat rejection to the atmosphere and to other ambient media, circulations arising in heated rooms, in the atmosphere, and in bodies of water, rise of buoyant flow to cause thermal stratification of the medium. Such a temperature inversion and many other such heat transfer process in our natural environment, as well as in many technological applications, are included in the area of natural convection. The flow may also arise owing to concentration differences such as those caused by salinity differences in the sea and by composition differences in chemical processing unit, and these cause a natural convection mass transfer.

Practically some time both processes, natural and forced convection are important and heat transfer is occurred by mixed convection, in which neither mode is truly predominant. The main difference between the two really lies in the word external. A heated body lying in still air loses energy by natural convection. But it also generates a buoyant flow above it and body placed in that flow is subjected to an external flow and it becomes necessary to determine the natural, as well as the forced convection effects in the regime in which the heat transfer mechanisms lie.

When MHD become a popular subject, it was normal that these flows would be investigated with the additional ponder motive body force as well as the buoyancy force. At a first glance there seems to be no practical applications for these MHD solutions, for most heat exchangers utilize liquids, whose conductively is so small that prohibitively large magnetic fields are necessary to influence the flow. But some nuclear plants employ heat exchangers with liquid metal coolants, so the application of moderate magnetic fields to change the convection pattern appears feasible. Another classical natural convection problem is the thermal instability that occurs in a liquid heated from below. This subject is of natural interest to geophysicists and astrophysicists, although some applications might arise in boiling heat transfer.

The basic concepts involved in employing the boundary layer approximation to natural convection flows are very similar to those in forced flows. The main difference lies in the fact is that the pressure in the region beyond the boundary layer is hydrostatic instead of being imposed by an external flow, and that the velocity out side the layer is zero. However the basic treatment and analysis remain the same, the book by Schlichting [41] is an excellent collection of the boundary layer analysis. There are several methods for the solution of the boundary layer equations namely the similarity variable method, the perturbation method, analytical method, numerical method etc. and their details are available in the books by Rosenberg [38], Patanker and Spalding [31].

1.8.3. Heat and Mass Transfer

The basic heat and mass transfer problem is governed by the combined buoyancy effects arising from the simultaneous diffusion of thermal energy and chemical species. Therefore the equations of continuity, momentum, energy, mass diffusion are coupled through the buoyancy terms alone, if there are other effects, such as the Soret and Duffor effects, they are neglected. This would again valid for low species concentration levels. These additional effects have also been considered in several investigations, for example, the work of the Caldwell [9], Groots and Mozur [18], Hurel and Jakeman [21] and Legros, et al. [26].

Somers [45] considered combined buoyancy mechanisms for flow adjacent to a wet isothermal vertical surface in an unsaturated environment. Uniform temperature and uniform species concentration at the surface were assumed and an integral analysis was carried out to obtain the result which is expected to be valid for P_r and Sc values around 1.0 with one buoyancy effect being small compared with the other. Adams and McFadden [1] presented experimental measurements of heat and mass transfer parameters, with opposed buoyancy effects. Gebhart and Pere [16] studied laminar vertical natural convection flows resulting from the combined buoyancy mechanisms in terms of similarity solutions.

Nanousis and Goudas [29] have studied the effects of mass transfer on free convection problem in the stokes problem in the stokes problem for an infinite vertical limiting surface. Georgantopolous and Nanousis [17] have considered the effects of the mass transfer on free convection flow of an electrically conducting viscous fluid (e.g. of a stellar atmosphere, of star) in the presence of transverse magnetic field. Solution for the velocity and skin friction in closed form are obtained with the help of the Laplace transformation technique, and the results obtained for the various values of the parameters Sc, P_r and M are given in graphical form. Raptis and Kafoussias [35] presented the analysis of free convection and mass transfer steady hydro magnetic flow of an electrically conducting viscous incompressible fluid, through a porous medium, occupying a semi infinite region of the space bounded by an infinite vertical porous plate under the action of transverse magnetic field. Agrawal et al.[2] have investigated the effect of Hall current on the combined effect of thermal and mass diffusion of an electrically conducting liquid past an infinite vertical porous plate, when the free stream oscillates about constant non zero mean. The velocity and temperature distributions are shown on graphs for different of parameters.

1.8.4. Thermal Diffusion

In the above mentioned studies heat and mass transfer occur simultaneously in a moving fluid, where the relations between the fluxes and driving potentials are of more complicated nature. In general the thermal diffusion effects is of a small order of magnitude, described by Fourier or Flick's law, is often neglected in heat and mass transfer processes. Mass fluxes can also be created by temperature gradients and this is Soret or thermal diffusion effect. There are however, exceptions. The thermal diffusion effect for instance has been utilized for isotope separation and in mixtures between gases with very light molecular weight (H_2, He) and of medium molecular weight (N_2, air). Kafoussias [23] studied the MHD free convection and mass transfer flow, past an infinite vertical plate moving on its own plane, taken into account the thermal diffusion when (i) the boundary surface is impulsively started moving in its own plane (I.S.P) and (ii) it is uniformly accelerated (U.A.P). The problem is solved with the help of Laplace transformation method and analytical expressions are given for the velocity field as well as for the skin friction for the above-mentioned two different cases. The effect of the velocity and skin friction of the various dimensionless parameters entering into the problem is discussed with the help of graph. For the both cases, it is seen from the figure that the effect of magnetic parameters M is to decreases the fluid (water) velocity inside the boundary layer. This influence of the magnetic field on the velocity field is more evident in the presence of the thermal diffusion. From the same figures is also concluded that the fluid velocity rises due to greater thermal diffusion. Hence, the velocity field is considerably affected by the magnetic field and the thermal diffusion. Nanousis [28] extended the work of Kafoussias [23] to the case of rotating fluid taking into account the Soret effect. The plate is assumed to be moving on its own plane with arbitrary velocity $U_0 f(t')$, where U_0 is a constant velocity and $f(t')$ is a non dimensional function of time t'. The solution of the problem is also obtained with the help of Laplace transformation technique. Analytical expression is given for the velocity field and for the skin friction for two different cases; **Case-I:** When the plate is impulsively started, moving on it own plane and **Case-II**: When the plate is uniformly accelerated, the effect on the velocity field and skin friction of various dimensionless parameters entering into the problem, specially of the Soret number S_0, are discussed with the help of graphs. In the case of an impulsively started plate and uniformly accelerated plate, it is seen the primary velocity to increase of S_0 and the magnetic parameter M. the mass fluxes can also be created by temperature gradients and this is the Soret or the thermal diffusion effect.

CHAPTER II

Basic Equations and Transformations

Cobble [11] showed the condition under which similarity solutions exist to a hydromagnetic flow over a semi-infinite plate in presence of a magnetic field and pressure gradient with or without injection and suction. The heat transfer aspect of this problem has been studied by Soundalgekar and Ramanamurthy [46] taking the effects of suction and injection.

Following the above studies, Singh [43] studied the problem of Soundalgekar and Ramanamurthy [46] for large values of suction parameter by making use of the perturbation technique as has been done by Nanbu [27]. Later Singh and Dikshit [44] studied the hydromagnetic flow past a continuously moving semi-infinite porous plate employing the same perturbation technique. They also derived similarity solutions for large suction. This large suction, in fact, enabled them to obtain analytical solutions and indeed these analytical solutions are of immense value and complement various numerical solutions.

2.1. Governing Equations of the flow

We consider the steady free convection heat and mass transfer flow of a viscous, incompressible, electrically conducting fluid past a semi-infinite vertical electrically non-conducting porous plate. Following the Cartesian coordinate system, the flow is assumed to be in the x-direction, which is taken along the vertical plate in the upward direction, where, y-direction normal to the plate. The schematic view of the flow configuration and coordinates system of the problem are shown in fig

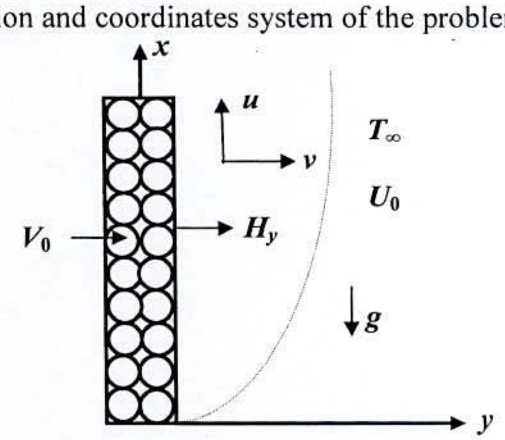

Figure 2.1 : Schematic representation and coordinate system of the problem.

A uniform transverse magnetic field is applied normal to the flow region. Based on the assumptions that the magnetic Reynolds number of the flow is not taken to be of considerable magnitude so that the induced magnetic field is taken into account. The magnetic field is of the form $\mathbf{H} = (H_x, H_y, 0)$ on of electric charge is $\nabla.\mathbf{J} = 0$ where $\mathbf{J} = (J_x, J_y, J_z)$. Since the direction of propagation of electric charge is along the y-axis and the plate is electrically non-conducting, $J_y = 0$ every where within the flow. It is also assumed that the Joule heating effect is small enough and divergence equation for the magnetic field $\nabla.\mathbf{H} = 0$ is of the form $H_y = H_0$. Further, as the plate is infinite extent, all physical variables depend on y only and therefore the equation of continuity is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

whose solution gives $v = -V_0$, where V_0 is the constant velocity of suction normal to the plate and the negative sign indicates that the suction velocity is directed towards the plate surface. In accordance with the above assumptions and initiating the concept of usual Boussinesq's approximation, the basic equations related to the problem incorporating with the Maxwell's equations and generalized Oham's law can be put in the following form:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty) + g\beta^*(C - C_\infty) + \frac{\mu_e H_0}{\rho}\frac{\partial H_x}{\partial y} \tag{2.2}$$

$$u\frac{\partial H_x}{\partial x} + v\frac{\partial H_x}{\partial y} = H_x\frac{\partial u}{\partial x} + H_0\frac{\partial u}{\partial y} + \frac{1}{\sigma\mu_e}\frac{\partial^2 H_x}{\partial y^2} \tag{2.3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2} + \frac{\upsilon}{C_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{\sigma\rho C_p}\left(\frac{\partial H_x}{\partial y}\right)^2 \tag{2.4}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_m\frac{\partial^2 C}{\partial y^2} + D_T\frac{\partial^2 T}{\partial y^2} \tag{2.5}$$

The relevant boundary conditions on the vertical surface and in the uniform stream are defined as follows:

$$\left.\begin{aligned} &u = U_0, v = V_0, T = T_w, C = C_w,\ H_x = H_w,\ at\ \ y = 0 \\ &u = 0, v = 0, T = T_\infty, C = C_\infty, H_x = 0\ \ when\ \ y \to \infty \end{aligned}\right\} \tag{2.6}$$

where g is the acceleration due to gravity, β is the coefficient of thermal expansion, T denotes fluid temperature, C is concentration of species, T_∞ and C_∞ are the temperature and species concentration of the uniform flow, β^* is the concentration expansion coefficient, υ is the Newtonian kinematics viscosity of the fluid, μ_e is the magnetic permeability, H_0 is the applied constant magnetic field, H_x is induced magnetic field, ρ is the density of the fluid, σ is the electrical conductivity, k is the thermal conductivity, D_m and D_T are the chemical molecular and thermal diffusivity, C_p is the specific heat capacity of the fluid at constant pressure, C_w is concentration of species at the wall, k_T is the thermal diffusion ratio, U_0 is the uniform velocity, V_0 is the non-zero suction velocity perpendicular to the wall and H_w is the induced magnetic field at the wall, respectively. In order to simplify the above equations (2.1) – (2.5) with boundary conditions (2.6), we introduce the following transformations, viz:

$$\eta = y\sqrt{\frac{U_0}{2\upsilon x}},\ f'(\eta)=\frac{u}{U_0},\ \theta=\frac{T-T_\infty}{T_w-T_\infty},\ \varphi=\frac{C-C_\infty}{x(C_0-C_\infty)},\ H_x=\sqrt{\frac{\rho}{\mu_e}}\sqrt{\frac{\upsilon U_0}{2x}}H(\eta) \tag{2.7}$$

Then by equation (2.1) we obtain $v=\sqrt{\frac{\upsilon U_0}{2x}}\left[\eta f'(\eta)-f(\eta)\right]$

2.2. Mathematical Calculations

For equation (2.2):

$$\eta = y\sqrt{\frac{U_0}{2\upsilon x}},\quad f'(\eta)=\frac{u}{U_0}\quad \Rightarrow u = f'(\eta)U_0$$

$$\therefore \frac{\partial \eta}{\partial y}=\sqrt{\frac{U_0}{2\upsilon x}},$$

$$\begin{aligned}\frac{\partial \eta}{\partial x} &= \frac{y\sqrt{U_0}}{\sqrt{2\upsilon}}\left(-\frac{1}{2}\right)x^{-3/2}\\ &=\left(-\frac{1}{2}\right)y\sqrt{\frac{U_0}{2\upsilon x}}x^{-1}\\ &=-\frac{1}{2x}\eta\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\{U_0 f'(\eta)\}\\ &=\frac{\partial}{\partial \eta}\{U_0 f'(\eta)\}\frac{\partial \eta}{\partial x}\\ &=U_0 f''(\eta)\left(-\frac{1}{2x}\eta\right)\end{aligned}$$

$$=-\frac{U_0\eta}{2x}f''(\eta)$$

$$\therefore u\frac{\partial u}{\partial x}=U_0f'(\eta)\left[-\frac{U_0\eta}{2x}f''(\eta)\right]$$

$$or, u\frac{\partial u}{\partial x}=-\frac{U_0{}^2\eta}{2x}f''(\eta)f'(\eta) \tag{2.8}$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\{U_0f'(\eta)\}$$

$$=\frac{\partial}{\partial\eta}\{U_0f'(\eta)\}\frac{\partial\eta}{\partial y}$$

$$=U_0f''(\eta)\sqrt{\frac{U_0}{2\upsilon x}}$$

$$=\sqrt{\frac{U_0}{2\upsilon x}}\,U_0f''(\eta)$$

$$\therefore v\frac{\partial u}{\partial y}=\sqrt{\frac{\upsilon U_0}{2x}}\left[\eta f'(\eta)-f(\eta)\right]\sqrt{\frac{U_0}{2\upsilon x}}\,U_0f''(\eta)$$

$$=\frac{U_0{}^2}{2x}\{\eta f'(\eta)-f(\eta)\}f''(\eta) \tag{2.9}$$

$$\frac{\partial^2u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$=\frac{\partial}{\partial\eta}\left(\frac{\partial u}{\partial y}\right)\frac{\partial\eta}{\partial y}$$

$$=\frac{\partial}{\partial y}\left\{\frac{U_0\sqrt{U_0}}{\sqrt{2\upsilon x}}f''(\eta)\right\}\sqrt{\frac{U_0}{2\upsilon x}}$$

$$=\frac{U_0{}^2}{2\upsilon x}f'''(\eta)$$

$$\therefore \upsilon\frac{\partial^2u}{\partial y^2}=\frac{U_0{}^2}{2x}f'''(\eta) \tag{2.10}$$

$$H_x=\sqrt{\frac{\rho}{\mu_e}}\sqrt{\frac{U_0\upsilon}{2x}}H(\eta)$$

$$\frac{\partial H_x}{\partial y}=\sqrt{\frac{\rho U_0 \upsilon}{2\mu_e x}}\; H'(\eta)\;\sqrt{\frac{U_0}{2\upsilon x}}$$

$$=\frac{U_0}{2x}\sqrt{\frac{\rho}{\mu_e}}\, H'(\eta)$$

$$\frac{\partial H_x}{\partial y} \;=\frac{\partial}{\partial y}\left(\sqrt{\frac{\rho}{\mu_e}}\sqrt{\frac{U_0\upsilon}{2x}}\; H(\eta)\right)$$

$$=\frac{\partial}{\partial \eta}\left(\sqrt{\frac{\rho U_0 \upsilon}{2\mu_e x}}\; H(\eta)\right)\frac{\partial \eta}{\partial y}$$

$$\therefore \frac{\mu_e H0}{\rho}\,\frac{\partial Hx}{\partial y}=\frac{\mu_e H_0}{\rho}\;\frac{U_0}{2x}\sqrt{\frac{\rho}{\mu_e}}\, H'(\eta) \;=\frac{H_0 U_0}{2x}\sqrt{\frac{\mu_e}{\rho}}\; H'(\eta) \tag{2.11}$$

Now using equations (2.8) – (2.11) in equation (2.2) we get

$$-\frac{U_0^{\,2}\eta}{2x} f''(\eta) f'(\eta)+\frac{U_0^{\,2}}{2x}\{\eta f'(\eta)-f(\eta)\}f''(\eta) = \frac{U_0^{\,2}}{2x} f'''(\eta) + g\beta(T-T_\infty)$$

$$+ g\beta^*(C-C_\infty)+\frac{H_0 U_0}{2x}\sqrt{\frac{\mu_e}{\rho}}\; H'(\eta)$$

$$\text{or, } -\frac{U_0^{\,2}\eta}{2x} f''(\eta) f'(\eta)+\frac{U_0^{\,2}\eta}{2x} f''(\eta) f'(\eta)-\frac{U_0^{\,2}}{2x} f(\eta) f''(\eta)=\frac{U_0^{\,2}}{2x} f'''(\eta)+g\beta(T-T_\infty)$$

$$+ g\beta^*(C-C_\infty)+\frac{H_0 U_0}{2x}\sqrt{\frac{\mu_e}{\rho}}\; H'(\eta)$$

Dividing both sides by $\dfrac{U_0^{\,2}}{2x}$

$$\text{or, } -f(\eta) f''(\eta)=f'''(\eta)+\frac{2x}{U_0^{\,2}}\, g\beta(T-T_\infty)+\frac{2x}{U_0^{\,2}}\, g\beta^*(C-C_\infty)+\frac{H_0}{U_0}\sqrt{\frac{\mu_e}{\rho}}\; H'(\eta)$$

$$\text{or, } f'''(\eta)+f(\eta) f''(\eta)+\frac{g\beta(T_w-T_\infty)2x}{U_0^{\,2}}\,\frac{(T-T_\infty)}{(T_w-T_\infty)}+\frac{g\beta^*(C_0-C_\infty)2x^2}{U_0^{\,2}}\,\frac{(C-C_\infty)}{(C_0-C_\infty)x}$$

$$+\frac{H_0}{U_0}\sqrt{\frac{\mu_e}{\rho}}\; H'(\eta)=0 \tag{2.12}$$

For equation (2.3):

$$H_x = \sqrt{\frac{\rho}{\mu_e}}\sqrt{\frac{U_0\upsilon}{2x}}H(\eta)$$

$$\frac{\partial H_x}{\partial x} = \sqrt{\frac{\rho}{\mu_e}}\sqrt{\frac{U_0\upsilon}{2}}\frac{\partial}{\partial x}\left\{x^{-1/2}H(\eta)\right\}$$

$$= \sqrt{\frac{\rho}{\mu_e}}\sqrt{\frac{U_0\upsilon}{2}}\left\{x^{-1/2}\frac{\partial}{\partial\eta}H(\eta)\frac{\partial\eta}{\partial x} + H(\eta)\frac{\partial}{\partial x}x^{-1/2}\right\}$$

$$= \sqrt{\frac{\rho}{\mu_e}}\sqrt{\frac{U_0\upsilon}{2}}\left\{x^{-1/2}H'(\eta)\left(-\frac{\eta}{2x}\right) + H(\eta)(-1/2)x^{-3/2}\right\}$$

$$= -\sqrt{\frac{\rho U_0\upsilon}{2\mu_e}}\frac{1}{2x^{3/2}}\{\eta H'(\eta) + H(\eta)\}$$

$$= \sqrt{\frac{\rho}{\mu_e}}\sqrt{\frac{U_0\upsilon}{2}}\left\{\frac{-H'(\eta).\eta}{2x^{3/2}} - \frac{H(\eta)}{2x^{3/2}}\right\}$$

$$\therefore u\frac{\partial H_x}{\partial x} = -\sqrt{\frac{\rho U_0\upsilon}{2\mu_e}}\frac{U_0 f'(\eta)}{2x^{3/2}}\{\eta H'(\eta) + H(\eta)\} \tag{2.13}$$

$$\therefore v\frac{\partial H_x}{\partial y} = \sqrt{\frac{\upsilon U_0}{2x}}[\eta f'(\eta) - f(\eta)]\frac{U_0}{2x}\sqrt{\frac{\rho}{\mu_e}}H'(\eta)$$

$$= \frac{U_0}{2x}\sqrt{\frac{\upsilon U_0\rho}{2x\mu_e}}[\eta f'(\eta) - f(\eta)]H'(\eta) \tag{2.14}$$

$$H_x\frac{\partial u}{\partial x} = -\frac{U_0\eta}{2x}\sqrt{\frac{U_0\rho\upsilon}{2\mu_e x}}H(\eta)f''(\eta) \tag{2.15}$$

$$H_0\frac{\partial u}{\partial y} = H_0\sqrt{\frac{U_0}{2\upsilon x}}U_0 f''(\eta) \tag{2.16}$$

$$\frac{\partial^2 H_x}{\partial y^2} = \frac{U_0}{2x}\sqrt{\frac{U_0\rho}{2\mu_e\upsilon x}}H''(\eta)$$

$$\therefore \frac{1}{\sigma\mu_e}\frac{\partial^2 H_x}{\partial y^2} = \frac{U_0}{2\sigma\mu_e x}\sqrt{\frac{U_0\rho}{2\mu_e\upsilon x}}H''(\eta) \tag{2.17}$$

Now using equations (2.13) – (2.17) in equation (2.3) we get

$$-\sqrt{\frac{\rho U_0 \upsilon}{2\mu_e}}\ \frac{U_0 f'(\eta)}{2x^{3/2}}\ \{\eta\, H'(\eta)+H(\eta)\} + \frac{U_0}{2x}\sqrt{\frac{\upsilon U_0 \rho}{2x\mu_e}}\ [\eta f'(\eta)-f(\eta)]H'(\eta)$$

$$= -\frac{U_0 \eta}{2x}\sqrt{\frac{U_0 \rho \upsilon}{2\mu_e x}} H(\eta) f''(\eta) + \ H_0\sqrt{\frac{U_0}{2\upsilon x}}\ U_0 f''(\eta) + \frac{U_0}{2\sigma\mu_e x}\sqrt{\frac{U_0 \rho}{2\mu_e \upsilon x}}\ H''(\eta)$$

or, $$-\sqrt{\frac{\rho U_0 \upsilon}{2\mu_e}}\ \frac{U_0}{2x^{3/2}}\ \{f(\eta)\, H'(\eta)+H(\eta)\, f'(\eta)\} = -\frac{U_0 \eta}{2x}\sqrt{\frac{U_0 \rho \upsilon}{2\mu_e x}} H(\eta) f''(\eta)$$

$$+ \ H_0\sqrt{\frac{U_0}{2\upsilon x}}\ U_0 f''(\eta) + \frac{U_0}{2\sigma\mu_e x}\sqrt{\frac{U_0 \rho}{2\mu_e \upsilon x}}\ H''(\eta)$$

Dividing both sides by $\frac{U_0}{2\sigma\mu_e x}\sqrt{\frac{U_0 \rho}{2\mu_e \upsilon x}}$

or, $$-\ \mu_e \sigma \upsilon\ \{f(\eta)\, H'(\eta)+H(\eta)\, f'(\eta)\} = -\ \mu_e \sigma \upsilon\ .\ \eta H(\eta) f''(\eta)$$

$$+\sqrt{\frac{\mu_e}{\rho}}\frac{H_0}{U_0}\ \mu_e \sigma \upsilon\ \frac{2U_0 x}{\upsilon} f''(\eta) + H''(\eta)$$

or, $$H''(\eta) +\ \mu_e \sigma \upsilon\ \{f(\eta)\, H'(\eta)+H(\eta)\, f'(\eta)\}\ -\mu_e \sigma \upsilon\ .\eta H(\eta) f''(\eta)$$

$$+2\sqrt{\frac{\mu_e}{\rho}}\frac{H_0}{U_0}\ \mu_e \sigma \upsilon\ \ f''(\eta) = 0 \qquad (2.18)$$

For equation (2.4):

$$\theta = \frac{T-T_\infty}{T_w - T_\infty}$$

$$T = T_\infty + (T_w - T_\infty)\theta$$

$$\frac{\partial T}{\partial x} = (T_w - T_\infty)\frac{\partial \theta}{\partial \eta}.\frac{\partial \eta}{\partial x}$$

$$= (T_w - T_\infty)\theta'(\eta)\left(-\frac{\eta}{2x}\right)$$

$$= -\frac{\eta}{2x}(T_w - T_\infty)\theta'(\eta)$$

$$\therefore u\ \frac{\partial T}{\partial x} = -\frac{\eta U_0}{2x}(T_w - T_\infty)\theta'(\eta)\, f'(\eta) \qquad (2.19)$$

$$\frac{\partial T}{\partial y} = (T_w - T_\infty)\frac{\partial \theta}{\partial \eta}.\frac{\partial \eta}{\partial y}$$

$$= (T_w - T_\infty)\theta'(\eta)\sqrt{\frac{U_0}{2\upsilon x}}$$

$$\therefore v\frac{\partial T}{\partial y} = \sqrt{\frac{\upsilon U_0}{2x}}\left[\eta f'(\eta) - f(\eta)\right](T_w - T_\infty)\theta'(\eta)\sqrt{\frac{U_0}{2\upsilon x}}$$

$$= \frac{U_0}{2x}\left[\eta f'(\eta) - f(\eta)\right](T_w - T_\infty)\theta'(\eta) \tag{2.20}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial}{\partial y}\left\{\frac{\partial T}{\partial y}\right\}$$

$$= \frac{\partial}{\partial \eta}\left\{\frac{\partial T}{\partial y}\right\}\frac{\partial \eta}{\partial y}$$

$$= \frac{\partial}{\partial \eta}\left\{(T_w - T_\infty)\theta'(\eta)\sqrt{\frac{U_0}{2\upsilon x}}\right\}\sqrt{\frac{U_0}{2\upsilon x}}$$

$$= (T_w - T_\infty)\frac{U_0}{2\upsilon x}\theta''(\eta)$$

$$\therefore \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2} = \frac{k}{\rho C_p}(T_w - T_\infty)\frac{U_0}{2\upsilon x}\theta''(\eta) \tag{2.21}$$

$$\frac{\upsilon}{Cp}\left(\frac{\partial u}{\partial y}\right)^2 = \frac{U_0^{\,3}}{2C_p x}\{f''(\eta)\}^2 \tag{2.22}$$

$$\frac{1}{\rho C_p \sigma}\left\{\frac{\partial H_x}{\partial y}\right\}^2 = \frac{U_0^{\,2}}{4C_p \sigma \mu_e x^2}\{H'(\eta)\}^2 \tag{2.23}$$

Now using equations (2.19) – (2.23) in equation (2.4) we get

$$-\frac{\eta U_0}{2x}(T_w - T_\infty)\theta'(\eta)f'(\eta) + \frac{U_0}{2x}\left[\eta f'(\eta) - f(\eta)\right](T_w - T_\infty)\theta'(\eta)$$

$$= \frac{k}{\rho C_p}(T_w - T_\infty)\frac{U_0}{2\upsilon x}\theta''(\eta) + \frac{U_0^{\,3}}{2C_p x}\{f''(\eta)\}^2 + \frac{U_0^{\,2}}{4C_p \sigma \mu_e x^2}\{H'(\eta)\}^2$$

or, $-\frac{U_0}{2x}(T_w - T_\infty)\theta'(\eta)f(\eta) = \frac{k}{\rho C_p}(T_w - T_\infty)\frac{U_0}{2\upsilon x}\theta''(\eta) + \frac{U_0^{\,3}}{2C_p x}\{f''(\eta)\}^2 + \frac{U_0^{\,2}}{4C_p \sigma \mu_e x^2}\{H'(\eta)\}^2$

Dividing both sides by $\frac{k}{\rho C_p}(T_w - T_\infty)\frac{U_0}{2\upsilon x}$

or, $-\frac{\rho C_p \upsilon}{k}\theta'(\eta)f(\eta) = \theta''(\eta) + \frac{U_0^{\,2}\upsilon\rho}{k(T_w - T_\infty)}\{f''(\eta)\}^2 + \frac{U_0 \rho \upsilon}{2\sigma \mu_e x k(T_w - T_\infty)}\{H'(\eta)\}^2$

$$\text{or, } -\frac{\rho C_p \upsilon}{k}\,\theta'(\eta)\,f(\eta) = \theta''(\eta) + \frac{\rho C_p \upsilon}{k}\frac{U_0^{\,2}}{C_p(T_w - T_\infty)}\{f''(\eta)\}^2 + \frac{\rho C_p \upsilon}{k}\frac{U_0^{\,2}}{C_p(T_w - T_\infty)}\frac{1}{2\sigma\mu_e x U_0}\{H'(\eta)\}^2$$

$$\text{or, } \theta''(\eta) + \frac{\rho C_p \upsilon}{k}\,\theta'(\eta)\,f(\eta) + \frac{\rho C_p \upsilon}{k}\frac{U_0^2}{C_p(T_w - T_\infty)}\{f''(\eta)\}^2 + \frac{\rho C_p \upsilon}{k}\frac{U_0^2}{C_p(T_w - T_\infty)}\frac{1}{\sigma\mu_e\upsilon}\frac{\upsilon}{2xU_0}\{H'(\eta)\}^2 = 0$$

$$\text{or, } \theta''(\eta) + \frac{\rho C_p \upsilon}{k}\,\theta'(\eta)\,f(\eta) + \frac{\rho C_p \upsilon}{k}\frac{U_0^2}{C_p(T_w - T_\infty)}\{f''(\eta)\}^2 + \frac{1}{2}\frac{\rho C_p \upsilon}{k}\frac{U_0^2}{C_p(T_w - T_\infty)}\frac{1}{\sigma\mu_e\upsilon}\{H'(\eta)\}^2 = 0 \quad (2.24)$$

For equation (2.5):

$$\varphi = \frac{C - C_\infty}{x(C_0 - C_\infty)} \quad \Rightarrow \quad C = C_\infty + \frac{xU_0}{\upsilon}\varphi(C_0 - C_\infty)$$

$$\frac{\partial C}{\partial x} = \frac{U_0}{\upsilon}(C_0 - C_\infty)\varphi + \frac{xU_0}{\upsilon}(C_0 - C_\infty)\frac{\partial \varphi}{\partial x}$$

$$= \frac{U_0}{\upsilon}(C_0 - C_\infty)\varphi + \frac{xU_0}{\upsilon}(C_0 - C_\infty)\frac{\partial \varphi}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= \frac{U_0}{\upsilon}(C_0 - C_\infty)\varphi + \frac{xU_0}{\upsilon}(C_0 - C_\infty)\varphi'(\eta)\left(-\frac{\eta}{2x}\right)$$

$$= \frac{U_0}{\upsilon}(C_0 - C_\infty)\varphi - \frac{\eta U_0}{2\upsilon}(C_0 - C_\infty)\varphi'(\eta)$$

$$\therefore u\frac{\partial C}{\partial x} = \frac{U_0^{\,2}}{\upsilon}(C_0 - C_\infty)\varphi(\eta)f'(\eta) - \frac{\eta U_0^{\,2}}{2\upsilon}(C_0 - C_\infty)\varphi'(\eta)f'(\eta) \quad (2.25)$$

$$\frac{\partial C}{\partial y} = \frac{xU_0}{\upsilon}(C_0 - C_\infty)\frac{\partial \varphi}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= \frac{xU_0}{\upsilon}(C_0 - C_\infty)\varphi'(\eta)\sqrt{\frac{U_0}{2\upsilon x}}$$

$$\therefore v\frac{\partial C}{\partial y} = \sqrt{\frac{\upsilon U_0}{2x}}\left[\eta f'(\eta) - f(\eta)\right]\frac{xU_0}{\upsilon}(C_0 - C_\infty)\varphi'(\eta)\sqrt{\frac{U_0}{2\upsilon x}}$$

$$= \frac{U_0^{\,2}}{2\upsilon}\left[\eta f'(\eta) - f(\eta)\right](C_0 - C_\infty)\varphi'(\eta) \quad (2.26)$$

$$\frac{\partial^2 C}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial C}{\partial y}\right)$$

$$= \frac{\partial}{\partial \eta}\left(\frac{\partial C}{\partial y}\right)\frac{\partial \eta}{\partial y}$$

$$= \frac{\partial}{\partial \eta}\left\{\frac{xU_0}{\upsilon}(C_0 - C_\infty)\varphi'(\eta)\sqrt{\frac{U_0}{2\upsilon x}}\right\}\sqrt{\frac{U_0}{2\upsilon x}}$$

$$= \frac{xU_0}{\upsilon}(C_0 - C_\infty)\frac{U_0}{2\upsilon x}\varphi''(\eta)$$

$$\therefore \frac{\partial^2 C}{\partial y^2} = \frac{U_0^{\,2}}{2\upsilon^2}(C_0 - C_\infty)\varphi''(\eta) \tag{2.27}$$

$$\frac{\partial^2 T}{\partial y^2} = (T_w - T_\infty)\frac{U_0}{2\upsilon x}\theta''(\eta) \tag{2.28}$$

Now using equations (2.25) – (2.28) in equation (2.5) we get

$$\frac{U_0^{\,2}}{\upsilon}(C_0 - C_\infty)\varphi(\eta)f'(\eta) - \frac{\eta U_0^{\,2}}{2\upsilon}(C_0 - C_\infty)\varphi'(\eta)f'(\eta) + \frac{U_0^{\,2}}{2\upsilon}\left[\eta f'(\eta) - f(\eta)\right](C_0 - C_\infty)\varphi'(\eta)$$

$$= D_m\frac{U_0^{\,2}}{2\upsilon^2}(C_0 - C_\infty)\varphi''(\eta) + D_T\,(T_w - T_\infty)\frac{U_0}{2\upsilon x}\theta''(\eta)$$

or, $$\frac{U_0^{\,2}}{\upsilon}(C_0 - C_\infty)\varphi(\eta)f'(\eta) - \frac{U_0^{\,2}}{2\upsilon}(C_0 - C_\infty)\varphi'(\eta)f(\eta) = D_m\frac{U_0^{\,2}}{2\upsilon^2}(C_0 - C_\infty)\varphi''(\eta)$$

$$+ D_T\,(T_w - T_\infty)\frac{U_0}{2\upsilon x}\theta''(\eta)$$

Dividing both sides by $D_m\dfrac{U_0^{\,2}}{2\upsilon^2}(C_0 - C_\infty)$

or, $$\frac{\upsilon}{D_m}2f'(\eta)\varphi(\eta) - \frac{\upsilon}{D_m}f(\eta)\varphi'(\eta) = \varphi''(\eta) + \frac{T_w - T_\infty}{C_0 - C_\infty}\frac{D_T}{xU_0}\frac{\upsilon}{D_m}\theta''(\eta)$$

or, $$\varphi''(\eta) + \frac{\upsilon}{D_m}f(\eta)\varphi'(\eta) - \frac{\upsilon}{D_m}2f'(\eta)\varphi(\eta) + \frac{T_w - T_\infty}{C_0 - C_\infty}\frac{D_T}{xU_0}\frac{\upsilon}{D_m}\theta''(\eta) = 0 \tag{2.29}$$

Boundary Conditions:

For $\eta = 0$

$$u = U_0 \;\Rightarrow f'(\eta) = \frac{u}{U_0} = \frac{U_0}{U_0} = 1$$

$$v = V_0 \;\Rightarrow \sqrt{\frac{\upsilon U_0}{2x}}\left(\eta f'(\eta) - f(\eta)\right) = V_0$$

or, $$\eta f'(\eta) - f(\eta) = V_0\sqrt{\frac{2x}{\upsilon U_0}}$$

$$\text{as } f'(\eta)=1,\ f_w=-V_0\sqrt{\frac{2x}{\upsilon U_0}}$$

$$\eta-f(\eta)=-f_w \quad \text{as } \eta=0, \qquad f(0)=f_w$$

$$T=T_w \quad \Rightarrow \theta(0)=\frac{T-T_\infty}{T_w-T_\infty}=\frac{T_w-T_\infty}{T_w-T_\infty}=1$$

$$C=C_w \quad \Rightarrow \varphi(0)=\frac{C-C_\infty}{\bar{x}(C_0-C_\infty)}=\frac{C_w-C_\infty}{(C_0-C_\infty)}\frac{C_0-C_\infty}{(C_w-C_\infty)}=1 \ \ as\ \ \bar{x}=\frac{C_w-C_\infty}{(C_0-C_\infty)}$$

$H_x=H_w$

For $\eta\to\infty$

$$u=0 \quad \Rightarrow f'(\infty)=0$$

$$v=0 \qquad \Rightarrow\sqrt{\frac{\upsilon U_0}{2x}}\left(\eta f'(\eta)-f(\eta)\right)=0$$

$$\text{or}, \eta f'(\eta)-f(\eta)=0$$

$$\text{or}, f'(\eta)=\frac{f(\eta)}{\eta} \qquad \therefore f'(\infty)=0 \quad \text{as} \quad \eta\to\infty$$

$$T=T_\infty \quad \Rightarrow \theta(\infty)=\frac{T-T_\infty}{T_w-T_\infty}=\frac{T_\infty-T_\infty}{T_w-T_\infty}=0$$

$$C=C_\infty \quad \Rightarrow \varphi(\infty)=\frac{C-C_\infty}{\bar{x}(C_0-C_\infty)}=\frac{C_\infty-C_\infty}{\bar{x}(C_0-C_\infty)}=0$$

$H_x=0$

Now we define the following dimensionless parameters

Grashoff number : $G_r=\dfrac{g\beta(T_w-T_\infty)2x}{U_0{}^2}$

Modified Grashoff number : $G_m=\dfrac{g\beta^*(C_0-C_\infty)2x^2}{U_0^2}$

Magnetic Parameter : $M=\dfrac{H_0}{U_0}\sqrt{\dfrac{\mu_e}{\rho}}$

Magnetic diffusivity : $P_m=\sigma\upsilon\mu_e$

Prandtl number : $P_r=\dfrac{\rho\upsilon C_p}{k}$

Eckert number : $E_c = \dfrac{U_0^{\ 2}}{C_p(T_w - T_\infty)}$

Schmidt number : $S_c = \dfrac{\upsilon}{D_m}$

Soret number : $S_0 = \dfrac{(T_w - T_\infty)D_T}{(C_0 - C_\infty)U_0 x}$

Therefore, substituting equations (2.7), (2.12), (2.18), (2.24) and (2.29) into (2.2) – (2.5) with boundary conditions (2.6) and introducing the above non-dimensional quantities and ignoring the asterisk (*), we obtain

$$f'''(\eta) + f(\eta)f''(\eta) + G_r\theta + G_m\varphi + M\ H'(\eta) = 0 \tag{2.30}$$

$$H''(\eta) + P_m\{f(\eta)H'(\eta) + H(\eta)f'(\eta)\} - P_m.\eta H(\eta)f''(\eta) + 2M\ P_m\ f''(\eta) = 0 \tag{2.31}$$

$$\theta''(\eta) + \mathrm{P_r}\,\theta'(\eta)f(\eta) + \mathrm{P_r}E_c[\{f''(\eta)\}^2 + \frac{1}{2P_m}\{H'(\eta)\}^2] = 0 \tag{2.32}$$

$$\varphi''(\eta) + S_c f(\eta)\varphi'(\eta) - 2S_c f'(\eta)\varphi(\eta) + S_0 S_c \theta''(\eta) = 0 \tag{2.33}$$

with transformed boundary conditions

$$\left.\begin{array}{l} f(0) = f_w,\ f'(0) = 1,\ \theta(0) = 1\ \varphi(0) = 1,\ H(0) = h = 1 \\ f'(\infty) = 0,\ \theta(\infty) = 0,\ \varphi(\infty) = 0,\ H(\infty) = 0 \end{array}\right\} \tag{2.34}$$

where θ is the dimensionless temperature and $f_w = -V_0\sqrt{\dfrac{2x}{\upsilon U_0}}$, $h = \dfrac{\sqrt{2}\,H_w M}{H_0}$, $\dfrac{U_0 x}{\upsilon} = 1$.

To obtain the more convenient form of above equations (2.30) – (2.33), we further introduce the following transformations:

$$\xi = \eta f_w,\ f(\eta) = f_w F(\xi),\ H(\eta) = f_w L(\xi),\ \theta(\eta) = f_w^{\ 2}G(\xi) \text{ and } \varphi(\eta) = f_w^{\ 2}P(\xi) \tag{2.35}$$

For equation (2.12):

$$\xi = \eta f_w \Rightarrow \frac{\partial\xi}{\partial\eta} = f_w$$

$$f(\eta) = f_w F(\xi)$$

$$f'(\eta) = f_w F'(\xi)\frac{\partial\xi}{\partial\eta} = f_w^{\ 2}F'(\xi)$$

$$f''(\eta) = f_w^{\,2} F''(\xi)\frac{\partial \xi}{\partial \eta} = f_w^{\,3} F''(\xi)$$

$$f'''(\eta) = f_w^{\,3} F'''(\xi)\frac{\partial \xi}{\partial \eta} = f_w^{\,4} F'''(\xi) \tag{2.36}$$

$$f(\eta)f''(\eta) = f_w F(\xi) f_w^{\,3} F''(\xi) = f_w^{\,4} F(\xi)F''(\xi) \tag{2.37}$$

$$G_r\theta(\eta) = G_r f_w^{\,2} G(\xi) \tag{2.38}$$

$$G_m\varphi(\eta) = G_m f_w^{\,2} P(\xi) \tag{2.39}$$

$$MH'(\eta) = Mf_w L'(\xi)\frac{\partial \xi}{\partial \eta} = Mf_w^{\,2} L'(\xi) \tag{2.40}$$

Now using equations (2.36) – (2.40) in equation (2.30) we get

$$f_w^{\,4} F'''(\xi) + f_w^{\,4} F(\xi)F''(\xi) + G_r f_w^{\,2} G(\xi) + G_m f_w^{\,2} P(\xi) + Mf_w^{\,2} L'(\xi) = 0$$

Dividing both sides by $f_w^{\,4}$

$$\text{or, } F'''(\xi) + F(\xi)F''(\xi) + \frac{1}{f_w^{\,2}}[G_r G(\xi) + G_m P(\xi) + ML'(\xi)] = 0 \tag{2.41}$$

For equation (2.13):

$$H(\eta) = f_w L(\xi) \Rightarrow H'(\eta) = f_w L'(\xi)\frac{\partial \xi}{\partial \eta} = f_w^{\,2} L'(\xi)$$

$$H''(\eta) = f_w^{\,2} L''(\xi)\frac{\partial \xi}{\partial \eta} = f_w^{\,3} L''(\xi) \tag{2.42}$$

$$f'(\eta)H(\eta) = f_w^{\,2} F'(\xi) f_w L(\xi) = f_w^{\,3} F'(\xi)L(\xi) \tag{2.43}$$

$$f(\eta)H'(\eta) = f_w F(\xi) f_w^{\,2} L'(\xi) = f_w^{\,3} F(\xi)L'(\xi) \tag{2.44}$$

$$H(\eta)f''(\eta) = f_w L(\xi) f_w^{\,3} F''(\xi) = f_w^{\,4} L(\xi)F''(\xi) \tag{2.45}$$

$$f''(\eta) = f_w^{\,3} F''(\xi) \tag{2.46}$$

Now using equations (2.42) – (2.46) into equations (2.31) – (2.33) we get

$$f_w^{\,3} L''(\xi) + P_m\,[f_w^{\,3} F'(\xi)L(\xi) + f_w^{\,3} F(\xi)L'(\xi)] - P_m \frac{\xi}{f_w} f_w^{\,4} L(\xi)F''(\xi) + 2M\,P_m f_w^{\,3} F''(\xi) = 0$$

Dividing both sides by $f_w^{\,3}$

$$L''(\xi) + P_m[F'(\xi)L(\xi) + F(\xi)L'(\xi)] - P_m\xi\, L(\xi)F''(\xi) + 2M\,P_m F''(\xi) = 0 \tag{2.47}$$

For equation (2.14):

$$\theta(\eta) = f_w^{\,2}G(\xi) \;\Rightarrow \theta'(\eta) = f_w^{\,2}G'(\xi)\frac{\partial \xi}{\partial \eta} = f_w^{\,3}G'(\xi)$$

$$\theta''(\eta) = f_w^{\,3}G''(\xi)\frac{\partial \xi}{\partial \eta} = f_w^{\,4}G''(\xi) \tag{2.48}$$

$$f(\eta)\theta'(\eta) = f_w F(\xi) f_w^{\,3}G'(\xi) = f_w^{\,4}F(\xi)G'(\xi) \tag{2.49}$$

$$\left(f''(\eta)\right)^2 = f_w^{\,6}\left(F''(\xi)\right)^2 \tag{2.50}$$

$$\left(H'(\eta)\right)^2 = f_w^{\,4}\left(L'(\xi)\right)^2 \tag{2.51}$$

Now using equations (2.48) – (2.51) in equation (2.32) we get

$$f_w^{\,4}G''(\xi) + \mathrm{P_r}\, f_w^{\,4}F(\xi)G'(\xi) + \mathrm{P_r}\, E_c\,[\, f_w^{\,6}\left(F''(\xi)\right)^2 + \frac{1}{2P_m}\, f_w^{\,4}\left(L'(\xi)\right)^2\,] = 0$$

Dividing both sides by $f_w^{\,4}$

$$G''(\xi) + \mathrm{P_r}\, F(\xi)G'(\xi) + \mathrm{P_r}\, E_c\left[\frac{1}{f_w^{\,2}}\left(F''(\xi)\right)^2 + \frac{1}{2P_m}\left(L'(\xi)\right)^2\right] = 0 \tag{2.52}$$

For equation (2.15):

$$\varphi(\eta) = f_w^{\,2}P(\xi) \;\Rightarrow \varphi'(\eta) = f_w^{\,2}P'(\xi)\frac{\partial \xi}{\partial \eta} = f_w^{\,3}P'(\xi)$$

$$\varphi''(\eta) = f_w^{\,3}P''(\xi)\frac{\partial \xi}{\partial \eta} = f_w^{\,4}P''(\xi) \tag{2.53}$$

$$f'(\eta)\varphi(\eta) = f_w^{\,2}F'(\xi) f_w^{\,2}P(\xi) = f_w^{\,4}F'(\xi)P(\xi) \tag{2.54}$$

$$f(\eta)\varphi'(\eta) = f_w^{\,3}F(\xi) f_w P'(\xi) = f_w^{\,4}F(\xi)P'(\xi)$$

$$\theta''(\eta) = f_w^{\,3}G''(\xi)\frac{\partial \xi}{\partial \eta} = f_w^{\,4}G''(\xi) \tag{2.55}$$

Now using equations (2.53) – (2.55) in equation (2.33) we get

$$f_w^{\,4}P''(\xi) - 2S_c\, f_w^{\,4}F'(\xi)P(\xi) + S_c\, f_w^{\,4}F(\xi)P'(\xi) + S_0\, S_c\; f_w^{\,4}G''(\xi) = 0$$

Dividing both sides by $f_w^{\,4}$

$$P''(\xi) - 2S_c\, F'(\xi)P(\xi) + S_c\, F(\xi)P'(\xi) + S_0\, S_c\; G''(\xi) = 0 \tag{2.56}$$

Boundary Conditions:

For $\xi = 0$

$$f(\eta) = f_w \Rightarrow f(\eta) = f_w F(\xi) \text{ or, } f_w = f_w F(\xi) \Rightarrow F(\xi) = 1$$

$$f'(\eta) = 1 \Rightarrow f_w^{\,2} F'(\xi) = 1 \text{ or, } F'(\xi) = \frac{1}{f_w^{\,2}} = \varepsilon$$

$$\theta(\eta) = 1 \Rightarrow f_w^{\,2} G(\xi) = 1 \text{ or, } G(\xi) = \frac{1}{f_w^{\,2}} = \varepsilon$$

$$\varphi(\eta) = 1 \Rightarrow f_w^{\,2} P(\xi) = 1, \text{ or, } P(\xi) = \frac{1}{f_w^{\,2}} = \varepsilon$$

$$H_0 = 1 = H(\eta) \Rightarrow f_w L(\xi) = 1, \text{ or, } L(\xi) = \frac{1}{f_w} = \sqrt{\varepsilon}$$

For $\xi \to \infty$

$$f'(\eta) = 0 \Rightarrow F'(\xi) = 0$$

$$\theta(\eta) = 0 \Rightarrow G(\xi) = 0$$

$$\varphi(\eta) = 0 \Rightarrow P(\xi) = 0$$

$$\mathrm{H}(\eta) = 0 \Rightarrow \mathrm{L}(\xi) = 0$$

Therefore, substituting equation (2.35) into equations (2.41), (2.47), (2.52), (2.56) with boundary conditions (2.34)

$$F'''(\xi) + F(\xi)F''(\xi) + \varepsilon\,[G_r G(\xi) + G_m P(\xi) + ML'(\xi)] = 0 \tag{2.57}$$

$$L''(\xi) + P_m [F'(\xi)L(\xi) + F(\xi)L'(\xi)] - P_m \xi\, L(\xi)F''(\xi) + 2M\,P_m F''(\xi) = 0 \tag{2.58}$$

$$G''(\xi) + \mathrm{P_r}\, F(\xi)G'(\xi) + \mathrm{P_r}\, E_c [\,\varepsilon (F''(\xi))^2 + \frac{1}{2P_m} (L'(\xi))^2\,] = 0 \tag{2.59}$$

$$P''(\xi) - 2S_c\, F'(\xi)P(\xi) + S_c F(\xi)P'(\xi) + S_0 S_c\, G''(\xi) = 0 \tag{2.60}$$

with transformed boundary conditions

$$\left.\begin{array}{l} F(\xi) = 1,\ F'(\xi) = \varepsilon,\ G(\xi) = \varepsilon,\ P(\xi) = \varepsilon\ \ L(\xi) = \sqrt{\varepsilon} \quad at \quad \xi = 0 \\ F'(\xi) = 0,\ G(\xi) = 0,\ P(\xi) = 0\ \ L(\xi) = 0 \quad at \quad \xi \to \infty \end{array}\right\} \tag{2.61}$$

CHAPTER III

Perturbation Solution

To obtain a complete solution of the coupled nonlinear system of equations (2.57) – (2.60) under boundary conditions (2.61), we introduce the perturbation approximation. Since the dependent variables F, L, G and P are mostly dependent on ξ only and the fluid is purely incompressible one, we expand the dependent variables F, L, G and P in powers of small perturbation quantity ε such that the terms in ε^4 and its higher order can be neglected. Thus we assume

$$F(\xi)=1+\varepsilon F_1(\xi)+\varepsilon^2 F_2(\xi)+\varepsilon^3 F_3(\xi)+............ \qquad (3.1)$$

$$L(\xi)=\varepsilon L_1(\xi)+\varepsilon^2 L_2(\xi)+\varepsilon^3 L_3(\xi)+.......... \qquad (3.2)$$

$$G(\xi)=\varepsilon G_1(\xi)+\varepsilon^2 G_2(\xi)+\varepsilon^3 G_3(\xi)+........... \qquad (3.3)$$

$$P(\xi)=\varepsilon P_1(\xi)+\varepsilon^2 P_2(\xi)+\varepsilon^3 P_3(\xi)+........... \qquad (3.4)$$

$$\therefore F'(\xi)=\varepsilon F_1'+\varepsilon^2 F_2'+\varepsilon^3 F_3'+..............$$

$$F''(\xi)=\varepsilon F_1''+\varepsilon^2 F_2''+\varepsilon^3 F_3''+..............$$

$$F'''(\xi)=\varepsilon F_1'''+\varepsilon^2 F_2'''+\varepsilon^3 F_3'''+..............$$

From equation (2.57):

$$\varepsilon F_1'''+\varepsilon^2 F_2'''+\varepsilon^3 F_3'''+......+(1+\varepsilon F_1+\varepsilon^2 F_2+\varepsilon^3 F_3+....)(\varepsilon F_1''+\varepsilon^2 F_2''+\varepsilon^3 F_3''+...)$$

$$+\varepsilon[G_r(\varepsilon G_1+\varepsilon^2 G_2+\varepsilon^3 G_3+...)+G_m(\varepsilon P_1+\varepsilon^2 P_2+\varepsilon^3 P_3+...)$$

$$+M(\varepsilon L_1'+\varepsilon^2 L_2'+\varepsilon^3 L_3'+...]=0$$

For first order $O(\varepsilon)$:

$$\varepsilon F_1'''+\varepsilon F_1''=0 \Rightarrow F_1'''+F_1''=0$$

For second order $O(\varepsilon^2)$:

$$\varepsilon^2 F_2'''+\varepsilon^2 F_1 F_1''+\varepsilon^2 F_2''+\varepsilon^2 G_r G_1+\varepsilon^2 G^m P_1+\varepsilon^2 M L_1'=0$$

$$\Rightarrow F_2'''+F_1 F_1''+F_2''+G_r G_1+G_m P_1+M L_1'=0$$

Again

$$L'=\varepsilon L_1'+\varepsilon^2 L_2'+\varepsilon^3 L_3'+......$$

$$L''=\varepsilon L_1''+\varepsilon^2 L_2''+\varepsilon^3 L_3''+.....$$

From equation (2.58):

$$\varepsilon L_1'' + \varepsilon^2 L_2'' + \varepsilon^3 L_3'' + ... + P_m(\varepsilon F_1' + \varepsilon^2 F_2' + \varepsilon^3 F_3' + ...)(\varepsilon L_1 + \varepsilon^2 L_2 + \varepsilon^3 L_3 + ...)$$

$$+ P_m(1 + \varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 F_3 + ...)(\varepsilon L_1' + \varepsilon^2 L_2' + \varepsilon^3 L_3' + ...)$$

$$- P_m \xi(\varepsilon L_1 + \varepsilon^2 L_2 + \varepsilon^3 L_3 + ...)(F_3'' + ...) + 2MP_m(\varepsilon F_1'' + \varepsilon^2 F_2'' + \varepsilon^3 F_3'' + ...) = 0$$

For first order $O(\varepsilon)$:

$$L_1'' + P_m L_1' + 2MP_m F_1'' = 0$$

For second order $O(\varepsilon^2)$:

$$L_2'' + P_m(F_1' L_1 + F_1 L_1' + L_2') - P_m \xi L_1 F_1'' + 2MP_m F_2'' = 0$$

Again

$$G' = \varepsilon G_1' + \varepsilon^2 G_2' + \varepsilon^3 G_3' + ...$$

$$G'' = \varepsilon G_1'' + \varepsilon^2 G_2'' + \varepsilon^3 G_3'' + ...$$

From equation (2.59):

$$\varepsilon G_1'' + \varepsilon^2 G_2'' + \varepsilon^3 G_3'' + ... + P_r(1 + \varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 F_3 + ..)(\varepsilon G_1' + \varepsilon^2 G_2' + \varepsilon^3 G_3' + ..)$$

$$+ P_r E_c \left\{ \varepsilon(\varepsilon F_1'' + \varepsilon^2 F_2'' + \varepsilon^3 F_3'' + ..)^2 + \frac{1}{2P_m}(\varepsilon L_1' + \varepsilon^2 L_2' + \varepsilon^3 L_3' + ..)^2 \right\} = 0$$

For first order $O(\varepsilon)$:

$$G_1'' + P_r G_1' = 0$$

For second order $O(\varepsilon^2)$:

$$G_2'' + P_r(G_2' + F_1 G_1') + \frac{E_c P_r}{2P_m}(L_1')^2 = 0$$

Again

$$P' = \varepsilon P_1' + \varepsilon^2 P_2' + \varepsilon^3 P_3' + ...$$

$$P'' = \varepsilon P_1'' + \varepsilon^2 P_2'' + \varepsilon^3 P_3'' + ...$$

From equation (2.60):

$$\varepsilon P_1'' + \varepsilon^2 P_2'' + \varepsilon^3 P_3'' + ... - 2S_c(\varepsilon F_1' + \varepsilon^2 F_2' + \varepsilon^3 F_3' + ..)(\varepsilon P_1 + \varepsilon^2 P_2 + \varepsilon^3 P_3 + ..)$$

$$+ S_c(\varepsilon P_1' + \varepsilon^2 P_2' + \varepsilon^3 P_3' + ..)(1 + \varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 F_3 + ..)$$

$$+ S_o S_c(\varepsilon G_1'' + \varepsilon^2 G_2'' + \varepsilon^3 G_3'' + ..) = 0$$

For first order $O(\varepsilon)$:

$$P_1'' + S_c P_1' + S_o S_c G_1'' = 0$$

For second order $O(\varepsilon^2)$:

$$P_2'' - 2S_c P_1 F_1' + S_c(P_1'F_1 + P_2') + S_o S_c G_2'' = 0$$

So that the first order equations are:

$$F_1''' + F_1'' = 0 \tag{3.5}$$

$$L_1'' + P_m L_1' + 2MP_m F_1'' = 0 \tag{3.6}$$

$$G_1'' + P_r G_1' = 0 \tag{3.7}$$

$$P_1'' + S_c P_1' + S_o S_c G_1'' = 0 \tag{3.8}$$

with boundary conditions

$$\left.\begin{aligned} &F_1 = 0, F_1' = 1, G_1 = 1, P_1 = 1, L_1 = \frac{1}{\sqrt{\varepsilon}} \ at\ \xi = 0 \\ &F_1' = 0, G_1 = 0, P_1 = 0, L_1 = 0\ \ at\ \xi \to \infty \end{aligned}\right\} \tag{3.9}$$

and the second order equations are :

$$F_2''' + F_1 F_1'' + F_2'' + G_r G_1 + G_m P_1 + ML_1' = 0 \tag{3.10}$$

$$L_2'' + P_m(F_1'L_1 + F_1 L_1' + L_2') - P_m \xi L_1 F_1'' + 2MP_m F_2'' = 0 \tag{3.11}$$

$$G_2'' + P_r(G_2' + F_1 G_1') + \frac{E_c P_r}{2P_m}(L_1')^2 = 0 \tag{3.12}$$

$$P_2'' - 2S_c P_1 F_1' + S_c(P_1'F_1 + P_2') + S_o S_c G_2'' = 0 \tag{3.13}$$

with boundary conditions

$$\left.\begin{aligned} &F_2 = 0, F_2' = 0, G_2 = 0, P_2 = 0, L_2 = 0\ at\ \xi = 0 \\ &F_2' = 0, G_2 = 0, P_2 = 0, L_2 = 0\ \ at\ \xi \to \infty \end{aligned}\right\} \tag{3.14}$$

Now we are interested to solve equation (3.5) – (3.8) with boundary conditions (3.9) and equation (3.10) – (3.13) with boundary conditions (3.14)

From equation (3.5) we have

$$F_1''' + F_1'' = 0$$

The general solution of equation (3.5) is given by

$$F_1 = c_1 + c_2\xi + c_3 e^{-\xi}$$

Applying boundary conditions:

$F_1 = 0,\ F_1' = 1$ as $\xi = 0$ and $F_1' = 0$ as $\xi \to \infty$

$\therefore 0 = c_1 + c_3,\ F_1' = c_2 - c_3 e^{-\xi},\quad 1 = c_2 - c_3$ and $0 = c_2$ so that $c_1 = 0,\ c_2 = 0,\ c_3 = -1$

Hence the complete solution of equation (3.5) $F_1(\xi) = 1 - e^{-\xi}$ (3.15)

Again from equation (3.6) we have

$$L_1'' + P_m L_1' + 2MP_m F_1'' = 0$$

Here $F_1 = 1 - e^{-\xi} \quad \therefore F_1'' = -e^{-\xi}$

So that $L_1'' + P_m L_1' = 2MP_m e^{-\xi}$

The complementary function is obtained by

$$L_{1c} = c_1 + c_2 e^{-P_m \xi}$$

Now the particular integral

$$L_{1p} = \frac{1}{D^2 + P_m D} 2MP_m e^{-\xi}$$
$$= \frac{2MP_m e^{-\xi}}{1 - P_m}$$

The general solution is $L_1 = c_1 + c_2 e^{-P_m \xi} + \dfrac{2MP_m e^{-\xi}}{1 - P_m}$

Using boundary conditions:

$$L_1 = \frac{1}{\sqrt{\xi}} \text{ as } \xi = 0 \text{ and } L_1 = 0 \text{ as } \xi \to \infty$$

$$\therefore \frac{1}{\sqrt{\xi}} = c_1 + c_2 + \frac{2MP_m}{1 - P_m} \text{ and } c_1 = 0, \text{ so that } c_1 = 0,\ c_2 = \frac{1}{\sqrt{\xi}} - \frac{2MP_m}{1 - P_m}$$

Hence the complete solution of equation (3.6) is

$$L_1 = \frac{2MP_m}{1 - P_m} e^{-\xi} + \left(\frac{1}{\sqrt{\xi}} - \frac{2MP_m}{1 - P_m} \right) e^{-P_m \xi}$$

$$\text{or, } L_1 = \frac{2MP_m}{1 - P_m} \left(e^{-\xi} - e^{-P_m \xi} \right) + \frac{1}{\sqrt{\xi}} e^{-P_m \xi}$$

$$\text{or, } L_1(\xi) = A_1 \left(e^{-\xi} - e^{-P_m \xi} \right) + K_1 e^{-P_m \xi} \tag{3.16}$$

where $A_1 = \dfrac{2MP_m}{1 - P_m}$, $K_1 = \dfrac{1}{\sqrt{\xi}}$

Again from equation (3.7) we have

$$G_1'' + P_r G_1' = 0$$

The general solution is $G_1 = c_1 + c_2 e^{-P_r \xi}$

Using boundary conditions:

$$G_1 = 1 \text{ as } \xi = 0 \text{ and } G_1 = 0 \text{ as } \xi \to \infty$$

$\therefore 1 = c_1 + c_2$ and $0 = c_1$ so that $c_1 = 0,\ c_2 = 1$

Hence the complete solution of equation (3.7) is

$$G_1(\xi) = e^{-P_r\xi} \tag{3.17}$$

Again from equation (3.8) we have

$P_1'' + S_c P_1' + S_o S_c G_1'' = 0$

Here $G_1 = e^{-P_r\xi}$ $\therefore G_1' = -P_r e^{-P_r\xi}, G_1'' = P_r^2 e^{-P_r\xi}$

$\therefore P_1'' + S_c P_1' = -S_o S_c P_r^2 e^{-P_r\xi}$

The complementary function is obtained by

$P_{1c} = c_1 + c_2 e^{-S_c\xi}$

and the particular integral is

$$P_{1p} = \frac{1}{D^2 + S_c D}\left(-S_o S_c P_r^2 e^{-P_r\xi}\right)$$

or, $P_{1p} = \dfrac{-S_o S_c P_r e^{-P_r\xi}}{P_r - S_c}$

The general solution is $P_1 = c_1 + c_2 e^{-S_c\xi} - \dfrac{S_o S_c P_r e^{-P_r\xi}}{P_r - S_c}$

Using boundary conditions:

$P_1 = 1$ as $\xi = 0$ and $P_1 = 0$ as $\xi \to \infty$

$\therefore 1 = c_1 + c_2 - \dfrac{S_o S_c P_r}{P_r - S_c}$ and $0 = c_1$ so that $c_1 = 0,\ c_2 = 1 + \dfrac{S_o S_c P_r}{P_r - S_c}$

Hence the complete solution of equation (3.8) is

$$P_1 = \left(1 + \frac{S_o S_c P_r}{P_r - S_c}\right) e^{-S_c\xi} - \frac{S_o S_c P_r}{P_r - S_c} e^{-P_r\xi}$$

$$\text{or, } P_1(\xi) = A_2 e^{-S_c\xi} + A_3 e^{-P_r\xi} \tag{3.18}$$

where $A_2 = 1 - A_3,\ A_3 = -\dfrac{S_o S_c P_r}{P_r - S_c}$

Again from equation (3.10) we have

$F_2''' + F_1 F_1'' + F_2'' + G_r G_1 + G_m P_1 + M L_1' = 0$

Here $F_1 = 1 - e^{-\xi},\ F_1'' = -e^{-\xi},\ G_1 = e^{-P_r\xi}, P_1 = A_2 e^{-S_c\xi} + A_3 e^{-P_r\xi}$,

$L_1 = A_1\left(e^{-\xi} - e^{-P_m\xi}\right) + K_1 e^{-P_m\xi}$,

$L_1' = A_1\left(P_m e^{-P_m\xi} - e^{-\xi}\right) - K_1 P_m e^{-P_m\xi}$

$$\therefore F_2''' + F_2'' = \left(1-e^{-\xi}\right)e^{-\xi} - G_r e^{-P_r\xi} - G_m A_2 e^{-S_c\xi} - G_m A_3 e^{-P_r\xi} - MA_1\left(P_m e^{-P_m\xi} - e^{-\xi}\right) + MK_1 P_m e^{-P_m\xi}$$

The complementary function is obtained by

$$F_{2c} = c_1 + c_2\xi + c_3 e^{-\xi}$$

and the particular integral is

$$F_{2p} = \frac{1}{D^3+D^2}\left(e^{-\xi}+e^{-2\xi}\right) - \frac{G_r}{D^3+D^2}e^{-P_r\xi} - \frac{G_m A_2}{D^3+D^2}e^{-S_c\xi} - \frac{G_m A_3}{D^3+D^2}e^{-P_r\xi}$$
$$-\frac{MA_1P_m}{D^3+D^2}e^{-P_m\xi} + \frac{MA_1}{D^3+D^2}e^{-\xi} + \frac{MK_1P_m}{D^3+D^2}e^{-P_m\xi}$$

$$= \frac{\xi}{3D^2+2D}e^{-\xi} + \frac{1}{4}e^{-2\xi} - \frac{G_r e^{-P_r\xi}}{-P_r^3+P_r^2} - \frac{G_m A_2 e^{-S_c\xi}}{-S_c^{\ 3}+S_c^{\ 2}} - \frac{G_m A_3 e^{-P_r\xi}}{-P_r^3+P_r^2} - \frac{MA_1P_m e^{-P_m\xi}}{-P_m^{\ 3}+P_m^{\ 2}} + \frac{MA_1\xi}{3D^2+2D}e^{-\xi}$$
$$+\frac{MK_1P_m e^{-P_m\xi}}{-P_m^{\ 3}+P_m^{\ 2}}$$

$$= \xi e^{-\xi} + \frac{1}{4}e^{-2\xi} + \frac{G_r e^{-P_r\xi}}{P_r^{\ 2}(P_r-1)} + \frac{G_m A_2 e^{-S_c\xi}}{S_c^{\ 2}(S_c-1)} + \frac{G_m A_3 e^{-P_r\xi}}{P_r^{\ 2}(P_r-1)} + \frac{MA_1 e^{-P_m\xi}}{P_m(P_m-1)} + MA_1\xi e^{-\xi} - \frac{MK_1 e^{-P_m\xi}}{P_m(P_m-1)}$$

The general solution is

$$F_2 = c_1 + c_2\xi + c_3 e^{-\xi} + \frac{1}{4}e^{-2\xi} + (1+MA_1)\xi e^{-\xi} + \frac{(G_r+G_mA_3)e^{-P_r\xi}}{P_r^2(P_r-1)} + \frac{G_mA_2e^{-S_c\xi}}{S_c^{\ 2}(S_c-1)} + \frac{(MA_1-MK_1)e^{-P_m\xi}}{P_m(P_m-1)}$$

Using boundary conditions:

$F_2 = 0,\ F_2' = 0$ as $\xi = 0$ and $F_2' = 0$ as $\xi \to \infty$

$$0 = c_1 + c_3 + \frac{1}{4} + \frac{(G_r+G_mA_3)}{P_r^{\ 2}(P_r-1)} + \frac{G_mA_2}{S_c^{\ 2}(S_c-1)} + \frac{(MA_1-MK_1)}{P_m(P_m-1)}$$

$$F_2' = c_2 - c_3e^{-\xi} - \frac{1}{2}e^{-2\xi} + (1+MA_1)e^{-\xi} - (1+MA_1)\xi e^{-\xi} - \frac{(G_r+G_mA_3)e^{-P_r\xi}}{P_r(P_r-1)} - \frac{G_mA_2e^{-S_c\xi}}{S_c(S_c-1)}$$
$$-\frac{(MA_1-MK_1)e^{-P_m\xi}}{(P_m-1)}$$

$$\therefore 0 = c_2 - c_3 - \frac{1}{2} + (1+MA_1) - \frac{(G_r+G_mA_3)}{P_r(P_r-1)} - \frac{G_mA_2}{S_c(S_c-1)} - \frac{(MA_1-MK_1)}{(P_m-1)}$$

$$c_2 = 0$$

$$c_3 = -\frac{1}{2} + (1+MA_1) - \frac{(G_r+G_mA_3)}{P_r(P_r-1)} - \frac{G_mA_2}{S_c(S_c-1)} - \frac{(MA_1-MK_1)}{(P_m-1)}$$

$$\therefore c_2 = \frac{1}{2} - (1+MA_1) + \frac{(G_r+G_mA_3)}{P_r(P_r-1)} + \frac{G_mA_2}{S_c(S_c-1)} + \frac{(MA_1-MK_1)}{(P_m-1)} - \frac{1}{4} - \frac{(G_r+G_mA_3)}{P_r^2(P_r-1)}$$
$$-\frac{G_mA_2}{S_c^{\ 2}(S_c-1)} - \frac{(MA_1-MK_1)}{P_m(P_m-1)}$$

$$\therefore\ F_2 = \frac{1}{2} - (1 + MA_1) + \frac{(G_r + G_m A_3)}{P_r(P_r - 1)} + \frac{G_m A_2}{S_c(S_c - 1)} + \frac{(MA_1 - MK_1)}{(P_m - 1)} - \frac{1}{4} - \frac{(G_r + G_m A_3)}{P_r^2(P_r - 1)}$$

$$- \frac{G_m A_2}{S_c^2(S_c - 1)} - \frac{(MA_1 - MK_1)}{P_m(P_m - 1)} \left[-\frac{1}{2} + (1 + MA_1) - \frac{(G_r + G_m A_3)}{P_r(P_r - 1)} - \frac{G_m A_2}{S_c(S_c - 1)} \right.$$

$$\left. - \frac{(MA_1 - MK_1)}{(P_m - 1)} \right] e^{-\xi} + \frac{1}{4} e^{-2\xi} + (1 + MA_1)\xi e^{-\xi} + \frac{(G_r + G_m A_3) e^{-P_r \xi}}{P_r^2(P_r - 1)}$$

$$+ \frac{G_m A_2 e^{-S_c \xi}}{S_c^2(S_c - 1)} + \frac{(MA_1 - MK_1) e^{-P_m \xi}}{P_m(P_m - 1)}$$

Hence the complete solution of equation (3.10) is

$$F_2(\xi) = \frac{1}{4} e^{-2\xi} + D_1 \xi e^{-\xi} + D_9 e^{-P_r \xi} + D_{10} e^{-S_c \xi} + D_7 e^{-P_m \xi} + D_8 e^{-\xi} + D_{11} \tag{3.19}$$

where $D_1 = 1 + MA_1$, $D_2 = G_r + G_m A_3$, $D_3 = G_m A_2$, $D_4 = \frac{D_2}{P_r(P_r - 1)}$, $D_5 = \frac{D_3}{S_c(S_c - 1)}$,

$D_6 = \frac{D_{12}}{P_m - 1}$

$D_{12} = MA_1 - MK_1$, $D_7 = \frac{D_{12}}{P_m(P_m - 1)}$, $D_8 = -\frac{1}{2} + D_1 - D_4 - D_5 - D_6$, $D_9 = \frac{D_2}{P_r^2(P_r - 1)}$,

$D_{10} = \frac{D_3}{S_c^2(S_c - 1)}$, $D_{11} = -\frac{1}{4} - D_8 - D_9 - D_{10} - D_7$

Again from equation (3.11) we have

$L_2'' + P_m L_2' + P_m(F_1' L_1 + F_1 L_1') - P_m \xi L_1 F_1'' + 2MP_m F_2'' = 0$

Here $F_1 = 1 - e^{-\xi}$, $F_1' = e^{-\xi}$, $F_1'' = -e^{-\xi}$

$$F_2 = \frac{1}{4} e^{-2\xi} + D_1 \xi e^{-\xi} + D_9 e^{-P_r \xi} + D_{10} e^{-S_c \xi} + D_7 e^{-P_m \xi} + D_8 e^{-\xi} + D_{11}$$

$$F_2' = -\frac{1}{2} e^{-2\xi} - D_1 \xi e^{-\xi} + D_1 e^{-\xi} - D_9 P_r e^{-P_r \xi} - D_{10} S_c e^{-S_c \xi} - D_7 P_m e^{-P_m \xi} - D_8 e^{-\xi}$$

$$F_2'' = e^{-2\xi} + D_1 \xi e^{-\xi} + (D_8 - 2D_1) e^{-\xi} + D_9 P_r^2 e^{-P_r \xi} + D_{10} S_c^2 e^{-S_c \xi} + D_7 P_m^2 e^{-P_m \xi}$$

$$L_1 = A_1 \left(e^{-\xi} - e^{-P_m \xi}\right) + K_1 e^{-P_m \xi}, \quad L_1' = A_1 \left(P_m e^{-P_m \xi} - e^{-\xi}\right) - K_1 P_m e^{-P_m \xi}$$

$$\therefore\ L_2'' + P_m L_2' + P_m \left[e^{-\xi} \left\{ A_1 \left(e^{-\xi} - e^{-P_m \xi}\right) + K_1 e^{-P_m \xi} \right\} + (1 - e^{-\xi}) \left\{ A_1 \left(e^{-\xi} + P_m e^{-P_m \xi}\right) - K_1 P_m e^{-P_m \xi} \right\} \right]$$

$$- P_m \xi \left\{ A_1 \left(e^{-\xi} - e^{-P_m \xi}\right) + K_1 e^{-P_m \xi} \right\} \left(- e^{-\xi}\right) + 2MP_m \left[e^{-2\xi} + D_1 \xi e^{-\xi} + (D_8 - 2D_1) e^{-\xi} + D_9 P_r^2 e^{-P_r \xi} \right.$$

$$\left. + D_{10} S_c^2 e^{-S_c \xi} + D_7 P_m^2 e^{-P_m \xi} \right] = 0$$

$$\therefore L_2'' + P_m L_2' + P_m[A_1 e^{-2\xi} - A_1 e^{-(P_m+1)\xi} + K_1 e^{-(P_m+1)\xi} + A_1 e^{-\xi} + P_m A_1 e^{-P_m\xi} - K_1 P_m e^{-P_m\xi}$$
$$- A_1 e^{-2\xi} - A_1 P_m e^{-(P_m+1)\xi} + K_1 P_m e^{-(P_m+1)\xi}] + A_1 P_m \xi e^{-2\xi} - A_1 P_m \xi e^{-(P_m+1)\xi} + K_1 P_m \xi e^{-(P_m+1)\xi}$$
$$+ 2MP_m e^{-2\xi} + 2MP_m(D_8 - 2D_1)e^{-\xi} + 2MP_m D_1 \xi e^{-\xi} + 2MP_m D_9 P_r^{\ 2} e^{-P_r\xi}$$
$$+ 2MP_m D_{10} S_c^{\ 2} e^{-S_c\xi} + 2MP_m D_7 P_m^{\ 2} e^{-P_m\xi} = 0$$

Or, $$L_2'' + P_m L_2' + P_m\left(K_1 - A_1 - A_1 P_m + K_1 P_m\right)e^{-(P_m+1)\xi} + P_m^{\ 2}(A_1 - K_1)e^{-P_m\xi} + A_1 P_m e^{-\xi}$$
$$+ A_1 P_m \xi e^{-2\xi} + \left(K_1 - A_1\right)\xi e^{-(P_m+1)\xi} + 2MP_m e^{-2\xi} + 2MP_m(D_8 - 2D_1)e^{-\xi} + 2MP_m D_1 \xi e^{-\xi}$$
$$+ 2MP_m D_9 P_r^{\ 2} e^{-P_r\xi} + 2MP_m D_{10} S_c^{\ 2} e^{-S_c\xi} + 2MP_m^{\ 3} D_7 e^{-P_m\xi} = 0$$

The complementary function is obtained by

$$L_{2c} = c_1 + c_2 e^{-P_m\xi}$$

Now the particular integral is

$$L_{2p} = -\frac{1}{D^2 + P_m D}[P_m\left(K_1 - A_1 - A_1 P_m + K_1 P_m\right)e^{-(P_m+1)\xi} + P_m^{\ 2}(A_1 - K_1)e^{-P_m\xi} + A_1 P_m e^{-\xi}$$
$$+ A_1 P_m \xi e^{-2\xi} + \left(K_1 - A_1\right)\xi e^{-(P_m+1)\xi} + 2MP_m e^{-2\xi} + 2MP_m(D_8 - 2D_1)e^{-\xi} + 2MP_m D_1 \xi e^{-\xi}$$
$$+ 2MP_m D_9 P_r^{\ 2} e^{-P_r\xi} + 2MP_m D_{10} S_c^{\ 2} e^{-S_c\xi} + 2MP_m^{\ 3} D_7 e^{-P_m\xi}]$$

$$= -\frac{P_m\left(K_1 - A_1 - A_1 P_m + K_1 P_m\right)e^{-(P_m+1)\xi}}{(P_m+1)^2 - P_m(P_m+1)} - \frac{\xi P_m^{\ 2}(A_1 - K_1)}{2D + P_m}e^{-P_m\xi} - \frac{A_1 P_m}{1 - P_m}e^{-\xi} - \frac{A_1 P_m \xi}{D^2 + P_m D}e^{-2\xi}$$
$$+ \frac{A_1 P_m(2D + P_m)}{\left(D^2 + P_m D\right)^2}e^{-2\xi} - \frac{\xi(K_1 - A_1)}{D^2 + P_m D}e^{-(P_m+1)\xi} + \frac{(K_1 - A_1)(2D + P_m)}{\left(D^2 + P_m D\right)^2}e^{-(P_m+1)\xi} - 2\frac{MP_m}{4 - 2P_m}e^{-2\xi}$$
$$- \frac{2MP_m(D_8 - 2D_1)}{1 - P_m}e^{-\xi} - \frac{2M\xi P_m D_1}{D^2 + P_m D}e^{-\xi} + \frac{2MP_m D_1(2D + P_m)}{\left(D^2 + P_m D\right)^2}e^{-\xi} - \frac{2MP_m D_9 P_r^{\ 2}}{P_r^{\ 2} - P_m P_r}e^{-P_r\xi}$$
$$- \frac{2MP_m D_{10} S_c^{\ 2}}{S_c^{\ 2} - P_m S_c}e^{-S_c\xi} - \frac{2M\xi P_m^{\ 3} D_7}{2D + P_m}e^{-P_m\xi}$$

$$= -\frac{P_m\left(K_1 - A_1\right)(P_m+1)e^{-(P_m+1)\xi}}{(P_m+1)(P_m+1-P_m)} + \xi P_m(A_1 - K_1)e^{-P_m\xi} - \frac{A_1 P_m}{1 - P_m}e^{-\xi} - \frac{A_1 P_m \xi}{4 - 2P_m}e^{-2\xi}$$
$$+ \frac{A_1 P_m(2D + P_m)}{\left(4 - 2P_m\right)^2}e^{-2\xi} - \frac{\xi(K_1 - A_1)}{(P_m+1)^2 - P_m(P_m+1)}e^{-(P_m+1)\xi} + \frac{(K_1 - A_1)(2D + P_m)}{\left\{(P_m+1)^2 - P_m(P_m+1)\right\}^2}e^{-(P_m+1)\xi}$$
$$+ \frac{MP_m}{P_m - 2}e^{-2\xi} + \frac{2MP_m(D_8 - 2D_1)}{P_m - 1}e^{-\xi} + \frac{2M\xi P_m D_1}{P_m - 1}e^{-\xi} + \frac{2MP_m D_1(2D + P_m)}{\left(P_m - 1\right)^2}e^{-\xi}$$
$$- \frac{2MP_m D_9 P_r}{P_r - P_m}e^{-P_r\xi} - \frac{2MP_m D_{10} S_c}{S_c - P_m}e^{-S_c\xi} + 2M\xi P_m^{\ 2} D_7 e^{-P_m\xi}$$

$$= P_m\left(A_1 - K_1\right)e^{-(P_m+1)\xi} + \xi P_m(A_1 - K_1)e^{-P_m\xi} - \frac{A_1P_m}{1-P_m}e^{-\xi} + \frac{A_1P_m\xi}{2\left(P_m-2\right)}e^{-2\xi}$$

$$+\frac{A_1P_m(P_m-4)}{4\left(P_m-2\right)^2}e^{-2\xi} + \frac{\xi(A_1-K_1)}{(P_m+1)}e^{-(P_m+1)\xi} + \frac{(A_1-K_1)(P_m+2)}{(P_m+1)^2}e^{-(P_m+1)\xi} + \frac{MP_m}{P_m-2}e^{-2\xi}$$

$$+\frac{2MP_m(D_8-2D_1)}{P_m-1}e^{-\xi} + \frac{2MP_mD_1\xi e^{-\xi}}{P_m-1} + \frac{2MP_mD_1(P_m-2)}{\left(P_m-1\right)^2}e^{-\xi} + \frac{2MP_mD_9P_re^{-P_r\xi}}{P_m-P_r}$$

$$+\frac{2MP_mD_{10}S_ce^{-S_c\xi}}{P_m-S_c} + 2M\xi {P_m}^2D_7e^{-P_m\xi}$$

The general solution is

$$L_2 = c_1 + c_2e^{-P_m\xi} + P_m\left(A_1 - K_1\right)e^{-(P_m+1)\xi} + \xi P_m(A_1 - K_1)e^{-P_m\xi} - \frac{A_1P_m}{1-P_m}e^{-\xi} + \frac{A_1P_m\xi}{2\left(P_m-2\right)}e^{-2\xi}$$

$$+\frac{A_1P_m(P_m-4)}{4\left(P_m-2\right)^2}e^{-2\xi} + \frac{\xi(A_1-K_1)}{(P_m+1)}e^{-(P_m+1)\xi} + \frac{(A_1-K_1)(P_m+2)}{(P_m+1)^2}e^{-(P_m+1)\xi} + \frac{MP_m}{P_m-2}e^{-2\xi}$$

$$+\frac{2MP_m(D_8-2D_1)}{P_m-1}e^{-\xi} + \frac{2MP_mD_1\xi e^{-\xi}}{P_m-1} + \frac{2MP_mD_1(P_m-2)}{\left(P_m-1\right)^2}e^{-\xi} + \frac{2MP_mD_9P_re^{-P_r\xi}}{P_m-P_r}$$

$$+\frac{2MP_mD_{10}S_ce^{-S_c\xi}}{P_m-S_c} + 2M\xi {P_m}^2D_7e^{-P_m\xi}$$

Using boundary conditions:

$L_2 = 0,$ as $\xi = 0$ and $L_2 = 0$ as $\xi \to \infty$

$$\therefore 0 = c_1 + c_2 + P_m\left(A_1 - K_1\right) + \frac{A_1P_m}{P_m-1} + \frac{A_1P_m(P_m-4)}{4\left(P_m-2\right)^2} + \frac{(A_1-K_1)(P_m+2)}{(P_m+1)^2} + \frac{MP_m}{P_m-2}$$

$$+\frac{2MP_m(D_8-2D_1)}{P_m-1} + \frac{2MP_mD_1(P_m-2)}{\left(P_m-1\right)^2} + \frac{2MP_mD_9P_r}{P_m-P_r} + \frac{2MP_mD_{10}S_c}{P_m-S_c}$$

and $c_1 = 0$, so that

$$L_2 = -[P_m\left(A_1 - K_1\right) + \frac{A_1P_m}{P_m-1} + \frac{A_1P_m(P_m-4)}{4\left(P_m-2\right)^2} + \frac{(A_1-K_1)(P_m+2)}{(P_m+1)^2} + \frac{MP_m}{P_m-2} + \frac{2MP_m(D_8-2D_1)}{P_m-1}$$

$$+\frac{2MP_mD_1(P_m-2)}{\left(P_m-1\right)^2} + \frac{2MP_mD_9P_r}{P_m-P_r} + \frac{2MP_mD_{10}S_c}{P_m-S_c}]e^{-P_m\xi} + (A_1-K_1)P_me^{-(P_m+1)\xi}$$

$$+(A_1-K_1)P_m\xi e^{-P_m\xi} - \frac{A_1P_m}{1-P_m}e^{-\xi} + \frac{A_1P_m\xi}{2\left(P_m-2\right)}e^{-2\xi} + \frac{A_1P_m(P_m-4)}{4\left(P_m-2\right)^2}e^{-2\xi} + \frac{\xi(A_1-K_1)}{(P_m+1)}e^{-(P_m+1)\xi}$$

$$+\frac{(A_1-K_1)(P_m+2)}{(P_m+1)^2}e^{-(P_m+1)\xi} + \frac{MP_m}{P_m-2}e^{-2\xi} + \frac{2MP_m(D_8-2D_1)}{P_m-1}e^{-\xi} + \frac{2MP_mD_1\xi e^{-\xi}}{P_m-1}$$

$$+\frac{2MP_mD_1(P_m-2)}{(P_m-1)^2}e^{-\xi}+\frac{2MP_mD_9P_re^{-P_r\xi}}{P_m-P_r}+\frac{2MP_mD_{10}S_ce^{-S_c\xi}}{P_m-S_c}+2M\xi P_m^{\ 2}D_7e^{-P_m\xi}$$

Hence the complete solution of equation (3.11) is

$$L_2(\xi)=B_1e^{-(P_m+1)\xi}+B_1\xi e^{-P_m\xi}+B_2e^{-\xi}+B_3e^{-2\xi}+B_4e^{-(P_m+1)\xi}+B_5e^{-2\xi}+B_6e^{-\xi}+B_7e^{-\xi}$$
$$+B_8e^{-P_r\xi}+B_9e^{-S_c\xi}+\ B_{10}e^{-P_m\xi}+B_{11}\xi e^{-2\xi}+B_{12}\xi e^{-(P_m+1)\xi}+B_{13}\xi e^{-\xi}+B_{14}\xi e^{-P_m\xi} \tag{3.20}$$

where $B_1=P_m(A_1-K_1), B_2=\dfrac{A_1P_m}{P_m-1},\ B_3=\dfrac{A_1P_m(P_m-4)}{4(P_m-2)^2},\ B_4=\dfrac{(A_1-K_1)(P_m+2)}{(P_m+1)^2},\ B_5=\dfrac{MP_m}{P_m-2},$

$B_6=\dfrac{2MP_m(D_8-2D_1)}{P_m-1},\ B_7=\dfrac{2MP_mD_1(P_m-2)}{(P_m-1)^2},\ B_8=\dfrac{2MP_mD_9P_r}{P_m-P_r},\ B_9=\dfrac{2MP_mD_{10}S_c}{P_m-S_c}$

$B_{10}=-[B_1+B_2+B_3+B_4+B_5+B_6+B_7+B_8+B_9],$

$B_{11}=\dfrac{A_1P_m}{2(P_m-2)},\ B_{12}=\dfrac{A_1-K_1}{P_m+1},\ B13=\dfrac{2MP_mD_1}{P_m-1},\quad B_{14}=2MP_m^{\ 2}D_7$

Again from equation (3.12) we have

$$G_2''+P_r(G_2'+F_1G_1')+\frac{E_cP_r}{2P_m}(L_1')^2=0$$

Here $G_1=e^{-P_r\xi},\ G_1'=-P_re^{-P_r\xi}$

$F_1=1-e^{-\xi},\ \ L_1=A_1\left(e^{-\xi}-e^{-P_m\xi}\right)+K_1e^{-P_m\xi},\ \ L_1'=A_1\left(P_me^{-P_m\xi}-e^{-\xi}\right)-K_1P_me^{-P_m\xi}$

$$\begin{aligned}(L_1')^2&=A_1^{\ 2}(P_me^{-P_m\xi}-e^{-\xi})^2+K_1^{\ 2}P_m^{\ 2}e^{-2P_m\xi}-2A_1K_1P_m(P_me^{-P_m\xi}-e^{-\xi})e^{-P_m\xi}\\&=A_1^{\ 2}(P_m^{\ 2}e^{-2P_m\xi}-2P_me^{-P_m\xi}e^{-\xi}+e^{-2\xi})+K_1^{\ 2}P_m^{\ 2}e^{-2P_m\xi}-2A_1K_1P_m^{\ 2}e^{-2P_m\xi}+2A_1K_1P_me^{-\xi}e^{-P_m\xi}\\&=\left(A_1^{\ 2}P_m^{\ 2}+K_1^{\ 2}P_m^{\ 2}-2A_1K_1P_m^{\ 2}\right)e^{-2P_m\xi}+\left(2A_1K_1P_m-2P_mA_1^{\ 2}\right)e^{-(P_m+1)\xi}+A_1^{\ 2}e^{-2\xi}\\&=P_m^{\ 2}(A_1-K_1^{\ 2})e^{-2P_m\xi}+\left(2A_1K_1P_m-2A_1^{\ 2}P_m\right)e^{-(P_m+1)\xi}+A_1^{\ 2}e^{-2\xi}\end{aligned}$$

The complementary function is obtained by

$$G_{2c}=c_1+c_2e^{-P_r\xi}$$

and the particular integral is

$$G_{2p}=\frac{1}{D^2+P_rD}(P_r^{\ 2}(1-e^{-\xi})e^{-P_r\xi}-\frac{P_rE_c}{2P_m}P_m^{\ 2}(A_1-K_1)e^{-2P_m\xi}-\frac{P_rE_c}{2P_m}(2A_1K_1P_m-2A_1^{\ 2}P_m)e^{-(P_m+1)\xi}$$
$$-\frac{P_rE_c}{2P_m}A_1^{\ 2}e^{-2\xi})$$

$$=\frac{P_r^2}{D^2+P_rD}e^{-P_r\xi}-\frac{P_r^2}{D^2+P_rD}e^{-(P_r+1)\xi}-\frac{1}{D^2+P_rD}\frac{P_rP_mE_c}{2}(A_1-K_1)^2e^{-2P_m\xi}$$

$$-\frac{P_rE_c}{D^2+P_rD}\frac{(2A_1K_1P_m-2A_1^2P_m)}{2P_m}e^{-(P_m+1)\xi}-\frac{P_rE_c}{D^2+P_rD}\frac{A_1^2}{2P_m}e^{-2\xi}$$

$$=\frac{P_r^{\,2}\xi}{2D+P_r}e^{-P_r\xi}-\frac{P_r^{\,2}e^{-(P_r+1)\xi}}{P_r^{\,2}+2P_r+1-P_r^{\,2}-P_r}-\frac{P_rP_mE_c(A_1-K_1)^2}{2(4P_m^{\,2}-2P_rP_m)}e^{-2P_m\xi}$$

$$-\frac{P_rE_c}{P_m^{\,2}+2P_m+1-P_rP_m-P_r}\times\frac{2A_1K_1P_m-2A_1^2P_m}{2P_m}e^{-(P_m+1)\xi}-\frac{1}{4-2P_r}\frac{P_rE_c}{2P_m}A_1^{\,2}e^{-2\xi}$$

$$=-P_r\xi e^{-P_r\xi}-\frac{P_r^{\,2}}{(P_r+1)}e^{-(P_r+1)\xi}-\frac{P_rE_c(A_1-K_1)^2}{4(2P_m-P_r)}e^{-2P_m\xi}+\frac{A_1P_rE_c(A_1-K_1)}{(P_m+1)(P_m-P_r+1)}e^{-(P_m+1)\xi}$$

$$-\frac{P_rE_cA_1^{\,2}}{4P_m(2-P_r)}e^{-2\xi}$$

The general solution is

$$G_2=c_1+c_2e^{-P_r\xi}-P_r\xi e^{-P_r\xi}-\frac{P_r^2}{(P_r+1)}e^{-(P_r+1)\xi}-\frac{P_rE_c(A_1-K_1)^2}{4(2P_m-P_r)}e^{-2P_m\xi}+\frac{A_1P_rE_c(A_1-K_1)}{(P_m+1)(P_m-P_r+1)}e^{-(P_m+1)\xi}$$

$$-\frac{P_rE_cA_1^{\,2}}{4P_m(2-P_r)}e^{-2\xi}$$

Using boundary conditions:

$G_2=0,$ as $\xi=0$ and $G_2=0$ as $\xi\to\infty$

$$\therefore 0=c_1+c_2-\frac{P_r^2}{(P_r+1)}-\frac{P_rE_c(A_1-K_1)^2}{4(2P_m-P_r)}+\frac{A_1P_rE_c(A_1-K_1)}{(P_m+1)(P_m-P_r+1)}-\frac{P_rE_cA_1^{\,2}}{4P_m(2-P_r)}$$

and $c_1=0$, so that

$$G_2=\left[\frac{P_r^2}{(P_r+1)}+\frac{P_rE_c(A_1-K_1)^2}{4(2P_m-P_r)}-\frac{A_1P_rE_c(A_1-K_1)}{(P_m+1)(P_m-P_r+1)}+\frac{P_rE_cA_1^2}{4P_m(2-P_r)}\right]e^{-P_r\xi}-P_r\xi e^{-P_r\xi}-\frac{P_r^2}{(P_r+1)}e^{-(P_r+1)\xi}$$

$$-\frac{P_rE_c(A_1-K_1)^2}{4(2P_m-P_r)}e^{-2P_m\xi}+\frac{A_1P_rE_c(A_1-K_1)}{(P_m+1)(P_m-P_r+1)}e^{-(P_m+1)\xi}-\frac{P_rE_cA_1^{\,2}}{4P_m(2-P_r)}e^{-2\xi}$$

or, $G_2(\xi)=A_4\xi e^{-P_r\xi}-A_5e^{-(P_r+1)\xi}-A_6e^{-2P_m\xi}+A_7e^{-(P_m+1)\xi}-A_8e^{-2\xi}+A_9e^{-P_r\xi}$ (3.21)

where $A_4=-P_r$, $A_5=\dfrac{P_r^{\,2}}{P_r+1}$, $A_6=\dfrac{P_rE_c(A_1-K_1)^2}{4(2P_m-P_r)}$, $A_7=\dfrac{A_1P_rE_c(A_1-K_1)}{(P_m+1)(P_m-P_r+1)}$,

$A_8=\dfrac{P_rE_cA_1^{\,2}}{4P_m(2-P_r)}$, $A_9=A_5+A_6-A_7+A_8$

Again from equation (3.13) we have

$P_2''-2S_cP_1F_1'+S_c(P_1'F_1+P_2')+S_oS_cG_2''=0$

Here

$$F_1 = 1 - e^{-\xi}$$

$$\therefore F_1' = e^{-\xi},$$

$$P_1 = A_2 e^{-S_c\xi} + A_3 e^{-P_r\xi}$$

$$\therefore P_1' = -A_2 S_c e^{-S_c\xi} - A_3 P_r e^{-P_r\xi}$$

$$G_2 = A_4 \xi e^{-P_r\xi} - A_5 e^{-(P_r+1)\xi} - A_6 e^{-2P_m\xi} + A_7 e^{-(P_m+1)\xi} - A_8 e^{-2\xi} + A_9 e^{-P_r\xi}$$

$$\therefore G_2' = A_4 e^{-P_r\xi} - A_4 P_r \xi e^{-P_r\xi} + A_5(P_r+1)e^{-(P_r+1)\xi} + 2A_6 P_m e^{-2P_m\xi}$$
$$- A_7(P_m+1)e^{-(P_m+1)\xi} + 2A_8 e^{-2\xi} - A_9 P_r e^{-P_r\xi}$$

and

$$G_2'' = -A_4 P_r e^{-P_r\xi} - A_4 P_r e^{-P_r\xi} + A_4 \xi P_r^{\ 2} e^{-P_r\xi} - A_5 (P_r+1)^2 e^{-(P_r+1)\xi}$$
$$- 4A_6 P_m^{\ 2} e^{-2P_m\xi} + A_7 (P_m+1)^2 e^{-(P_m+1)\xi} - 4A_8 e^{-2\xi} + A_9 P_r^{\ 2} e^{-P_r\xi}$$

The complementary function is obtained by

$$P_{2c} = c_1 + c_2 e^{-S_c\xi}$$

and the particular integral

$$P_{2p} = \frac{2A_2 S_c}{D^2 + S_c D} e^{-(S_c+1)\xi} + \frac{2A_3 S_c}{D^2 + S_c D} e^{-(P_r+1)\xi} + \frac{A_2 S_c^{\ 2}}{D^2 + S_c D} e^{-S_c\xi} + \frac{A_3 P_r S_c}{D^2 + S_c D} e^{-P_r\xi}$$
$$- \frac{A_2 S_c^{\ 2}}{D^2 + S_c D} e^{-(S_c+1)\xi} - \frac{A_3 P_r S_c}{D^2 + S_c D} e^{-(P_r+1)\xi} + \frac{2S_o S_c A_4 P_r}{D^2 + S_c D} e^{-P_r\xi} - \frac{S_o S_c A_4 P_r^{\ 2}}{D^2 + S_c D} \xi e^{-P_r\xi}$$
$$+ \frac{A_5 S_o S_c (P_r+1)^2}{D^2 + S_c D} e^{-(P_r+1)\xi} + \frac{4A_6 P_m^{\ 2} S_o S_c}{D^2 + S_c D} e^{-2P_m\xi} - \frac{A_7 S_o S_c (P_m+1)^2}{D^2 + S_c D} e^{-(P_m+1)\xi}$$
$$+ \frac{4A_8 S_o S_c}{D^2 + S_c D} e^{-2\xi} - \frac{A_9 P_r^{\ 2} S_o S_c}{D^2 + S_c D} e^{-P_r\xi}$$
$$= \frac{2A_2 S_c - A_2 S_c^{\ 2}}{(S_c+1)^2 - S_c(S_c+1)} e^{-(S_c+1)\xi} + \frac{2A_3 S_c - A_3 P_r S_c + A_5 S_o S_c (P_r+1)^2}{(P_r+1)^2 - S_c(P_r+1)} e^{-(P_r+1)\xi}$$
$$- A_2 S_c \xi e^{-S_c\xi} + \frac{A_3 P_r S_c + 2S_o S_c A_4 P_r - A_9 S_o S_c P_r^2}{P_r^2 - S_c P_r} e^{-P_r\xi} - \frac{A_4 S_c S_o P_r^2}{P_r^2 - S_c P_r} \xi e^{-P_r\xi} + \frac{A_4 S_c S_o P_r^2}{\left(P_r^2 - S_c P_r\right)^2} (S_c - 2P_r) e^{-P_r\xi}$$
$$+ \frac{4A_6 S_c S_o P_m^{\ 2}}{4P_m^{\ 2} - 2S_c P_m} e^{-2P_m\xi} - \frac{A_7 S_o S_c (P_m+1)^2}{(P_m+1)^2 - S_c(P_m+1)} e^{-(P_m+1)\xi} + \frac{4S_o S_c A_8}{4 - 2S_c} e^{-2\xi}$$
$$= \frac{2A_2 S_c - A_2 S_c^{\ 2}}{(S_c+1)} e^{-(S_c+1)\xi} + \frac{2A_3 S_c - A_3 P_r S_c + A_5 S_o S_c (P_r+1)^2}{(P_r+1)(P_r - S_c + 1)} e^{-(P_r+1)\xi} - A_2 S_c \xi e^{-S_c\xi}$$

$$+\frac{A_3S_c+2S_oS_cA_4-A_9S_oS_cP_r}{P_r-S_c}e^{-P_r\xi}-\frac{A_4S_cS_oP_r}{P_r-S_c}\xi e^{-P_r\xi}+\frac{A_4S_cS_o}{(P_r-S_c)^2}(S_c-2P_r)e^{-P_r\xi}$$

$$+\frac{2A_6S_cS_oP_m}{2P_m-S_c}e^{-2P_m\xi}-\frac{A_7S_oS_c(P_m+1)}{(P_m-S_c+1)}e^{-(P_m+1)\xi}+\frac{2S_oS_cA_8}{2-S_c}e^{-2\xi}$$

The general solution is

$$P_2=c_1+c_2e^{-S_c\xi}+\frac{2A_2S_c-A_2S_c^{\,2}}{(S_c+1)}e^{-(S_c+1)\xi}+\frac{2A_3S_c-A_3P_rS_c+A_5S_oS_c(P_r+1)^2}{(P_r+1)(P_r-S_c+1)}e^{-(P_r+1)\xi}$$

$$-A_2S_c\xi e^{-S_c\xi}+\frac{A_3S_c+2S_oS_cA_4-A_9S_oS_cP_r}{P_r-S_c}e^{-P_r\xi}-\frac{A_4S_cS_oP_r}{P_r-S_c}\xi e^{-P_r\xi}+\frac{A_4S_cS_o}{(P_r-S_c)^2}(S_c-2P_r)e^{-P_r\xi}$$

$$+\frac{2A_6S_cS_oP_m}{2P_m-S_c}e^{-2P_m\xi}-\frac{A_7S_oS_c(P_m+1)}{(P_m-S_c+1)}e^{-(P_m+1)\xi}+\frac{2S_oS_cA_8}{2-S_c}e^{-2\xi}$$

Using boundary conditions:

$P_2=0$ as $\xi=0$ and $G_2=0$ as $\xi\to\infty$

$$\therefore 0=c_1+c_2+\frac{2A_2S_c-A_2S_c^{\,2}}{(S_c+1)}+\frac{2A_3S_c-A_3P_rS_c+A_5S_oS_c(P_r+1)^2}{(P_r+1)(P_r-S_c+1)}+\frac{A_3S_c+2S_oS_cA_4-A_9S_oS_cP_r}{P_r-S_c}$$

$$+\frac{A_4S_cS_o}{(P_r-S_c)^2}(S_c-2P_r)+\frac{2A_6S_cS_oP_m}{2P_m-S_c}-\frac{A_7S_oS_c(P_m+1)}{(P_m-S_c+1)}+\frac{2S_oS_cA_8}{2-S_c}$$

and $c_1=0$, so that

$$P_2=\left[-\frac{2A_2S_c-A_2S_c^{\,2}}{(S_c+1)}-\frac{2A_3S_c-A_3P_rS_c+A_5S_oS_c(P_r+1)^2}{(P_r+1)(P_r-S_c+1)}-\frac{A_3S_c+2S_oS_cA_4-A_9S_oS_cP_r}{P_r-S_c}\right.$$

$$\left.-\frac{A_4S_cS_o}{(P_r-S_c)^2}(S_c-2P_r)-\frac{2A_6S_cS_oP_m}{2P_m-S_c}+\frac{A_7S_oS_c(P_m+1)}{(P_m-S_c+1)}-\frac{2S_oS_cA_8}{2-S_c}\right]e^{-S_c\xi}$$

$$+\frac{2A_2S_c-A_2S_c^{\,2}}{(S_c+1)}e^{-(S_c+1)\xi}+\frac{2A_3S_c-A_3P_rS_c+A_5S_oS_c(P_r+1)^2}{(P_r+1)(P_r-S_c+1)}e^{-(P_r+1)\xi}-A_2S_c\xi e^{-S_c\xi}$$

$$+\frac{A_3S_c+2S_oS_cA_4-A_9S_oS_cP_r}{P_r-S_c}e^{-P_r\xi}-\frac{A_4S_cS_oP_r}{P_r-S_c}\xi e^{-P_r\xi}+\frac{A_4S_cS_o}{(P_r-S_c)^2}(S_c-2P_r)e^{-P_r\xi}$$

$$+\frac{2A_6S_cS_oP_m}{2P_m-S_c}e^{-2P_m\xi}-\frac{A_7S_oS_c(P_m+1)}{(P_m-S_c+1)}e^{-(P_m+1)\xi}+\frac{2S_oS_cA_8}{2-S_c}e^{-2\xi}$$

or, $P_2(\xi)=E_1e^{-P_r\xi}+E_2e^{-(P_r+1)\xi}+E_3e^{-(S_c+1)\xi}+E_4e^{-2P_m\xi}-E_5e^{-(P_m+1)\xi}$

$$+E_6e^{-2\xi}+E_7e^{-P_r\xi}+E_8e^{-S_c\xi}-E_9\xi e^{-S_c\xi}-E_{10}\xi e^{-P_r\xi} \tag{3.22}$$

The solution of the equations (3.5) – (3.8) and (3.10) – (3.13) up to order 2 under the prescribed boundary conditions are obtained in a straightforward and are:

$$F_1(\xi) = 1 - e^{-\xi} \tag{3.23}$$

$$L_1(\xi) = A_1\left(e^{-\xi} - e^{-P_m\xi}\right) + K_1 e^{-P_m\xi} \tag{3.24}$$

$$G_1(\xi) = e^{-P_r\xi} \tag{3.25}$$

$$P_1(\xi) = A_2 e^{-S_c\xi} + A_3 e^{-P_r\xi} \tag{3.26}$$

$$F_2(\xi) = \frac{1}{4}e^{-2\xi} + D_1\xi e^{-\xi} + D_9 e^{-P_r\xi} + D_{10}e^{-S_c\xi} + D_7 e^{-P_m\xi} + D_8 e^{-\xi} + D_{11} \tag{3.27}$$

$$\begin{aligned} L_2(\xi) = {} & B_1 e^{-(P_m+1)\xi} + B_1\xi e^{-P_m\xi} + B_2 e^{-\xi} + B_3 e^{-2\xi} + B_4 e^{-(P_m+1)\xi} + B_5 e^{-2\xi} + B_6 e^{-\xi} + B_7 e^{-\xi} + B_8 e^{-P_r\xi} \\ & + B_9 e^{-S_c\xi} + B_{10}e^{-P_m\xi} + B_{11}\xi e^{-2\xi} + B_{12}\xi e^{-(P_m+1)\xi} + B_{13}\xi e^{-\xi} + B_{14}\xi e^{-P_m\xi} \end{aligned} \tag{3.28}$$

$$G_2(\xi) = A_4\xi e^{-P_r\xi} - A_5 e^{-(P_r+1)\xi} - A_6 e^{-2P_m\xi} + A_7 e^{-(P_m+1)\xi} - A_8 e^{-2\xi} + A_9 e^{-P_r\xi} \tag{3.29}$$

$$P_2(\xi) = E_1 e^{-P_r\xi} + E_2 e^{-(P_r+1)\xi} + E_3 e^{-(S_c+1)\xi} + E_4 e^{-2P_m\xi} - E_5 e^{-(P_m+1)\xi} + E_6 e^{-2\xi} + E_7 e^{-P_r\xi} + E_8 e^{-S_c\xi} - E_9\xi e^{-S_c\xi} - E_{10}\xi e^{-P_r\xi} \tag{3.30}$$

where the constants A_i, B_i, D_i, and K_1 are shown in Appendix 3.A

The above solutions (3.15) – (3.22) are however valid for $P_r = S_c \neq 1$ and $P_r \neq S_c$

The velocity, temperature induced magnetic field and the concentration can now be calculated from (3.1) – (3.4) as follows:

$$\frac{u}{U_0} = f'(\eta) = F_1' + \varepsilon F_2' + \varepsilon^2 F_3' \tag{3.31}$$

$$H(\eta) = \sqrt{\varepsilon}L_1 + \varepsilon^{3/2}L_2 \tag{3.32}$$

$$\theta(\eta) = G_1 + \varepsilon G_2 + \varepsilon^2 G_3 \tag{3.33}$$

$$\varphi(\eta) = P_1 + \varepsilon P_2 + \varepsilon^2 P_3 \tag{3.34}$$

Thus with the help of the solutions (3.23) – (3.30) the velocity, temperature, induced magnetic field and concentration distributions are calculated from (3.31) – (3.34). However for different values of the established parameters, the results of the velocity, temperature, induced magnetic field and concentration distributions are plotted graphically and the coefficient of skin friction and heat transfer are given in tabular form in CHAPTER IV.

Appendix 3.A

$$A_1 = \frac{2MP_m}{1-P_m}, \qquad K_1 = \frac{1}{\sqrt{\xi}} \qquad A_2 = 1 - A_3, \qquad A_3 = -\frac{S_o S_c P_r}{P_r - S_c}$$

$$D_1 = 1 + MA_1, \qquad D_2 = G_r + G_m A_3, \qquad D_3 = G_m A_2, \qquad D_4 = \frac{D_2}{P_r(P_r - 1)},$$

$$D_5 = \frac{D_3}{S_c(S_c - 1)}, \qquad D_6 = \frac{D_{12}}{P_m - 1}, \qquad D_7 = \frac{D_{12}}{P_m(P_m - 1)},$$

$$D_8 = -\frac{1}{2} + D_1 - D_4 - D_5 - D_6, \qquad D_9 = \frac{D_2}{P_r^{\,2}(P_r - 1)}, \qquad D_{10} = \frac{D_3}{S_c^{\,2}(S_c - 1)},$$

$$D_{11} = -\frac{1}{4} - D_8 - D_9 - D_{10} - D_7, \qquad D_{12} = MA_1 - MK_1, \qquad B_1 = P_m(A_1 - K_1),$$

$$B_2 = \frac{A_1 P_m}{P_m - 1}, \qquad B_3 = \frac{A_1 P_m (P_m - 4)}{4(P_m - 2)^2}, \qquad B_4 = \frac{(A_1 - K_1)(P_m + 2)}{(P_m + 1)^2},$$

$$B_5 = \frac{MP_m}{P_m - 2}, \qquad B_6 = \frac{2MP_m(D_8 - 2D_1)}{P_m - 1}, \qquad B_7 = \frac{2MP_m D_1 (P_m - 2)}{(P_m - 1)^2},$$

$$B_8 = \frac{2MP_m D_9 P_r}{P_m - P_r}, \qquad B_9 = \frac{2MP_m D_{10} S_c}{P_m - S_c}, \qquad B_{10} = -[B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7 + B_8 + B_9],$$

$$B_{11} = \frac{A_1 P_m}{2(P_m - 2)}, \qquad B_{12} = \frac{A_1 - K_1}{P_m + 1}, \qquad B_{13} = \frac{2MP_m D_1}{P_m - 1}, \qquad B_{14} = 2MP_m^{\,2} D_7$$

$$A_4 = -P_r, \qquad A_5 = \frac{P_r^{\,2}}{P_r + 1}, \qquad A_6 = \frac{P_r E_c (A_1 - K_1)^2}{4(2P_m - P_r)}, \quad A_7 = \frac{A_1 P_r E_c (A_1 - K_1)}{(P_m + 1)(P_m - P_r + 1)},$$

$$A_8 = \frac{P_r E_c A_1^{\,2}}{4P_m(2 - P_r)}, \qquad A_9 = A_5 + A_6 - A_7 + A_8$$

$$E_1 = \frac{A_3 S_c + 2S_o S_c A_4 - A_9 S_o S_c P_r}{P_r - S_c}, \qquad E_2 = \frac{2A_3 S_c - A_3 P_r S_c + A_5 S_o S_c (P_r + 1)^2}{(P_r + 1)(P_r - S_c + 1)},$$

$$E_3 = \frac{2A_2 S_c - A_2 S_c^{\,2}}{(S_c + 1)} e^{-(S_c + 1)\xi}, \qquad E_4 = \frac{2A_6 S_c S_o P_m}{2P_m - S_c}, \qquad E_5 = \frac{A_7 S_o S_c (P_m + 1)}{(P_m - S_c + 1)},$$

$$E_6 = \frac{2S_o S_c A_8}{2 - S_c}, \qquad E_7 = \frac{A_4 S_c S_o}{(P_r - S_c)^2}(S_c - 2P_r),$$

$$E_8 = -E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7, \qquad E_9 = A_2 S_c, \qquad E_{10} = \frac{A_4 S_c S_o P_r}{P_r - S_c}$$

CHAPTER IV

Perturbation Solutions and Results Discussions

The system of coupled, nonlinear, ordinary differential equations (3.5) – (3.8) and (3.10) – (3.13) governed by the boundary conditions (3.9) and (3.14), respectively are obtained by using perturbation technique. In order to get insight into the physical phenomena of the problem, it is required to find the approximate numerical results of the first order solutions (3.23) – (3.26) along with the second order solutions (3.27) – (3.30), concerning the velocity, temperature, induced magnetic field and concentration. For more consistent results the numerical approximation of the second order solutions (3.27) – (3.30), with the assist of the first order approximation, have been carried out here for small values of Eckert number $E_c = 0.2$ (which is the measure of the heat produced by friction) with different selected values of the established dimensionless parameters like Soret number (S_0), Grashof number (G_r), modified Grashof number (G_m) for mass transfer, suction parameter f_w, magnetic parameter (M), etc. Since the two most important fluids are atmospheric air and water, the values of the Prandtl number (P_r) are limited to 0.71 for air (at 20°C) and 7.0 for water (at 20°C) for numerical investigation. The other parameters like magnetic diffusivity (P_m) and Schmidt number (S_c) are chosen to be fixed values 3.0, 0.6, respectively. With the above mentioned parameters the velocity and temperature profiles, the variation of induced magnetic field and mass concentration are presented in the following Figure 4.1 through Figure 4.20.

Figures 4.1 and 4.2 show the effect of Soret number (S_0) on the velocity and temperature fields respectively. It is observed that velocity increases with the increase of S_0 but there is no remarkable effect of S_0 on the temperature field. Also the velocity decreases more with increasing S_0 for negative values of modified Grashof number G_m and it becomes negative and asymptotically tends to zero far away from the plate surface.

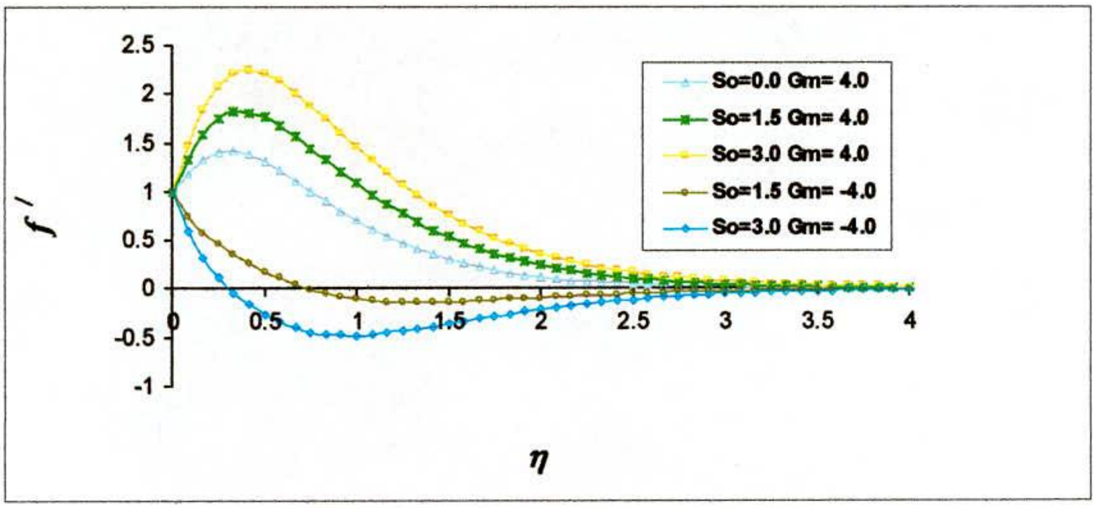

Figure 4.1: Velocity profiles for different values of S_0 (with fixed values of $P_r = 0.71$, $M = 1.5$, $G_r = 10.0, f_w = 3.0$) taking $G_m = 4.0$ and -4.0.

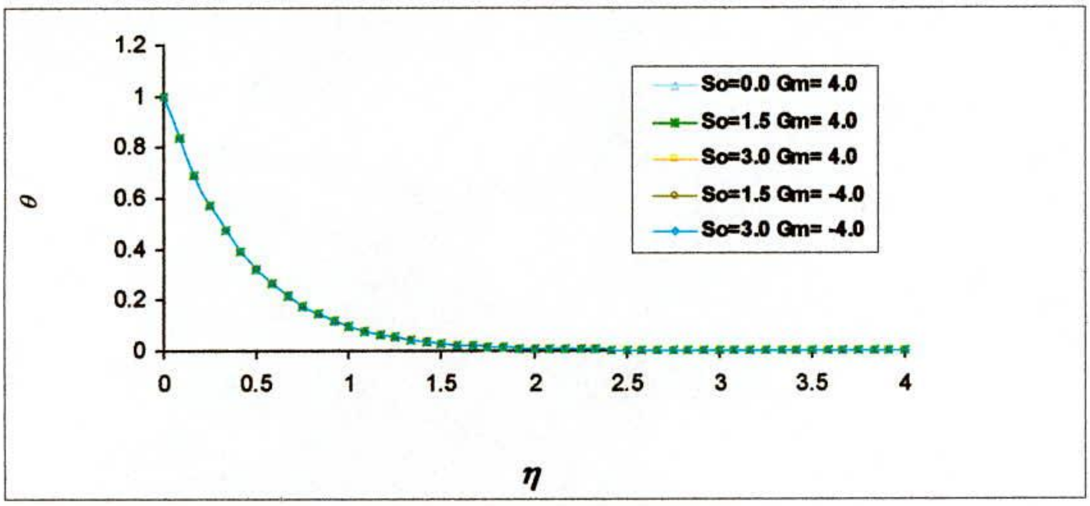

Figure 4.2: Temperature profiles for different values of S_0 (with fixed values of $P_r = 0.71$, $M = 1.5$, $G_r = 10.0, f_w = 3.0$) taking $G_m = 4.0$ and -4.0.

The variation of induced magnetic field and mass concentration with S_0 are shown in Figure 4.3 and Figure 4.4 respectively. It is seen that with the increase in S_0, the induced magnetic field decreases but the reverse effect is observed for the mass concentration, that is, concentration increases with increasing S_0. The induced magnetic field becomes higher for negative values of G_m but show increasing with increasing S_0 as observed in Figure 4.3.

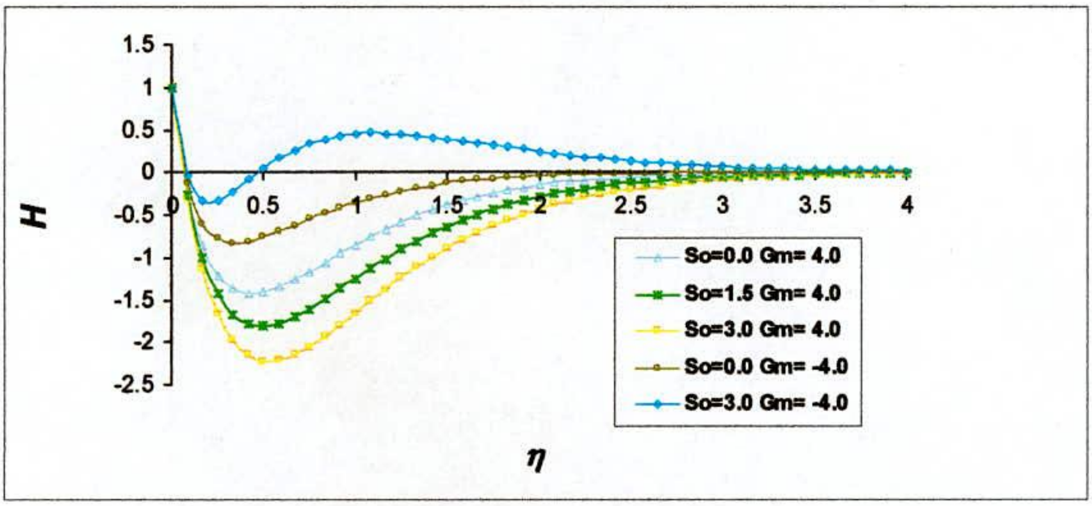

Figure 4.3: Variation of induced magnetic field for different values of S_0 (with fixed values of $P_r = 0.71$, $M = 1.5$, $G_r = 10.0$, $f_w = 3.0$) taking $G_m = 4.0$ and -4.0.

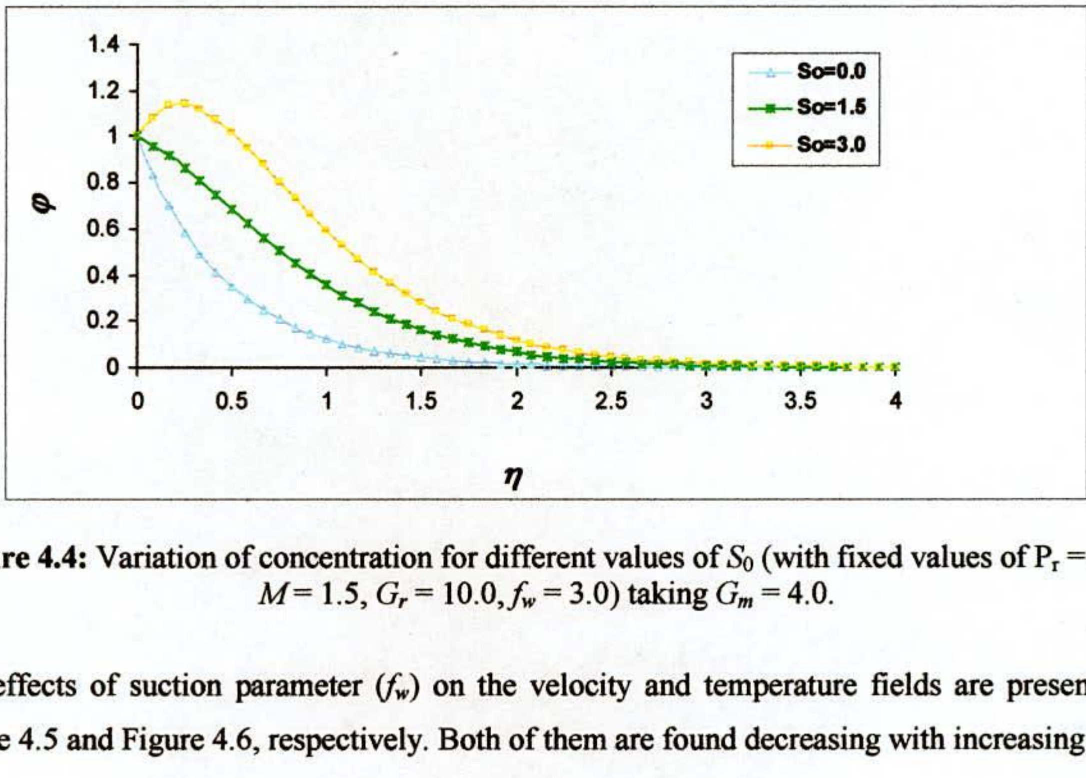

Figure 4.4: Variation of concentration for different values of S_0 (with fixed values of $P_r = 0.71$, $M = 1.5$, $G_r = 10.0$, $f_w = 3.0$) taking $G_m = 4.0$.

The effects of suction parameter (f_w) on the velocity and temperature fields are presented in Figure 4.5 and Figure 4.6, respectively. Both of them are found decreasing with increasing f_w.

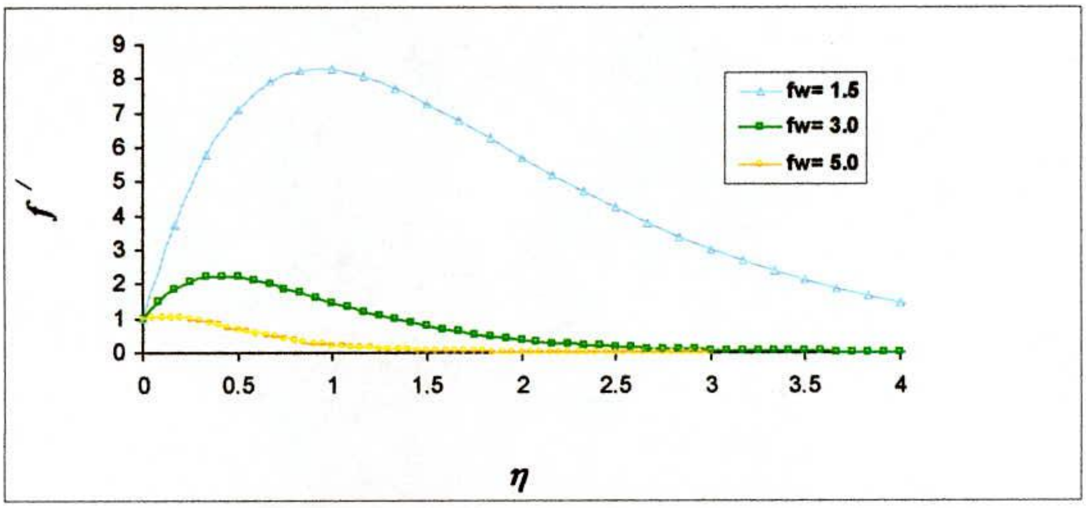

Figure 4.5: Velocity profiles for different values of f_w (with fixed values of $P_r = 0.71$, $S_0 = 3.0$, $M = 1.5$, $G_r = 10.0$ and $G_m = 4.0$).

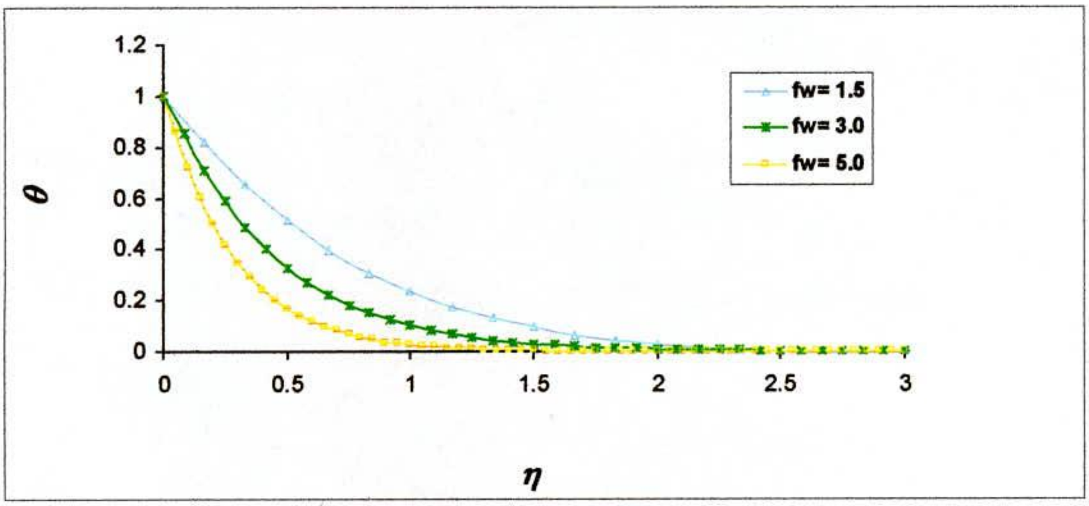

Figure 4.6: Temperature profiles for different values of f_w (with fixed values of $P_r = 0.71$, $S_0 = 3.0$, $M = 1.5$, $G_r = 10.0$ and $G_m = 4.0$).

Figure 4.7 and 4.8 respectively show the effect of f_w on the induced magnetic field and mass concentration. The induced magnetic field rapidly increases with the increase of f_w, but concentration first increases very close to the plate surface and then found to decrease further with increasing f_w away from the surface.

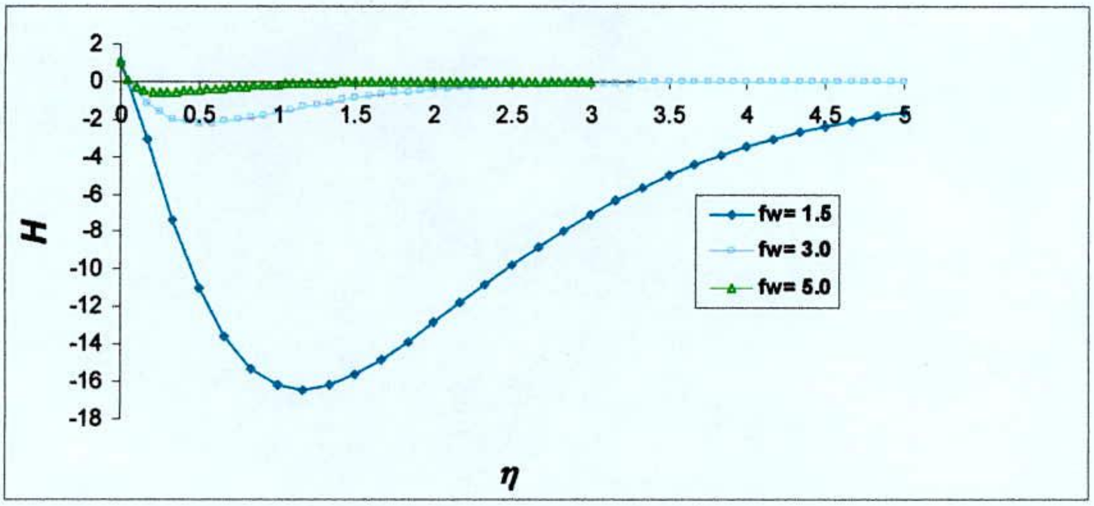

Figure 4.7: Variation of induced magnetic field for different values of f_w (with fixed values of $P_r = 0.71$, $S_0 = 3.0$, $M = 1.5$, $G_r = 10.0$ and $G_m = 4.0$).

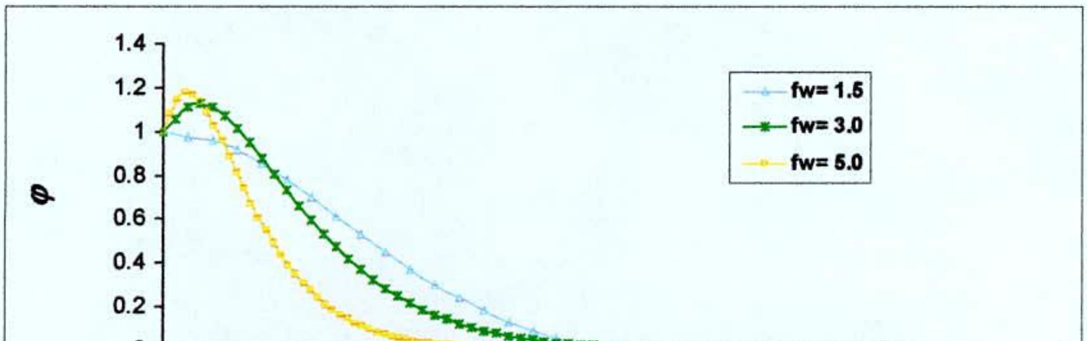

Figure 4.8: Variation of concentration for different values of f_w (with fixed values of $P_r = 0.71$, $S_0 = 3.0$, $M = 1.5$, $G_r = 10.0$ and $G_m = 4.0$).

The effect of Grashof number (G_r) on the velocity and temperature fields, induced magnetic field and mass concentration are displayed in Figures 4.9 and 4.10 and Figures 4.11 and 4.12, respectively. Figures show that velocity increases with the increase of G_r but the induced magnetic field decreases with increasing values of G_r. No considerable effect of G_r on the temperature and concentration is found as seen in Figure 4.10 and Figure 4.12, respectively.

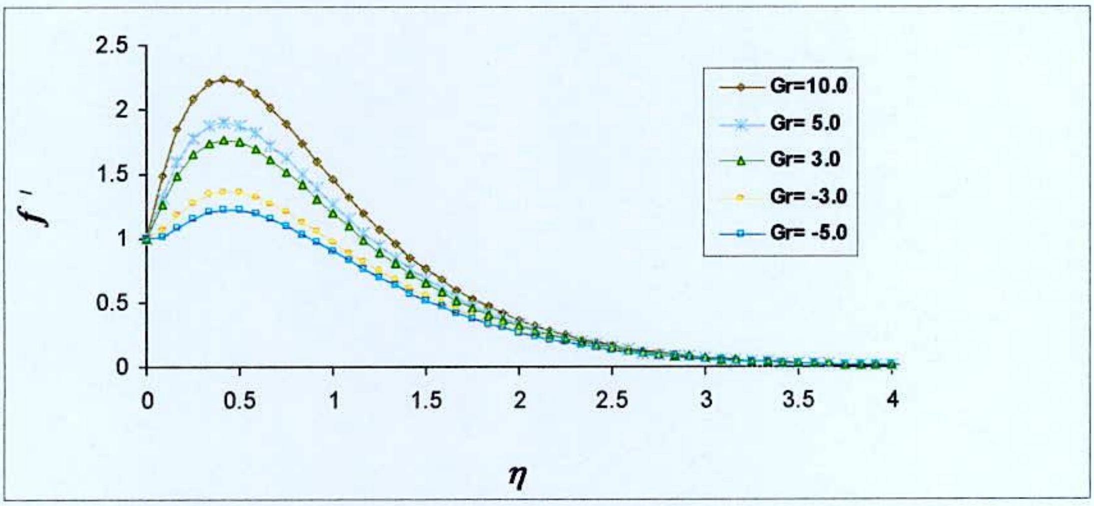

Figure 4.9: Velocity profiles for different values of G_r (with fixed values of $P_r = 0.71$, $S_0 = 3.0$, $M = 1.5, f_w = 3.0$ and $G_m = 4.0$).

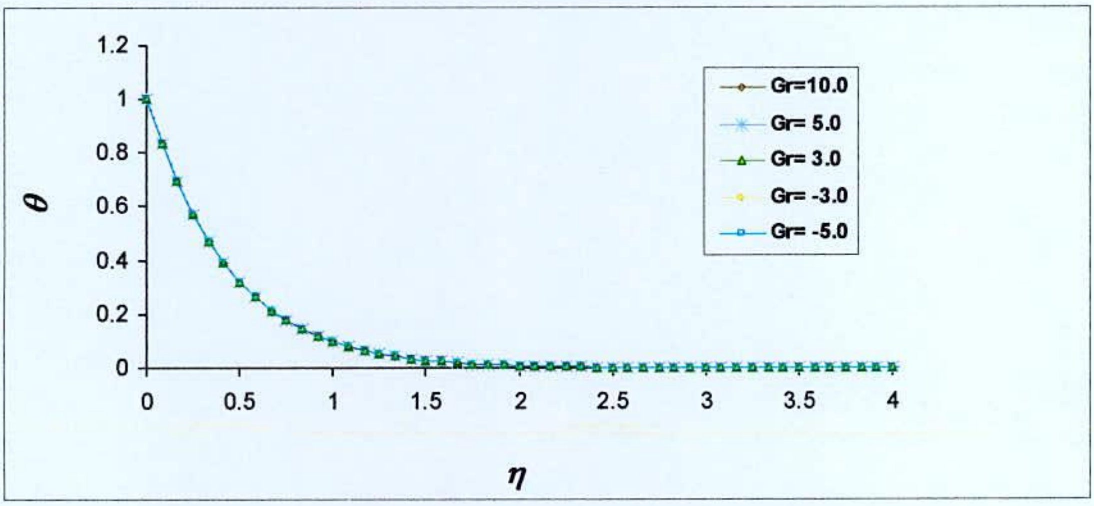

Figure 4.10: Temperature profiles for different values of G_r (with fixed values of $P_r = 0.71$, $S_0 = 3.0, M = 1.5, f_w = 3.0$ and $G_m = 4.0$).

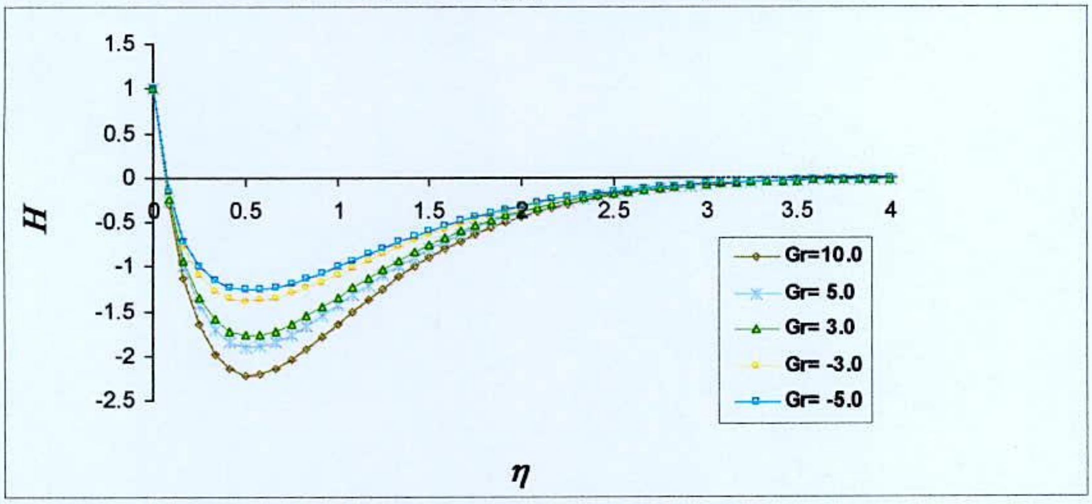

Figure 4.11: Variation of induced magnetic field for different values of G_r (with fixed values of $P_r = 0.71$, $S_0 = 3.0$, $M = 1.5, f_w = 3.0$ and $G_m = 4.0$).

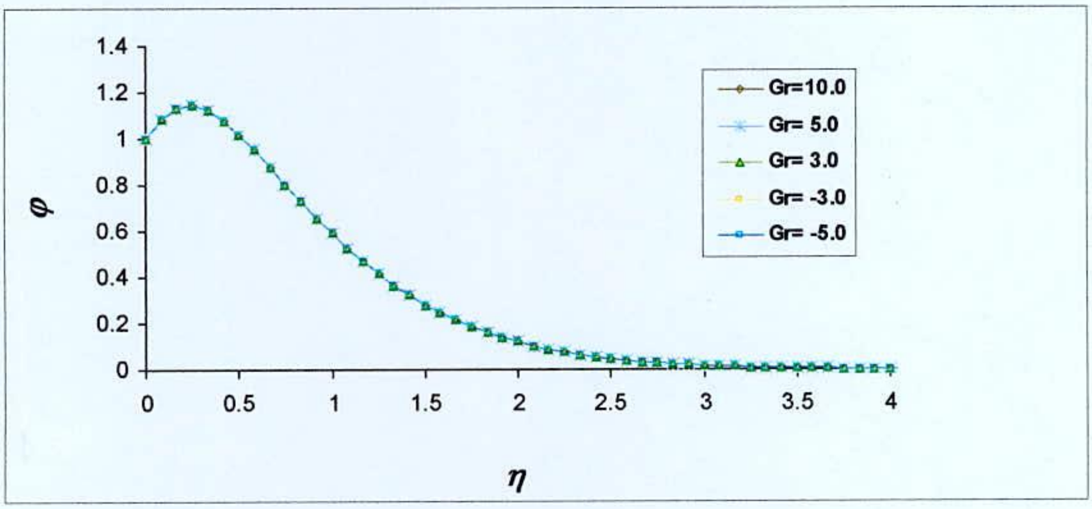

Figure 4.12: Variation of concentration for different values of G_r (with fixed values of $P_r = 0.71$, $S_0 = 3.0$, $M = 1.5$, $f_w = 3.0$ and $G_m = 4.0$).

Figure 4.13 and Figure 4.14 observed the effect of magnetic parameter (M) on the velocity and temperature fields. The effect of magnetic parameter (M) on induced magnetic field and mass concentration are shown in Figures 4.15 and 4.16, respectively. As M increases, velocity increases but the induced magnetic field decreases as shown in Figures 4.13 and 4.15, respectively. But no significant effect is observed on the temperature and concentration gradient for the variation of magnetic parameter as observed in Figure 4.14 and Figure 4.16, respectively.

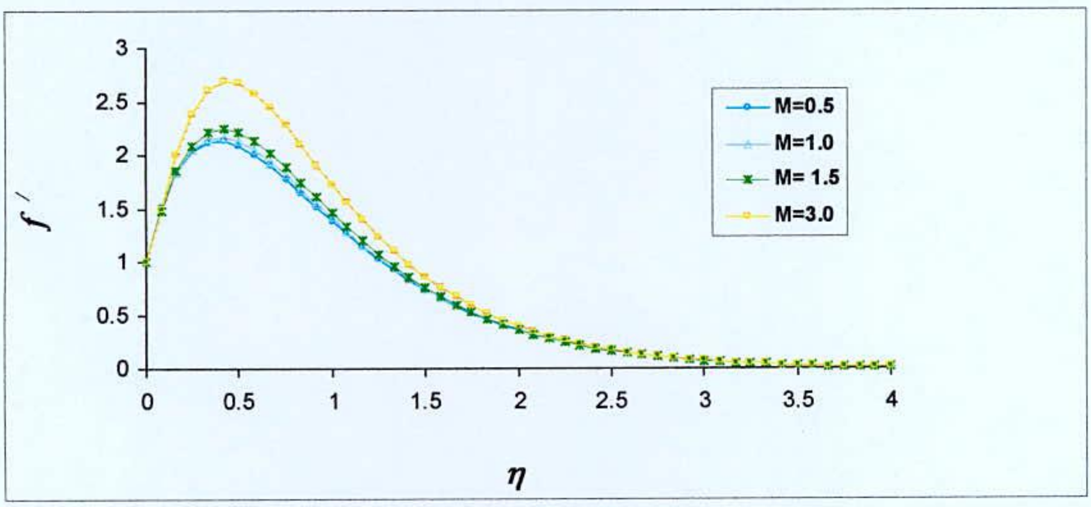

Figure 4.13: Velocity profiles for different values of M (with fixed values of $P_r = 0.71$, $S_0 = 3.0$, $f_w = 3.0$, $G_r = 10.0$ and $G_m = 4.0$).

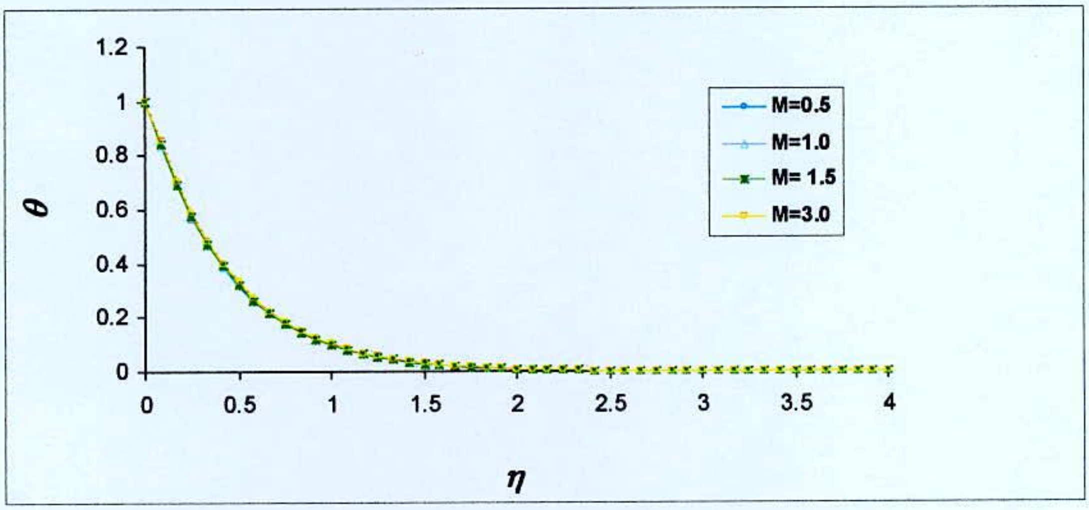

Figure 4.14: Temperature profiles for different values of M (with fixed values of $P_r = 0.71$, $S_0 = 3.0, f_w = 3.0$, $G_r = 10.0$ and $G_m = 4.0$).

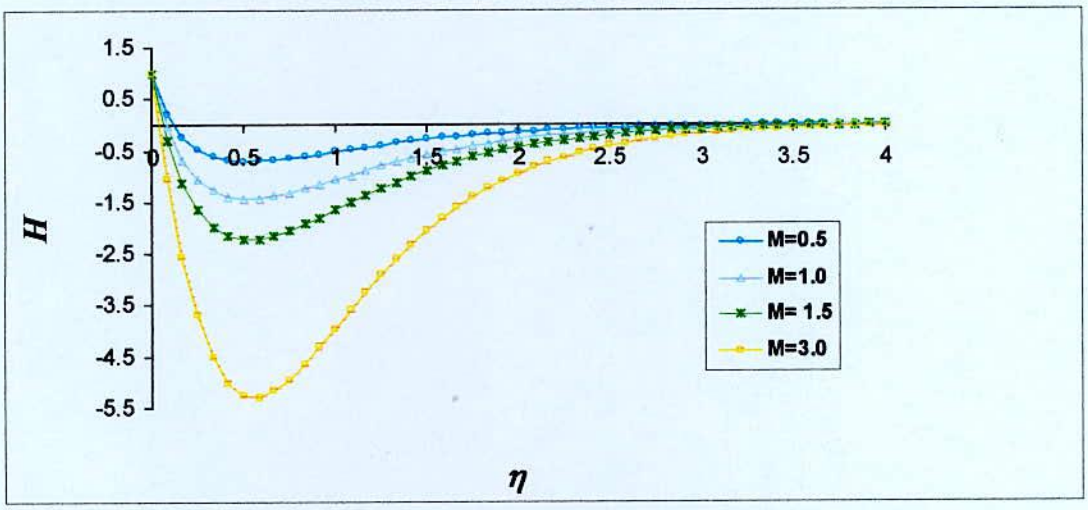

Figure 4.15: Variation of induced magnetic field for different values of M (with fixed values of $P_r = 0.71$, $S_0 = 3.0, f_w = 3.0$, $G_r = 10.0$ and $G_m = 4.0$).

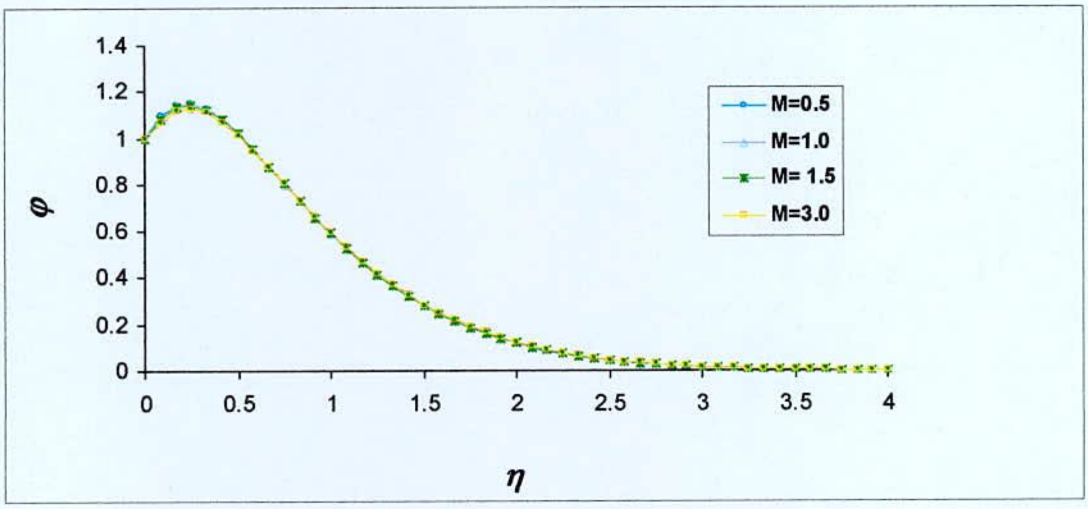

Figure 4.16: Variation of concentration for different values of M (with fixed values of $P_r = 0.71$, $S_0 = 3.0, f_w = 3.0, G_r = 10.0$ and $G_m = 4.0$).

Displayed Figure 4.17 and Figure 4.18 show the effect of Prandtl number (P_r) on the velocity and temperature fields where as Figure 4.19 and Figure 4.20 show the effect of P_r on the induced magnetic field and concentration, respectively. Here we see that both velocity and temperature decrease with the increase of P_r. A reverse effect is observed for the induced magnetic field. Here the induced magnetic field increases with increasing P_r as is seen in Figure 4.19.

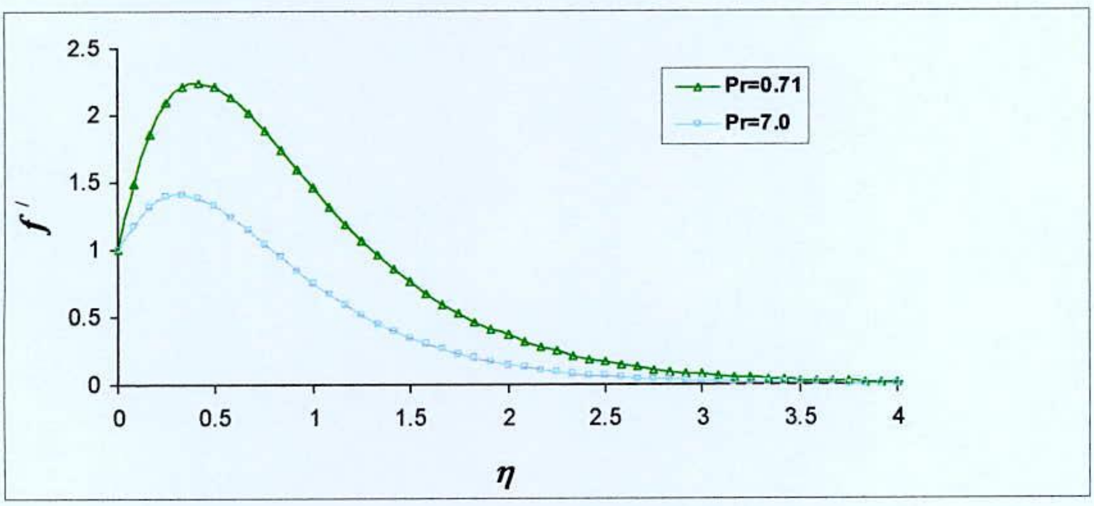

Figure 4.17: Velocity profiles for different values of P_r (with fixed values of $S_0 = 3.0, f_w = 3.0$, $M = 1.5, G_r = 10.0$ and $G_m = 4.0$).

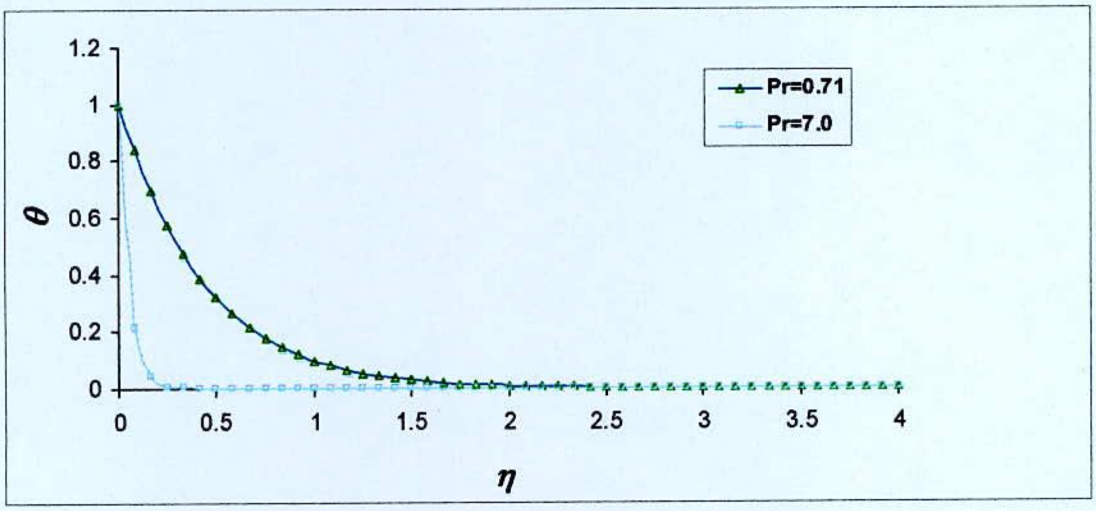

Figure 4.18: Temperature profiles for different values of P_r (with fixed values of $S_0 = 3.0$, $f_w = 3.0$, $M = 1.5$, $G_r = 10.0$ and $G_m = 4.0$).

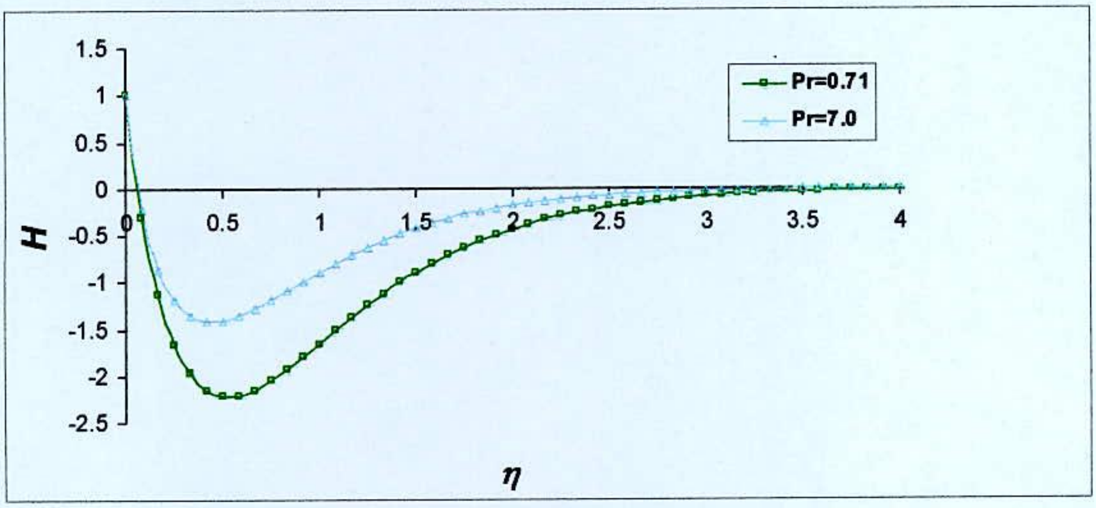

Figure 4.19: Variation of induced magnetic field for different values of P_r (with fixed values of $S_0 = 3.0, f_w = 3.0, M = 1.5, G_r = 10.0$ and $G_m = 4.0$).

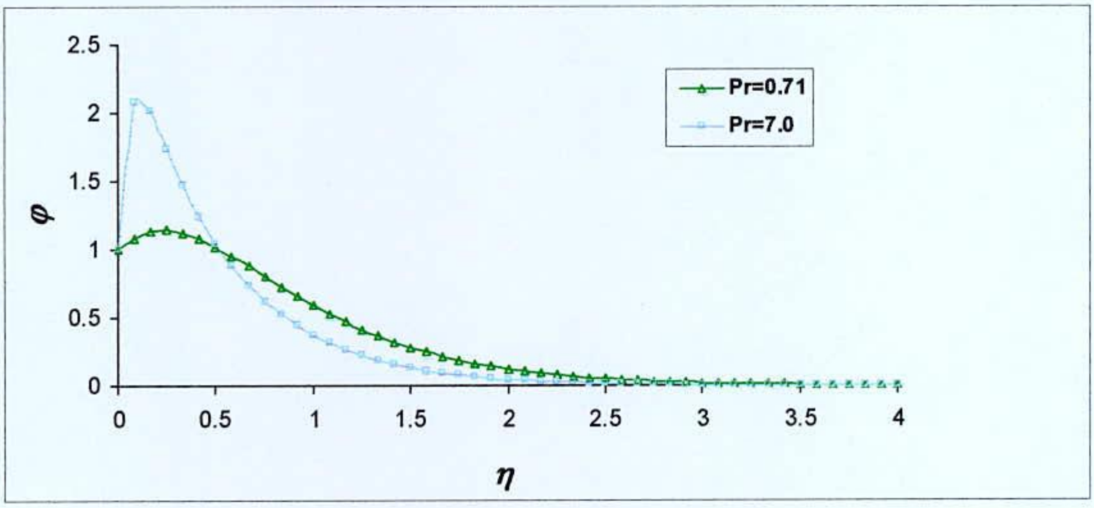

Figure 4.20: Variation of concentration for different values of P_r (with fixed values of $S_0 = 3.0$, $f_w = 3.0$, $M = 1.5$, $G_r = 10.0$ and $G_m = 4.0$).

Again with increasing P_r, the concentration is found to increase first very close to the plate surface and after that it further decreases and asymptotically approaches to zero away from the plate surface (Figure 4.20).

The variations of the values proportional to the coefficients of skin friction $f^{//}(0)$ and heat transfer $-\vartheta^{/}(0)$ with the variation of the values of different selected established dimensionless parameters are tabulated in Table (4.1) – (4.4).

Table 4.1: Variations of the values proportional to the coefficients of skin-friction ($f^{//}(0)$) and heat transfer ($-\vartheta^{/}(0)$) with the variation of S_0 (for fixed values of $P_r = 0.71$, $S_c = 0.6$, $f_w = 3.0$, $M = 1.5$, $G_r = 10.0$ and $G_m = 4.0$).

S_0	$f^{//}(0)$	$-\vartheta^{/}(0)$
3.0	6.250391	1.92228
1.5	4.250391	1.92228
0.0	2.250391	1.92228

From Table 4.1, it is observed that with the increase in S_o, the coefficient of skin friction increases but no effect of S_0 on the coefficient of heat transfer is perceived.

Table 4.2: Variations of the values proportional to the coefficients of skin-friction ($f^{\prime\prime}(0)$) and heat transfer ($-\vartheta^{\prime}(0)$) with the variation of f_w (for fixed values of $P_r = 0.71$, $S_c = 0.6$, $S_0 = 3.0$, $M = 1.5$, $G_r = 10.0$ and $G_m = 4.0$).

f_w	$f^{\prime\prime}(0)$	$-\vartheta^{\prime}(0)$
5.0	-0.04977	3.24787
3.0	6.250391	1.92228
1.5	18.50078	0.96905

From Table 4.2, it is seen that, with the increase in f_w, the coefficient of skin friction highly decrease and the rate of heat transfer increases. The usual stabilizing effect of the suction parameter on the boundary layer growth is also evident from this Table.

Table 4.3: Variations of the values proportional to the coefficients of skin-friction ($f^{\prime\prime}(0)$) and heat transfer ($-\vartheta^{\prime}(0)$) with the variation of G_r (for fixed values of $P_r = 0.71$, S_c =0.6, $S_0 = 3.0$, $f_w = 3.0$, $M = 1.5$ and $G_m = 4.0$).

G_r	$f^{\prime\prime}(0)$	$-\vartheta^{\prime}(0)$
10.0	6.250391	1.92228
5.0	3.902973	1.92228
3.0	2.964006	1.92228
–3.0	0.147105	1.92228
–5.0	–0.791860	1.92228

Table 4.3 shows that with the decrease in G_r, the coefficient of skin friction gradually decreases but there is no effect of G_r on the coefficient of heat transfer as is seen in the Table.

Table 4.4: Variations of the values proportional to the coefficients of skin-friction ($f^{\prime\prime}(0)$) and heat transfer ($-\vartheta^{\prime}(0)$) with the variation of M (for fixed values of $P_r = 0.71$, $S_c = 0.6$, $S_0 = 3.0$, $f_w = 3.0$, $G_r = 10.0$ and $G_m = 4.0$).

M	$f^{\prime\prime}(0)$	$-\vartheta^{\prime}(0)$
3.0	4.750391	1.52290
1.5	6.250391	1.92228
1.0	6.750391	2.01990
0.5	7.250391	2.09978

From Table 4.4, we see that with the increase of induced magnetic field M, both the coefficient of skin friction and the coefficient of heat transfer reduce significantly.

Table 4.5: Variations of the values proportional to the coefficients of skin-friction ($f^{//}(0)$) and heat transfer ($-\vartheta^{/}(0)$) with the variation of P_r (for fixed values of $S_c = 0.6$, $S_0 = 3.0$, $f_w = 3.0$, $M = 1.5$, $G_r = 10.0$, and $G_m = 4.0$).

P_r	$f^{//}(0)$	$-\vartheta^{/}(0)$
0.71	6.250391	1.92228
3.0	2.666667	7.78750
7.0	2.031746	17.87920

It is observed from Table 4.5 that, with the increase of the Prandtl number P_r, the coefficient of skin friction decreases but the rate of heat transfer extensively increases.

Table 4.6: Variations of the values proportional to the coefficients of skin-friction ($f^{//}(0)$) and heat transfer ($-\vartheta^{/}(0)$) with the variation of S_c (for fixed values of, $P_r = 0.71$, $S_0 = 3.0$ $f_w = 3.0$, $M = 1.5$, $G_r = 10.0$ and $G_m = 4.0$).

S_c	$f^{//}(0)$	$-\vartheta^{/}(0)$
5.0	4.294836	1.92228
3.0	4.472613	1.92228
1.5	4.917085	1.92228
0.6	6.250391	1.92228
0.1	17.361500	1.92228

Table 4.6 shows that with the decrease in S_c, the coefficient of skin friction sharply decreases but no through effect of S_c on the coefficient of heat transfer is seen here.

Table 4.7: Variations of the values proportional to the coefficients of skin-friction ($f^{//}(0)$) and heat transfer ($-\vartheta^{/}(0)$) with the variation of G_m (for fixed values of $P_r = 0.71$, $S_c = 0.6$, $S_0 = 3.0$, $f_w = 3.0$, $M = 1.5$, and $G_r = 10.0$).

S_c	$f^{//}(0)$	$-\vartheta^{/}(0)$
4.0	6.250391	1.92228
–4.0	–6.194050	1.92228

The coefficient of skin friction decrease with the decreasing values of S_c from positive to negative but no significant effect of S_c on the coefficient of heat transfer is observed as is seen in Table 4.7.

CHAPTER V

Numerical Scheme and Procedure

In Chapter IV, we have obtained the approximate analytical solutions of the system of coupled nonlinear ordinary differential equations (2.30) – (2.33) with boundary conditions (2.34) by using the perturbation technique. In this Chapter, our aim is to solve the same equations (2.30) – (2.33) together with the boundary conditions (2.34) numerically using a standard initial value solver numerical procedure based on the sixth order Runge-Kutta integration scheme along with Nachtsheim-Swigert iteration technique. Also for more accuracy of the solutions, a comparison will be made of the numerical results adopting the aforementioned numerical techniques with the obtained analytical approximate results through graphs and tables.

Nachtsheim-Swigert iteration technique

To obtain the solution of the system of ordinary differential equations (2.30) – (2.33) with boundary conditions (2.34), an extension of the Nachtsheim-Swigert shooting iteration technique (guessing the missing value) (Nachtsheim and Swigert (1965)) together with Runge-kutta sixth order integration scheme is implemented.

It is clear that the numbers of initial conditions are not sufficient to obtain the particular solution of the differential equations, so we require assuming additional missing/unspecified initial conditions. Thus, in this method, the missing initial conditions at the initial point of the interval are assumed and with all the initial conditions (given and assumed) the equations are integrated numerically in steps as an initial value problem to the terminal point. These are to be so assumed that the solution of the outer prescribed points also matches. The accuracy of the assumed missing initial condition is checked by comparing the calculated value of the dependent variable at the terminal point with its given value there. If match is not found (a difference exists) at the outer end then another set of missing initial conditions are considered and the process is repeated. This trial and error process is taken care through Nachtsheim-Swigert iteration technique and the process is continued until the agreement between the calculated and the given condition at the terminal point is within the specified degree of accuracy. For this type of iterative approach, one naturally inquires whether or not there is a systematic way of finding each succeeding (assumed) value of the missing initial condition.

The boundary conditions (2.34) associated with the system are of the two-point asymptotic class. Two-point boundary conditions have values of the dependent variable specified at two different values of the independent variable, where the outer boundary conditions are specified at infinity. There are four asymptotic boundary conditions and five known surface conditions as well as four unknown surface conditions $f''(0),\ H'(0),\ \theta'(0),\ \varphi'(0)$ here. Specification of asymptotic boundary condition implies that the value of velocity approaches to zero, the value of induced magnetic field approaches from unity to zero, the value of temperature approaches from unity to zero, and the value of concentration approaches from unity to zero as the outer specified value of the independent variable η is approached infinity. The governing differential equations are then integrated with these assumed surface boundary conditions. If the required outer boundary condition is satisfied, a solution has been achieved. However, this is not generally the case. Hence a method must be devised to logically estimate the new surface boundary conditions for the next trial integration. Asymptotic boundary value problems such as those governing the boundary layer equations are further complicated by the fact that the outer boundary conditions are specified at infinity. In the trial integrations, infinity is numerically approximated by some large specified value of the independent variable. There is not a priori general method of estimating this value. Selection of too small a maximum value for the independent variable may not allow the solution to asymptotically converge to the required accuracy. Selecting a large value may result in divergence of the trial integration or in slow convergence of surface boundary conditions required satisfying the asymptotic outer boundary condition. Selecting too large a value of the independent variable is expensive in terms of computer time. Nachtsheim-Swigert developed an iteration method, which overcomes these difficulties.

Within the context of the initial value method and Nachtsheim-Swigert iteration technique the outer boundary conditions may be functionally represented as

$$f'(\xi_{\max}) = f'(f''(0),\ H'(0),\ \theta'(0),\ \varphi'(0)) = \delta_1 \tag{5.1}$$

$$H(\xi_{\max}) = H(f''(0),\ H'(0),\ \theta'(0),\ \varphi'(0)) = \delta_2 \tag{5.2}$$

$$\theta(\xi_{\max}) = \theta(f''(0),\ H'(0),\ \theta'(0),\ \varphi'(0)) = \delta_3 \tag{5.3}$$

$$\varphi(\xi_{\max}) = \varphi(f''(0),\ H'(0),\ \theta'(0),\ \varphi'(0)) = \delta_4 \tag{5.4}$$

With the asymptotic convergence criteria is given by

$$f''(\xi_{\max}) = f''(f''(0),\ H'(0),\ \theta'(0),\ \varphi'(0)) = \delta_5 \tag{5.5}$$

$$H'(\xi_{\max}) = H'(f''(0),\ H'(0),\ \theta'(0),\ \varphi'(0)) = \delta_6 \tag{5.6}$$

$$\theta'(\xi_{\max}) = \theta'(f''(0),\, H'(0),\, \theta'(0),\, \varphi'(0)) = \delta_7 \tag{5.7}$$

$$\varphi'(\xi_{\max}) = \varphi'(f''(0),\, H'(0),\, \theta'(0),\, \varphi'(0)) = \delta_8 \tag{5.8}$$

Let us choose $f''(0) = g_1$, $H'(0) = g_2$, $\theta'(0) = g_3$, $\varphi'(0) = g_4$, and expanding first order Taylor series expansion after using the above equations (5.1) – (5.8), yields

$$f'(\xi_{\max}) = f_c'(\xi_{\max}) + \frac{\partial f'}{\partial g_1}\Delta g_1 + \frac{\partial f'}{\partial g_2}\Delta g_2 + \frac{\partial f'}{\partial g_3}\Delta g_3 + \frac{\partial f'}{\partial g_4}\Delta g_4 = \delta_1 \tag{5.9}$$

$$H(\xi_{\max}) = H_c(\xi_{\max}) + \frac{\partial H}{\partial g_1}\Delta g_1 + \frac{\partial H}{\partial g_2}\Delta g_2 + \frac{\partial H}{\partial g_3}\Delta g_3 + \frac{\partial H}{\partial g_4}\Delta g_4 = \delta_2 \tag{5.10}$$

$$\theta(\xi_{\max}) = \theta_c(\xi_{\max}) + \frac{\partial \theta}{\partial g_1}\Delta g_1 + \frac{\partial \theta}{\partial g_2}\Delta g_2 + \frac{\partial \theta}{\partial g_3}\Delta g_3 + \frac{\partial \theta}{\partial g_4}\Delta g_4 = \delta_3 \tag{5.11}$$

$$\varphi(\xi_{\max}) = \varphi_c(\xi_{\max}) + \frac{\partial \varphi}{\partial g_1}\Delta g_1 + \frac{\partial \varphi}{\partial g_2}\Delta g_2 + \frac{\partial \varphi}{\partial g_3}\Delta g_3 + \frac{\partial \varphi}{\partial g_4}\Delta g_4 = \delta_4 \tag{5.12}$$

$$f''(\xi_{\max}) = f_c''(\xi_{\max}) + \frac{\partial f''}{\partial g_1}\Delta g_1 + \frac{\partial f''}{\partial g_2}\Delta g_2 + \frac{\partial f''}{\partial g_3}\Delta g_3 + \frac{\partial f''}{\partial g_4}\Delta g_4 = \delta_5 \tag{5.13}$$

$$H'(\xi_{\max}) = H_c'(\xi_{\max}) + \frac{\partial H'}{\partial g_1}\Delta g_1 + \frac{\partial H'}{\partial g_2}\Delta g_2 + \frac{\partial H'}{\partial g_3}\Delta g_3 + \frac{\partial H'}{\partial g_4}\Delta g_4 = \delta_6 \tag{5.14}$$

$$\theta'(\xi_{\max}) = \theta_c''(\xi_{\max}) + \frac{\partial \theta'}{\partial g_1}\Delta g_1 + \frac{\partial \theta'}{\partial g_2}\Delta g_2 + \frac{\partial \theta'}{\partial g_3}\Delta g_3 + \frac{\partial \theta'}{\partial g_4}\Delta g_4 = \delta_7 \tag{5.15}$$

$$\varphi'(\xi_{\max}) = \varphi_c'(\xi_{\max}) + \frac{\partial \varphi'}{\partial g_1}\Delta g_1 + \frac{\partial \varphi'}{\partial g_2}\Delta g_2 + \frac{\partial \varphi'}{\partial g_3}\Delta g_3 + \frac{\partial \varphi'}{\partial g_4}\Delta g_4 = \delta_8 \tag{5.16}$$

where subscript 'c' indicates the value of the function at $\xi_{\max}$ determined from the trial integration. Solution of these equations in a least square sense requires determining the minimum value of the error as

$$E = \delta_1^{\,2} + \delta_2^{\,2} + \delta_3^{\,2} + \delta_4^{\,2} + \delta_5^{\,2} + \delta_6^{\,2} + \delta_7^{\,2} + \delta_8^{\,2} \tag{5.17}$$

with respect to g_1, g_2, g_3 and g_4.

Now differentiating equation (4.17) with respect to g_1, g_2, g_3 and g_4

$$\delta_1\frac{\partial \delta_1}{\partial g_1} + \delta_2\frac{\partial \delta_2}{\partial g_1} + \delta_3\frac{\partial \delta_3}{\partial g_1} + \delta_4\frac{\partial \delta_4}{\partial g_1} + \delta_5\frac{\partial \delta_5}{\partial g_1} + \delta_6\frac{\partial \delta_6}{\partial g_1} + \delta_7\frac{\partial \delta_7}{\partial g_1} + \delta_8\frac{\partial \delta_8}{\partial g_1} = 0 \tag{5.18}$$

$$\delta_1\frac{\partial \delta_1}{\partial g_2} + \delta_2\frac{\partial \delta_2}{\partial g_2} + \delta_3\frac{\partial \delta_3}{\partial g_2} + \delta_4\frac{\partial \delta_4}{\partial g_2} + \delta_5\frac{\partial \delta_5}{\partial g_2} + \delta_6\frac{\partial \delta_6}{\partial g_2} + \delta_7\frac{\partial \delta_7}{\partial g_2} + \delta_8\frac{\partial \delta_8}{\partial g_2} = 0 \tag{5.19}$$

$$\delta_1\frac{\partial \delta_1}{\partial g_3} + \delta_2\frac{\partial \delta_2}{\partial g_3} + \delta_3\frac{\partial \delta_3}{\partial g_3} + \delta_4\frac{\partial \delta_4}{\partial g_3} + \delta_5\frac{\partial \delta_5}{\partial g_3} + \delta_6\frac{\partial \delta_6}{\partial g_3} + \delta_7\frac{\partial \delta_7}{\partial g_3} + \delta_8\frac{\partial \delta_8}{\partial g_3} = 0 \tag{5.20}$$

$$\delta_1\frac{\partial\delta_1}{\partial g_4}+\delta_2\frac{\partial\delta_2}{\partial g_4}+\delta_3\frac{\partial\delta_3}{\partial g_4}+\delta_4\frac{\partial\delta_4}{\partial g_4}+\delta_5\frac{\partial\delta_5}{\partial g_4}+\delta_6\frac{\partial\delta_6}{\partial g_4}+\delta_7\frac{\partial\delta_7}{\partial g_4}+\delta_8\frac{\partial\delta_8}{\partial g_4}=0 \tag{5.21}$$

Now using the equations (5.9) – (5.16) in the equation (5.18)

$$\left[f_c'+\frac{\partial f'}{\partial g_1}\Delta g_1+\frac{\partial f'}{\partial g_2}\Delta g_2+\frac{\partial f'}{\partial g_3}\Delta g_3+\frac{\partial f'}{\partial g_4}\Delta g_4\right]\frac{\partial f'}{\partial g_1}+\left[H_c+\frac{\partial H}{\partial g_1}\Delta g_1+\frac{\partial H}{\partial g_2}\Delta g_2+\frac{\partial H}{\partial g_3}\Delta g_3+\frac{\partial H}{\partial g_4}\Delta g_4\right]\frac{\partial H}{\partial g_1}$$
$$+\left[\theta_c+\frac{\partial \theta}{\partial g_1}\Delta g_1+\frac{\partial \theta}{\partial g_2}\Delta g_2+\frac{\partial \theta}{\partial g_3}\Delta g_3+\frac{\partial \theta}{\partial g_4}\Delta g_4\right]\frac{\partial \theta}{\partial g_1}+\left[\varphi_c+\frac{\partial \varphi}{\partial g_1}\Delta g_1+\frac{\partial \varphi}{\partial g_2}\Delta g_2+\frac{\partial \varphi}{\partial g_3}\Delta g_3+\frac{\partial \varphi}{\partial g_4}\Delta g_4\right]\frac{\partial \varphi}{\partial g_1}$$
$$+\left[f_c''+\frac{\partial f''}{\partial g_1}\Delta g_1+\frac{\partial f''}{\partial g_2}\Delta g_2+\frac{\partial f''}{\partial g_3}\Delta g_3+\frac{\partial f''}{\partial g_4}\Delta g_4\right]\frac{\partial f''}{\partial g_1}$$
$$+\left[H_c'+\frac{\partial H'}{\partial g_1}\Delta g_1+\frac{\partial H'}{\partial g_2}\Delta g_2+\frac{\partial H'}{\partial g_3}\Delta g_3+\frac{\partial H'}{\partial g_4}\Delta g_4\right]\frac{\partial H'}{\partial g_1}$$
$$+\left[\theta_c'+\frac{\partial \theta'}{\partial g_1}\Delta g_1+\frac{\partial \theta'}{\partial g_2}\Delta g_2+\frac{\partial \theta'}{\partial g_3}\Delta g_3+\frac{\partial \theta'}{\partial g_4}\Delta g_4\right]\frac{\partial \theta'}{\partial g_1}$$
$$+\left[\varphi_c'+\frac{\partial \varphi'}{\partial g_1}\Delta g_1+\frac{\partial \varphi'}{\partial g_2}\Delta g_2+\frac{\partial \varphi'}{\partial g_3}\Delta g_3+\frac{\partial \varphi'}{\partial g_4}\Delta g_4\right]\frac{\partial \varphi'}{\partial g_1}=0$$

$$\text{or,}\left[\left(\frac{\partial f'}{\partial g_1}\right)^2+\left(\frac{\partial H}{\partial g_1}\right)^2+\left(\frac{\partial \theta}{\partial g_1}\right)^2+\left(\frac{\partial \varphi}{\partial g_1}\right)^2+\left(\frac{\partial f''}{\partial g_1}\right)^2+\left(\frac{\partial H'}{\partial g_1}\right)^2+\left(\frac{\partial \theta'}{\partial g_1}\right)^2+\left(\frac{\partial \varphi'}{\partial g_1}\right)^2\right]\Delta g_1$$
$$+\left[\frac{\partial f'}{\partial g_2}\frac{\partial f'}{\partial g_1}+\frac{\partial H}{\partial g_2}\frac{\partial H}{\partial g_1}+\frac{\partial \theta}{\partial g_2}\frac{\partial \theta}{\partial g_1}+\frac{\partial \varphi}{\partial g_2}\frac{\partial \varphi}{\partial g_1}+\frac{\partial f''}{\partial g_2}\frac{\partial f''}{\partial g_1}+\frac{\partial H'}{\partial g_2}\frac{\partial H'}{\partial g_1}+\frac{\partial \theta'}{\partial g_2}\frac{\partial \theta'}{\partial g_1}+\frac{\partial \varphi'}{\partial g_2}\frac{\partial \varphi'}{\partial g_1}\right]\Delta g_2$$
$$+\left[\frac{\partial f'}{\partial g_3}\frac{\partial f'}{\partial g_1}+\frac{\partial H}{\partial g_3}\frac{\partial H}{\partial g_1}+\frac{\partial \theta}{\partial g_3}\frac{\partial \theta}{\partial g_1}+\frac{\partial \varphi}{\partial g_3}\frac{\partial \varphi}{\partial g_1}+\frac{\partial f''}{\partial g_3}\frac{\partial f''}{\partial g_1}+\frac{\partial H'}{\partial g_3}\frac{\partial H'}{\partial g_1}+\frac{\partial \theta'}{\partial g_3}\frac{\partial \theta'}{\partial g_1}+\frac{\partial \varphi'}{\partial g_3}\frac{\partial \varphi'}{\partial g_1}\right]\Delta g_3$$
$$+\left[\frac{\partial f'}{\partial g_4}\frac{\partial f'}{\partial g_1}+\frac{\partial H}{\partial g_4}\frac{\partial H}{\partial g_1}+\frac{\partial \theta}{\partial g_4}\frac{\partial \theta}{\partial g_1}+\frac{\partial \varphi}{\partial g_4}\frac{\partial \varphi}{\partial g_1}+\frac{\partial f''}{\partial g_4}\frac{\partial f''}{\partial g_1}+\frac{\partial H'}{\partial g_4}\frac{\partial H'}{\partial g_1}+\frac{\partial \theta'}{\partial g_4}\frac{\partial \theta'}{\partial g_1}+\frac{\partial \varphi'}{\partial g_4}\frac{\partial \varphi'}{\partial g_1}\right]\Delta g_4$$
$$=-\left[f_c'\frac{\partial f'}{\partial g_1}+H_c\frac{\partial H}{\partial g_1}+\theta_c\frac{\partial \theta}{\partial g_1}+\varphi_c\frac{\partial \varphi}{\partial g_1}+f_c''\frac{\partial f''}{\partial g_1}+H_c'\frac{\partial H'}{\partial g_1}+\theta_c'\frac{\partial \theta'}{\partial g_1}+\varphi_c'\frac{\partial \varphi'}{\partial g_1}\right] \tag{5.22}$$

Similarly by using the equations (5.9) – (5.16) in the equation (5.19), (5.20) and (5.21)

$$\left[\frac{\partial f'}{\partial g_1}\frac{\partial f'}{\partial g_2}+\frac{\partial H}{\partial g_1}\frac{\partial H}{\partial g_2}+\frac{\partial \theta}{\partial g_1}\frac{\partial \theta}{\partial g_2}+\frac{\partial \varphi}{\partial g_1}\frac{\partial \varphi}{\partial g_2}+\frac{\partial f''}{\partial g_1}\frac{\partial f''}{\partial g_2}+\frac{\partial H'}{\partial g_1}\frac{\partial H'}{\partial g_2}+\frac{\partial \theta'}{\partial g_1}\frac{\partial \theta'}{\partial g_2}+\frac{\partial \varphi'}{\partial g_1}\frac{\partial \varphi'}{\partial g_2}\right]\Delta g_1$$
$$+\left[\left(\frac{\partial f'}{\partial g_2}\right)^2+\left(\frac{\partial H}{\partial g_2}\right)^2+\left(\frac{\partial \theta}{\partial g_2}\right)^2+\left(\frac{\partial \varphi}{\partial g_2}\right)^2+\left(\frac{\partial f''}{\partial g_2}\right)^2+\left(\frac{\partial H'}{\partial g_2}\right)^2+\left(\frac{\partial \theta'}{\partial g_2}\right)^2+\left(\frac{\partial \varphi'}{\partial g_2}\right)^2\right]\Delta g_2$$
$$+\left[\frac{\partial f'}{\partial g_3}\frac{\partial f'}{\partial g_2}+\frac{\partial H}{\partial g_3}\frac{\partial H}{\partial g_2}+\frac{\partial \theta}{\partial g_3}\frac{\partial \theta}{\partial g_2}+\frac{\partial \varphi}{\partial g_3}\frac{\partial \varphi}{\partial g_2}+\frac{\partial f''}{\partial g_3}\frac{\partial f''}{\partial g_2}+\frac{\partial H'}{\partial g_3}\frac{\partial H'}{\partial g_2}+\frac{\partial \theta'}{\partial g_3}\frac{\partial \theta'}{\partial g_2}+\frac{\partial \varphi'}{\partial g_3}\frac{\partial \varphi'}{\partial g_2}\right]\Delta g_3$$
$$+\left[\frac{\partial f'}{\partial g_4}\frac{\partial f'}{\partial g_2}+\frac{\partial H}{\partial g_4}\frac{\partial H}{\partial g_2}+\frac{\partial \theta}{\partial g_4}\frac{\partial \theta}{\partial g_2}+\frac{\partial \varphi}{\partial g_4}\frac{\partial \varphi}{\partial g_2}+\frac{\partial f''}{\partial g_4}\frac{\partial f''}{\partial g_2}+\frac{\partial H'}{\partial g_4}\frac{\partial H'}{\partial g_2}+\frac{\partial \theta'}{\partial g_4}\frac{\partial \theta'}{\partial g_2}+\frac{\partial \varphi'}{\partial g_4}\frac{\partial \varphi'}{\partial g_2}\right]\Delta g_4$$

$$= -\left[f_c^{\prime} \frac{\partial f^{\prime}}{\partial g_2} + H_c \frac{\partial H}{\partial g_2} + \theta_c \frac{\partial \theta}{\partial g_2} + \varphi_c \frac{\partial \varphi}{\partial g_2} + f_c^{\prime\prime} \frac{\partial f^{\prime\prime}}{\partial g_2} + H_c^{\prime} \frac{\partial H^{\prime}}{\partial g_2} + \theta_c^{\prime} \frac{\partial \theta^{\prime}}{\partial g_2} + \varphi_c^{\prime} \frac{\partial \varphi^{\prime}}{\partial g_2} \right] \quad (5.23)$$

and

$$\left[\frac{\partial f^{\prime}}{\partial g_1}\frac{\partial f^{\prime}}{\partial g_3} + \frac{\partial H}{\partial g_1}\frac{\partial H}{\partial g_3} + \frac{\partial \theta}{\partial g_1}\frac{\partial \theta}{\partial g_3} + \frac{\partial \varphi}{\partial g_1}\frac{\partial \varphi}{\partial g_3} + \frac{\partial f^{\prime\prime}}{\partial g_1}\frac{\partial f^{\prime\prime}}{\partial g_3} + \frac{\partial H^{\prime}}{\partial g_1}\frac{\partial H^{\prime}}{\partial g_3} + \frac{\partial \theta^{\prime}}{\partial g_1}\frac{\partial \theta^{\prime}}{\partial g_3} + \frac{\partial \varphi^{\prime}}{\partial g_1}\frac{\partial \varphi^{\prime}}{\partial g_3} \right] \Delta g_1$$

$$+ \left[\frac{\partial f^{\prime}}{\partial g_2}\frac{\partial f^{\prime}}{\partial g_3} + \frac{\partial H}{\partial g_2}\frac{\partial H}{\partial g_3} + \frac{\partial \theta}{\partial g_2}\frac{\partial \theta}{\partial g_3} + \frac{\partial \varphi}{\partial g_2}\frac{\partial \varphi}{\partial g_3} + \frac{\partial f^{\prime\prime}}{\partial g_2}\frac{\partial f^{\prime\prime}}{\partial g_3} + \frac{\partial H^{\prime}}{\partial g_2}\frac{\partial H^{\prime}}{\partial g_3} + \frac{\partial \theta^{\prime}}{\partial g_2}\frac{\partial \theta^{\prime}}{\partial g_3} + \frac{\partial \varphi^{\prime}}{\partial g_2}\frac{\partial \varphi^{\prime}}{\partial g_3} \right] \Delta g_2$$

$$+ \left[\left(\frac{\partial f^{\prime}}{\partial g_3}\right)^2 + \left(\frac{\partial H}{\partial g_3}\right)^2 + \left(\frac{\partial \theta}{\partial g_3}\right)^2 + \left(\frac{\partial \varphi}{\partial g_3}\right)^2 + \left(\frac{\partial f^{\prime\prime}}{\partial g_3}\right)^2 + \left(\frac{\partial H^{\prime}}{\partial g_3}\right)^2 + \left(\frac{\partial \theta^{\prime}}{\partial g_3}\right)^2 + \left(\frac{\partial \varphi^{\prime}}{\partial g_3}\right)^2 \right] \Delta g_3$$

$$+ \left[\frac{\partial f^{\prime}}{\partial g_4}\frac{\partial f^{\prime}}{\partial g_3} + \frac{\partial H}{\partial g_4}\frac{\partial H}{\partial g_3} + \frac{\partial \theta}{\partial g_4}\frac{\partial \theta}{\partial g_3} + \frac{\partial \varphi}{\partial g_4}\frac{\partial \varphi}{\partial g_3} + \frac{\partial f^{\prime\prime}}{\partial g_4}\frac{\partial f^{\prime\prime}}{\partial g_3} + \frac{\partial H^{\prime}}{\partial g_4}\frac{\partial H^{\prime}}{\partial g_3} + \frac{\partial \theta^{\prime}}{\partial g_4}\frac{\partial \theta^{\prime}}{\partial g_3} + \frac{\partial \varphi^{\prime}}{\partial g_4}\frac{\partial \varphi^{\prime}}{\partial g_3} \right] \Delta g_4$$

$$= -\left[f_c^{\prime} \frac{\partial f^{\prime}}{\partial g_3} + H_c \frac{\partial H}{\partial g_3} + \theta_c \frac{\partial \theta}{\partial g_3} + \varphi_c \frac{\partial \varphi}{\partial g_3} + f_c^{\prime\prime} \frac{\partial f^{\prime\prime}}{\partial g_3} + H_c^{\prime} \frac{\partial H^{\prime}}{\partial g_3} + \theta_c^{\prime} \frac{\partial \theta^{\prime}}{\partial g_3} + \varphi_c^{\prime} \frac{\partial \varphi^{\prime}}{\partial g_3} \right] \quad (5.24)$$

and

$$\left[\frac{\partial f^{\prime}}{\partial g_1}\frac{\partial f^{\prime}}{\partial g_4} + \frac{\partial H}{\partial g_1}\frac{\partial H}{\partial g_4} + \frac{\partial \theta}{\partial g_1}\frac{\partial \theta}{\partial g_4} + \frac{\partial \varphi}{\partial g_1}\frac{\partial \varphi}{\partial g_4} + \frac{\partial f^{\prime\prime}}{\partial g_1}\frac{\partial f^{\prime\prime}}{\partial g_4} + \frac{\partial H^{\prime}}{\partial g_1}\frac{\partial H^{\prime}}{\partial g_4} + \frac{\partial \theta^{\prime}}{\partial g_1}\frac{\partial \theta^{\prime}}{\partial g_4} + \frac{\partial \varphi^{\prime}}{\partial g_1}\frac{\partial \varphi^{\prime}}{\partial g_4} \right] \Delta g_1$$

$$+ \left[\frac{\partial f^{\prime}}{\partial g_2}\frac{\partial f^{\prime}}{\partial g_4} + \frac{\partial H}{\partial g_2}\frac{\partial H}{\partial g_4} + \frac{\partial \theta}{\partial g_2}\frac{\partial \theta}{\partial g_4} + \frac{\partial \varphi}{\partial g_2}\frac{\partial \varphi}{\partial g_4} + \frac{\partial f^{\prime\prime}}{\partial g_2}\frac{\partial f^{\prime\prime}}{\partial g_4} + \frac{\partial H^{\prime}}{\partial g_2}\frac{\partial H^{\prime}}{\partial g_4} + \frac{\partial \theta^{\prime}}{\partial g_2}\frac{\partial \theta^{\prime}}{\partial g_4} + \frac{\partial \varphi^{\prime}}{\partial g_2}\frac{\partial \varphi^{\prime}}{\partial g_4} \right] \Delta g_2$$

$$+ \left[\frac{\partial f^{\prime}}{\partial g_3}\frac{\partial f^{\prime}}{\partial g_4} + \frac{\partial H}{\partial g_3}\frac{\partial H}{\partial g_4} + \frac{\partial \theta}{\partial g_3}\frac{\partial \theta}{\partial g_4} + \frac{\partial \varphi}{\partial g_3}\frac{\partial \varphi}{\partial g_4} + \frac{\partial f^{\prime\prime}}{\partial g_3}\frac{\partial f^{\prime\prime}}{\partial g_4} + \frac{\partial H^{\prime}}{\partial g_3}\frac{\partial H^{\prime}}{\partial g_4} + \frac{\partial \theta^{\prime}}{\partial g_3}\frac{\partial \theta^{\prime}}{\partial g_4} + \frac{\partial \varphi^{\prime}}{\partial g_3}\frac{\partial \varphi^{\prime}}{\partial g_4} \right] \Delta g_3$$

$$+ \left[\left(\frac{\partial f^{\prime}}{\partial g_4}\right)^2 + \left(\frac{\partial H}{\partial g_4}\right)^2 + \left(\frac{\partial \theta}{\partial g_4}\right)^2 + \left(\frac{\partial \varphi}{\partial g_4}\right)^2 + \left(\frac{\partial f^{\prime\prime}}{\partial g_4}\right)^2 + \left(\frac{\partial H^{\prime}}{\partial g_4}\right)^2 + \left(\frac{\partial \theta^{\prime}}{\partial g_4}\right)^2 + \left(\frac{\partial \varphi^{\prime}}{\partial g_4}\right)^2 \right] \Delta g_4$$

$$= -\left[f_c^{\prime} \frac{\partial f^{\prime}}{\partial g_4} + H_c \frac{\partial H}{\partial g_4} + \theta_c \frac{\partial \theta}{\partial g_4} + \varphi_c \frac{\partial \varphi}{\partial g_4} + f_c^{\prime\prime} \frac{\partial f^{\prime\prime}}{\partial g_4} + H_c^{\prime} \frac{\partial H^{\prime}}{\partial g_4} + \theta_c^{\prime} \frac{\partial \theta^{\prime}}{\partial g_4} + \varphi_c^{\prime} \frac{\partial \varphi^{\prime}}{\partial g_4} \right] \quad (5.25)$$

We can write the equations (5.22) – (5.25) in system of linear equations in the following form as:

$$a_{11}\Delta g_1 + a_{12}\Delta g_2 + a_{13}\Delta g_3 + a_{14}\Delta g_4 = b_{11} \quad (5.26)$$

$$a_{21}\Delta g_1 + a_{22}\Delta g_2 + a_{23}\Delta g_3 + a_{24}\Delta g_4 = b_{22} \quad (5.27)$$

$$a_{31}\Delta g_1 + a_{32}\Delta g_2 + a_{33}\Delta g_3 + a_{34}\Delta g_4 = b_{33} \quad (5.28)$$

$$a_{41}\Delta g_1 + a_{42}\Delta g_2 + a_{43}\Delta g_3 + a_{44}\Delta g_4 = b_{44} \quad (5.29)$$

where

$$a_{11}=\left(\frac{\partial f'}{\partial g_1}\right)^2+\left(\frac{\partial H}{\partial g_1}\right)^2+\left(\frac{\partial \theta}{\partial g_1}\right)^2+\left(\frac{\partial \varphi}{\partial g_1}\right)^2+\left(\frac{\partial f''}{\partial g_1}\right)^2+\left(\frac{\partial H'}{\partial g_1}\right)^2+\left(\frac{\partial \theta'}{\partial g_1}\right)^2+\left(\frac{\partial \varphi'}{\partial g_1}\right)^2$$

$$a_{22}=\left(\frac{\partial f'}{\partial g_2}\right)^2+\left(\frac{\partial H}{\partial g_2}\right)^2+\left(\frac{\partial \theta}{\partial g_2}\right)^2+\left(\frac{\partial \varphi}{\partial g_2}\right)^2+\left(\frac{\partial f''}{\partial g_2}\right)^2+\left(\frac{\partial H'}{\partial g_2}\right)^2+\left(\frac{\partial \theta'}{\partial g_2}\right)^2+\left(\frac{\partial \varphi'}{\partial g_2}\right)^2$$

$$a_{33}=\left(\frac{\partial f'}{\partial g_3}\right)^2+\left(\frac{\partial H}{\partial g_3}\right)^2+\left(\frac{\partial \theta}{\partial g_3}\right)^2+\left(\frac{\partial \varphi}{\partial g_3}\right)^2+\left(\frac{\partial f''}{\partial g_3}\right)^2+\left(\frac{\partial H'}{\partial g_3}\right)^2+\left(\frac{\partial \theta'}{\partial g_3}\right)^2+\left(\frac{\partial \varphi'}{\partial g_3}\right)^2$$

$$a_{44}=\left(\frac{\partial f'}{\partial g_4}\right)^2+\left(\frac{\partial H}{\partial g_4}\right)^2+\left(\frac{\partial \theta}{\partial g_4}\right)^2+\left(\frac{\partial \varphi}{\partial g_4}\right)^2+\left(\frac{\partial f''}{\partial g_4}\right)^2+\left(\frac{\partial H'}{\partial g_4}\right)^2+\left(\frac{\partial \theta'}{\partial g_4}\right)^2+\left(\frac{\partial \varphi'}{\partial g_4}\right)^2$$

$$a_{12}=a_{21}=\frac{\partial f'}{\partial g_1}\frac{\partial f'}{\partial g_2}+\frac{\partial H}{\partial g_1}\frac{\partial H}{\partial g_2}+\frac{\partial \theta}{\partial g_1}\frac{\partial \theta}{\partial g_2}+\frac{\partial \varphi}{\partial g_1}\frac{\partial \varphi}{\partial g_2}+\frac{\partial f''}{\partial g_1}\frac{\partial f''}{\partial g_2}+\frac{\partial H'}{\partial g_1}\frac{\partial H'}{\partial g_2}+\frac{\partial \theta'}{\partial g_1}\frac{\partial \theta'}{\partial g_2}+\frac{\partial \varphi'}{\partial g_1}\frac{\partial \varphi'}{\partial g_2}$$

$$a_{13}=a_{31}=\frac{\partial f'}{\partial g_1}\frac{\partial f'}{\partial g_3}+\frac{\partial H}{\partial g_1}\frac{\partial H}{\partial g_3}+\frac{\partial \theta}{\partial g_1}\frac{\partial \theta}{\partial g_3}+\frac{\partial \varphi}{\partial g_1}\frac{\partial \varphi}{\partial g_3}+\frac{\partial f''}{\partial g_1}\frac{\partial f''}{\partial g_3}+\frac{\partial H'}{\partial g_1}\frac{\partial H'}{\partial g_3}+\frac{\partial \theta'}{\partial g_1}\frac{\partial \theta'}{\partial g_3}+\frac{\partial \varphi'}{\partial g_1}\frac{\partial \varphi'}{\partial g_3}$$

$$a_{14}=a_{41}=\frac{\partial f'}{\partial g_1}\frac{\partial f'}{\partial g_4}+\frac{\partial H}{\partial g_1}\frac{\partial H}{\partial g_4}+\frac{\partial \theta}{\partial g_1}\frac{\partial \theta}{\partial g_4}+\frac{\partial \varphi}{\partial g_1}\frac{\partial \varphi}{\partial g_4}+\frac{\partial f''}{\partial g_1}\frac{\partial f''}{\partial g_4}+\frac{\partial H'}{\partial g_1}\frac{\partial H'}{\partial g_4}+\frac{\partial \theta'}{\partial g_1}\frac{\partial \theta'}{\partial g_4}+\frac{\partial \varphi'}{\partial g_1}\frac{\partial \varphi'}{\partial g_4}$$

$$a_{23}=a_{32}=\frac{\partial f'}{\partial g_2}\frac{\partial f'}{\partial g_3}+\frac{\partial H}{\partial g_2}\frac{\partial H}{\partial g_3}+\frac{\partial \theta}{\partial g_2}\frac{\partial \theta}{\partial g_3}+\frac{\partial \varphi}{\partial g_2}\frac{\partial \varphi}{\partial g_3}+\frac{\partial f''}{\partial g_2}\frac{\partial f''}{\partial g_3}+\frac{\partial H'}{\partial g_2}\frac{\partial H'}{\partial g_3}+\frac{\partial \theta'}{\partial g_2}\frac{\partial \theta'}{\partial g_3}+\frac{\partial \varphi'}{\partial g_2}\frac{\partial \varphi'}{\partial g_3}$$

$$a_{24}=a_{42}=\frac{\partial f'}{\partial g_2}\frac{\partial f'}{\partial g_4}+\frac{\partial H}{\partial g_2}\frac{\partial H}{\partial g_4}+\frac{\partial \theta}{\partial g_2}\frac{\partial \theta}{\partial g_4}+\frac{\partial \varphi}{\partial g_2}\frac{\partial \varphi}{\partial g_4}+\frac{\partial f''}{\partial g_2}\frac{\partial f''}{\partial g_4}+\frac{\partial H'}{\partial g_2}\frac{\partial H'}{\partial g_4}+\frac{\partial \theta'}{\partial g_2}\frac{\partial \theta'}{\partial g_4}+\frac{\partial \varphi'}{\partial g_2}\frac{\partial \varphi'}{\partial g_4}$$

$$a_{34}=a_{43}=\frac{\partial f'}{\partial g_3}\frac{\partial f'}{\partial g_4}+\frac{\partial H}{\partial g_3}\frac{\partial H}{\partial g_4}+\frac{\partial \theta}{\partial g_3}\frac{\partial \theta}{\partial g_4}+\frac{\partial \varphi}{\partial g_3}\frac{\partial \varphi}{\partial g_4}+\frac{\partial f''}{\partial g_3}\frac{\partial f''}{\partial g_4}+\frac{\partial H'}{\partial g_3}\frac{\partial H'}{\partial g_4}+\frac{\partial \theta'}{\partial g_3}\frac{\partial \theta'}{\partial g_4}+\frac{\partial \varphi'}{\partial g_3}\frac{\partial \varphi'}{\partial g_4}$$

$$b_{11}=-\left[f_c'\frac{\partial f'}{\partial g_1}+H_c\frac{\partial H}{\partial g_1}+\theta_c\frac{\partial \theta}{\partial g_1}+\varphi_c\frac{\partial \varphi}{\partial g_1}+f_c''\frac{\partial f''}{\partial g_1}+H_c'\frac{\partial H'}{\partial g_1}+\theta_c'\frac{\partial \theta'}{\partial g_1}+\varphi_c'\frac{\partial \varphi'}{\partial g_1}\right]$$

$$b_{22}=-\left[f_c'\frac{\partial f'}{\partial g_2}+H_c\frac{\partial H}{\partial g_2}+\theta_c\frac{\partial \theta}{\partial g_2}+\varphi_c\frac{\partial \varphi}{\partial g_2}+f_c''\frac{\partial f''}{\partial g_2}+H_c'\frac{\partial H'}{\partial g_2}+\theta_c'\frac{\partial \theta'}{\partial g_2}+\varphi_c'\frac{\partial \varphi'}{\partial g_2}\right]$$

$$b_{33}=-\left[f_c'\frac{\partial f'}{\partial g_3}+H_c\frac{\partial H}{\partial g_3}+\theta_c\frac{\partial \theta}{\partial g_3}+\varphi_c\frac{\partial \varphi}{\partial g_3}+f_c''\frac{\partial f''}{\partial g_3}+H_c'\frac{\partial H'}{\partial g_3}+\theta_c'\frac{\partial \theta'}{\partial g_3}+\varphi_c'\frac{\partial \varphi'}{\partial g_3}\right]$$

$$b_{44}=-\left[f_c'\frac{\partial f'}{\partial g_4}+H_c\frac{\partial H}{\partial g_4}+\theta_c\frac{\partial \theta}{\partial g_4}+\varphi_c\frac{\partial \varphi}{\partial g_4}+f_c''\frac{\partial f''}{\partial g_4}+H_c'\frac{\partial H'}{\partial g_4}+\theta_c'\frac{\partial \theta'}{\partial g_4}+\varphi_c'\frac{\partial \varphi'}{\partial g_4}\right].$$

Now solving the equations (5.26) – (5.29) by Cramer's rule, we have

$\Delta g_1=\frac{\det A_1}{\det A}$, $\Delta g_2=\frac{\det A_2}{\det A}$, $\Delta g_3=\frac{\det A_3}{\det A}$ and $\Delta g_4=\frac{\det A_4}{\det A}$, where

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix},$$

$$\det A_1 = \begin{vmatrix} b_{11} & a_{12} & a_{13} & a_{14} \\ b_{22} & a_{22} & a_{23} & a_{24} \\ b_{33} & a_{32} & a_{33} & a_{34} \\ b_{44} & a_{42} & a_{43} & a_{44} \end{vmatrix}, \ \det A_2 = \begin{vmatrix} a_{11} & b_{11} & a_{13} & a_{14} \\ a_{21} & b_{22} & a_{23} & a_{24} \\ a_{31} & b_{33} & a_{33} & a_{34} \\ a_{41} & b_{44} & a_{43} & a_{44} \end{vmatrix},$$

$$\det A_3 = \begin{vmatrix} a_{11} & a_{12} & b_{11} & a_{14} \\ a_{21} & a_{22} & b_{22} & a_{24} \\ a_{31} & a_{32} & b_{33} & a_{34} \\ a_{41} & a_{42} & b_{44} & a_{44} \end{vmatrix}, \ \det A_4 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & b_{11} \\ a_{21} & a_{22} & a_{23} & b_{22} \\ a_{31} & a_{32} & a_{33} & b_{33} \\ a_{41} & a_{42} & a_{43} & b_{44} \end{vmatrix}.$$

Then we obtain the missing (guess) values as

$$\left.\begin{aligned} g_1 &\leftarrow g_1 + \Delta g_1 \\ g_2 &\leftarrow g_2 + \Delta g_2 \\ g_3 &\leftarrow g_3 + \Delta g_3 \\ g_4 &\leftarrow g_4 + \Delta g_4 \end{aligned}\right\} \tag{5.30}$$

Based on the integration done with the aforementioned numerical technique, the velocity f_η, the induced magnetic field H, temperature θ and concentration function φ are determined in terms of the similarity variable η for some selected values of the established parameters. In the process of calculation, the skin friction coefficient $f_{\eta\eta}(0)$ and the heat transfer rate $-\theta_\eta(0)$ are also evaluated. The numerical results thus obtained for velocity and temperature fields in terms of the similarity variable are plotted in graphs together with the analytical approximate results to make a comparison of the solutions below:

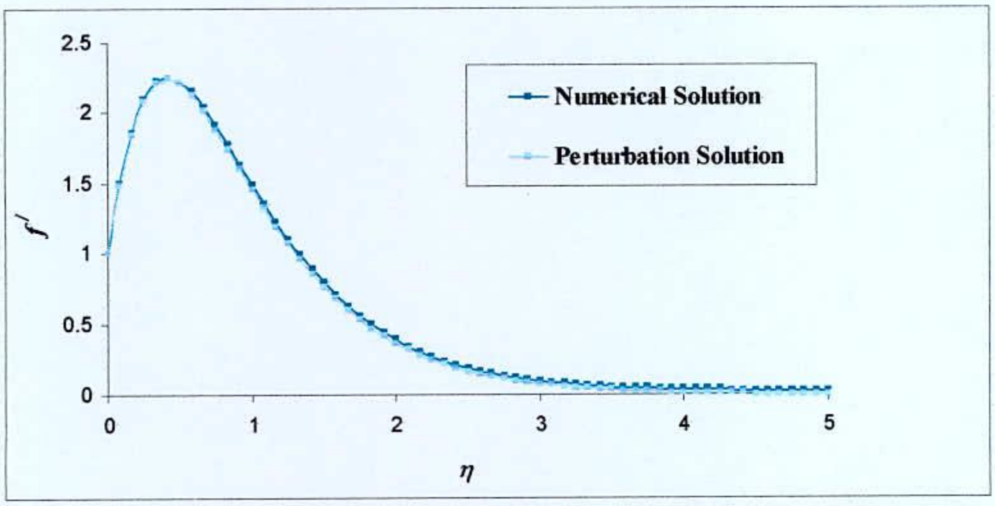

Figure 5.1: Comparison between velocity profiles of numerical solution and perturbation solution ($P_r = 0.71$, $S_0 = 3.0$, $f_w = 3.0$, $M = 1.5$, $G_r = 10.0$ and $G_m = 4.0$).

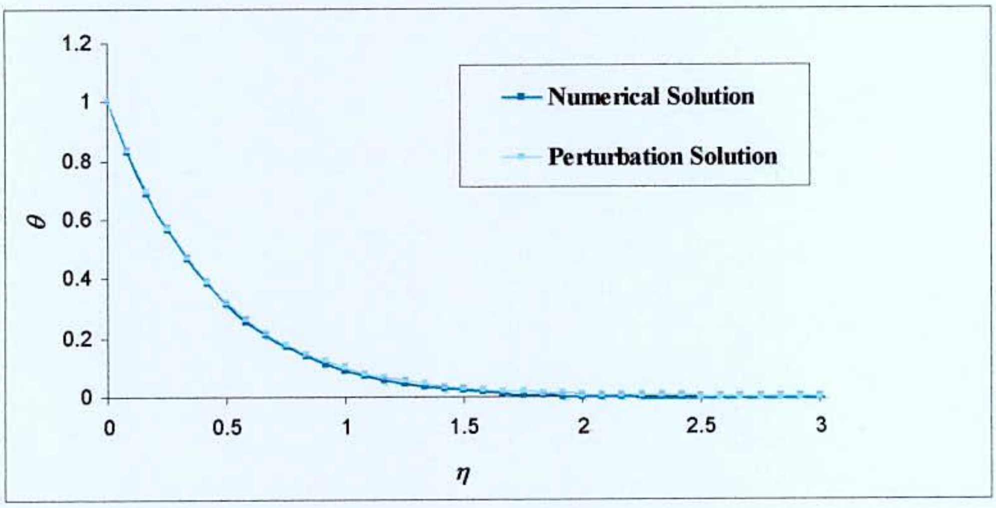

Figure 5.2: Comparison between temperature profiles of numerical solution and perturbation solution ($P_r = 0.71$, $S_0 = 3.0$, $f_w = 3.0$, $M = 1.5$, $G_r = 10.0$ and $G_m = 4.0$).

The values proportional to the coefficient of skin friction and the rate of heat transfer are compared by putting them in tabular form given below:

Table 5.1: Comparison of the values proportional to the coefficient of skin friction and the rate of transfer heat between numerical solution and perturbation solution ($P_r = 0.71$, $S_0 = 3.0$, $f_w = 3.0$, $M = 1.5$, $G_r = 10.0$ and $G_m = 4.0$).

	Perturbation solution	Numerical solution
$f''(0)$	6.250391	6.209126
$-\theta'(0)$	1.922228	1.918692

Therefore, a very good agreement is found between the numerical results and the analytical approximate results as seen in the above Figures and Tables.

CHAPTER VI

Conclusions

An analysis of the steady MHD free convection heat and mass transfer flow of viscous incompressible electrically conducting fluid above a vertical porous plate is presented under the action of a transverse applied magnetic. The thermo diffusion (Soret) effect is taken into account. Approximate numerical results for the second order solutions incorporation with the first order solutions regarding the velocity, temperature, induced magnetic field and mass concentration are presented for different selected values of the established dimensionless parameters. A numerical solution have been obtained by using standard initial value solver numerical procedure based on the sixth order Runge-Kutta integration scheme along with Nachtsheim-Swigert iteration technique to measure the accuracy of the approximate results.

On the basis of the figures, it is concluded that:

a. The velocity increases with the increase of Soret number (S_0)
b. For negative values of modified Grashof number (G_m), the velocity decreases more with increasing S_0.
c. There is no remarkable effect of S_0 on the temperature field.
d. The induced magnetic field decreases with the increase of S_0.
e. Concentration increases with increasing S_0.
f. Both the velocity and temperature are decreased with the increase of suction parameter (f_w).
g. The induced magnetic field increases rapidly with the increase of f_w.
h. Concentration increases very close to the plate surface and then decrease away from the surface with the increase of f_w.
i. The velocity increases with the increase of Grashof number (G_r).
j. No considerable effect of G_r on the velocity.
k. The induced magnetic field decreases with increasing values of G_r.
l. No considerable effect of G_r on the temperature and concentration are found.
m. The velocity increases with the increase of magnetic parameter (M).
n. The induced magnetic field decreases with the increase of magnetic parameter (M).
o. No significant effect is observed on the temperature and concentration with the increase of M.
p. The velocity decreases with the increase of Prandtl number (P_r).

q. The temperature decreases with the increase of P_r.
r. The induced magnetic field increases with increasing P_r.
s. The concentration increases with increasing P_r.

On the basis of the tables, it is concluded that:

a. With the increase in S_0, the coefficient of skin friction increases but no effect of S_o on the coefficient of heat transfer is perceived.
b. With the increase in f_w, the coefficient of skin friction highly decreases and the rate of heat transfer increases.
c. With the decrease in G_r, the coefficient of skin friction gradually decreases but there is no effect of G_r on the coefficient of heat transfer.
d. With the increase of induced magnetic field M, both the coefficient of skin friction and the coefficient of heat transfer reduce significantly.
e. With the increase of the Prandtl number P_r, the coefficient of skin friction decreases but the rate of heat transfer extensively increases.
f. With the decrease in S_c, the coefficient of skin friction sharply decreases but no through effect of S_c on the coefficient of heat transfer.
g. The coefficient of skin friction decrease with the decreasing values of S_c from positive to negative but no significant effect of S_c on the coefficient of heat transfer.

From the comparison of numerical solution and perturbation solution a very good agreement is found between the numerical and analytical approximate results.

REFERENCES

[1] Adams, J. A. and McFadden, P. W. (1966). Journal of American Industrial Chemical Engineering 12, 842.

[2] Agrawal, H.L., Ram, P.C. and Singh, V. (1983): Astrophysics Space Science, 91, 445.

[3] Alam, MD. M. (1995) "Steady MHD free convection and mass transfer flow with thermal diffusion and large suction", Ph. D. Thesis, Ch 7, 134.

[4] Alam, M. S. Rahman, M. M. Ferdows, M. Kaino, Koji, Mureithi, Eunice and Postelnicu, A. (2007) "Diffusion-thermo and thermal-diffusion effects on free convective heat and mass transfer flow in a porous medium with time dependent temperature and concentration", Int. J. Appl. Engg. Res., 2(1), 81–96.

[5] Alam, M. S. Rahman M. M. and Samad, M. A. (2006) "Dufour and Soret effects on unsteady MHD free convection and mass transfer flow past a vertical porous plate in a porous medium, Nonlinear Analysis". Modelling and Control, 11(3), 217–226.

[6] Alfven, H. (1942). On the existence of electromagnetic Hydromagnetic waves, Arkiv F. Mat. Astro. O. Fysik. Bd., 295 No. 2.

[7] Anghel, M. Takhar H. S. and Pop, I. (2000) "Dufour and Soret effects on free convection boundary layer over a vertical surface embedded in a porous medium", Studia Universitatis Babes-Bolyai, Mathematica XLV(4), pp. 11–21.

[8] Bestman, A. R. (1990). International J. Energy Resch. 14, 384.

[9] Caldwell, D.R. (1974). J. Fluid Mech., 64, 347.

[10] Chaudhary, R.C. and Sharma, B.K. (2006) "Combined heat and mass transfer by laminar mixed convection flow from a vertical surface with induced magnetic field". J. Appl. Phys., 99, 034901–10.

[11] Cobble, M. H. (1977). J. Engg. Maths. 11, 249.

[12] Cramer, K. R. and Pai, S.I. (1973). Magneto fluid Dynamics for Engineers and applied physicists, McGraw Hill, New York.

[13] Djukic, Dj. S. (1973). "On Unsteady Magnetic Low-Speed Slip Flow in the Bounbary Layer" Acta Mech., 18(1-2), 35 – 48.

[14] Dursunkaya, Z. and Worek, W. M. (1992) "Diffusion-thermo and thermal-diffusion effects in transient and steady natural convection from a vertical surface", Int. J. Heat Mass Transfer, 35, 2060–2065.

[15] Farady, M. (1832). Experimental Researches in electrically Phill, Trans. 15, 175.

[16] Gebhart, B. and Pera, L. (1971). Int. J. Heat Mass Transfer, 14, 2025.

[17] Georgantopolous, G. A. and Nanousis, N. D.(1980). Astrophys. Space Sci., 67(1), 229.

[18] Groots, S. R. T. and Mozur, P. (1962). Non-equilibrium thermodynamics, North Holland, Amsterdam.

[19] Hasimoto, H. (1957). J. of the Physical Society of Japan, 12(1), 68.

[20] Hossain M. M. T. and Khatun, M. (2012) "Study of Diffusion - Thermo Effect on Laminar Mixed Convection Flow and Heat Transfer from a Vertical Surface with Induced Magnetic Field". Int. J. of Appl. Math and Mech., Vol. 8(5), 40 –60.

[21] Hurel, D. T. J. and Jakeman, E. (1971). J. Fluid Mech., 47, 667.

[22] Ingar, G. R. and Swearn, T. F. (1975). AIAA, J. 13(5), 616.

[23] Kafoussias, N. G. (1992). Astrophys. Space Sci. 192,11.

[24] Kafoussias, N. G. and Williams, E. W. (1995) "Thermal-diffusion and diffusion-thermo effects on mixed free-forced convective and mass transfer boundary layer flow with temperature dependent viscosity", Int. J. Engng. Sci., 33, 1369–1384.

[25] Kim, Y. J., (2004) "Heat and mass transfer in MHD micropolar flow over a vertical moving porous plate in a porous medium, Transport in Porous Media", 56(1), 17-37.

[26] Legros, J. G.,Van Hook, W. K. and Thomas, G.(1968). Chem. Phys. Lett., 2, 696.

[27] Nanbu, K. (1971). AIAA, J. 9, 1642.

[28] Nanousis, N. (1992) Astrophys. Space Sci., 191,313.

[29] Nanousis, N. D. and Goudas, C. L. (1979). Astrophys. Space Sci., 66(1), 13.

[30] Pantokratoras, A. (2007). Comment on "Combined heat and mass transfer by laminar mixed convection flow from a vertical surface with induced magnetic field", Journal of Applied Physics, 102, 076113.

[31] Patanker, S. V. and Spalding, D. B. (1970). Heat and Mass Transfer in Boundary Layers, 2nd Edn., Intertext Books, London.

[32] Postelnicu, A. (2004). "Influence of magnetic field on heat and mass transfer by natural convection from vertical surfaces in porous media considering Soret and Dufour effects", Int J. Heat and Mass Transfer, 47, 1467-1472.

[33] Postelnicu, A. (2007) "Influence of chemical reaction on heat and mass transfer by natural convection from vertical surfaces in porous media considering Soret and Dufour effects", Heat and Mass Transfer Journal, 43 (6), 595-602.

[34] Raptis, A. (1986). Flow through a porous medium in the presence of magnetic field, Int. J. Energy Res., 10, 97-101.

[35] Raptis, A. and Kafoussias, N.G. (1982). "Magnetohydrodynamic free convection flow and mass transfer through porous medium bounded by an infinite vertical porous plate with constant heat flux", Can. J. Phys., 60(12), 1725–1729.

[36] Rawat, S. and Bhargava, R. (2009). "Finite element study of natural convection heat and mass transfer in a micropolar fluid-saturated porous regime with Soret/Dufour effects". Int. J. of Appl. Math and Mech., 5(2), 58–71.

[37] Reddy B. P. and Rao, J. A. (2011). "Numerical solution of thermal diffusion effect on an unsteady MHD free convective mass transfer flow past a vertical porous plate with Ohmic dissipation". Int. J. of Appl. Math. And Mech., 7(8), 78-97.

[38] Rosenberg, D. U. V. (1969). Method for numerical solutions of partial differential equations, American Elsevier, New York.

[39] Sattar, M.A. (1993). "Unsteady hydromagnetic free convection flow with hall current mass transfer and variable suction through a porous medium near an infinite vertical porous plate with constant heat flux", Int. J. Energy Research, 17, 1–5.

[40] Sattar, M. A. Hossain, M. M. (1992). "Unsteady hydromagnetic free convection flow with hall current and mass transfer along an accelerated porous plate with time-dependent temperature and concentration". Can. J. Phys., 70, 369-374.

[41] Schlichting, H. (1968). Boundary Layer Theory, McGraw-hill, New York.

[42] Sharma, R. and Singh, G. (2008). "Unsteady MHD free convective flow and heat Transfer along a vertical porous plate with variable suction and internal heat generation", Int. J. of Appl. Math. and Mech., 4(5), 1–8.

[43] Singh, A. K. (1980). Astrophys. Space Science, 115, 387.

[44] Singh, A. K. and Dikshit, G. K. (1988). "Hydromagnetic flow past a continuously moving semi-infinite plate for large suction", Astrophysics and Space Sci. 148, 249–256.

[45] Somers, F. V. (1956). J. Appl. Mech., 23, 295.

[46] Soundalgekar, V. M. and Ramanamurthy, T. V. (1980). J. Engg. Maths. 14, 155.

[47] Spalding D. B. (1977).GENMIX, a general computer program for two dimensional parabolic phenomena, Pergamon Press, Oxford, UK.