Similarity Solution of Unsteady Convective Laminar Boundary Layer Flow about a Vertical Porous Surface with Suction and Blowing

by

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Declaration

This is to certify that the thesis work entitled "*Similarity Solution of Unsteady Convective Laminar Boundary Layer Flow about a Vertical Pours Surface with Suction and Blowing*" has been carried out by *Bichakshan Mandal* in the Department of *Mathematics*, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

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Approval

This to certify that the thesis work submitted by ***Bichakshan Mandal*** entitled "***Similarity Solution of Unsteady Convective Laminar Boundary Layer Flow about a Vertical Pours Surface with Suction and Blowing***" has been approved by the board of examiners for the partial fulfillment of the requirements for the degree of ***Master of Philosophy (M. Phil)*** in the Department of ***Mathematics***, Khulna University & Engineering Technology, Khulna, Bangladesh in May 2013.

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ABSTRACT

In this dissertation, similarity solution of unsteady laminar combined free and forced convective boundary layer flow past a vertical porous plate in viscous incompressible fluid with suction and blowing has been investigated Firstly, the governing boundary layer partial differential equations have been made dimensionless and then simplified by using Boussinesq approximation. Secondly, similarity transformations are then introduced on the basis of detailed analysis in order to transform the simplified coupled partial differential equations into a set of ordinary differential equations. The transformed complete similarity equations are solved numerically by using Nachtscheim-Swigert shooting iteration technique along with sixth order Runga-Kutta method. The flow phenomenon has been characterized with the help of obtained flow controlling parameters such as suction parameter, buoyancy parameter, Prandtl number and other driving parameters. Finally the effects of involved parameters on the velocity and temperature distributions are presented graphically. It is found that a small suction or blowing can play a significant role on the patterns of flow and temperature fields.

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Nomenclature

x, y	Cartesian coordinates
t	Time
f_w	Suction parameter
u	Fluid velocity in the x direction
v	Fluid velocity in the y direction
k	Thermal conductivity
M	Mach number
μ	Kinematic viscosity
η	Similarity variable
υ	Coefficient of kinematics viscosity
ψ	Stream function
ρ	Fluid density
θ	Dimensionless temperature
γ	Boundary layer thickness
τ_w	Skin friction coefficient
φ	Velocity potential
p	Pressure
Q	Universal gas constant
m	Molecular weight of the gas
$f(\eta)$	Similarity function
β	Coefficient of thermal expansion
q_w	Heat transfer coefficient
u_e	External velocity
L_c	Characteristic length

T_e Ambient temperature

g_x x component body force

R_e Reynolds number, $R_e = \frac{UL}{\upsilon_0}$

C_p Specific heat

G_r Grashop number

Fr Froude number, $Fr = \frac{U^2}{gL}$

Pr Prandtl number, $\Pr = \frac{\mu_0 C_{p_0}}{k_0}$

E_c Eckert number, $E_c = \frac{U^2}{C_{p_0}\Delta T}$

$\frac{U_F^2}{u_0^2}$ Buoyancy parameter

CHAPTER I

Introduction and Literature Review

Fluid dynamics is a subject of widespread interest to researcher and it become an obvious challenge for the scientists, engineers as well as users to understand more about fluid motion. An important contribution to the fluid dynamics is the concept of boundary layer flow introduced first by L. Prandtl [42]. The concept of the boundary layer is the consequence of the fact that flows at high Reynolds numbers can be divided into unequally spaced regions. A very thin layer (called boundary layer) in the vicinity (of the object) in which the viscous effects dominate, must be taken into account, and for the bulk of the flow region, the viscosity can be neglected and the flow corresponds to the inviscid outer flow. Although the boundary layer is very thin, it plays a vital role in the fluid dynamics. Boundary layer theory has become an essential study now-a-days in analyzing the complex behaviors of real fluids. The concept of boundary layer can be used to simplify the Navier-Stocks' equations to such an extent that the viscous effects of flow parameters are evaluated, and these are useable in many practical problems (viz. the drag on ships and missiles, the efficiency of compressors and turbines in jet engines, the effectiveness of air intakes for ram and turbojets and so on).

Furthermore the boundary layer effects caused by free convection are frequently observed in our environmental happenings and engineering devices. We know that if externally induced flow is provided and flows arising naturally solely due to the effect of the differences in density, caused by temperature or concentration differences in the body force field (such as gravitational field), this type of flow is called `free convection' or 'natural convection' flow. The density difference causes buoyancy effects and these effects act as 'driving forces' due to which the flow is generated. Hence free convection is the process of heat transfer which occurs due to movement of the fluid particles by density differences associated with temperature differences in a fluid. In such case, free stream velocity falls away, in deed, no reference velocity does a prior exist. If the density in the vicinity of the object is kept constant, natural convection flow can not be formed. Thus, this is an effect of variable properties, where there is a mutual coupling between momentum and heat transport. The direct origin of the formation of natural convection flows is a heat transfer via conduction through the fixed surfaces surrounding the fluid. If the surface temperature is greater than that of ambient fluid, heat is transferred from the plate to the fluid leads to an increase in temperature of the fluid close the surfaces and to a change in the density, because it is temperature dependent. If the density decreases with increasing temperature, buoyancy forces arise close to the surface and warmer fluid moves upwards. Such buoyancy forces are proportional to the coefficient of thermal expansion β_T, defined as $\beta_T = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{p=\text{constant}}$,where ρ, T and p are density, temperature and pressure respectively. It is observed that $\beta_T = \frac{1}{T}$ for a perfect gas, and we see that stream is well approximated by the perfect-gas result $\beta_T T = 1$ at low pressure and high temperature. Also $\beta_T < \frac{1}{T}$ for a liquid and may even be negative, and $\beta_T > \frac{1}{T}$ for imperfect gas, particularly at high pressure. β_T is also useful in estimating the dependence of enthalpy '*h*' on pressure, from the thermodynamic relation $dh = c_p dT + (1-\beta_T T)\frac{dp}{\rho}$, where T is the absolute temperature. For the perfect gas, the second term vanishes, so that $h = h(T)$ only. The natural convection studies began in the year 1881 with Lorentz and continued at a relatively constant rate until recently. This mode of heat transfer occurs very commonly, the cooling of transmission lines, electric transformers and rectifiers, the heating of rooms by use of radiators, the heat transfer from hot pipes and ovens surrounded by cooled air, cooling the reactor core (in nuclear power plant) and carry out the heat generated by nuclear fission etc and the Mixed convection flows, combined forced and free convection flows, arise in many transport processes in engineering devices and in nature. This flows are characterized by the buoyancy parameter (measure of the influence of the free convection in comparison with that of forced convection on the fluid flow) which depends on the flow configuration and the surface heating conditions. Bulks of information are now available in literature about the boundary layer form of natural convection flows over bodies of different shapes. The theoretical, experimental and numerical analysis for the natural and the mixed convection boundary layer flow about isothermal, vertical porous flat plates have been carried out widely by many authors (viz. [13, 33, 35, 38, 48, 51, 54, 55]).

Schmidt [47] was apparently the first researcher who investigated experimentally the behavior of the flow near the leading edge above a flat horizontal surface. The theoretical analysis of the laminar, two-dimensional, steady natural convection boundary layer flow on a semi-infinite horizontal flat plate was first considered by Stewartson [52] (later corrected by Gill, Zeh and Del-Caeal [16]). In that analysis Buossinesq approximation had been used to show how the boundary layer analysis could be incorporated with the natural convection on rectangular plates.

Rotem and Claassen [45] investigated the boundary layer equation over a semi-infinite horizontal surface of uniform temperature and results were presented for some specific values of Prandtl number with its limits from zero to infinity. The effect of buoyancy forces that exist in boundary layer flow, over a horizontal surface, where the surface temperature differs from that of ambient fluid, was studied by Sparrow and Minkowycz [50]. The free convection above a heated and almost horizontal plate was treated by Jones [28].

Mixed convection flows, or combined forced and free convection floes, arise in many transport processes in engineering devices and in nature. These flows are characterized by the buoyancy parameter (measure of the influence of the free convection in comparison with that of forced convection on the fluid flow) which depends on the flow configuration and the surface heating conditions. The problem of free, mixed and forced convection over a horizontal porous plate has been attracted the interest of many investigators (Viz. Clark and Riley [11], Schneider [54] and Merkin and Ingham [33] among several others) in view of its applications in many engineering and geophysical problems. Ramanaiah et al. [43, 44] considered the problem of mixed convection over a horizontal plate subjected to a temperature or surface heat flux varying as a power of x.

The problem of mixed convection due to a heated or cooled vertical flat plate provides one of the most basic scenarios for heat transfer theory and thus is of considerable theoretical and practical interest and has been extensively studied by Sparrow et al. [56], Wilks [58], Afzal and Banthiya [5], Hunt and Wilks [22], Lin and Chen [31], Hussain and Afzal [25], Merkin et al. [37] etc. However, the problem of forced, free and mixed convection flows past a heated or cooled body with porous wall is of interest in relation to the boundary layer control on airfoil, lubrication of ceramic machine parts and food processing. Watanabe [59] considered the mixed convection boundary layer flow past an isothermal vertical porous flat plate with uniform suction or injection. Satter [53] made analytical studies on the combined forced and free convection flow in a porous medium. Further, a vast literature of similarity solution has appeared in the area of fluid mechanics, heat transfer, and mass transfer, etc. as it is one of the important means for the reduction of a number of independent variable with simplifying assumptions. It is revealed that the similarity solution, which being attained for some suitable values of different parameters, might be thought of being the solution of the convective boundary-layer context either near the leading edge or far away in the downstream. Deswita et al. [12] obtained a similarity solution for the steady laminar free convection boundary layer flow on a horizontal plate with variable wall temperature. Hossain and Mojumder [19] presented the similarity solution for the steady laminar free convection boundary layer flow generated above a heated horizontal rectangular surface. The boundary layer type of the natural convection flow, which occurs on the upper surface of heated horizontal surface has been investigated theoretically and experimentally by among other, Rotem and Claassen [46], Pera and Gebhart [39 , 40] and Goldstein et al. [17]. It is seen from their experiments and also from the flow visualization of Husra and Sparrow [21] that a boundary layer starts from each edge of a plate edge, each boundary layer having its leading at a straight-side plate edge. The boundary layer development occur normal to the corresponding edge so that collisions between opposing boundary layer flows occur on the plate surface. After collision, the fluid cheeked in the boundary layer forms a rising buoyant plume.

Furthermore, the study of complete similarity solutions of the unsteady laminar natural convection boundary layer flow about a heated horizontal semi-infinite porous plate have been considered by Hossain *et al.* [23, 24] even though the solution of a system of coupled partial differential equations with boundary conditions is often difficult and some times impossible with the usual classical method. Thus, it is imperative to reduce the number of variables from the system which reached in a stage of great extent. Similarity solution is one of the important means for the reduction of a number of independent variables with simplifying assumptions and finally the system of partial differential equations reduces to a set of ordinary differential equations successfully. A vast literature of similarity solution has appeared in the area of fluid mechanics, heat transfer, and mass transfer, etc. The similarity solutions in the context of mixed convection boundary layer flow of steady viscous incompressible fluid over an impermeable vertical flat plate were discussed by Ishak et al. [26]. Ramanaiah *et al.* [44] studied the similarity solutions of free, mixed and forced convection problems in a saturated porous media.

In 1978, Johnson and Cheng [27] examined the necessary and sufficient conditions under which similarity solutions exist for free convection boundary layers adjacent to flat plates in porous media. The solutions obtained in their work were more general than those appearing in the previous studies. With a parameter associated with the body shapes a similarity solution on the natural convection flow has also been studied by Pop and Takhar [41]. Ferdows et al. [14] performed a similarity analysis for the forced as well as free convection boundary layer flow of an electrically conducting viscous incompressible fluid past a semi-infinite con-conducting vertical porous plate by introducing a time dependent suction.

Recently, Hossain and Mojumder [19] presented the similarity solution for the steady laminar free convection boundary layer flow generated above a heated horizontal rectangular surface. They investigated the effect of suction and blowing on fluid flow and heat transfer as well as skin friction coefficients. They also found that suction increased skin-friction and heat transfer coefficients whereas injection caused a decrease in both. In our present study, we confined our discussion about the unsteady, laminar, free convection boundary layer flow above a semi-infinite heated, horizontal porous plate and investigated the effects of suction and blowing on the flow and temperature fields and other important flow parameters like pressure distribution, skin friction and heat transfer coefficients.

Most of the above analysis were based on the Buossinesq approximation and have been concerned with the seeking of similarity solutions in which the plate temperature varies with the distance from plate leading edge. In this approximation thus density, viscosity, thermal conductivity and specific heat variations are ignored except for the necessary inclusion of the density-variation in the body force term. An analysis was performed by Chen et al. [9] to study the flow and heat transfer characteristic of laminar natural convection in boundary layer flows from horizontal, inclined and vertical plates with power law variation of the wall temperature.

In most of the above analysis the boundary layer of the natural convection flows were considered over heated or uniformly heated horizontal vertical, horizontal or near horizontal, semi-infinite, rectangular porous plates. The surface is impermeable to the fluid, so that there is no transpiration i.e., suction or blowing velocity normal to the surface. This led to the kinematics boundary condition $v_w = 0$.

The problem of boundary layer control has become very important factor; in actual application it is often necessary to prevent separation. The separation of the boundary layer is generally undesirable, since separated flow causes a great increase in the drag experienced by the body. So it is often necessary to prevent separation in order to reduce pressure drag and attain high lift.

In order to solve the boundary layer equation, it is anticipated that the v-component of the velocity is small quantity of the order of magnitude $O\left(\mathrm{Re}^{\frac{1}{2}}\right)$ and it is assumed that the suction (or blowing) velocity $v_w = 0$ normal to the surface has its magnitude of order (characteristic Reynolds number)$^{1/2}$. The consequence of this is that outer flow is independent of v_w and the boundary condition at the surface is given by $y = 0\ ;\ u = 0$, $v = v_w(x)$.

Suction (or blowing) is one of the useful means in preventing boundary layer separation. The effect of suction consists in the removal of decelerated particles from the boundary layer before they are given a chance to cause separation. The surface is considered to be permeable to the fluid, so that the surface will allow a non-zero normal velocity and fluid is either sucked or blown through it. In doing this however, no-slip condition $u_w = 0$ at the surface (non-moving) shall continue to remain valid.

Suction or blowing causes double effects with respect to the heat transfer. On the one hand, the temperature profile is influenced by the changed velocity field in the boundary-layer, leading to a change in the heat conduction at the surface. On the other hand, convective heat transfer occurs at the surface along with the heat conduction for $v_w \neq 0$. A summary of flow separation and its control are found in Chang [6, 7].

The study of natural convection on a horizontal plate with suction and blowing is of huge interest in many engineering applications, for instance, transpiration cooling, boundary layer control and other diffusion operations. The effects of blowing and suction on forced or free convection flow over vertical as well as horizontal plates were analyzed in a symmetric way by Gortler [18], Sparrow and Cess [49], Koh and Hartnett [29], Gersten and Gross [15], Merkin [32, 34], Vedhanaygarm, Altenkirch and Eichhorn [57], Hasio-Tsung and Wen-Shing [20], Merkin [36] and Acharya, Shing and Dash [1] etc.

Using the usual asymptotic approach, the similar solutions of the steady natural convection boundary layer for a non-similar flow situation on a horizontal plate with large suction approximation has been developed by Afzal and Hussain [3]. A detailed study on similarity solutions for free convection boundary layer flow over a permeable wall in a fluid saturated porous media was carried out by Chaudhary et al. [8]. They have shown that the system depends on the power law exponent and the dimensionless surface mass transfer rate. They also examined the range of exponent under which the solution exists. With constant plate temperature and particular distribution of blowing rate Clarke and Riley [11] obtained a special case of similarity solution, allowing variable fluid density. But there is still a shortage of accurate data for a wide range of both suction and blowing rate. Lin and Yu [30] presented a non-similar solution for the laminar free convection flow over a semi-infinite heated upward-facing horizontal porous plate with suitable transpiration rate as a power-law variation. Emphasis was given for an isothermal plate under the condition of uniform blowing or suction. Lately, using a parameter concerned pseudo-similarity technique of time and position coordinates, Cheng and Huang [10] studied the unsteady laminar boundary layer flow and heat transfer in the presence and absence of heat source or sink on a continuous moving and stretching isothermal surface with suction and blowing. In their analysis they paid attention on the temporal developments of the hydrodynamic and thermal characteristics after the sudden simultaneous changes in momentum and heat transfer. Recently, an analysis is performed by Aydin and Kayato [4] for the laminar boundary layer flow over a porous horizontal flat plate, particularly, to study the effect of uniform suction/injection on the heat transfer. Using the constant surface temperature as thermal boundary condition they also investigated the effect of Prandtl number on heat transfer. The aim of the present paper is, therefore, to obtain a complete similarity solution of the unsteady laminar combined free and forced convection boundary layer flow about a heated vertical porous plate in viscous incompressible fluid and be attempted to investigate the effects of several involved parameters on the velocity and temperature fields and other flow parameters like skin friction, heat transfer coefficients across the boundary layer. We are also tried to predict the role of small suction or blowing velocity on these parameters as well.

In order to solve the laminar natural convection boundary layer equations, the general Navier-Stokes' and energy equations are transformed into convenient simplified forms using the usual method of dimensional analysis. At the outset attempts are made to incorporate the idea of similarity analysis. Because, the objectives of seeking similarity solutions are modified, the governing differential equations relevant to the problem have been solved by using the similarity technique. The Boussinesq approximation is employed to deal with the possible requirements of unsteady solution. Similarity requirements for an incompressible fluid are sought on the basis of detailed analysis in order to reduced the governing coupled partial differential equations into a set of ordinary differential equations Here we adopt the method of classical 'separation of variables' which is of the simplest and most straightforward method of determining similarity solutions. This method was first initiated by Abbott and Kline (1960). In this method, once form of similarity variable is chosen, the given PDE is changed under the selected co-ordinate system. The dependent variables are to be expressed in terms of the product of separable functions of the new independent variables where each function is dependent on the single variable. Substitution of the product from of the dependent variables in to the original PDE generally leads to an equation in which no functions of single variable can be isolated on the two sides of the equation unless certain parameters are to be specified. Usually, these parameters can be specified quite readily and "separation of the variables" is achieved. In this way the separation proceeds until the one side becomes an ODE. Four different similarity cases arise here, viz. Case A, Case B, Case C and Case D, on the basis of our assumptions.

Thus, this dissertation is composed of Six Chapters. An introduction of basic principles of boundary layer theory, natural convection flows, suction and blowing phenomena with historical review of earlier researches and background of our problem are presented in CHAPTER I.

Basic equations governing the problem, dimensional analysis with simplifying assumptions and similarity transformations with possible similarity case are given in CHAPTER II.

CHAPTER III is concerned with the study of "The Calculation Technique".

In CHAPTER IV; a detailed discussion of one of the four similarity cases, namely, Case A has given. Under the considered condition, the numerical solutions with graphs and tables have also been given here for some selected values of the established parameters. The effects of these parameters on several variables will also be exhibited in the analysis. CHAPTER V is concerned with the study of another similarity case (Case B). The numerical solutions with the graph and tables for this case are also displayed here. We also have predicted the role of small suction or blowing velocity on these parameters concerned.

In CHAPTER VI, the conclusions gained from this work and brief descriptions for further works related to our present research are discussed.

CHAPTER II

Mathematical Formulation of the present study

2.1 Basic Equations

A semi- infinite flat-plate extending vertically upwards and which is fixed with its leading edge horizontal is placed in an unsteady free stream. The plate is heated to a certain unsteady temperature above the ambient temperature T_e. Heat is supplied by diffusion from the plate. The density of the fluid near the plate is reduced so that the fluid there is buoyant compared with the fluid in the free stream at a large distance from the plate. Consequently layers of the fluid close to the plate begin to rise. It is supposed that the maximum velocity created in this buoyant layer at a distance L from the bottom of the plate is U. If the Reynolds number based on this velocity U is sufficiently large, buoyant flow is amenable to Prandtl's boundary layer analysis. From a mathematical view point the Froude number for such flows may no longer be large and the body force term must be retained in the boundary layer equations. There are two cases to consider. Case I is that in which the plate extends upwards and case II that in which it extends downwards. In the first case the buoyancy forces act in the direction of the free stream and in the second case they oppose the free stream.

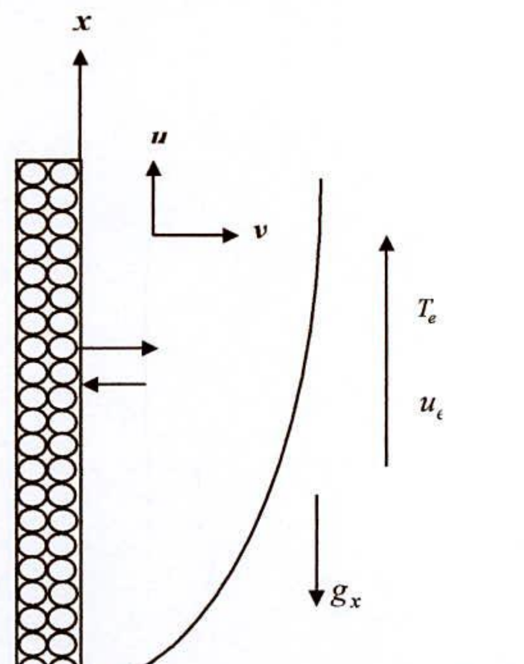

Figure 2.1: Schematic representation and coordinate system of the problem.

Considering the flow direction along the x-axis (as shown in the Fig. 2.1) the basic boundary layer equations of mass, momentum and energy for a viscous and heat conducting fluid of variable properties subject to a body force are

$$\frac{D\rho}{Dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0 \tag{2.1}$$

$$\rho\frac{Du}{Dt}=\rho g_x-\frac{\partial P}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2.2}$$

$$\rho C_p\frac{DT}{Dt}=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+T\beta_T\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{2.3}$$

Where, $$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y} \tag{2.4}$$

Here u, v denote velocity components in the x and y directions, ρ is the density, t denotes time, g_x is the x- component of body force per unit mass (here assumed due to gravity), μ is the dynamic viscosity coefficient, p is the pressure, T is the temperature, C_p is the specific heat at constant pressure, k is the thermal conductivity coefficient and β_T is the coefficient of thermal expansion. U_e and T_e represent the external velocity and the ambient temperature of the fluid flowing luminary parallel to the buoyancy effect. Equation (2.3) is the boundary layer form of one of the numerous basic forms of the energy equation deduced by Haworth (1953).

2.2 Dimensional Analysis

To obtain solutions of equations (2.1) to (2.3), it is proposed first to discover the dimensionless group upon which the solutions must depend. We begin by introducing dimensionless quantities into the equations, referring all lengths to some characteristic length L along the surface, velocities with reference to some characteristic velocity U and t by $\frac{U}{L}$. The density will be made dimensionless with respect to ρ_0, the pressure will be referred to $\rho_0 U^2$ and the temperature to the temperature difference between the wall and T_0. The other transport properties of the fluid μ, k, C_p and the gravitational component g_x will be made dimensionless by μ_0, k_0, C_{p_0} and g respectively. We use suffix '0' to refer to some convenient constant reference condition far from the surface. Hence the substitutions are as follows:

$$x = Lx', \quad y = \mathrm{Re}^{\frac{1}{2}} Ly', \quad u = Uu', \quad v = \mathrm{Re}^{\frac{1}{2}} Uv', \quad t = \frac{L}{U}t', \quad \rho = \rho_0\rho', \quad p = \rho_0 U^2 p',$$

$\theta = \dfrac{T - T_0}{T_w - T_0} = \dfrac{T - T_0}{\Delta T}$, $\mu = \mu_0\mu'$, $k = k_0 k'$, $C_p = C_{p_0} C_{p'}$ and $g_x = gg'_x$, where $\mathrm{Re} = \dfrac{UL}{\upsilon_0}$.

Substituting the above similarity transformations the dimensionless form of equations (2.1) to (2.3) are obtained as follows

$$\frac{\rho_0 U}{L}\frac{D\rho'}{Dt'} + \frac{\rho_0\rho'}{L}U\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) = 0 \qquad \text{(Calculations are shown in Appendix A)}$$

$$\text{or, } \frac{D\rho'}{Dt'} + \rho'\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) = 0 \tag{2.5}$$

$$\frac{\rho_0\rho' U^2}{L}\frac{Du'}{Dt'} = \rho_0\rho' gg_{x'} - \frac{\rho_0 U^2}{L}\frac{\partial p'}{\partial x'} + \frac{\mu_0 U^2}{LU\,\mathrm{Re}\,L}\frac{\partial}{\partial y'}\left(\mu'\frac{\partial u'}{\partial y'}\right)$$

$$\text{or, } \rho'\frac{Du'}{Dt'} = \frac{\rho' g_{x'}}{\dfrac{U^2}{gL}} - \frac{\partial p'}{\partial x'} + \frac{\mu_0}{UL\,\mathrm{Re}\,\rho_0}\frac{\partial}{\partial y'}\left(\mu'\frac{\partial u'}{\partial y'}\right)$$

$$\therefore \rho'\frac{Du'}{Dt'} = \frac{\rho' g_{x'}}{Fr} - \frac{\partial p'}{\partial x'} + \frac{\partial}{\partial y'}\left(\mu'\frac{\partial u'}{\partial y'}\right) \tag{2.6}$$

and

$$\rho_0\rho' C_{p_0} C_{p'}\Delta T\frac{U}{L}\left[\frac{D\theta}{Dt'} + \theta\left\{\frac{\partial}{\partial t}(\log\Delta T) + u'\frac{\partial}{\partial x'}(\log\Delta T)\right\}\right]$$

$$= \frac{\Delta T}{\mathrm{Re}\,L^2}\frac{\partial}{\partial y'}\left(k'\frac{\partial\theta}{\partial y'}\right) + T\beta_T\frac{U^3\rho_0}{L}\left(\frac{\partial p'}{\partial t'} + u'\frac{\partial p'}{\partial x'}\right) + \frac{\mu_0 U^2}{\mathrm{Re}\,L^2}\mu'\left(\frac{\partial u}{\partial y'}\right)^2$$

$$\therefore \rho' C_{p'}\left[\frac{D\theta}{Dt'} + \theta\{\frac{\partial}{\partial t}(\log\Delta T) + u'\frac{\partial}{\partial x'}(\log\Delta T)\right]$$

$$= \frac{1}{\mathrm{Pr}}\frac{\partial}{\partial y'}\left(k'\frac{\partial\theta}{\partial y'}\right) + ET\beta_T\left(\frac{\partial p'}{\partial t'} + u'\frac{\partial p'}{\partial x'}\right) + E\mu'\left(\frac{\partial u'}{\partial y'}\right)^2 \tag{2.7}$$

where $\mathrm{Pr} = \dfrac{\mu_0 C_{p_0}}{k_0}$, $E_c = \dfrac{U^2}{C_{p_0}\Delta T}$ and $Fr = \dfrac{U^2}{gL}$. If we restrict ourselves to thermally perfect gasses then $T\beta_T = 1$.

The solutions of dimensionless equations (2.5) to (2.7) depend on dimensionless groups Pr, E and Fr, that is on the Prandtl, Eckert and Froude numbers. Thus, for flows over geometrically similar bodies to be similar, the values of the above groups must be the same. That is to say that if these conditions are satisfied, the non-dimensional forms of the equations (2.5) – (2.7) governing the flows over geometrically similar bodies would be the same. There are, however, further conditions which must be imposed in order to achieve flow similarity:

1. For a solid surface the fluid must adhere to the plate (the no-slip condition of viscous flows) and the plate must be a streamline.
   Thus, $u(x',0,t') = v(x',0,t') = 0$.
2. The temperature of the fluid at the plate must be equal to the plate temperature. Hence, $\theta'(x',0,t') = \theta_w = 1$.
3. The fluid at a large distance from the plate must be undisturbed by the boundary layer, i.e. $u'(x',\infty,t') = \frac{u_e}{U}$, u_e being in general a function of both x and t.
4. The temperature at a large distance from the plate must be equal to the undisturbed fluid temperature i.e. $\theta(x',\infty,t') = 0$

Let us now focus our attention on the second dimensionless group E. This quantity leads directly to the temperature increase through adiabatic compression (Schlichting (1968). Hence one gets:

$$E_c = \frac{U^2}{C_p \Delta T} = \frac{\frac{U^2}{C_p}}{\Delta T} = \frac{2(\Delta T)ad}{\Delta T} \tag{2.8}$$

where $ad = \frac{U^2}{2C_p \Delta T}$

It is now possible to conclude that frictional heat and heat due to compression are important for calculation of temperature fields when the characteristic velocity is so large that the adiabatic temperature increase is of the same order of magnitude as the prescribed temperature difference between the wall and T_0. If this prescribed temperature difference is of the same order of magnitude as the absolute temperature of free stream, the Eckert number becomes equivalent to the square of the Mach number. That is

$$E_c = \frac{U^2}{C_p \Delta T}$$

$$= \frac{U^2}{C_p T_0} \cdot \frac{T_0}{\Delta T} \quad = (\gamma - 1) M_0^2 \frac{T_0}{\Delta T} \tag{2.9}$$

where, $M_0 = \frac{U}{a_0}$ is the Mach number and $C_p T_0 = \frac{a_0^2}{\gamma - 1}$.

The work of compression and that due to friction become important when the characteristic velocity (ultimately free stream velocity) is comparable with or much greater than that of sound, or when the prescribed temperature difference is small compared with the absolute temperature of the free stream. The former situation occurs usually in practice at high speed when buoyancy effects are ignored.

The third dimensionless group Fr is related to the second dimensionless group E_c by the expression

$$Fr = \frac{U^2}{gL} = E_c \frac{\Delta T}{T_0} \frac{C_p T_0}{gL} \tag{2.10}$$

For a perfect gas

$$\frac{C_p T_0}{gL} = \frac{1}{\gamma - 1} \frac{a_0^2}{gl} \tag{2.11}$$

so that $\frac{C_p T_0}{gL}$ is typically of order $\frac{10^5}{gL}$ with L in meters. Thus, when E is extremely small with, $\frac{\Delta T}{T_0} \sim 0(1)$, Fr can be of the order of unity. In these circumstances the stress work term in the energy equation are ignored but the body force term in the momentum equation must be retained.

As a result of the above discussions of the Eckert and Froude numbers (given by the equations (2.9) and (2.10)) we are in a position to begin our analysis on the following dimensional boundary layer equations:

$$\frac{D\rho}{Dt} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \tag{2.12}$$

$$\rho \frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \tag{2.13}$$

$$\rho C_p \frac{DT}{Dt} = \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) \tag{2.14}$$

Since at a particular station (x,t) the pressure p does not vary with y through the boundary layer we can write $p = p_e$. To eliminate the pressure term in equation (2.13) the condition outside the boundary layer

$U \to u_0$, $T \to T_e$ and $\dfrac{\partial}{\partial y} \to 0$ are imposed. Hence we get

$$\rho_e\left(\frac{\partial u_e}{\partial t} + u_e\frac{\partial u_e}{\partial x}\right) = \rho_e g_x - \frac{\partial p}{\partial x} \qquad \left|\because \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right. \tag{2.15}$$

$$\frac{\partial T_e}{\partial t} + u_e\frac{\partial T_e}{\partial x} = 0 \tag{2.16}$$

In view of (2.15) – (2.16) and writing

$$\frac{T - T_e}{T_W - T_e} = \theta \tag{2.17}$$

$$T_w - T_e = \Delta T \tag{2.18}$$

the equations (2.12) – (2.14) become

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{2.19}$$

$$\rho\frac{Du}{Dt} = (\rho - \rho_e)g_x + \rho_e\left(\frac{\partial u_e}{\partial t} + u_e\frac{\partial u_e}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2.20}$$

$$\rho C_p\frac{DT}{Dt} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$$

$$\text{or, } \rho C_p\left\{\frac{DT_e}{Dt} + \Delta T\frac{D\theta}{Dt} + \theta\frac{D}{Dt}(\Delta T)\right\} = \frac{\partial}{\partial y}\left(k\Delta T\frac{\partial\theta}{\partial y}\right)$$

$$\text{or, } \Delta T\rho C_p\left\{\frac{D\theta}{Dt} + \frac{\theta}{\Delta T}\frac{D}{Dt}(\Delta T)\right\} = \frac{\partial}{\partial y}\left(k\Delta T\frac{\partial\theta}{\partial y}\right) - \rho C_p\frac{DT_e}{Dt}$$

$$\text{or, } \Delta T\rho C_p\left[\frac{D\theta}{Dt} + \theta\left\{\frac{\partial}{\partial t}(\log\Delta T) + u\frac{\partial}{\partial x}(\log\Delta T)\right\}\right] = \Delta T\frac{\partial}{\partial y}\left(k\frac{\partial\theta}{\partial y}\right) - \rho C_p(u - u_e)\frac{\partial T_e}{\partial t}$$

$$\text{where, } \frac{DT_e}{Dt} = -u_e\frac{\partial T_e}{\partial x}$$

$$\therefore \rho C_p\left[\frac{D\theta}{Dt} + \theta\left\{\frac{\partial}{\partial t}(\log\Delta T) + u\frac{\partial}{\partial x}(\log\Delta T)\right\}\right] = \frac{\partial}{\partial y}\left(k\frac{\partial\theta}{\partial y}\right) - \frac{\rho C_p}{\Delta T}(u - u_e)\frac{\partial T_e}{\partial x} \tag{2.21}$$

where ΔT and T_e are functions of x and t. $T_e = T_0$ (= constant) is one of the solutions of (2.16)

2.3 Similar Solutions for the Boussinesq Approximation:

In this section, we shall begin our study of the unsteady combined free and forced convective laminar boundary layer equations by simplifying them using the Boussinesq approximation. In this approximation, density variations other than the variation in the body force term in the momentum equation are ignored. Thus the elimination of the first term $\frac{D\rho}{Dt}$ in the continuity equation will lead to great simplifications in the boundary layer equations, particularly when the later are expressed in terms of a stream function. Fluid property variations are ignored completely in this approximation and this factor, together with the removal of density variations in the convection terms, removes the requirement for the use of Howarth-Dorodnitsyn transformation. It is assumed here also that the fluid temperature outside the boundary layer, T_e, is constant.

2.4 Equation of State and Boussinesq Approximation

The equation of state plays an important role in solving the differential equations (2.19) – (2.21). In the thermodynamic theory of continuous media at rest it is shown that any one variable of state (p, ρ, t) can be expressed in terms of any other two. We assume that an equation of state exists in a form

$$p = p(\rho, t) \tag{2.22}$$

For gases, a good approximation is provided by kinetic theory for widely spaced molecules of simple structure:

$$p = \rho RT = \frac{Q}{m}\rho T \tag{2.23}$$

Where Q the universal gas constant and m is is the molecular weight of the gas. Equation (2.23) is said to be the equation of state for a thermally perfect gas. For a calorically and thermally perfect gas the specific heats at constant pressure and constant volume, C_p and C_v respectively, are constant so that

$$p = \frac{\gamma - 1}{\gamma}\rho h \tag{2.24}$$

where, $\gamma = \dfrac{C_p}{C_v}$ (2.25)

$$h = C_p T \quad (2.26)$$

$$C_p - C_v = R \quad (2.27)$$

For liquids no such simple equation of state exists. All that can be done is to consider the effect of small change in pressure and temperature. Writing $\rho = \rho(T, p)$ we can write

$$dp = \left(\frac{\partial \rho}{\partial T}\right)_p dp + \left(\frac{\partial \rho}{\partial p}\right)_T dT$$

$$= -\rho\left\{\frac{1}{-\rho}\left(\frac{\partial \rho}{\partial T}\right)_p dT\right\} + \left\{\rho\left(\frac{\partial \rho}{\partial p}\right)_T dp\right\}$$

$$= -\rho\beta_T dT + \rho k dp \quad (2.28)$$

Where, β_T and k are the coefficients of thermal expansion and isothermal compressibility given by

$$\beta_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \quad (2.29)$$

and $k = \rho\left(\dfrac{\partial \rho}{\partial p}\right)_T$ (2.30)

respectively.

In non-dimensional form equation (2.28) may be written as

$$\frac{dp}{\rho} = kdp - \beta_T dt$$

Or, $\dfrac{dp'}{\rho'} = kp_0 \dfrac{\rho_0 U^2}{p_0} dp' - \beta_T \Delta T d\theta$ (2.31)

where, $\rho = \rho_0 \rho'$

or, $d\rho = \rho_0 d\rho'$

or, $\dfrac{d\rho}{\rho} = \dfrac{\rho_0}{\rho} d\rho'$

$\therefore \dfrac{d\rho}{\rho} = \dfrac{1}{\rho} d\rho'$

$$P = \rho_0 U^2 p'$$

$$\text{or, } dp = \rho_0 U^2 dp^{/}$$

$$\text{or, } kdp = k\rho_0 U^2 dp^{/}$$

$$= kp_0 \frac{\rho_0 U^2}{p_0} dp^{/}$$

and $T - T_0 = \Delta T \quad \therefore dT = \Delta T d\theta$

In the case of slow motion one obtains for a gas $kp_0 \sim 0(1)$, $\rho \frac{U^2}{p_0} \propto M^2 << 1$ but for a liquid $\frac{\rho_0 U^2}{p_0}$ is more significant than for a gas . However for a liquid $kp_0 << 1$, the equation (2.31) becomes

$$\frac{d\rho^{/}}{\rho^{/}} = -\beta_T \Delta T d\theta \tag{2.32}$$

$$\text{Or, } \rho = \rho(T) \tag{2.33}$$

For small change from reference condition (denoted by suffix r) as a first approximation we can replace the equation of state (2.33) in total differential form by

$$\rho - \rho_r = -\rho_r \beta_T (T - T_r)$$

$$\rho = \rho_r (1 - \beta_T (T - T_r)) \tag{2.34}$$

Similarly we can write $\mu = \mu(T)$, from which one may obtain

$$\mu = \mu_r \{1 + a(T - T_r)\} \ ; \quad a = \left(\frac{1}{\mu}\frac{\partial \mu}{\partial T}\right)_r \tag{2.35}$$

Similar relation to (2.34) may be derived for λ, k, C_p etc. By virtue of exterior relations analogous to (2.34) we have

$$\rho - \rho_e = -\rho_r \beta_T \Delta T \theta \ ; \text{since } T_e = T_0 \quad \text{and} \quad T - T_r = \Delta T \theta \tag{2.36}$$

To the first order of small quantities equations (2.34) – (2.35) and similar equation for k and c_p provide

$$\rho \approx \rho_r, \ \mu \approx \mu_r, \quad k \approx k_r \ , \quad C_p \approx C_{p_r} \tag{2.37}$$

That is the fluid property variations other than density variation in the buoyancy term of the momentum equation are ignored. In view of (2.36) – (2.37) the boundary layer equations (2.19) – (2.21) are simplified as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.38}$$

$$\frac{Du}{Dt} = -g_x \beta_T \Delta T \theta + \frac{\partial u_e}{\partial t} + u_e \frac{\partial u_e}{\partial x} + \upsilon \frac{\partial^2 u}{\partial y^2} \tag{2.39}$$

$$\frac{D\theta}{Dt} + \theta \left\{ \frac{\partial}{\partial \theta}(\log \Delta T) + u \frac{\partial}{\partial x}(\log \Delta T) \right\} = \frac{\upsilon}{P_r} \frac{\partial^2 \theta}{\partial y^2} \tag{2.40}$$

Where, $\upsilon = \dfrac{\mu_r}{\rho_r}$, $\Pr = \dfrac{\mu_r C_{p_r}}{k_r}$, ν is the kinematic viscosity and Pr is the Prandtl number of the fluid.

2.5 Similarity Transformations

Equations (2.38) – (2.40) are non-linear, simultaneous partial differential equations and to obtain solutions for them is extremely difficult. Hence we now proceed to reduce equations (2.39) – (2.40) together with the continuity equation (2.38) to a corresponding pair of ordinary differential equations which includes permissible variations in ΔT and u_e.

Let us now change the variables *x, y and t* to a new set of variables (ξ, φ, τ), where the relations between the two sets are given by

$$\xi = x \tag{2.41}$$

$$\varphi = \frac{y}{\gamma(x,t)} \tag{2.42}$$

$$\tau = t \tag{2.43}$$

Here $\gamma(x,t)$ can be thought of as being proportional to the local boundary layer thickness. From (2.41) – (2.43) one may obtain

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \frac{\varphi}{\gamma} \gamma_\tau \frac{\partial}{\partial \varphi} \tag{2.44}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{\varphi}{\gamma} \gamma_\xi \frac{\partial}{\partial \varphi} \tag{2.45}$$

$$\frac{\partial}{\partial y} = \frac{1}{\gamma} \frac{\partial}{\partial \varphi} \tag{2.46}$$

The equation of continuity (2.19) permits us to write

$$u = \frac{\partial \psi}{\partial y} \tag{2.47}$$

and $$v = -\frac{\partial \psi}{\partial x} \tag{2.48}$$

where $\psi(x,y,t)$ is the stream function at any point (x, y, t). Using the equation (2.46), the equation (2.47) may be expressed as

$$\frac{u}{U(x,t)} = \frac{\partial}{\partial y}\left(\frac{\psi}{U(x,t)}\right)$$

$$= \frac{1}{\gamma}\frac{\partial}{\partial \varphi}\left(\frac{\psi}{U(x,t)}\right)$$

$$= \frac{\partial}{\partial \varphi}\left\{\frac{\psi(\xi, \varphi, \tau)}{\gamma(\xi, t)U(\xi, t)}\right\} \qquad (2.49)$$

Now on writing

$$\int_0^{\varphi}\frac{u}{U(x,t)}d\varphi = F(\xi, \varphi, \tau) \qquad (2.50)$$

equation (2.49) yields

$$\int_0^{\varphi}\frac{\partial}{\partial \varphi}\left\{\frac{\psi(\xi,\varphi,\tau)}{\gamma(\xi,t)U(\xi,t)}\right\}d\varphi = \int_0^{\varphi}\frac{u}{U(x,t)} = F(\xi,\varphi.\tau)$$

$$\text{or, } \psi(\xi,\varphi,\tau)\Big]_0^{\varphi} = \gamma(\xi,\tau)U(\xi,\tau)F(\xi,\varphi,\tau)$$

$$\text{or, } \psi(\xi,\varphi,\tau) - \psi(\xi,0,\tau) = \gamma(\xi,\tau)U(\xi,\tau)F(\xi,\varphi,\tau)$$

$$\therefore \quad \psi = \gamma UF + \psi(\xi,0,\tau) \qquad (2.51)$$

In view of (2.44) – (2.46), (2.47) – (2.48) and (2.51), the velocity component may be found as

$$-v = \frac{\partial \psi}{\partial x}$$

$$= \frac{\partial \psi}{\partial \xi} - \frac{\varphi}{\gamma}\gamma_x\frac{\partial \psi}{\partial \varphi} \qquad \text{by (2.45)}$$

$$= (\gamma UF)_{\xi} + \psi_{\xi}(\xi,0,\tau) - \frac{\varphi}{\gamma}\gamma_x\gamma UF_{\varphi}$$

$$= (\gamma UF)_{\xi} - \varphi\gamma_x UF_{\varphi} - v_w \qquad (2.52)$$

where, $-v_w = \psi_{\xi}(\xi,0,\tau)$

Now, since the surface is porous, v_w represents the suction or injection velocity to the surface. Subscripts ξ, φ denote here partial differentiation with respect to corresponding arguments. Since the external velocity u_e is independent of y, it must also be independent of φ, yielding $(u_e)_\varphi = 0$. Thus the convective operator in terms of the new set of variables

is, $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x} + v\dfrac{\partial}{\partial y}$

$$= \frac{\partial}{\partial \tau} - \frac{\varphi}{\gamma}\gamma_\tau \frac{\partial}{\partial \varphi} + UF_\varphi \left\{ \frac{\partial}{\partial \xi} - \frac{\varphi}{\gamma}\gamma_\xi \frac{\partial}{\partial \varphi} \right\} - \left\{ (\gamma UF)_\xi - \varphi U \gamma_\xi F_\varphi - v_w \right\} \frac{1}{\gamma}\frac{\partial}{\partial \varphi}$$

(By equations 2.44, 2.45, 2.46)

$$\therefore \frac{D}{Dt} = \frac{\partial}{\partial \tau} + UF_\varphi \frac{\partial}{\partial \xi} - \left[\varphi\gamma_\tau + (\gamma UF)_\xi - v_w \right] \frac{1}{\gamma}\frac{\partial}{\partial \varphi} \tag{2.53}$$

In attempting separation of variables for $F(\xi,\varphi,\tau)$ and $\theta(\xi,\varphi,\tau)$ we assume

$$F(\xi,\varphi,\tau) = L(\xi,\tau)\tilde{f}(\varphi) \tag{2.54}$$

and $\theta(\xi,\varphi,\tau) = m(\xi,\tau)\vartheta(\varphi)$ (2.55)

Where $\tilde{f}$ and ϑ are functions of the single variable φ

Here, $u = UF_\varphi = UL\tilde{f}_\varphi$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(UL\tilde{f}_\varphi\right) = UL\frac{\partial \tilde{f}_\varphi}{\partial \varphi}.\frac{\partial \varphi}{\partial y} = \frac{UL}{\gamma}\tilde{f}_{\varphi\varphi}$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{UL}{\gamma^2}\tilde{f}_{\varphi\varphi\varphi}$$

Now from equation (2.39)

$$\frac{Du}{Dt} = -g_x\beta_T\Delta T\theta + \frac{\partial u_e}{\partial t} + u_e\frac{\partial u_e}{\partial x} + \upsilon\frac{\partial^2 u}{\partial y^2}$$

Or, $\dfrac{\partial u}{\partial \tau} + UF_\varphi\dfrac{\partial u}{\partial \xi} - \left[\varphi\gamma_\tau + (\gamma UF)_\xi - v_w\right]\dfrac{1}{\gamma}\dfrac{\partial u}{\partial \varphi} = -g_x\beta_T\Delta T\theta + \dfrac{\partial u_e}{\partial t} + u_e\dfrac{\partial u_e}{\partial x} + \dfrac{\upsilon UL}{\gamma^2}\tilde{f}_{\varphi\varphi\varphi}$

Or, $(UL)_\tau\tilde{f}_\varphi + UL(UL)_\xi\tilde{f}_\varphi^2 - \left[\varphi\gamma_\tau + \left(\gamma UL\tilde{f}\right)_\xi - v_w\right]\dfrac{UL}{\gamma}\tilde{f}_{\varphi\varphi}$

$$= -g_x\beta_T\Delta T\theta + \frac{\partial u_e}{\partial t} + u_e\frac{\partial u_e}{\partial x} + \frac{\upsilon UL}{\gamma^2}\tilde{f}_{\varphi\varphi\varphi}$$

Or, $\upsilon\dfrac{UL}{\gamma^2}\tilde{f}_{\varphi\varphi\varphi} + \left[(u_e)_\tau + u_e(u_e)_\xi\right] - g_x\beta_T\Delta T\theta + \dfrac{u}{\gamma}(\gamma UL)_\xi\tilde{f}\tilde{f}_{\varphi\varphi} + \left[\varphi\gamma_\tau - v_w\right]\dfrac{UL}{\gamma}\tilde{f}_{\varphi\varphi}$

$$-UL(UL)_\xi\tilde{f}_\varphi^2 - (UL)_\tau\tilde{f}_\varphi = 0$$

$$\therefore\ \upsilon\tilde{f}_{\varphi\varphi\varphi} + \frac{\gamma^2}{UL}\left[(u_e)_\tau + u_e(u_e)_\xi\right] - \frac{\gamma^2}{UL}g_x\beta_T\Delta T m\vartheta + \gamma(\gamma UL)_\xi\tilde{f}\tilde{f}_{\varphi\varphi} + \left[\varphi\gamma\gamma_\tau - \gamma v_w\right]\tilde{f}_{\varphi\varphi}$$

$$-\gamma^2(UL)_\xi \tilde{f}_\varphi^2 - \frac{\gamma^2}{UL}(UL)_\tau \tilde{f}_\varphi = 0 \text{ [Dividing both sides by } \frac{UL}{\gamma^2}] \quad (2.56)$$

Again from equation (2.40)

$$\frac{D\theta}{Dt} + \theta\left\{\frac{\partial}{\partial t}(\log \Delta T) + u\frac{\partial}{\partial x}(\log \Delta T)\right\} = \frac{\upsilon}{\Pr}\frac{\partial^2 \theta}{\partial y^2}$$

$$\text{Or, } \frac{\partial \theta}{\partial \tau} + UF_\varphi \frac{\partial \theta}{\partial \xi} - \left[\varphi\gamma_\tau + (\gamma UF)_\xi - v_w\right]\frac{1}{\gamma}\frac{\partial \theta}{\partial \varphi} + \theta\left\{\frac{\partial}{\partial t}(\log \Delta T) + u\frac{\partial}{\partial x}(\log \Delta T)\right\} = \frac{\upsilon}{\Pr}\frac{\partial^2 \theta}{\partial y^2}$$

$$\text{Or, } m_\tau \vartheta + UL\tilde{f}_\varphi m_\xi \vartheta - \left[\varphi\gamma_\tau + (\gamma UL)_\xi \tilde{f} - v_w\right]\frac{1}{\gamma} m\vartheta_\varphi +$$

$$mv\left\{(\log \Delta T)_\tau + UL\tilde{f}_\varphi (\log \Delta T)\right\} = \frac{\upsilon}{\gamma^2 \Pr} m\vartheta_{\varphi\varphi}$$

$$\therefore\ \frac{\upsilon}{\Pr}\vartheta_{\varphi\varphi} + \left[\varphi\gamma\gamma_\tau - \gamma v_w\right]\vartheta_\varphi + \gamma(\gamma UL)_\xi \tilde{f}\vartheta_\varphi = \gamma^2(\log(m\Delta T)_\tau + \gamma^2 UL(\log(m\Delta T)_\xi \vartheta\tilde{f}_\varphi \qquad (2.57)$$

[Multiplying both sides by $\frac{\gamma^2}{m}$]

Now putting

(i) $\gamma\gamma_\tau = a_0$

(ii) $\gamma(\gamma UL)_\xi = \frac{1}{2}\left\{(\gamma^2 UL)_\xi + \gamma^2(UL)_\xi\right\} = \frac{1}{2}(a_1 + a_2)$

(iii) $-\gamma v_w = a_3$

(iv) $\frac{\gamma^2 (UL)_\tau}{UL} = a_4$

(v) $-\frac{\gamma^2}{UL} m\Delta T \beta_T g_x = a_5$

(vi) $\frac{\gamma^2}{UL}\left\{(u_e)_\tau + u_e(u_e)_\xi\right\} = a_6$

(vii) $\gamma^2(\log(m\Delta T))_\tau = a_7$

(viii) $\gamma^2 UL(\log(m\Delta T))_\xi = a_8$ in equation (2.56) and (2.57), we get

$$\upsilon\tilde{f}_{\varphi\varphi\varphi} + (a_0\varphi + a_3)\tilde{f}_{\varphi\varphi} + \frac{1}{2}(a_1 + a_2)\tilde{f}\tilde{f}_{\varphi\varphi} - a_2\tilde{f}_\varphi^2 - a_4\tilde{f}_\varphi + a_5\vartheta + a_6 = 0 \qquad (2.58)$$

$$\frac{\upsilon}{\Pr}\vartheta_{\varphi\varphi} + (a_0\varphi + a_3)\vartheta_\varphi + \frac{1}{2}(a_1 + a_2)\tilde{f}\vartheta_\varphi = (a_7 + a_8\tilde{f}_\varphi)\vartheta \qquad (2.59)$$

Where $a_0, a_1, a_2, \ldots\ldots\ldots, a_8$ are constant.

There are, however, boundary conditions which must be satisfied in order to determine the solutions of the transformed boundary layer equations (2.58) and (2.59):

(a) The fluid must adhere to the plate and the plate must be a streamline. However, if the surface be porous mathematically on the surface

$$u(\xi,0,\tau) = 0 = \tilde{f}_\varphi \ , \ v(\xi,0,\tau) = v_w$$

(b) The temperature of the fluid at the plate must be equal to the plate temperature, i.e.

$$\theta(\xi,0,\tau) = m(\xi,\tau)\vartheta(0) = 1$$

(c) The fluid at a large distance from the plate must be undisturbed by the presence of the boundary layer, I, e.: $u(\xi,\varphi,\tau) = u_e \Rightarrow UL\tilde{f}_\varphi(\infty) = u_e$

(d) The temperature at large distance from the plate must be equal to the undisturbed fluid temperature, i, e.: $\theta(\xi,\infty,\tau) = m(\xi,\tau)\vartheta(\infty) = 0$.

In conditions (b) and (c) if general boundary conditions $\tilde{f}_\varphi(\infty) = \vartheta(0) = 1$ be introduced, without loss of generality, we may write $UL = u_e$ (2.60)

$$m(\xi,\tau) = 1 \qquad (2.61)$$

By virtue of equations (2.60) and (2.61) the co- efficient which are functions of ξ and τ may now be expressed in the following forms:

(i) $\gamma\gamma_\tau = a_0$

(ii) $\gamma(\gamma u_e)_\xi = \frac{1}{2}\left\{(\gamma^2 u_e)_\xi + \gamma^2 (u_e)_\xi\right\} = \frac{1}{2}(a_1 + a_2)$

(iii) $-\gamma v_w = a_3$

(iv) $\dfrac{\gamma^2 (u_e)_\tau}{u_e} = a_4$ (2.62)

(v) $-\dfrac{\gamma^2}{u_e}\Delta T \beta_T g_x = a_5$

(vi) $\dfrac{\gamma^2}{u_e}\left\{(u_e)_\tau + u_e (u_e)_\xi\right\} = a_6$

(vii) $\gamma^2(\log\Delta T)_\tau = a_7$

(viii) $\gamma^2 u_e(\log\Delta T)_\xi = a_8$

Similar solutions for (2.58) and (2.59) exist only when all the a's are finite and independent of ξ and τ; that is to say that all the's must be constants. Thus, the equations (2.58) and (2.59) will become non- linear ordinary differential equations. Hence the relations stated by equations (2.62) may be considered to be the conditions which furnish equations for $u_e(\xi,\tau)$ and $\gamma(\xi,\tau)$ in deferent situations and consequently the suction velocity v_w for the possible requirements of similarity solution in the case of Boussinesq fluid.

The first part of similarity requirement of equation 2.62(ii) yields

$$\left(\gamma^2 u_e\right)_\xi = a_1$$

$$\text{Or, } \gamma^2 u_e = a_1\xi + A(\tau) \tag{2.63}$$

here $A(\tau)$ is either a function of τ or constant.

Differentiating (2.63) with respect to τ one obtains

$$\gamma^2(u_e)_\tau + u_e(\gamma^2)_\tau = \frac{dA(\tau)}{d\tau}$$

$$\text{Or, } u_e\frac{\gamma^2(u_e)_\tau}{u_e} + u_e(\gamma^2)_\tau = \frac{dA(\tau)}{d\tau}$$

In view of conditions (i) and (iv) (similarity requirements) of (2.62) the above equation may be expressed as

$$u_e\left(a_4 + 2a_0\right) = \frac{dA(\tau)}{d\tau} \tag{2.64}$$

Similarly from the condition (i) of (2.62) one gets

$$\gamma\gamma_\tau = a_0$$

$$\text{Or, } \frac{\partial}{\partial\tau}(\gamma^2) = 2a_0$$

$$\therefore \gamma^2 = 2a_0\tau + B(\xi) \tag{2.65}$$

here $B(\xi)$ is either a function of ξ or constant.

Differentiating (2.65) with respect to ξ we have

$$\left(\gamma^2\right)_\xi = \frac{dB(\xi)}{d\xi}$$

By virtue of condition (ii) the above equation may be expressed as:

$$\frac{a_1 - a_0}{u_e} = \frac{dB(\xi)}{d\xi} \tag{2.66}$$

Taking the product of (2.63) and (2.64) we get,

$$\frac{dA(\tau)}{d\tau} \cdot \frac{dB(\xi)}{d\xi} = (a_1 - a_0)(a_4 + 2a_0) \tag{2.67}$$

The forms of the similarity equations, the scale factors $u_e(\xi, \tau)$ and $\gamma(\xi, \tau)$ depends wholly on the equation (2.67) and this situation leads to the following four possibilities:

A. Both $\frac{dA(\tau)}{d\tau}$ and $\frac{dB(\xi)}{d\xi}$ are finite constants;

B. Both $\frac{dA(\tau)}{d\tau}$ and $\frac{dB(\xi)}{d\xi}$ are zero;

C. $\frac{dA(\tau)}{d\tau} \neq 0$ but $\frac{dB(\xi)}{d\xi} = 0$;

D. $\frac{dA(\tau)}{d\tau} = 0$ but $\frac{dB(\xi)}{d\xi} \neq 0$

Of those above four cases two important cases, namely Case A and Case B, will be discussed in detail in subsequent CHAPTER IV and CHAPTER V.

CHAPTER III

Numerical Calculation Procedure

3.1 The Shooting Method

To solve the boundary layer equations (4.12) – (4.13) with boundary conditions (4.14) and (5.9)–(5.10) with boundary conditions (5.10) by using shooting method technique, an extension of the Nachtsheim-Swigert iteration scheme to the system of equations and boundary conditions is straightforward. Since there are two asymptotic boundary conditions, hence two unknown surface conditions, namely, $f''(0)$ and $\vartheta'(0)$ are to be assumed. Within the context of initial value method and Nachtsheim-Swigert iteration technique the outer boundary conditions may be functionally represented as

$$f'(\eta_{\max}) = f'(f''(0), \vartheta'(0)) = \delta_1 \tag{3.1}$$

$$\vartheta'(\eta_{\max}) = \vartheta'(f''(0), \vartheta'(0)) = \delta_2 \tag{3.2}$$

with asymptotic convergence criteria given by

$$f''(\eta_{\max}) = f''(f''(0), \vartheta'(0)) = \delta_3 \tag{3.3}$$

$$\vartheta'(\eta_{\max}) = \vartheta'(f''(0), \vartheta'(0)) = \delta_4 \tag{3.4}$$

Let us choose $f''(0) = g_1$, $\vartheta'(0) = g_2$.

Retaining only the first order terms from the Taylor's expansion from equations (3.1)–(3.4) we get

$$f'(\eta_{\max}) = f'_c(\eta_{\max}) + \frac{\partial f'}{\partial g_1}\nabla g_1 + \frac{\partial f'}{\partial g_2}\nabla g_2 = \delta_1 \tag{3.5}$$

$$\vartheta(\eta_{\max}) = \vartheta_c(\eta_{\max}) + \frac{\partial v}{\partial g_1}\nabla g_1 + \frac{\partial v}{\partial g_2}\nabla g_2 = \delta_2 \tag{3.6}$$

$$f''(\eta_{\max}) = f''_c(\eta_{\max}) + \frac{\partial f''}{\partial g_1}\nabla g_1 + \frac{\partial f''}{\partial g_2}\nabla g_2 = \delta_3 \tag{3.7}$$

$$\vartheta'(\eta_{\max}) = \vartheta'_c(\eta_{\max}) + \frac{\partial v'}{\partial g_1}\nabla g_1 + \frac{\partial v'}{\partial g_2}\nabla g_2 = \delta_4 \tag{3.8}$$

Where the subscripts 'c' indicates the value of the function at $\eta_{\max}$ determine from the trial integration. Solution of these equations in a least squares sense requires determining the minimum value of

$$E = \delta_1^{\ 2} + \delta_2^{\ 2} + \delta_3^{\ 2} + \delta_4^{\ 2} \qquad \rightarrow \ \text{Error} \tag{3.9}$$

with respect to g_1 and g_2.

Now differentiating E with respect to g_1 and g_2 we get

$$\delta_1 \frac{\partial \delta_1}{\partial g_1} + \delta_2 \frac{\partial \delta_2}{\partial g_1} + \delta_3 \frac{\partial \delta_3}{\partial g_1} + \delta_4 \frac{\partial \delta_4}{\partial g_1} = 0 \tag{3.10}$$

$$\delta_1 \frac{\partial \delta_1}{\partial g_2} + \delta_2 \frac{\partial \delta_2}{\partial g_2} + \delta_3 \frac{\partial \delta_3}{\partial g_2} + \delta_4 \frac{\partial \delta_4}{\partial g_2} = 0 \tag{3.11}$$

Applying equations (3.5)–(3.8) in equation (3.10), we obtain

$$\left[\left(\frac{\partial f'}{\partial g_1}\right)^2 + \left(\frac{\partial \vartheta}{\partial g_1}\right)^2 + \left(\frac{\partial f''}{\partial g_1}\right)^2 + \left(\frac{\partial \vartheta'}{\partial g_1}\right)^2\right]\Delta g_1 + \left(\frac{\partial f'}{\partial g_2}\frac{\partial f'}{\partial g_1} + \frac{\partial \vartheta}{\partial g_2}\frac{\partial \vartheta}{\partial g_1} + \frac{\partial f''}{\partial g_2}\frac{\partial f''}{\partial g_1} + \frac{\partial \vartheta'}{\partial g_2}\frac{\partial \vartheta'}{\partial g_1}\right)\Delta g_2 = -\left(f'_c \frac{\partial f'}{\partial g_1} + \vartheta_c \frac{\partial \vartheta}{\partial g_1} + f''_c \frac{\partial f''}{\partial g_1} + \vartheta'_c \frac{\partial \vartheta'}{\partial g_1}\right) \tag{3.12}$$

Similarly applying equations (3.5) – (3.8) in equation (3.11), we obtain

$$\left[\left(\frac{\partial f'}{\partial g_2}\right)^2 + \left(\frac{\partial \vartheta}{\partial g_2}\right)^2 + \left(\frac{\partial f''}{\partial g_2}\right)^2 + \left(\frac{\partial \vartheta'}{\partial g_2}\right)^2\right]\Delta g_2 + \left(\frac{\partial f'}{\partial g_1}\frac{\partial f'}{\partial g_2} + \frac{\partial \vartheta}{\partial g_1}\frac{\partial \vartheta}{\partial g_2} + \frac{\partial f''}{\partial g_1}\frac{\partial f''}{\partial g_2} + \frac{\partial \vartheta'}{\partial g_1}\frac{\partial \vartheta'}{\partial g_2}\right)\Delta g_1 = -\left(f'_c \frac{\partial f'}{\partial g_2} + \vartheta_c \frac{\partial \vartheta}{\partial g_2} + f''_c \frac{\partial f''}{\partial g_2} + \vartheta'_c \frac{\partial \vartheta'}{\partial g_2}\right) \tag{3.13}$$

The equation (3.12) and (3.13) can be written as

$$a_{11}\Delta g_1 + a_{12}\Delta g_2 = b_1 \tag{3.14}$$

$$a_{21}\Delta g_1 + a_{22}\Delta g_2 = b_2 \tag{3.15}$$

Where

$$a_{11} = \left(\frac{\partial f'}{\partial g_1}\right)^2 + \left(\frac{\partial \vartheta}{\partial g_1}\right)^2 + \left(\frac{\partial f''}{\partial g_1}\right)^2 + \left(\frac{\partial \vartheta'}{\partial g_1}\right)^2 \tag{3.16}$$

$$a_{12} = \frac{\partial f'}{\partial g_2}\frac{\partial f'}{\partial g_1} + \frac{\partial \vartheta}{\partial g_2}\frac{\partial \vartheta}{\partial g_1} + \frac{\partial f''}{\partial g_2}\frac{\partial f''}{\partial g_1} + \frac{\partial \vartheta'}{\partial g_2}\frac{\partial \vartheta'}{\partial g_1} \tag{3.17}$$

$$a_{21} = \left(\frac{\partial f'}{\partial g_2}\right)^2 + \left(\frac{\partial \vartheta}{\partial g_2}\right)^2 + \left(\frac{\partial f''}{\partial g_2}\right)^2 + \left(\frac{\partial \vartheta'}{\partial g_2}\right)^2 \tag{3.18}$$

$$a_{22} = \frac{\partial f'}{\partial g_1}\frac{\partial f'}{\partial g_2} + \frac{\partial \vartheta}{\partial g_1}\frac{\partial \vartheta}{\partial g_2} + \frac{\partial f''}{\partial g_1}\frac{\partial f''}{\partial g_2} + \frac{\partial \vartheta'}{\partial g_1}\frac{\partial \vartheta'}{\partial g_2} \tag{3.19}$$

$$b_1 = -\left[f_c'\frac{\partial f'}{\partial g_1} + \vartheta_c\frac{\partial \vartheta}{\partial g_1} + f_c''\frac{\partial f''}{\partial g_1} + \vartheta_c'\frac{\partial \vartheta'}{\partial g_1} \right] \tag{3.20}$$

$$b_2 = -\left[f_c'\frac{\partial f'}{\partial g_2} + \vartheta_c\frac{\partial \vartheta}{\partial g_2} + f_c''\frac{\partial f''}{\partial g_2} + \vartheta_c'\frac{\partial \vartheta'}{\partial g_2} \right] \tag{3.21}$$

In matrix form, equations (3.14) and (3.15) can be written as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \Delta g_1 \\ \Delta g_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{3.22}$$

Now we will solve the system of linear equations (3.22) by Cramer's rule and thus we have

$$\Delta g_1 = \frac{\det A_1}{\det A},\ \Delta g_2 = \frac{\det A_2}{\det A} \tag{3.23}$$

Where

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \left(a_{11}a_{22} - a_{12}a_{21}\right) \tag{3.24}$$

$$\det A = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = \left(b_1 a_{22} - b_2 a_{12}\right) \tag{3.25}$$

$$\det A = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = \left(b_2 a_{11} - b_1 a_{21}\right) \tag{3.26}$$

Then we obtained the (unspecified) missing g_1 and g_2 as

$$g_1 = g_1 + \Delta g_1 \tag{3.27}$$

$$g_2 = g_2 + \Delta g_2 \tag{3.28}$$

Thus adopting this type of numerical technique described above, a computer program will be setup for the solution of the base nonlinear differential equations of our problem where the integration technique will be adopted as the sixth order Range-Kutta method of integration. Based on the integrations done with the above mentioned numerical technique, the obtained results will be presented in the appropriate section.

CHAPTER IV

Similarity Solution: Study of Case A

In this chapter we will discuss the similarity case, viz., Case A which is obtained in the similarity analysis as given in CHAPTER II.

4.1 Case A

When $\frac{dA(\tau)}{d\tau}$ and $\frac{dB(\xi)}{d\xi}$ both are finite constants then from the equation (2.64) we have

$$u_e = u_0 \text{ constant} \tag{4.1}$$

From equation (2.64) we have

$$\frac{dA(\tau)}{d\tau} = u_e(a_4 + 2a_0)$$

$$\therefore A(\tau) = u_0(a_4 + 2a_0) + C_1$$

here C_1 is the constant of integration. Now differentiating the above equation with respect to τ, we have

$$\frac{dA(\tau)}{d\tau} = u_0(a_4 + 2a_0) \tag{4.2}$$

By virtue of (2.63) and (2.66) we obtain

$$B(\xi) = \frac{a_1 - a_2}{u_0}\xi + C_2$$

where C_2 is also a constant of integration and will be found later in terms of C_1.

Now by equation (4.1) the equation (2.63) become

$$\gamma^2 u_0 = a_1\xi + u_0(2a_0 + a_4) + C_1$$

Hence γ^2 is found to be

$$\gamma^2 = (2a_0 + a_4) + \frac{a_1}{u_0}\xi + \frac{C_1}{u_0} \quad \text{by (2.63)} \tag{A}$$

Again from (2.65)

$$\gamma^2 = 2a_0\tau + B(\xi)$$

$$= 2a_0\tau + \frac{a_1 - a_2}{u_0}\xi + C_2 \qquad \text{by (2.66) and (4.1)} \qquad \text{(B)}$$

Comparing the equations (A) and (B) for γ^2 we have to set

$$a_2 = a_4 = 0,\ C_2 = \frac{C_1}{u_0}$$

Finally, in view of equation (4.1), the equation for γ^2 is

$$\gamma^2 = 2a_0\tau + \frac{a_1}{u_0}\xi + \frac{C_1}{u_0} \qquad (4.3)$$

The similarity requirements [equations (2.62)] furnish us with the relations between the constants (a's). These relations are: a_0 and a_1 are arbitrary,

$$a_3 = -\gamma v_w$$

or, $a_3 = -\sqrt{2a_0\tau + \frac{a_1}{u_0}\xi + \frac{C_1}{u_0}}\, v_w,$

$$a_5 = -\beta_T g_x \Delta T\left(\frac{2a_0\tau}{u_0} + \frac{a_1\xi + C_1}{u_0^2}\right),$$

$a_6 = 0$ (since u_e = constant).

Also by (v) of (2.62)

$$\Delta T = -\frac{a_5 u_0}{\gamma^2 \beta_T g_x}$$, we obtain

$$a_7 = \gamma^2(\log \Delta T)_\tau$$

$$= \gamma^2 \frac{\partial}{\partial \tau}\left\{\log\left(-\frac{a_5 u_0}{\gamma^2 \beta_T g_x}\right)\right\}$$

$$= \gamma^2 \frac{\partial}{\partial \tau}\left\{\log(-a_5 u_0) - \log\left(\gamma^2 \beta_T g_x\right)\right\}$$

$$= \gamma^2 \frac{\partial}{\partial \tau}\left\{\log(-a_5 u_0) - \log \gamma^2 - \log(\beta_T g_x)\right\}$$

$$= -\gamma^2 . \frac{\partial}{\partial \tau}\left(\log \gamma^2\right)$$

$$= -\gamma^2 . \frac{\partial}{\partial \tau}(2\log \gamma)$$

$$= -\gamma^2 . \frac{\gamma_\tau}{\gamma}$$

$$= -\gamma\gamma_\tau$$

$$= -2a_0 ,$$

and $a_8 = \gamma^2 u_0 \dfrac{\partial}{\partial \xi}(\log \Delta T)$

$$= \gamma^2 u_0 \frac{\partial}{\partial \xi}\left\{\log(-a_5 u_0) - \log \gamma^2 - \log(\beta_T g_x)\right\}$$

$$= \gamma^2 u_0 \frac{\partial}{\partial \xi}\left\{-\frac{1}{\gamma^2}\frac{\partial}{\partial \xi}\left(\gamma^2\right)\right\}$$

$$= -u_0 \frac{\partial}{\partial \xi}\left(2a_0\tau + \frac{a_1}{a_0}\xi + \frac{c_1}{u_0}\right)$$

$$= a_1 .$$

Hence the general equation (2.58) and (2.59) reduces to:

$$\upsilon \tilde{f}_{\varphi\varphi\varphi} + \left(a_0\varphi + a_3\right)\tilde{f}_{\varphi\varphi} + \frac{a_1}{2}\tilde{f}\tilde{f}_{\varphi\varphi} + a_5\vartheta = 0 \tag{4.4}$$

$$\frac{\upsilon}{\Pr}\vartheta_{\varphi\varphi} + \left(a_0\varphi + a_3\right)\vartheta_\varphi + \frac{a_1}{2}\tilde{f}\vartheta_\varphi + \left(2a_0 + a_1\tilde{f}_\varphi\right)\vartheta = 0 \tag{4.5}$$

Subject to the boundary conditions $\tilde{f}(0) = \tilde{f}_\varphi(0) = 0, \tilde{f}_\varphi(\infty) = 1$ and $\vartheta(0)=1,\ \vartheta(\infty)=0$ (4.6) for the dimensionless stream function and the dimensionless temperature function respectively.

Let us substitute $\tilde{f} = \alpha_1 f$ and $\varphi = \alpha_2\eta$ in the above equations. Then as yet arbitrary constants α_1, α_2 can be defined later so as to provide convenient simplifications in the above forms of equations.

Here

$$\tilde{f}_\varphi = \frac{\partial\tilde{f}}{\partial\varphi} = \frac{\partial\tilde{f}}{\partial\eta}.\frac{\partial\eta}{\partial\varphi} = \frac{\alpha_1\partial f}{\partial\eta}.\frac{\partial\eta}{\alpha_2\partial\varphi} = \frac{\alpha_1}{\alpha_2}f_\eta$$

$$\tilde{f}_{\varphi\varphi} = \frac{\partial^2 \tilde{f}}{\partial \varphi^2} = \frac{\partial \tilde{f}_\varphi}{\partial \varphi} = \frac{\alpha_1}{\alpha_2}\frac{\partial f_\eta}{\partial \eta}.\frac{\partial \eta}{\partial \varphi} = \frac{\alpha_1}{\alpha_2}\frac{\partial f_\eta}{\alpha_2 \partial \eta} = \frac{\alpha_1}{\alpha_2^2} f_{\eta\eta}$$

Similarly, $\tilde{f}_{\varphi\varphi\varphi} = \frac{\alpha_1}{\alpha_2^3} f_{\eta\eta\eta}$

$$\vartheta_\varphi = \frac{\partial \vartheta}{\partial \varphi} = \frac{\partial \vartheta}{\partial \eta}.\frac{\partial \eta}{\partial \varphi} = \frac{1}{\alpha_2}\vartheta_\eta$$

$$\vartheta_{\varphi\varphi} = \frac{\partial^2 \vartheta}{\partial \varphi^2} = \frac{\partial \vartheta_\varphi}{\partial \varphi} = \frac{1}{\alpha_2}\frac{\partial \vartheta_\eta}{\partial \eta}.\frac{\partial \eta}{\partial \varphi} = \frac{1}{\alpha_2}\frac{\partial \vartheta_\eta}{\partial \eta}.\frac{1}{\alpha_2} = \frac{1}{\alpha_2^2}\vartheta_{\eta\eta}$$

Thus the above equations are changed to:

$$\upsilon \frac{\alpha_1}{\alpha_2^3} f_{\eta\eta\eta} + \left(a_0 \alpha_2 \eta + a_3\right)\frac{\alpha_1}{\alpha_2^2} f_{\eta\eta} + \frac{a_1 \alpha_1 \alpha_1}{2\alpha_2^2} ff_{\eta\eta} + a_5 \vartheta = 0$$

Or, $f_{\eta\eta\eta} + \frac{\alpha_2}{\upsilon}(a_0 \alpha_2 \eta + a_3) f_{\eta\eta} + \frac{a_1}{2\upsilon}\alpha_1 \alpha_2 ff_{\eta\eta} + \frac{a_5 \alpha_2^3}{\upsilon \alpha_1}\vartheta = 0$ (4.7)

and

$$\frac{\upsilon}{\Pr}\frac{1}{\alpha_2^2}\vartheta_{\eta\eta} + \frac{1}{\alpha_2}\left(a_0 \alpha_2 \eta + a_3\right)\vartheta_\eta + \frac{a_1 \alpha_1}{2\alpha_2} f \vartheta_\eta + \left(2a_0 + \frac{a_1 \alpha_1}{\alpha_2} f_\eta\right)\vartheta = 0$$

or, $\frac{1}{\Pr}\vartheta_{\eta\eta} + \frac{\alpha_2}{\upsilon}\left(\alpha_2 \eta a_0 + a_3\right)\vartheta_\eta + \frac{a_1}{2\upsilon}\alpha_1 \alpha_2 f \vartheta_\eta + \frac{2a_0 \alpha_2^2}{\upsilon}\left(1 + \frac{a_1}{2a_0}\frac{\alpha_1}{\alpha_2} f_\eta\right)\vartheta = 0$ (4.8)

Choosing $\alpha_1 = \alpha_2$, $\frac{a_0 \alpha_2^2}{\upsilon} = 1$ and writing $\frac{a_1}{a_0} = 2\beta$ the above equations are further simplified to

$f_{\eta\eta\eta} + \left(\eta + \frac{a_3}{\sqrt{a_0 \upsilon}}\right) f_{\eta\eta} + \beta ff_{\eta\eta} + \frac{a_5}{a_0}\vartheta = 0$ (4.9)

and $\frac{1}{\Pr}\vartheta_{\eta\eta} + \left(\eta + \frac{a_3}{\sqrt{a_0 \upsilon}}\right)\vartheta_\eta + \beta f \vartheta_\eta + 2\left(1 + \beta f_\eta\right)\vartheta = 0$ (4.10)

Where $\frac{a_5}{a_0} = -g_x \beta_T \Delta T\left\{2u_0 \tau + 2\beta\xi + \frac{C_1}{a_0}\right\}\frac{1}{u_0^2}$

$= -g_x \beta_T \Delta T 2\left\{\beta\left(\xi + \xi_0\right) + u_0\left(\tau + \tau_0\right)\right\}\frac{1}{u_0^2}$, where $\frac{C_1}{a_0} = 2\left(\beta \xi_0 + u_0 \tau_0\right)$

$$= \frac{U_F^2}{u_0^2} \text{ (say)}$$

Where $U_F^2 = -g_x \beta_T \Delta T \times$characteristic length L_c $\left(L_c = 2\left\{\beta\left(\xi + \xi_0\right) + u_0\left(\tau + \tau_0\right)\right\}\right)$ (4.11)

As $\frac{a_3 \alpha_2}{\upsilon} = a_3 . \frac{\sqrt{\upsilon}}{\sqrt{a_0}} \frac{1}{\upsilon} = -\gamma v_w . \frac{1}{\sqrt{a_0 \upsilon}} = -\frac{\gamma v_w}{\sqrt{a_0 \upsilon}} = f_w$

$$\frac{a_3}{\sqrt{a_0 \upsilon}} = -\frac{\gamma v_w}{\sqrt{a_0 \upsilon}}$$

$$= -\frac{v_w}{\sqrt{a_0 \upsilon}} \sqrt{2a_0 \tau + \frac{a_1}{u_0} \xi + \frac{C_1}{u_0}}$$

$$= -\frac{v_w}{\sqrt{a_0 \upsilon}} \sqrt{\frac{a_0}{u_0}\left(2u_0 \tau + 2\beta\xi + 2u_0 \tau_0 + 2\beta\xi_0\right)}$$

$$= -\frac{v_w}{\sqrt{a_0 \upsilon}} \sqrt{\frac{2a_0}{u_0}\{\beta(\xi + \xi_0) + u_0(\tau + \tau_0)\}}$$

$$= -\frac{v_w}{\sqrt{a_0 \upsilon}} . \sqrt{\frac{a_0 L_c}{u_0}}$$

$$= -\frac{v_w}{u_0} . \sqrt{\frac{u_0 L_c}{\upsilon}}$$

$$= -\frac{v_w L_c}{\upsilon} . \sqrt{\frac{\upsilon}{u_0 L_c}}$$

$$= -\frac{v_w L_c}{\upsilon} \text{Re}^{-\frac{1}{2}}$$

$$= f_w$$

$$\therefore v_w = -\frac{\upsilon}{L_c} \text{Re}^{\frac{1}{2}} f_w$$

Thus in the Characteristic length $\left(= 2\{\beta(\xi + \xi_0) + u_0(\tau + \tau_0)\}\right)$, $u_0(\tau + \tau_0)$ forms effectively another length in unsteady flow. The terms $\frac{a_5}{a_0} \vartheta = \left(\frac{U_F^2}{u_0^2} \vartheta\right)$ and $\frac{a_3}{\sqrt{a_0 \upsilon}}$ in the momentum equation indicates how important buoyancy effects are compared with the forced flow effects. Hence the transformed equations to be solved with controlling parameters Pr, β, $\frac{U_F^2}{u_0^2}$ and f_w are

$$f_{\eta\eta\eta}+(\eta+f_w)f_{\eta\eta}+\beta ff_{\eta\eta}+\frac{U_F^2}{u_0^2}\vartheta=0 \tag{4.12}$$

$$\frac{1}{\Pr}v_{\eta\eta}+(\eta+f_w)\vartheta_\eta+\beta f\vartheta_\eta+2(1+\beta f_\eta)\vartheta=0$$

$$\text{or, } \vartheta_{\eta\eta}+\Pr\left[(\eta+f_w)\vartheta_\eta+\beta f\vartheta_\eta+2(1+\beta f_\eta)\vartheta\right]=0 \tag{4.13}$$

The boundary conditions are

$$f(0)=f_\eta(0)=0,\ f_\eta(\infty)=1,\ \vartheta(0)=1,\ \vartheta(\infty)=0 \tag{4.14}$$

The similarity function $f(\eta)$ the similarity variable η the velocity components (= u, v), the skin friction and heat transfer coefficients τ_w, q_w associated with the equations (4.12) and (4.13) with the boundary conditions (4.14) are as follows:

$$\begin{aligned}\psi &= \gamma UF-\psi(\xi,0,\tau)\\ &=\gamma UL\tilde{f}(\varphi)-\psi(\xi,0,\tau)\\ &=\gamma u_0\alpha_2 f(\eta)+\psi(\xi,0,\tau)\\ &=\sqrt{2a_0\tau+\frac{a_1}{u_0}\xi+\frac{C_1}{u_0}}u_0\alpha_2 f(\eta)+\psi(\xi,0,\tau)\\ &=\sqrt{\frac{a_0}{u_0}\left(2u_0\tau+\frac{a_1}{a_0}\xi+\frac{C_1}{a_0}\right)}u_0\alpha_2 f(\eta)+\psi(\xi,0,\tau)\\ &=\sqrt{\frac{a_0}{u_0}(2u_0\tau+2\beta\xi+2u_0\tau_0+2\beta\xi_0)}u_0\alpha_2 f(\eta)-\psi(\xi,0,\tau)\\ &=\sqrt{\frac{2a_0}{u_0}\{\beta(\xi+\xi_0)+u_0(\tau+\tau_0)\}}u_0\alpha_2 f(\eta)+\psi(\xi,0,\tau)\\ &=\sqrt{\upsilon u_0 L_c}f(\eta)+\psi(\xi,0,\tau),\ \text{where } L_c=2\{\beta(\xi+\xi_0)+u_0(\tau+\tau_0)\}\\ &=\upsilon \mathrm{Re}^{\frac{1}{2}}f(\eta)+\psi(\xi,0,\tau),\ \text{where } \mathrm{Re}^{\frac{1}{2}}=\frac{u_0L_c}{\upsilon}\end{aligned} \tag{4.15}$$

Here Re is the dimensionless Renold's number based on free convection velocity U_F given by (3.11) and the local characteristic length $L_c=2\{\beta(\xi+\xi_0)+u_0(\tau+\tau_0)\}$

$$\varphi = \alpha_2 \eta$$

$$\therefore \eta = \frac{\varphi}{\alpha_2} = \frac{1}{\alpha_2}\frac{y}{\gamma}$$

$$= \frac{y}{\sqrt{\frac{a_0}{u_0}}\alpha_2 . \sqrt{\frac{\upsilon}{a_0}}}$$

$$= \left(\frac{u_0}{\upsilon L_c}\right)^{\frac{1}{2}} y$$

$$= \sqrt{\frac{u_0 L_c}{\upsilon}} . \frac{y}{L_c}$$

$$= \mathrm{Re}^{\frac{1}{2}} \frac{y}{L_c} \tag{4.16}$$

$$u = UF_\varphi = UL\tilde{f}_\varphi$$

$$= u_0 \alpha_2 f_\varphi(\eta)$$

$$= u_0 \alpha_2 . \frac{1}{\alpha_2} f_\eta(\eta) \qquad \left(\text{Since } \frac{\partial f}{\partial \varphi} = \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial \varphi} = \frac{1}{\alpha_2}\frac{\partial f}{\partial \eta} = \frac{1}{\alpha_2} f_\eta \right)$$

$$= u_0 f_\eta(\eta) \tag{4.17}$$

$$\psi = \gamma UF - \psi(\xi, 0, \tau)$$

$$\text{or, } -v = (\gamma UF)_\xi - \psi_\xi(\xi, 0, \tau) - \varphi \gamma_\xi UF_\varphi$$

$$\therefore -v = (\gamma UF)_\xi - \varphi \gamma_\xi UF_\varphi - v_w$$

$$= \frac{\partial}{\partial \xi}\left(\sqrt{\frac{a_0 L_c}{u_0}}\sqrt{\frac{\upsilon}{a_0}} u_0 f\right) - \gamma_\xi \alpha_2 \eta u_0 \frac{\alpha_1}{\alpha_2} f_\eta(\eta) - v_w$$

$$= \frac{\partial}{\partial \xi}\left(\sqrt{u_0 L_c \upsilon} f\right) - u_0 . \gamma_\xi . \alpha_2 \frac{\alpha_1}{\alpha_2} \eta f_\eta(\eta) - v_w$$

$$= \beta \sqrt{\frac{\upsilon u_0}{L_c}} f - \sqrt{\frac{a_0}{u_0}} . \frac{\beta}{\sqrt{L_c}} . \sqrt{\frac{\upsilon}{a_0}} u_0 \eta f_\eta(\eta) - v_w$$

$$= \beta \sqrt{\frac{\upsilon u_0}{L_c}} f - \beta \sqrt{\frac{\upsilon u_0}{L_c}} \eta f_\eta(\eta) - v_w$$

$$= \beta \left(\frac{\upsilon u_0}{L_c}\right)^{\frac{1}{2}} \left(f - \eta f_\eta(\eta)\right) - v_w$$

$$= \beta \frac{\upsilon}{L_c}\sqrt{\frac{u_0 L_c}{\upsilon}}\left(f - \eta f_\eta\right) + \frac{\upsilon}{L_c}\mathrm{Re}^{\frac{1}{2}} f_w \qquad \left(\text{Since } f_w = -\frac{v_w}{\eta} L_c \,\mathrm{Re}^{\frac{1}{2}}\right)$$

$$= \frac{\upsilon}{L_c}\mathrm{Re}^{\frac{1}{2}}\left[\beta\left(f - \eta f_\eta\right) + f_w\right] \tag{4.18}$$

where $\gamma^2 = 2a_0\tau + \frac{a_1}{u_0}\xi + \frac{C_1}{u_0}$

or, $2\gamma\gamma_\xi = \frac{a_1}{u_0}$

or, $\gamma_\xi = \frac{a_1}{u_0}\frac{1}{2\gamma} = \frac{a_1}{u_0}\frac{1}{2\sqrt{\frac{a_0 L_c}{u_0}}} = \frac{a_1}{2u_0}\sqrt{\frac{u_0}{a_0 L_c}}\beta\sqrt{\frac{a_0}{u_0 L_c}}$

The skin friction coefficient $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$

$$= \mu \left.\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}\right|_{\eta=0}$$

$$= \mu \left.\frac{\partial u}{\partial \eta}\frac{\mathrm{Re}^{\frac{1}{2}}}{L_c}\right|_{\eta=0} \qquad \left(\text{Since } \frac{\partial \eta}{\partial y} = \frac{\mathrm{Re}^{\frac{1}{2}}}{L_c}\right)$$

$$= \mu u_0 \frac{\mathrm{Re}^{\frac{1}{2}}}{L_c} . f_{\eta\eta}(0)$$

$$= \mu u_0 \frac{\sqrt{\frac{u_0 L_c}{\upsilon}}}{L_c} . f_{\eta\eta}(0)$$

$$= \mu u_0 \sqrt{\frac{u_0}{\upsilon L_c}} . f_{\eta\eta}(0)$$

$$= \frac{\mu u_0}{L_c}\mathrm{Re}^{\frac{1}{2}} f_{\eta\eta}(0) \tag{4.19}$$

$$q_w = -k \left.\frac{\partial T}{\partial y}\right|_{y=0}$$

$$= -k \left.\frac{\partial T}{\partial \eta} . \frac{\partial \eta}{\partial y}\right|_{y=0} \qquad \left(\text{Since } T = \Delta T\theta(\eta) = \Delta T m v(\eta) = \Delta T v(\eta); m = 1\right)$$

$$= -k\sqrt{\frac{u_0}{\upsilon L_c}}\frac{\partial T}{\partial \eta}\bigg|_{y=0} \qquad \left(\text{Since } \frac{\partial T}{\partial \eta} = \Delta T\vartheta_\eta(\eta) = \Delta T\vartheta_\eta(0) \text{ at } \eta = 0\right)$$

$$= -k\Delta T\sqrt{\frac{u_0}{\upsilon L_c}}.\vartheta_\eta(0) \qquad (4.20)$$

respectively, where L_c (= characteristic length) is given by

$$L_c = 2\{\beta(x+x_0) + u_0(t+t_0)\} \qquad (4.21)$$

ΔT variation for this case is

$$\Delta T \propto \left[2\{\beta(x+x_0)+u_0(t+t_0)\}\right]^{-1} \qquad (4.22)$$

Thus, we have $\Delta T \propto \dfrac{1}{L_c}$ (4.23)

4.2 Numerical Scheme and Procedure

The set of ordinary differential equations (4.12) and (4.13) with the boundary conditions (4.14) are non linear coupled, which are difficult to solve. Therefore, the numerical procedure based on the standard initial-value solver, namely, the sixth order Runge-Kutta method in collaboration with the Runge-Kutta Merson method is adopted to obtain the solution of the problem. An extension of the Nachtsheim-Swigert shooting iteration technique (guessing the missing value) (Nachtsheim & Swigert (1965)) together with Runge-kutta sixth order integration scheme is implemented. The detailed descriptions of the procedure of numerical solution of the problems are given in CHAPTER III. It is clear that the numbers of initial conditions are insufficient to obtain the particular solution of the differential equations. So we require assuming additional missing/unspecified initial conditions. Thus, in this method, the initial conditions at the initial point of the interval are assumed and with all the initial conditions (given and assumed) the equations are integrated numerically in steps as an initial value problem to the terminal point. These are to be assumed that the solution of the outer prescribed points also matches. The accuracy of the assumed missing initial condition is checked by comparing the calculated value of the dependent variable at the terminal point with its given value there. If match is not found (a difference exists) at the outer end then another set of missing initial conditions are considered and the process is repeated. This trial and error process is taken care through Nachtsheim-Swigert shooting iteration technique and the process is continued until the agreement between the calculated and the given condition at the terminal point is within the specified degree of accuracy. For this type of iterative approach, one naturally inquires whether or not there is a systematic way of finding each succeeding (assumed) value of missing initial condition.

The boundary conditions (4.14) associated with the system are of the two-point asymptotic class. Two-point boundary conditions have values of independent variable specified at two different values of the independent variable, where the outer boundary conditions are specified at infinity. There are two asymptotic boundary conditions as well as two unknown surface conditions $f_{\eta\eta}(0)$, $\vartheta_{\eta}(0)$ here. Specification of asymptotic boundary condition implies that the value of velocity approaches to zero and the value of temperature approaches from unity to zero as the outer specified value of the independent variable is approached. The governing differential equations are then integrated with these assumed surface boundary conditions. If the required outer boundary condition is satisfied, a solution has been achieved. However, this is not generally the case. Hence a method must be devised to logically estimate the new surface boundary conditions for the next trial integration. Asymptotic boundary value problems such as those governing the boundary layer equations are further complicated by the fact that the outer boundary condition is specified at infinity. In the trial integrations, infinity is numerically approximated by some large specified value of the independent variable. There is not a priori general method of estimating this value. Selection of too small a maximum value for the independent variable may not allow the solution to asymptotically converge to the required accuracy. Selecting a large value may result in divergence of the trial integration or in slow convergence of surface boundary conditions required satisfying the asymptotic outer boundary condition. Selecting too large a value of the independent variable is expensive in terms of computer time. Nachtsheim-Swigert developed an iteration method, which overcomes these difficulties. Extension of the Nachtsheim-Swigert iteration shell to above system of differential equations (4.12) and (4.13) is very straightforward. The effects of various parameters on the flow and temperature fields have been determined for different values of the suction/blowing parameter f_w, the driving parameter β (the ratio between the changes of local boundary-layer thickness with regard to position and time, i.e., $\frac{\gamma_x}{\gamma_t}$ (since u_e being constant)), the buoyancy parameter $\frac{U_F^2}{u_0^2}$ (the square of the ratio between the fluid velocity caused by buoyancy effects and external velocity for the forced flow) and the Prandtl number Pr. Since there are four parameters of interest in the present problem which can be varied, to observe the effect of one, the other three parameters are kept as constants. Under these conditions the solutions to the problem thus obtained finally by employing the above mentioned numerical technique are plotted and tabulated in terms of the similarity variables.

4.3 **Numerical Results and Discussion**

The effects of f_w on the velocity and temperature profiles are plotted in Fig 4.1 and Fig. 4.2, respectively. From Fig. 4.1 we see that, for the case of suction ($f_w > 0$), the velocity profiles increase with the increase of η near the surface, become maximum and then decrease and finally become zero asymptotically. But for the case of blowing ($f_w < 0$) the velocity decreases near the surface and then increases with the increase in η and finally leads to zero asymptotically. Further, velocity profiles decrease with increasing suction case, but the magnitude of the velocity decreases with the increase of blowing. The usual stabilizing effect of the suction parameter on the boundary layer growth is also evident from this figure.

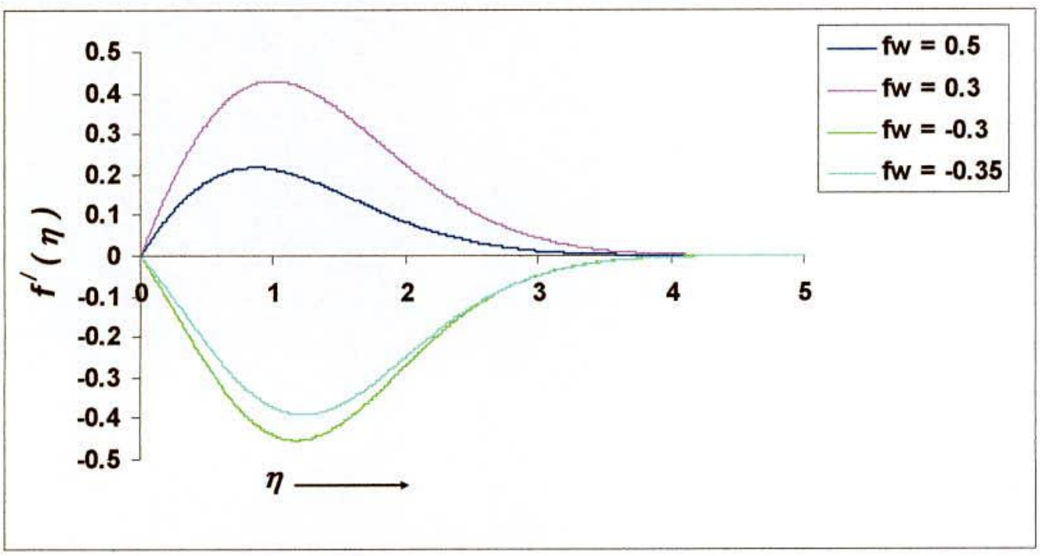

Figure 4.1: Velocity profiles for different values of f_w (with fixed values of $\frac{U_F^2}{u_0^2} = 0.3$, $\beta = -0.5$ and $\Pr = 0.72$).

From Fig. 4.2 we see that the temperature profiles increase close to the plate surface and away from the surface they decrease asymptotically and finally become zero with the increase of η, for the case of suction $(f_w > 0)$. A reverse situation is found for the case of blowing ($f_w < 0$), that is temperature profiles decrease first close to the plate surface and away from the surface they increase asymptotically and finally become zero with the increase of η. It is also observed that the temperature decreases with the increases of suction but increase with the increase of blowing.

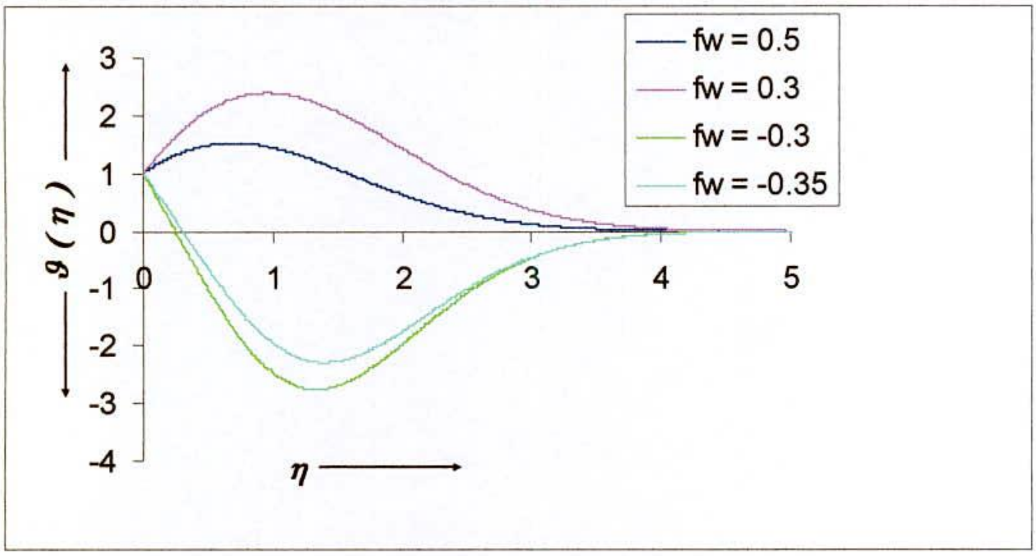

Figure 4.2: Temperature profiles for different values of f_w (with fixed values of $\frac{U_F^2}{u_0^2} = 0.3$, $\beta = -0.5$ and $\Pr = 0.72$).

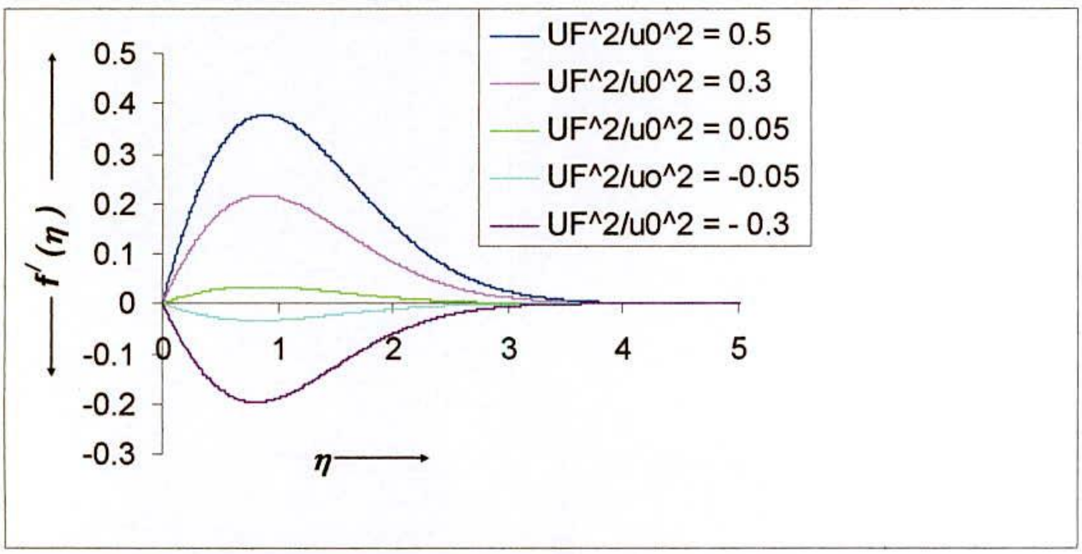

Figure 4.3: Velocity profiles for different values of $\frac{U_F^2}{u_0^2}$ (with fixed values of $\beta = -0.5$, $f_w = -0.5$ and $\Pr = 0.72$).

Fig. 4.3 and Fig. 4.4 show the effects of bouncy parameter $\frac{U_F^2}{u_0^2}$ on the velocity and temperature profiles. Physically, the flow is said to be aided when $\frac{U_F^2}{u_0^2} > 0$ and is called an opposing flow when $\frac{U_F^2}{u_0^2} < 0$. Further, when $U_F^2 << u_0^2$, the flow becomes a forced flow, whereas for $u_0^2 << U_F^2$ the flow becomes a free convection flow. We see from the Fig. 4.3 that with the decrease of $\frac{U_F^2}{u_0^2}$ the maximum velocity reduces and thus the velocity becomes zero within short range for smaller values of $\frac{U_F^2}{u_0^2} > 0$. Further, for $\frac{U_F^2}{u_0^2} < 0$, the magnitude of the velocity profiles increase with the increase of the magnitude of $\frac{U_F^2}{u_0^2}$. The unusual shape of temperature profiles as shown in Fig. 4.4 indicates that the wall receives more and more heat from the fluid as the buoyancy parameter $\frac{U_F^2}{u_0^2}$ decreases. This is due to possessing an infinite source of heat at the leading edge, that is, $\Delta T \propto \frac{1}{t}$ at $x=0$, hence $T_w \to 0$ as $t \to 0$ at leading edge.

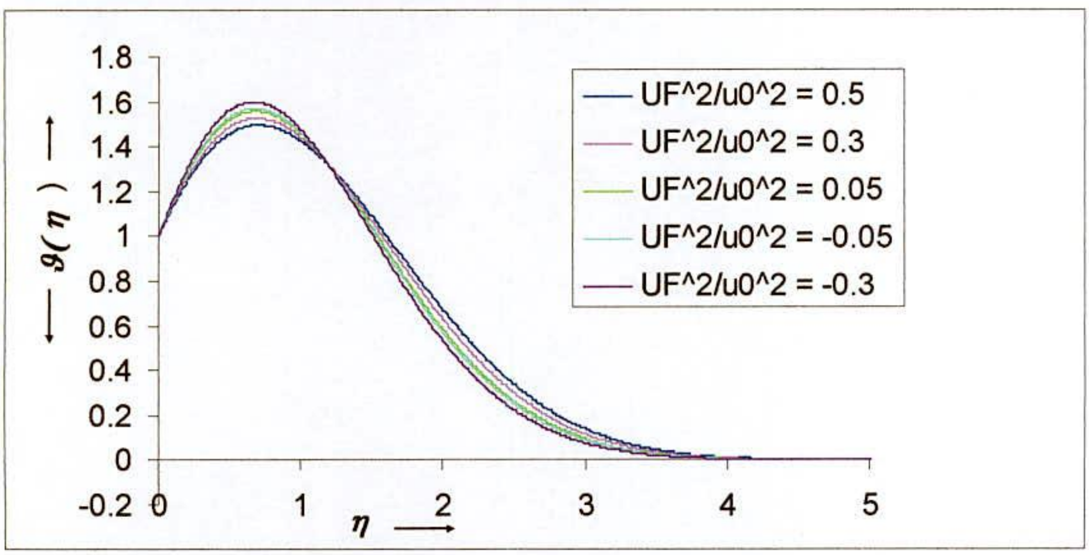

Figure 4.4: Temperature profiles for different values of $\frac{U_F^2}{u_0^2}$ (with fixed values of $\beta = -0.5$, $f_w = -0.5$ and $\Pr = 0.72$).

The effects of the controlling parameter β on the velocity and temperature profiles are observed in Fig. 4.5 and Fig. 4.6, respectively. Since $\Delta T \propto (L_c)^{-1}$ (as equation (4.23)), the parameter β controls the steadiness over unsteady effects in the additive form of characteristic length, caused by u_e and ΔT-variations. More precisely, as the boundary layer is characterized by ΔT-variation only (since u_e being constant), the parameter β specifies the temperature variation (i.e., equation (4.22)).

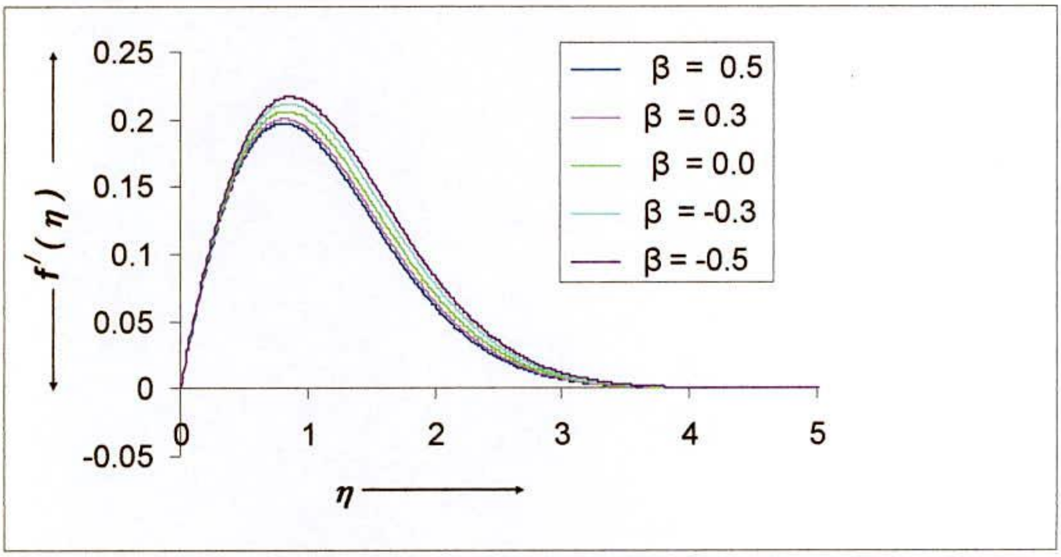

Figure 4.5: Velocity profiles for different values of β (with fixed values of $\frac{U_F^2}{u_0^2} = 0.3$, $f_w = 0.5$ and $\Pr = 0.72$).

From Fig. 4.5 we observe that, velocity profiles become maximum at about $\eta = 0.8$ and they become zero asymptotically at about $\eta = 3.25$ for all values of β. Here the maximum velocity also decreases with the increase of β.

Fig. 4.6 exhibits the effect of β on the temperature profiles. Temperature first increases with the increase of η and then decreases again and reduces to zero asymptotically as $\eta = 3.5$. In the range of $\eta = 1.2$ temperature increases with the increase in β and after that it decreases with the increase of β.

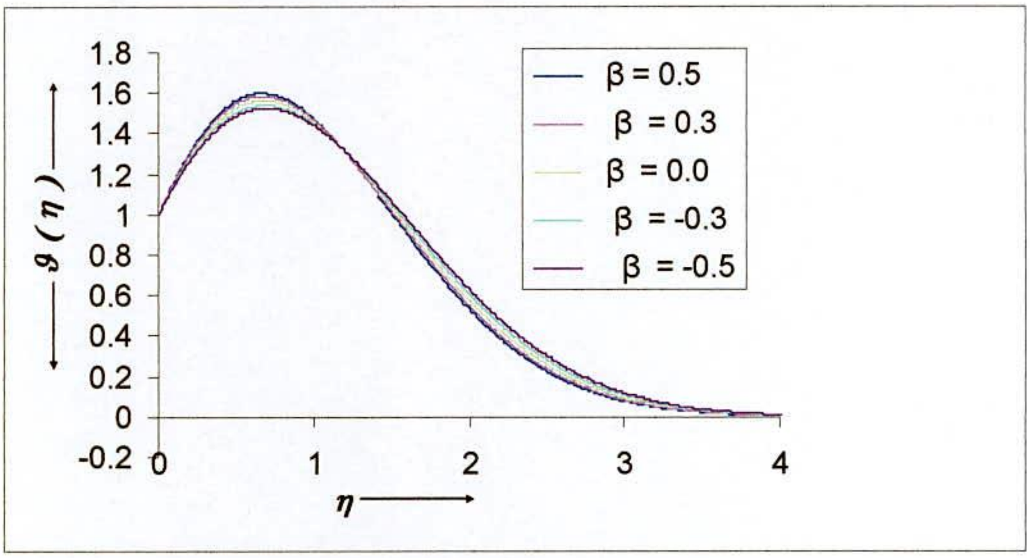

Figure 4.6: Temperature profiles for different values of β (with fixed values of $\frac{U_F^2}{u_0^2} = 0.3$, $f_w = 0.5$ and $\Pr = 0.72$).

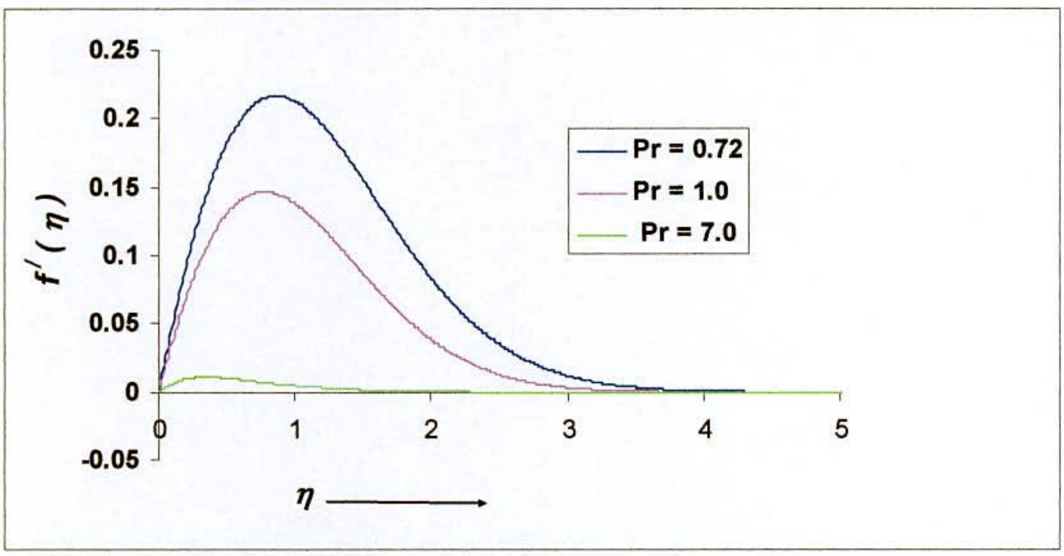

Figure 4.7: Velocity profiles for different values of Pr (with fixed values of $\frac{U_F^2}{u_0^2} = 0.3$, $\beta = -0.5$ and $f_w = 0.5$).

The last controlling parameter is the Prandtl number $\Pr\left(=\frac{\mu C_p}{k}\right)$ which depends on the properties of medium. The velocity and temperature profiles exhibit remarkable changes with the variation of $\Pr$ as observed from Fig. 4.7 and Fig.4.8. It is observed from Fig. 4.7 that with the increase in η for different values of $\Pr$, the pick values of velocity shift a little but they decrease rapidly with the increase in $\Pr$.

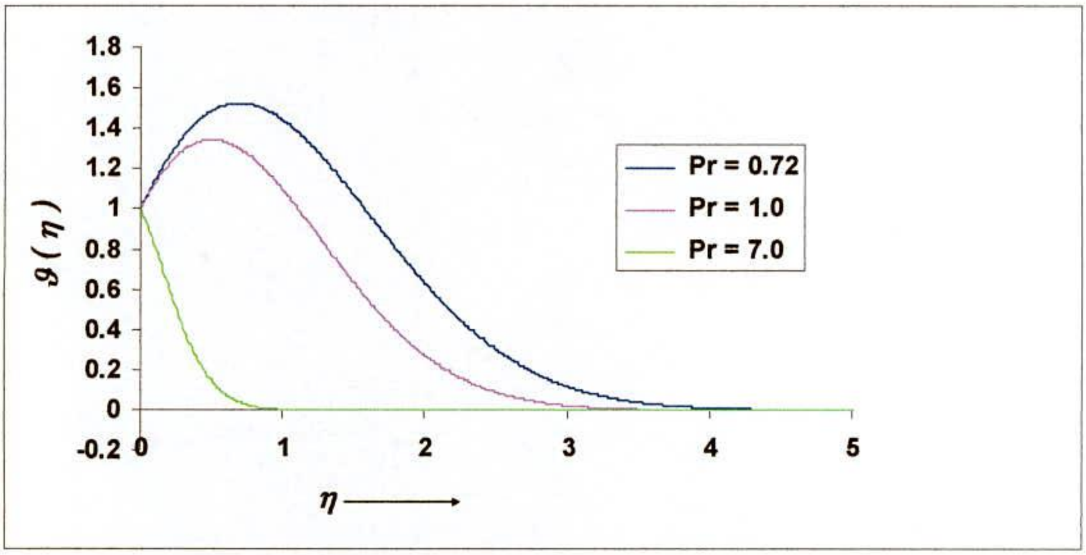

Figure 4.8: Temperature profiles for different values of Pr (with fixed values of $\frac{U_F^2}{u_0^2} = 0.3$, $\beta = -0.5$ and f_w=0.5).

Like before, temperature decreases faster with the increase of Pr as seen in Fig. 4.8.

The values proportional to the coefficients of skin friction $f''(0)$ and heat transfer - $\vartheta'(0)$ are tabulated in Table (4.1) – (4.4).

From Table 4.1, it is seen that with the increase in f_w, the coefficient of skin friction decrease but the coefficient of heat transfer increases for $f_w > 0$ but when $f_w < 0$, the coefficient of skin friction increases with the decreasing f_w but the coefficient of heat transfer decreases with decreasing f_w. Whereas both the coefficient of skin friction and the coefficient of heat transfer increase with increasing $\frac{U_F^2}{u_0^2}$ as seen in Table 4.2

Table 4.1: Values proportional to the coefficients of skin-friction ($f''(0)$) and heat transfer ($-\vartheta'(0)$) with the variation of suction parameter f_w for fixed $\frac{U_F^2}{u_0^2} = 0.3$, $\beta = -0.5$ and $\Pr = 0.72$.

f_w	$f''(0)$	$-\vartheta'(0)$
0.50	0.52341	-1.52677
0.30	0.840835	-2.839873
-0.30	-0.49478	3.80024
-0.50	-0.38129	3.0243

Table 4.2: Values proportional to the coefficients of skin-friction ($f^{//}(0)$) and heat transfer ($-\vartheta^{/}(0)$) with the variation of buoyancy parameter $\frac{U_F^2}{u_0^2}$ for fixed $f_w = 0.5$, $\beta = -0.5$ and Pr=0.72.

U_F^2 / u_0^2	$f^{//}(0)$	$-\vartheta^{/}(0)$
0.5	0.890985	-1.443896
0.30	0.52341	-1.52677
0.05	0.85423	-1.62055
0.00	0.00218	-1.636134
-0.05	-0.08465	-1.656793
-0.30	-0.50013	-1.745835

Table 4.3: Values proportional to the coefficients of skin-friction $f^{//}(0)$ and heat transfer $-\vartheta^{/}(0)$ with the variation of driving parameter β for fixed $f_w = 0.5$, $\frac{U_F^2}{u_0^2} = 0.3$ and Pr=0.72.

β	$f^{//}(0)$	$-\vartheta^{/}(0)$
-0.50	0.52340	-1.52677
-0.30	0.517295	-1.571002
0.00	0.510231	-1.639965
0.30	0.503856	-1.70361
0.50	0.500049	-1.744686

The reverse condition is observed for β variation. Here both $f^{//}(0)$ and $-\vartheta^{/}(0)$ reduces with the increase of β.

Table 4.4: Values proportional to the coefficients of skin-friction ($f^{//}(0)$) and heat transfer ($-\vartheta^{/}(0)$) with the variation of Prandtl number Pr for fixed $f_w = 0.5$, $\frac{U_F^2}{u_0^2} = 0.3$ and $\beta = -0.5$.

Pr	$f^{//}(0)$	$-\vartheta^{/}(0)$
0.72	0.52341	-1.52677
1.00	0.399258	-1.402672
7.00	0.075913	1.50934

Again $f^{//}(0)$ reduces and $-\vartheta^{/}(0)$ increases with the increase in Pr. Unfortunately, no experimental data is available to us to compare our numerical results.

CHAPTER V

Similarity Solution: Study of Case B

In this chapter we will discuss the similarity case, viz., Case B which is obtained in the similarity analysis as given in Chapter II.

5.1 Case B

When both $\frac{dA(\tau)}{d(\tau)} = \frac{dB(\xi)}{d(\xi)} = 0$, we have

A = constant and B = constant.

In view of equation (2.63) and (2.65), γ^2 and u_e are found to be

$$\gamma^2 = 2a_0\tau + B(\xi) \tag{5.1}$$

$$u_e = \frac{a_1\xi + A(\tau)}{2a_0\tau + B(\xi)} \tag{5.2}$$

Substituting (5.1) and (5.2) in the similarity requirements stated in equations (2.62) one may obtain the following relations between the constants (a's):

a_0, a_1 are arbitrary.

From equation (2.64)

$$u_e(a_4 + 2a_0) = 0$$

$$\therefore\ a_4 = -2a_0$$

From equation (2.66)

$$\frac{a_1 - a_2}{u_e} = 0$$

$$\therefore a_1 = a_2$$

From (iii) of equation (2.62)

$$a_3 = -\gamma v_w$$

$$= -\sqrt{2a_0\tau + B}v_w$$

$$= -\sqrt{a_0(2\tau + 2\tau_0)}v_w \qquad (\text{ Since } \frac{B}{a_0} = = 2\tau_0)$$

$= \sqrt{a_0 X}v_w$, where $X = 2(\tau + \tau_0)$ is the characteristic length.

From (iv) of equation (2.62)

$$a_4 = \frac{\gamma^2 (u_e)_\tau}{u_e}$$

$$= (2a_0\tau + B)\frac{(a_1 + A)}{(2a_0\tau + B)}.(-2a_0).\frac{(2a_0\tau + B)}{(a_1\xi + A)}$$

$$= -2a_0$$

From (v) of equation (2.62)

$$a_5 = -\frac{\gamma^2}{u_e}\beta\Delta T g_x$$

$$= -\beta_T \Delta T g_x . \frac{(2a_0\tau + B)^2}{(a_1\xi + A)}$$

From (vi) of equation (2.62)

$$a_6 = \frac{\gamma^2}{u_e}\{(u_e)\tau + u_e(u_e)_\xi\}$$

$$= \frac{(2a_0\tau + B)^2}{a_1\xi + A}\left\{\frac{-2a_0(a_1\xi + A)}{(2a_0\tau + B)^2} + \frac{a_1\xi + A}{2a_0\tau + B}.\frac{a_1}{(2a_0\tau + B)}\right\}$$

$$= a_1 - 2a_0$$

$$\therefore a_6 = a_1 - 2a_0$$

From (vii) of equation (2.62)

$$a_7 = \gamma^2(\log\Delta T)_\tau \qquad \left(\text{Since} \quad \Delta T = \frac{-a_5 u_e}{g_x\beta_T\gamma^2} = \frac{-a_5(a_1\xi + A)}{g_x\beta_T(2a_0\tau + B)^2}\right)$$

$$= (2a_0\tau + B)\frac{\partial}{\partial\tau}\left\{\log(-a_5) + \log(a_1\xi + A) - \log(g_x) - \log\beta_T - \log(2a_0\tau + B)^2\right\}$$

$$= (2a_0\tau + B)\left[-\frac{2}{(2a_0\tau + B)}.2a_0\right]$$

$$= -4a_0$$

From (viii) of equation (2.62)

$$a_8 = \gamma^2 u_e(\log\Delta T)_\xi$$

$$= \gamma^2 u_e \frac{\partial}{\partial\xi}\left\{\left(\frac{-a_5(a_1\xi + A)}{g_x\beta_T(2a_0\tau + B)^2}\right)\right\}$$

$$= (a_1\xi + A)\frac{\partial}{\partial\xi}\{\log(-a_5) + \log((a_1\xi + A)) - \log(g_x\beta_T) - 2\log(a_0\tau + B)\}$$

$$= (a_1\xi + A)\left[\frac{a_1}{(a_1\xi + A)}\right]$$

$$= a_1$$

$$\therefore a_8 = a_1$$

Thus the general equation (2.58) and (2.59) takes the form

$$\upsilon\tilde{f}_{\varphi\varphi\varphi} + \left(a_0\varphi + a_3\right)\tilde{f}_{\varphi\varphi} + a_1\tilde{f}\tilde{f}_{\varphi\varphi} + a_1\left(1 - \tilde{f}_\varphi^2\right) - 2a_0\left(\tilde{f}_\varphi - 1\right) + a_5\vartheta = 0 \tag{5.3}$$

$$\upsilon\vartheta_{\varphi\varphi} + \Pr\left[\left(a_0\varphi + a_3\right)\vartheta_\varphi + a_1\tilde{f}\vartheta_\varphi + \left(4a_0 - a_1\tilde{f}_\varphi\right)\right]\vartheta = 0 \tag{5.4}$$

with boundary conditions $\tilde{f}(0) = \tilde{f}_\varphi(0) = 0$, $\tilde{f}_\varphi(\infty) = 1$, $\vartheta(0) = 1$, $\vartheta(\infty) = 0$ (5.5)

As in case A, substituting $\tilde{f} = \alpha_1 f$, $\varphi = \alpha_1\eta$, choosing $\frac{a_0\alpha_1^2}{\upsilon} = 1$ and letter writing $\frac{a_1}{a_0} = 2\beta$, (4.3) and (4.4) with their attached boundary conditions (4.5) are simplified to

$$\frac{\upsilon}{a_1^2}f_{\eta\eta\eta} + \left(\alpha_0\alpha_1\eta + a_3\right)\frac{1}{\alpha_1}f_{\eta\eta} + a_1\alpha_1 f\frac{1}{\alpha_1}f_{\eta\eta} + a_1\left(1 - f_\eta^2\right) + 2a_0\left(f_\eta - 1\right) + a_5\vartheta = 0$$

$$\text{or, } f_{\eta\eta\eta} + \left(\frac{a_0\alpha_1^2}{\upsilon}\eta + \frac{a_3\alpha_1}{\upsilon}\right)f_{\eta\eta} + 2\beta ff_{\eta\eta} + 2\beta\left(1 - f_\eta^2\right) + 2\left(f_\eta - 1\right) + \frac{a_5}{a_0}\vartheta = 0$$

$$\therefore f_{\eta\eta\eta} + \left(\eta + \frac{a_3\alpha_1}{\upsilon}\right)f_{\eta\eta} + 2\beta\left(ff_{\eta\eta} - f_\eta^2 + 1\right) + 2\left(f_\eta - 1\right) + \frac{a_5}{a_0}\vartheta = 0 \tag{5.6}$$

$$\frac{\upsilon}{\alpha_1^2}\vartheta_{\eta\eta} + \Pr\left[\left(a_0\alpha_1\eta + a_3\right)\frac{1}{\alpha_1}\vartheta_\eta + a_1\alpha_1 f\frac{1}{\alpha_1}\vartheta_\eta + \left(4a_0 - a_1 f_\eta\right)\vartheta\right] = 0$$

$$\therefore \frac{1}{\Pr}\vartheta_\eta + \left[\left(\frac{a_0\alpha_1^2}{\upsilon}\eta + \frac{a_3\alpha_1}{\upsilon}\right)\vartheta_\eta + \frac{a_1\alpha_1^2}{\upsilon}f\vartheta_\eta + \left(4\frac{a_0\alpha_1^2}{\upsilon} - a_1\alpha_1^2 f_\eta\right)\vartheta\right] = 0 \tag{5.7}$$

Now $\frac{a_3}{a_0\alpha_1} = \frac{a_3}{a_0\sqrt{\frac{\upsilon}{a_0}}} = \frac{a_3}{\sqrt{\upsilon a_0}} = \frac{-\gamma v_w}{\sqrt{\upsilon a_0}}$

$$= -\frac{\sqrt{2a_0\tau + B}v_w}{\sqrt{\upsilon a_0}}$$

$$= -\frac{\sqrt{2\tau + 2\tau_0}\, v_w}{\upsilon}$$

$$= -\sqrt{\frac{2\tau + 2\tau_0}{\upsilon}}\, v_w = \sqrt{\frac{L_c}{\upsilon}}\, v_w = f_w \text{ (say)}$$

$$\therefore v_w = \sqrt{\frac{\upsilon}{L_c}}\, f_w$$

$$\frac{a_5}{a_0} = -\frac{g_x \beta_T \Delta T (2a_0\tau + B)^2}{a_0 (a_1 \xi + A)}$$

$$= -\frac{g_x \beta_T \Delta T \left(2\tau + \dfrac{B}{a_0}\right)^2}{u_e}$$

$$= -\frac{g_x \beta_T \Delta T \{2u_e(\tau - \tau_0)\}}{u_e^2}$$

$$= \frac{U_F^2}{u_e^2} \text{ (say)}$$

By using the above relations the equation (5.6) and (5.7) are simplified to

$$f_{\eta\eta\eta} + (\eta + f_w) f_{\eta\eta} + 2\beta\left(ff_{\eta\eta} - f_\eta^2 + 1\right) + 2\left(f_\eta - 1\right) + \frac{U_F^2}{u_e^2}\vartheta = 0 \tag{5.8}$$

$$\Pr^{-1}\vartheta_{\eta\eta} + (\eta + f_w)\vartheta_\eta + 2\beta f \vartheta_\eta + \left(4 - 2\beta f_\eta\right)\vartheta = 0 \tag{5.9}$$

and the boundary conditions are

$$f(0) = f_\eta(0) = 0;\;\; f_\eta(\infty) = 1;\;\; \vartheta(0) = 1;\;\; \vartheta(\infty) = 0 \tag{5.10}$$

$$\text{where }\; U_F^2 = -g_x \beta_T \Delta T \{2u_e(\tau + \tau_0)\} \tag{5.11}$$

The parameters are again $\Pr$, β, f_w and $\frac{U_F^2}{u_0^2}$ but the last two convey different physical meanings from the previous case due to the difference in nature of ΔT and u_e variations. Both ΔT and u_e, upon which combined convection effects depend, may be specified from equations (5.2) and (5.11) as

$$\Delta T \propto \frac{x + x_0}{t + t_0} \tag{5.12}$$

$$u_e = \beta \frac{x+x_0}{t+t_0} \tag{5.13}$$

Thus the parameter β controls the degree of steady effects of the boundary layer equations (4.8) – (4.9). If $\beta = 0$, no flow occurs as the characteristic length is $2u_e(\tau+\tau_0)$ or $2\beta(x+x_0)$. The parameter $\frac{U_F^2}{u_0^2}$ is the ratio between the square of the free convection velocity U_F generated by ΔT variation (equation (5.12)) and the forcing velocity u_e. The similarity function $f(\eta)$, the similarity variable η, the velocity components (u, v), the skin friction and heat transfer coefficients (τ_w, q_w) associated with the equations (5.8) – (5.9) are now calculated as;

$$\begin{aligned}\psi &= \gamma UF + \psi(\xi,0,\tau)\\ &= \sqrt{2a_0(\tau+\tau_0)}\beta\frac{(x+x_0)}{t+t_0}\sqrt{\frac{\upsilon}{a_0}}f(\eta)+\psi(\xi,0,\tau)\\ &= \sqrt{2\beta}\sqrt{\vartheta(x+x_0)}.u_e f(\eta)+\psi(\tau,\xi,0)\end{aligned}$$

$$\varphi = \alpha_1 \eta \tag{5.14}$$

$$\begin{aligned}\eta &= \frac{\varphi}{\alpha_1} = \frac{y}{\gamma\alpha_1}\\ &= \frac{y}{\sqrt{2a_0(t+t_0)}}\sqrt{\frac{a_0}{\upsilon}}\\ &= \frac{y}{\sqrt{\frac{2\beta(x+x_0)}{u_e}}}\sqrt{\frac{1}{\upsilon}}\\ &= \frac{y}{\sqrt{2\beta}}\left\{\frac{u_e}{\upsilon(x+x_0)}\right\}^{\frac{1}{2}}\end{aligned} \tag{5.15}$$

$$\begin{aligned}u &= UF_\varphi = UL\tilde{f}_\varphi(\varphi)\\ &= u_e\frac{\partial f(\alpha_1\eta)}{\partial\eta}.\frac{\partial\eta}{\partial\varphi}\\ &= u_e\alpha_1\frac{1}{\alpha_1}f_\eta(\eta)\end{aligned}$$

$$= u_e f_\eta(\eta) \tag{5.16}$$

$$-v = (\gamma UF)_\xi - \varphi\gamma_\xi UF_\varphi - v_w$$

$$= (\gamma UF)_\xi - v_w$$

$$= \gamma (u_e)_\xi f - v_w$$

$$= \sqrt{2a_0(t+t_0)}\frac{\beta}{(t+t_0)} f - v_w$$

$$= \frac{\beta\sqrt{2a_0}}{\sqrt{t+t_0}}\alpha_1 f(\eta) - v_w$$

$$= \sqrt{2\beta}\left(\frac{\upsilon u_e}{x+x_0}\right)^{\frac{1}{2}} f(\eta) - v_w \tag{5.17}$$

The skin friction coefficient $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$

$$= \mu \left.\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}\right|_{y=0}$$

$$= \mu \frac{u_e}{\sqrt{2\beta}}\left\{\frac{u_e}{\upsilon(x+x_0)}\right\}^{\frac{1}{2}} f_{\eta\eta}(0) \tag{5.18}$$

The heat transfer coefficient $q_w = -k\left.\frac{\partial T}{\partial y}\right|_{y=0} = -k\left.\frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y}\right|_{y=0}$

$$= -\frac{k\Delta T}{\sqrt{2\beta}}\left(\frac{u_e}{\upsilon(x+x_0)}\right)^{\frac{1}{2}} \vartheta_\eta(0) \tag{5.19}$$

respectively.

The problem represented by the equations (5.8) – (5.9) for forced convection $\left(\text{i.e. } \frac{U_F^2}{u_e^2}\right)$ becomes that considered by Schuh (1953) and Yang (1958). Whereas Schuh obtained the solution for the momentum equation only, using integral methods, Yang obtained similarity solutions for both momentum and energy equations for the case of constant wall temperature. For constant wall temperature (i.e. $a_7 = a_8 = 0$), by the introduction of minor changes in the similarity function $f(\eta)$ and the similarity variable η as

$$f(\eta)=\sqrt{2\beta}\bar{f}(\bar{\eta}) \text{ and } \eta=\frac{\bar{\eta}}{\sqrt{2\beta}}$$

$$\therefore f_{\eta}(\eta)=\sqrt{2\beta}\bar{f}_{\bar{\eta}}(\bar{\eta})\sqrt{2\beta}=2\beta\bar{f}_{\bar{\eta}}(\bar{\eta})$$

$$\therefore f_{\eta\eta}(\eta)=2\beta\sqrt{2\beta}\bar{f}_{\bar{\eta}\bar{\eta}}(\bar{\eta})$$

$$\therefore f_{\eta\eta\eta}(\eta)=4\beta^2\bar{f}_{\bar{\eta}\bar{\eta}\bar{\eta}}(\bar{\eta})$$

and $\vartheta_{\eta}=\frac{\partial\vartheta}{\partial\eta}=\frac{\partial\vartheta}{\partial\bar{\eta}}.\frac{\partial\bar{\eta}}{\partial\eta}=\sqrt{2\beta}\vartheta_{\bar{\eta}}$

$$\therefore \vartheta_{\eta\eta}=\frac{\partial}{\partial\eta}(\vartheta_{\eta})=\frac{\partial}{\partial\bar{\eta}}\left(\sqrt{2\beta}\vartheta_{\bar{\eta}}\right).\frac{\partial\bar{\eta}}{\partial\eta}=2\beta\vartheta_{\bar{\eta}\bar{\eta}}$$

The equation (4.8) – (4.9) will be reduced to those dealt with by Yang (1958). His parameter 'α' may be restored replacing β by $-\frac{1}{\alpha}$ only. The reduced type of equations with their boundary conditions are

$$4\beta^2\bar{f}_{\bar{\eta}\bar{\eta}\bar{\eta}}(\bar{\eta})+\left(\frac{\bar{\eta}}{\sqrt{2\beta}}-\sqrt{\frac{L_c}{\upsilon}}v_w\right)2\beta\sqrt{2\beta}\bar{f}_{\bar{\eta}\bar{\eta}}(\bar{\eta})+4\beta^2.2\beta\left(\bar{f}(\bar{\eta})\bar{f}_{\bar{\eta}\bar{\eta}}(\bar{\eta})-\bar{f}^2_{\bar{\eta}\bar{\eta}}(\bar{\eta})\right)+$$
$$+4\beta\bar{f}_{\bar{\eta}}(\bar{\eta})+2\beta-2=0$$

$$\text{or, } \bar{f}_{\bar{\eta}\bar{\eta}\bar{\eta}}(\bar{\eta})+\left(\frac{\bar{\eta}}{\sqrt{2\beta}}-\sqrt{\frac{L_c}{\upsilon}}v_w\right)\frac{1}{\sqrt{2\beta}}\bar{f}_{\bar{\eta}\bar{\eta}}(\bar{\eta})+2\beta\left(\bar{f}(\bar{\eta})\bar{f}_{\bar{\eta}\bar{\eta}}(\bar{\eta})-\bar{f}^2_{\bar{\eta}}(\bar{\eta})\right)$$
$$+\frac{1}{\beta}\bar{f}_{\bar{\eta}}(\bar{\eta})+\frac{1}{2\beta}-\frac{1}{2\beta^2}=0$$

$$\text{or, } \bar{f}_{\bar{\eta}\bar{\eta}\bar{\eta}}(\bar{\eta})+\left(-\frac{\alpha\bar{\eta}}{2}-\sqrt{-\frac{\alpha}{2}}\sqrt{\frac{L_c}{\upsilon}}v_w\right)\bar{f}_{\bar{\eta}\bar{\eta}}(\bar{\eta})-\frac{2}{\alpha}\left(\bar{f}(\bar{\eta})\bar{f}_{\bar{\eta}\bar{\eta}}(\bar{\eta})-\bar{f}^2_{\bar{\eta}}(\bar{\eta})\right)$$
$$-\alpha\bar{f}_{\bar{\eta}}(\bar{\eta})-\frac{\alpha}{2}-\frac{\alpha^2}{2}=0 \quad (5.20)$$

$$\text{and } 2\beta\vartheta_{\bar{\eta}\bar{\eta}}+\Pr\left\{\left(\frac{\bar{\eta}}{\sqrt{2\beta}}-\sqrt{\frac{L_c}{\upsilon}}v_w\right)\sqrt{2\beta}\vartheta_{\bar{\eta}}+4\beta^2\bar{f}(\bar{\eta})\vartheta_{\bar{\eta}}+\left(4+4\beta^2\bar{f}_{\bar{\eta}}(\bar{\eta})\right)\vartheta\right\}=0$$

$$\text{or, } \vartheta_{\bar{\eta}\bar{\eta}}+\Pr\left\{\left(\frac{\bar{\eta}}{\sqrt{2\beta}}-\sqrt{\frac{L_c}{\upsilon}}v_w\right)\frac{1}{\sqrt{2\beta}}\vartheta_{\bar{\eta}}+2\beta\bar{f}(\bar{\eta})\vartheta_{\bar{\eta}}+\left(\frac{2}{\beta}+2\beta\bar{f}_{\bar{\eta}}(\bar{\eta})\right)\vartheta\right\}=0$$

$$\text{or, } \vartheta_{\bar{\eta}\bar{\eta}}+\Pr\left\{\left(-\frac{\alpha\bar{\eta}}{2}-\sqrt{-\frac{\alpha}{2}}\sqrt{\frac{L_c}{\upsilon}}v_w\right)\vartheta_{\bar{\eta}}+\frac{2}{\alpha}\bar{f}(\bar{\eta})\vartheta_{\bar{\eta}}+\left(-2\alpha-\frac{2}{\alpha}\bar{f}_{\bar{\eta}}(\bar{\eta})\right)\vartheta\right\}=0 \quad (5.21)$$

$$\bar{f}(0)=\bar{f}_{\bar{\eta}}(0)=0,\ \bar{f}_{\bar{\eta}}(\infty)=1,\ \vartheta(0)=1,\ \vartheta(\infty)=0 \quad (5.22)$$

The equations (5.20)–(5.22) are a special case of combined forced and free convection ordinary differential equations (5.8)–(5.9). When $\alpha = 0$, the momentum equation (5.20) becomes entirely the steady flow equation of Hiemenz (1911) for two dimensional stagnation point of flow.

Since it is not possible to find analytical solutions of (5.8)–(5.9) in terms of the controlling parameters, numerical solutions are obtained for the particular values of f_w, $\Pr$, β and $\frac{U_F^2}{u_0^2}$.

5.2 Numerical Scheme and Procedure

The set of differential equations (5.8)–(5.9) with the boundary conditions (5.10) are solved on the same numerical procedure as stated in Case A, that is, using the Nachtsheim-Swigert shooting iteration technique (guessing the missing value) (Nachtsheim & Swigert (1965)) together with Rangue-Kutta sixth order iteration technique. Like Case A, here the velocity f_η, temperature ϑ are determined as a function of coordinate η. The skin friction coefficient $f_{\eta\eta}(0)$ and the heat transfer rate $-\vartheta_\eta(0)$ are also evaluated for this case and numerical results thus obtained in terms of the similarity variables are displayed in graphs and tables for several selected values of the established parameters f_w, β, $\frac{U_F^2}{u_0^2}$ and $\Pr$ below.

5.3 Numerical Results and Discussions

To obtain the solution of differential equations (5.20)–(5.21) with the boundary conditions (5.22), a numerical procedure based on Nachtsheim-Swigert shooting iteration technique (guessing the missing value) (Nachtsheim & Swigert (1965)) together with Runge-Kutta sixth order integration scheme is implemented. The effects of suction parameter f_w, driving parameter β (the ratio between the changes of local boundary-layer thickness with regard to position and time), the buoyancy parameter $\frac{U_F^2}{u_0^2}$ (the square of the ratio between the fluid velocity caused by buoyancy effects and external velocity for the forced flow) and the prandtl number $\Pr$ are plotted in Figs. (5.1) – (5.8). Also their effects on the coefficient of skin-friction and heat transfer coefficients are tabulated on Tables (5.1) – (5.4), respectively. To observe the effect of f_w, the other three parameters β, $\frac{U_F^2}{u_0^2}$ and Pr are taken constants. Similarly, we observe the effect of the parameters β, $\frac{U_F^2}{u_0^2}$ and Pr by taking the rest three parameters constant respectively. In all cases the constant values are $f_w = -0.3, \frac{U_F^2}{u_0^2} = -0.3, \beta = 1.0$ and $\Pr = 0.72$. Now for the variation of f_w, it varies only in the range of $-0.5 \le f_w \le 0.34$. Similarly β varies in the range of $0.967 \le \beta \le 1.01$ and $\frac{U_F^2}{u_0^2}$ in the range $-1.1 \le \frac{U_F^2}{u_e^2} \le -1.3$.

The effect of f_w on the velocity and temperature profiles is plotted in Fig. 5.1 and Fig. 5.2. From Fig. 5.1, we observe that the velocity is increasing for the decreasing value of f_w in the region $\eta \le 1.02$. The maximum velocity appears at $\eta = 1.0$. Then the velocity profiles start decreasing and become negative when $\eta > 1.59$ again the velocities take the reverse direction and finally become zero at about $\eta = 5.1$. The magnitude of the velocities reaches the highest value when $\eta \approx 2.53$. Further we conclude that the velocity profiles increase with the decreasing value of f_w in the region $(0 \le \eta \le 2.53)$ and increasing with the increasing of f_w in the region $(2.53 \le \eta \le 5.1)$ for both suction and blowing.

Again from Fig. 5.2, we observe the effect of f_w on the temperature profiles. From this figure, it is observed that the wall lost its temperature to the fluid and after some times it receives the temperature from the fluid. In the region very close to the surface, the temperature falls sharply and decreases with the increase in f_w. When $\eta \approx 1.22$, the temperature profiles take the reverse direction and increase with increasing f_w. Here the temperature again decreases with the increase of f_w when $\eta > 3.5$ and finally approaches to zero when $\eta > 5.06$.

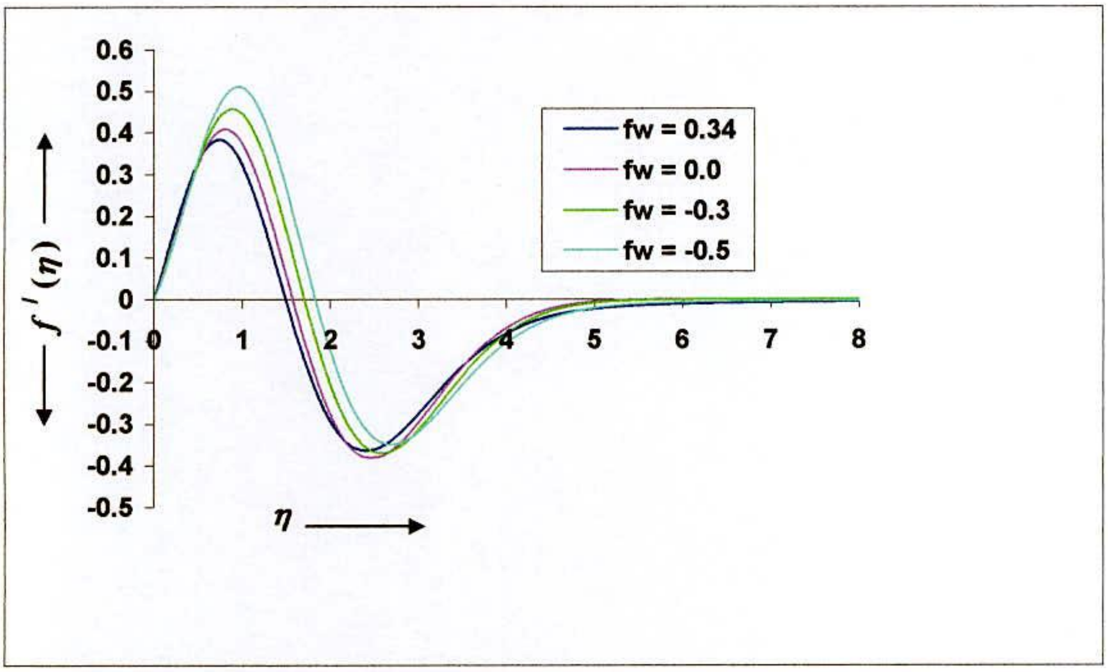

Figure 5.1: Velocity profiles for different values of f_w (with fixed values of $\frac{U_F^2}{u_0^2} = -1.3$, $\beta = 1.0$ and $P_r = 0.72$).

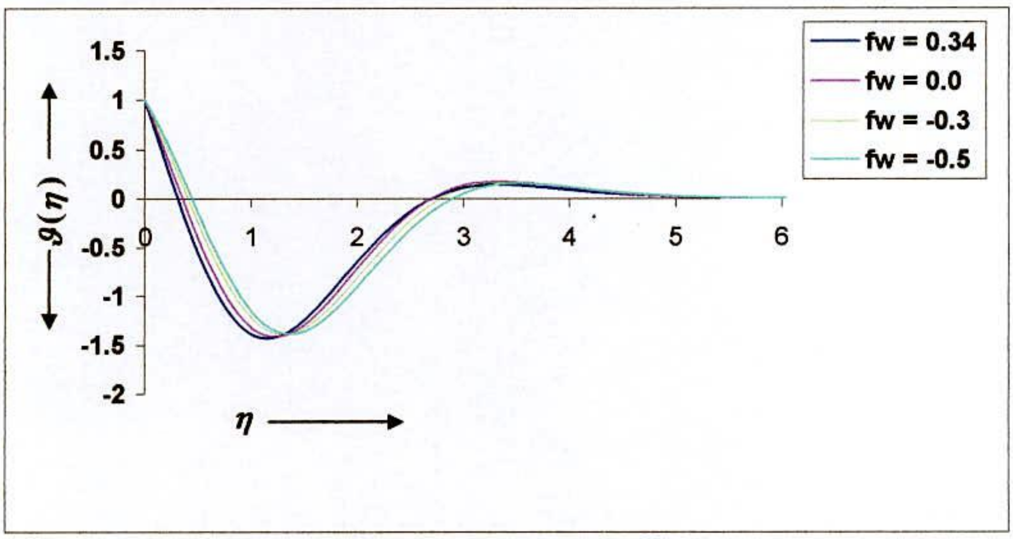

Figure 5.2: Temperature profiles for different values of f_w (with fixed values of $\frac{U_F^2}{u_0^2} = -1.3$, $\beta = 1.0$ and $\Pr = 0.72$).

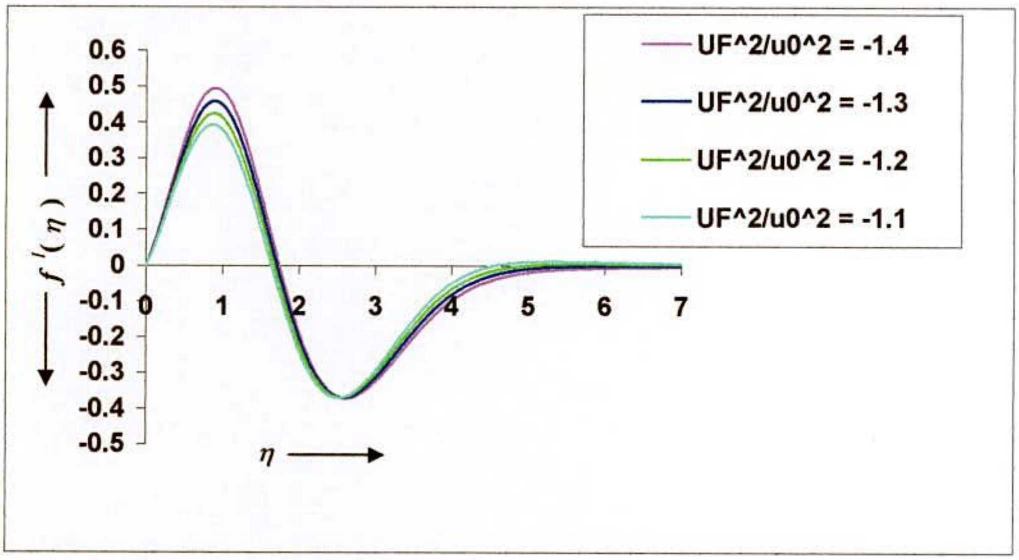

Figure 5.3: Velocity profiles for different values of $\frac{U_F^2}{u_0^2}$ (with fixed values of $\beta = 1.0$, $f_w = -0.3$ and $P_r = 0.72$).

Fig. 5.3 and Fig. 5.4 show the effects of $\frac{U_F^2}{u_0^2}$ on the velocity and temperature profiles. In this case we observe that the velocity profiles are increasing near the surface with the decreasing values of $\frac{U_F^2}{u_0^2}$ and obtained maximum value at $\eta \approx 0.99$. Then the velocity profiles change their directions and obtained negative values at $\eta > 1.65$ and finally become zero at $\eta = 6.5$. The magnitude of velocity obtained its highest value when $\eta \approx 2.65$ and after that a reverse characteristic is found. Here the magnitude of velocity is decreased with the increases of magnitude of $\frac{U_F^2}{u_0^2}$.

Again the effects of $\frac{U_F^2}{u_0^2}$ on the temperature profiles show that, very close to the wall the temperature falls sharply in the region $0 \le \eta < 1.35$. After that the temperature profiles changes their direction and become positive at $\eta \approx 2.71$ and then goes to the reverse direction before reaching zero as is seen in Fig. 5.4.

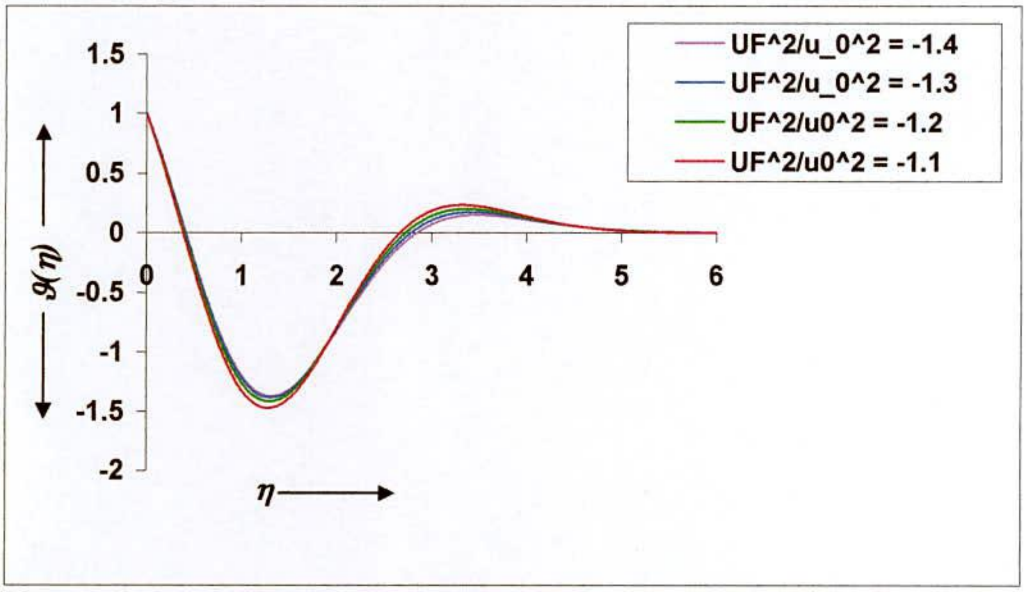

Figure 5.4: Temperature profiles for different values of $\frac{U_F^2}{u_0^2}$ (with fixed values of $\beta = 1.0$, $f_w = -0.3$ and $\Pr = 0.72$).

The unusual shape of the temperature profiles indicates that the wall rejects more and more heat to the fluid as the buoyancy parameter $\frac{U_F^2}{u_0^2}$ decreases. This is due to the plate possessing an infinite source of heat at the leading edge, that is, $\Delta T \propto \frac{1}{t}$ at $x = 0$, hence $T_w \to \infty$ as $t \to 0$ at the leading edge.

Fig. 5.5 and Fig. 5.6 exhibit the effect of the driving parameter β on the velocity and temperature profiles. The possible values of β are restricted to $0.967 \le \beta \le 1.01$. Two different behaviors are executed in the range when $0.967 \le \beta < 0.984$ and $0.984 \le \beta \le 1.01$, respectively. In the range of $0.967 \le \beta < 0.984$ the velocity is decreasing with the decreasing value of β and achieved the maximum magnitude at $\eta \approx 0.64$, then changes its directions and become positive in the region $1.0 \le \eta \le 3.51$. After that they changes their directions and asymptotically become zero when $\eta \approx 10.0$. The opposite properties are exhibited in the range $0.984 \le \beta \le 1.01$ for the increasing value of η. In the range $0.967 \le \beta < 0.984$ the velocity profiles take the negative values when $\eta \le 1.0$. The magnitude of velocity increased with decreasing β and achieved a maximum value at $\eta \approx 0.64$. When $\eta > 1.0$, the velocity becomes positive and a maximum velocity appears at $\eta \approx 2.35$ for minimum value of $\beta = 0.967$. After that the velocity profiles again change their directions and become negative values when $\eta \geq 3.25$ and asymptotically approaches zero far away from the plate surface.

A reverse phenomenon is observed in the range of $0.984 \leq \beta \leq 1.01$. Here the velocity profiles decreases with increasing β. A positive velocity is observed when $\eta < 1.6$ and after that the velocity become negative and approach to zero asymptotically.

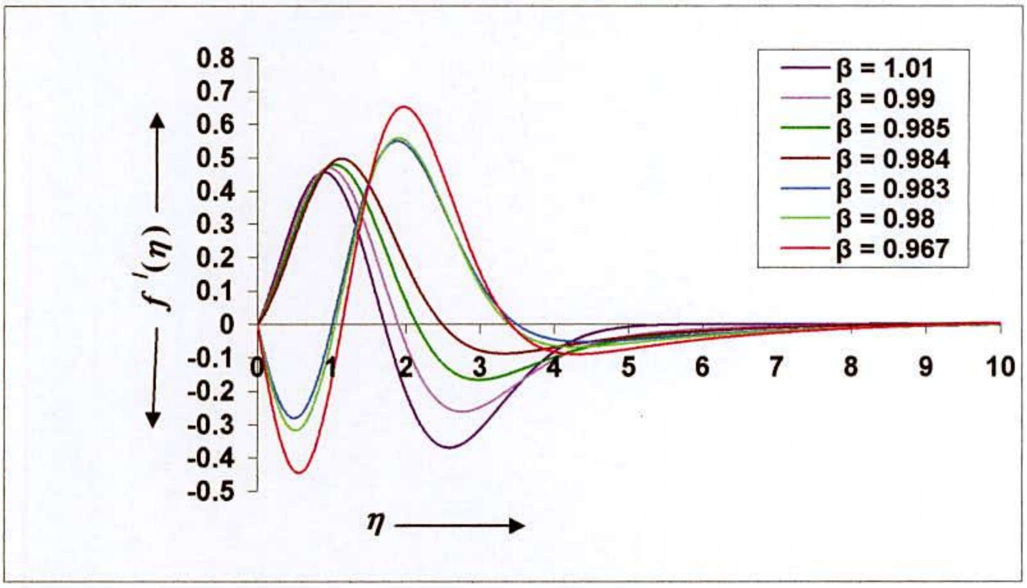

Fig 5.5: Velocity profiles for different values of β (with fixed values of $\frac{U_F^2}{u_0^2} = -1.3$, $f_w = -0.3$ and $\Pr = 0.72$).

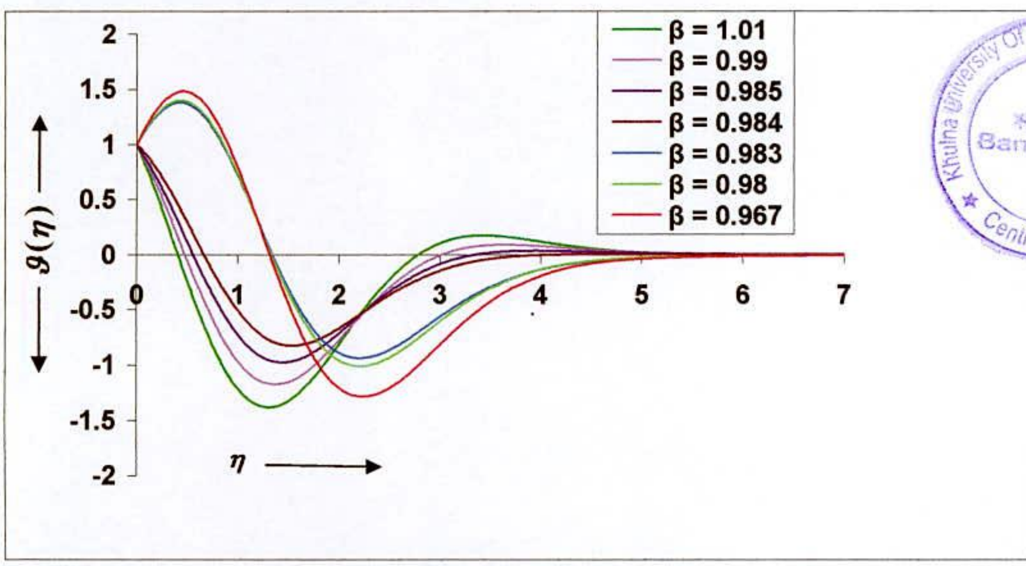

Figure 5.6: Temperature profiles for different values of β (with fixed values of $\frac{U_F^2}{u_0^2} = -1.3$, $f_w = -0.3$ and $P_r = 0.72$).

From the Fig. 5.6 we observe two different behaviors of temperature profiles in the region when $0.967 \le \beta < 0.984$ and $0.984 \le \beta \le 1.01$ respectively. In the range of $0.967 \le \beta < 0.984$ the temperature raises first very close to the surface and then falls sharply when $\eta > 0.45$. Temperature first increase with the decreasing values of β and then they again decrease with decreasing β $(0.967 \le \beta < 0.984)$. Further, in the region $0.984 \le \beta \le 1.0$, we observe that the temperature falls sharply and become negative in the region $0.47 \le \eta \le 2.86$. Here temperature first decreases with increasing β in the range of $0.984 \le \beta \le 1.0$, where $\eta < 1.36$. After that temperature profiles change direction and become increasing and in the region where $\eta > 2.3$ a reverse situation is observed. That is temperature again increases with decreasing β when $\eta > 2.3$ and asymptotically approaches to zero for all values of β for far away and then they are again increase with increasing β.

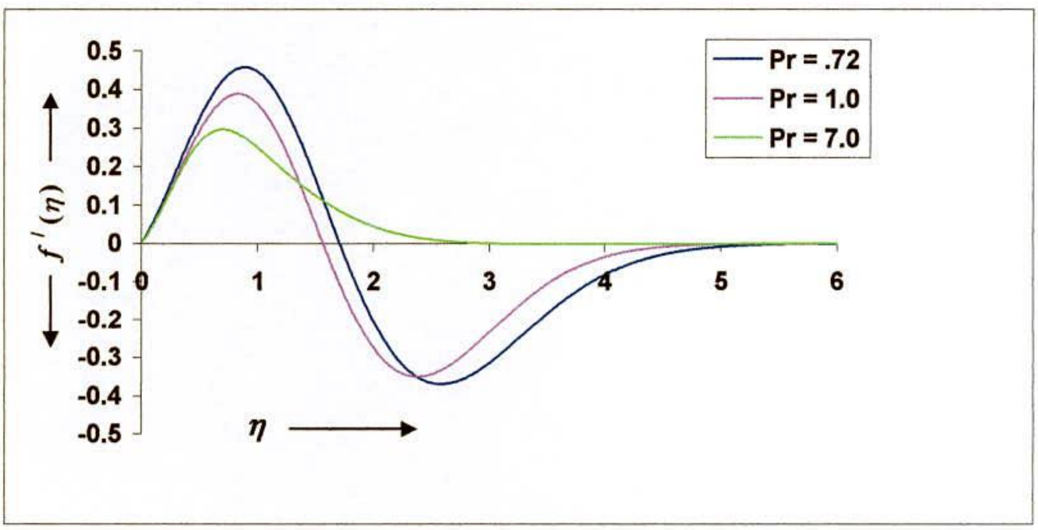

Fig. 5.7: Velocity profiles for different values of Pr (with fixed values of $\frac{U_F^2}{u_0^2} = -1.3$, $\beta = 1.0$ and $f_w = -0.3$).

From the Fig. 5.7 we observe that the velocity profiles decreases with the increase in Pr. The velocity is positive in the region $0 \le \eta \le 1.58$ and become maximum at $\eta = 0.97$. After that the velocity profiles changes their directions and become negative in the region $1.58 < \eta \le 5.1$ and finally reduced to zero asymptotically except for Pr = 7.0. For Pr = 7.0, velocity profile approaches zero directly from the positive side near $\eta \approx 3.0$ without changing its direction. This particular behave for $\Pr = 7.0$ is observed may due to the constituents of the fluid.

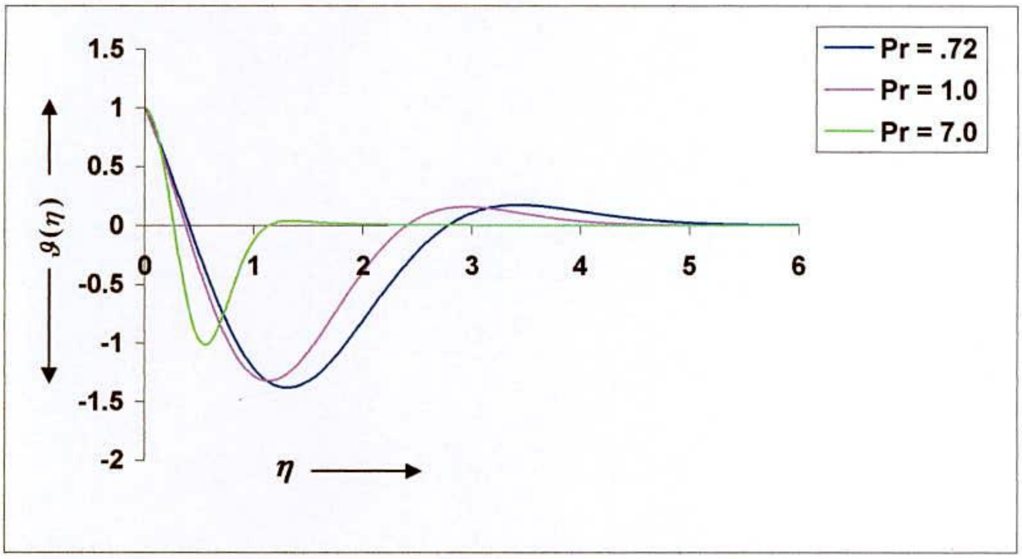

Figure 5.8: Velocity profiles for different values of Pr (with fixed values of $\frac{U_F^2}{u_0^2} = -1.3$, $\beta = 1.0$ and $f_w = -0.3$).

From the figure 5.8 we observe that the temperature profiles decreasing more with the decreasing in Pr close to the wall. For relatively higher values of η they changes there direction and become positive. Before reaching zero finally, the temperature become positive and never negative again.

The values proportional to the coefficients of skin friction ($f^{\prime\prime}(0)$) and heat transfer $\left(-\vartheta^{\prime}(0)\right)$ are tabulated in Table (5.1)–(5.4). From the table it is seen that with the increase in f_w, both the coefficients of skin friction and heat transfer increase. The coefficient of skin friction decreases whereas and coefficient of heat transfer increases with increasing $\frac{U_F^2}{u_0^2}$. Two different situations are observed for β variation. In the range of $0.967 \le \beta < 0.984$, the skin friction decrease but the coefficient of heat transfer increases whereas in the range of $0.984 \le \beta \le 1.01$ both the skin friction and heat transfer coefficients increase for the increase in β. Again both the skin friction ($f^{\prime\prime}(0)$) and heat transfer $\left(-\vartheta^{\prime}(0)\right)$ coefficients reduces with the increase in Pr. Unfortunately no experimental data is available to us to correspond our numerical results.

Table 5.1: Values proportional to the coefficients of skin friction ($f^{//}(0)$) and heat transfer ($-\vartheta^{/}(0)$) with the variation of suction parameter for fixed $\frac{U_F^2}{u_0^2}=-1.3$, $\beta=1.0$ and $\Pr=0.72$.

f_w	$f^{//}(0)$	$-\vartheta^{/}(0)$
0.34	0.64352	3.07080
0.30	0.63844	3.01001
0.00	0.545924	2.50976
- 0.30	0.63844	1.99617
- 0.50	0.446546	1.69337

Table 5.2: Values proportional to the coefficients of skin friction ($f^{//}(0)$) and heat transfer ($-\vartheta^{/}(0)$) with the variation of Buoyancy parameter $\frac{U_F^2}{u_0^2}$ for fixed $f_w=-0.3$, $\beta=1.0$ and $\Pr=0.72$.

U_F^2/u_0^2	$f^{//}(0)$	$-\vartheta^{/}(0)$
-1.1	0.438147	2.1618
-1.2	0.454657	2.0588
-1.3	0.477470	1.99617
-1.4	0.509787	1.96947

Table 5.3: Values proportional to the coefficients of skin friction ($f^{//}(0)$) and heat transfer ($-\vartheta^{/}(0)$) with the variation of driving parameter β for fixed $f_w=-0.3$, $\frac{U_F^2}{u_0^2}=-1.3$ and $\Pr=0.72$.

β	$f^{//}(0)$	$-\vartheta^{/}(0)$
0.967	- 1.568036	-1.992520
0.98	- 1.08877	-1.674788
0.983	-1.006547	-1.621930
0.984	0.268703	0.827137
0.985	0.331071	1.17268
0.99	0.411649	1.588687
1.01	0.528188	2.457179

Table 5.4: Values proportional to the coefficients of skin friction ($f''(0)$) and heat transfer ($-\vartheta'(0)$) with the variation of Prandtl's number Pr, for fixed $f_w = -0.3$, $\beta = 1.0$ and $\frac{U_F^2}{u_0^2} = -1.3$.

Pr	$f''(0)$	$-\vartheta'(0)$
0.72	0.47747	1.99617
1.00	0.422092	2.09212
7.00	0.4	0.3

CHAPTER VI

Conclusions and Recommendations

In this dissertation using the technique of similarity solutions, the governing boundary layer equations for the two-dimensional unsteady laminar combined free and forced convective flow over a semi infinite heated vertical porous plate have been analyzed in the present research works taking into account the effect of suction and blowing. Four different similarity cases arise with the choice of $\frac{dA}{d\tau}$ and $\frac{dB}{d\xi}$ either zero or constant. Similarity solutions for two of the cases, namely, Case A and Case B have been studied in this dissertation.

On the basis of the findings, it is observed for Case A that:

Case A :

(a) Velocity profiles decrease with the increase of suction but the magnitude of the velocity decreases with the increase of blowing.

(b) Temperature decrease with the increase of suction but increase with the increase of blowing.

(c) With the decrease in the buoyancy parameter $\frac{U_F^2}{u_0^2}$, the maximum velocity reduces and thus the velocity becomes zero within short range for the case of aided flow $\left(\frac{U_F^2}{u_0^2} > 0\right)$ but for the case of opposing flow $\left(\frac{U_F^2}{u_0^2} < 0\right)$ the magnitude of the velocity profiles increase with the increase of the magnitude of $\frac{U_F^2}{u_0^2}$.

(d) As the buoyancy parameter $\frac{U_F^2}{u_0^2}$ decreases, the wall receives more and more heat from the fluid.

(e) The maximum velocity decreases with the increase of β.

(f) Temperature increases with the increase of β very close to the plate surface and after that it decreases with the increase of β.

(g) Both velocity and temperature decrease rapidly with the increase of Pr.

(h) Suction decreases the coefficient of skin friction but increase the heat transfer rate while blowing increases the skin friction but decreases the heat transfer rate.

(i) Both the coefficients of skin friction and heat transfer decrease with decreasing $\frac{U_F^2}{u_0^2}$.

(j) Both the coefficients of skin friction and heat transfer reduces with reducing β and Pr.

But in the study of Case B it is observed that this case is very case sensitive relative to the values of controlling parameters and no systematic relationships of controlling parameters on the flow variables are observed here.

On the basis of findings, it is observed for Case B that;

Case B:

(a) **The** possible values of f_w are restricted to $-0.5 \le f_w \le 0.34$.

(b) The velocity profiles increase with the decreasing value of f_w in the region $0 \le \eta \le 2.53$ and increasing with the increasing of f_w in the region $2.53 \le \eta \le 5.1$ for both suction and blowing.

(c) The wall lost its temperature to the fluid and after some times it receives the temperature from the fluid.

(d) Here we observe only the opposing flow $\frac{U_F^2}{u_0^2}$ is restricted to $-1.4 \le \frac{U_F^2}{u_0^2} \le -1.1$ and no add flow found.

(e) As the buoyancy parameter decreases the wall rejects more and more heat to the fluid.

(f) Here the possible flow of β are restricted to $0.967 \le \beta \le 1.01$ the velocity profile decreased when $0.967 \le \beta \le 0.984$ and increased when $0.984 \le \beta \le 1.01$.

(g) In the region $0.967 \le \beta \le 0.984$, the temperature first increases and then decreases with the decreasing value of β.

(h) The temperature first falls sharply in the region $0.984 \le \beta \le 1.01$ and become negative in the region $0.47 \le \eta \le 2.86$.

(i) The velocity profile decreases with the increase in Pr in the region $0 \le \Pr \le 2.4$ and increase with the increasing value of Pr in the region $2.4 \le \Pr \le 5.0$ and shows the particular behave for $\Pr = 7.0$.

(j) The temperature profile decreases more with the decreasing in Pr and for the relative higher values of η they changes there directions.

(k) For $f_w \geq 0$ both the coefficient of skin friction and heat transfer increases and for $f_w < 0$
both the coefficient of skin friction and heat transfer decreases.

(l) For increasing of buoyancy parameter the coefficient of skin friction decreases and heat transfer increases.

(m) Both the coefficients of skin friction and heat transfer reduces with reducing β and increase with the reducing Pr.

Therefore, our conclusion is that, study of Case A is more suitable than Case B. Further study is necessary to solve rest of the cases.

CHAPTER VI

Appendix-A

Similarity Transformation $\rho = \rho_0 \rho^{/}$, $t = \frac{L}{U} t^{/}$

$$\therefore \frac{D\rho}{Dt} = \frac{D}{Dt}\left(\rho_0 \rho^{/}\right)$$

$$= \rho_0 \frac{Dp^{/}}{Dt}$$

$$= \rho_0 \frac{D\rho^{/}}{Dt^{/}} \frac{Dt^{/}}{Dt}$$

$$= \rho_0 \frac{D\rho^{/}}{Dt^{/}} \frac{D}{Dt}\left(\frac{L}{U} t\right)$$

$$= \frac{\rho_0 L}{U} \frac{D\rho^{/}}{Dt^{/}}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(Uu^{/})$$

$$= \frac{\partial}{\partial x^{/}}(Uu^{/}).\frac{\partial x^{/}}{\partial x}$$

$$= U \frac{\partial u^{/}}{\partial x^{/}} \frac{\partial}{\partial x}\left(\frac{x}{L}\right)$$

$$= \frac{U}{L} \frac{\partial u^{/}}{\partial x^{/}}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\mathrm{Re}^{\frac{1}{2}} U v^{/}\right)$$

$$= \frac{\partial}{\partial y^{/}}\left(\mathrm{Re}^{\frac{1}{2}} U v^{/}\right).\frac{\partial y^{/}}{\partial y}$$

$$= \mathrm{Re}^{\frac{1}{2}} U \frac{\partial v^{/}}{\partial y^{/}}.\frac{\partial}{\partial y}\left(\frac{y}{\mathrm{Re}^{\frac{1}{2}} L}\right)$$

$$= \frac{U}{L}\frac{\partial v'}{\partial y'}$$

$$\frac{Du}{Dt} = \frac{D}{Dt'}\left(Uu'\right)\frac{Dt'}{dt}$$

$$= U\frac{Du'}{Dt'}\frac{D}{Dt}\left(\frac{U}{L}t\right)$$

$$= \frac{U^2}{L}\frac{Du'}{Dt'}$$

$$\frac{Dp}{Dx} = \frac{\partial}{\partial x}\left(\rho_0 U^2 p'\right)$$

$$= \frac{\partial}{\partial x'}\left(\rho_0 U^2 p'\right)\frac{\partial x'}{\partial x}$$

$$= \rho_0 U^2 \frac{\partial p'}{\partial x'}.\frac{\partial}{\partial x}\left(\frac{x}{L}\right)$$

$$= \frac{\rho_0 U^2}{L}\frac{\partial p'}{\partial x'}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(Uu'\right)$$

$$= \frac{\partial}{\partial y'}\left(Uu'\right)\frac{\partial}{\partial y}\left(\frac{y}{\mathrm{Re}^{\frac{1}{2}}L}\right)$$

$$= \frac{U}{\mathrm{Re}^{\frac{1}{2}}L}\frac{\partial u'}{\partial y}$$

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y'}\left(\mu\frac{\partial}{\partial y}(Uu')\right)\frac{\partial y'}{\partial y}$$

or, $$\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y'}\left(\mu U\frac{\partial u'}{\partial y'}\frac{\partial y'}{\partial y}\right)\frac{\partial y'}{\partial y}$$

or, $$\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y'}\left(\frac{U}{\mathrm{Re}^{\frac{1}{2}}L}\mu'\mu_0\frac{\partial u'}{\partial y'}\right)\frac{\partial}{\partial y}\left(\frac{y}{\mathrm{Re}^{\frac{1}{2}}L}\right)$$

$$= \frac{U\mu_0}{\mathrm{Re}L^2}\frac{\partial}{\partial y'}\left(\mu'\frac{\partial u'}{\partial y'}\right)$$

$$k\frac{\partial T}{\partial y} = k\frac{\partial}{\partial y'}(\Delta T\theta)\frac{\partial y'}{\partial y}$$

$$\text{or, } k\frac{\partial T}{\partial y} = k\frac{\partial}{\partial y'}(\Delta T\theta)\frac{\partial}{\partial y}\left(\frac{y}{\mathrm{Re}^{\frac{1}{2}}L}\right)$$

$$= \frac{k\Delta T}{\mathrm{Re}^{\frac{1}{2}}L}\frac{\partial\theta}{\partial y'}$$

$$\therefore \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) = \frac{\Delta T}{\mathrm{Re}^{\frac{1}{2}}L^2}\frac{\partial}{\partial y'}\left(k'\frac{\partial\theta}{\partial y'}\right)$$

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial t'}\left(\rho_0 U^2 p'\right)\frac{\partial}{\partial t}\left(\frac{U}{L}t\right)$$

$$= \frac{\rho_0 U^3}{L}\frac{\partial p'}{\partial t'}$$

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\rho_0 U^2 p'\right)$$

$$= \frac{\partial}{\partial x'}\left(\rho_0 U^2 p'\right)\frac{\partial}{\partial x}\left(\frac{x}{L}\right)$$

$$= \frac{\rho_0 U^2}{L}\frac{\partial p'}{\partial x'}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y'}\left(Uu'\right)\frac{\partial}{\partial y}\left(\frac{y}{\mathrm{Re}^{\frac{1}{2}}L}\right)$$

$$= \frac{U}{\mathrm{Re}^{\frac{1}{2}}L}\frac{\partial u'}{\partial y'}$$

$$\frac{DT}{Dt} = \frac{D}{Dt}\left(\Delta T\theta\right)$$

$$= \Delta T\frac{D\theta}{Dt} + \theta\frac{D}{Dt}\left(\Delta T\right)$$

$$= \Delta T\frac{U}{L}\frac{D\theta}{Dt'} + \frac{U}{L}\theta\Delta T\left\{\frac{\partial}{\partial t'}\left(\log\Delta T\right) + u'\frac{\partial}{\partial x'}\left(\log\Delta T\right)\right\}$$

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