Numerical Solution of Mixed Convective Laminar Boundary Layer Flow around a Vertical Slender Body with Suction or Blowing

by

Md. Hasanuzzaman

Roll No. 1051505

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Philosophy in Mathematics

Khulna University of Engineering & Technology
Khulna-9203, Bangladesh.

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Declaration

This is to certify that the thesis work entitled "Numerical Solution of Mixed Convective Laminar Boundary Layer Flow around a Vertical Slender Body with Suction or Blowing" has been carried out by **Md. Hasanuzzaman** in the department of Mathematics, Khulna University of Engineering & Technology, Khulna-9203, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma

12.08.14

Signature of Supervisor
Name: Dr. M.M.Touhid Hossain
Designation: Professor of Mathematics
KUET

12.8.2014

Signature of Student
Name: Md. Hasanuzzaman
Designation: Student of Mathematics,
KUET

Approval

This is to certify that the thesis work submitted by **Md. Hasanuzzaman, Roll No. 1051505,** entitled "Numerical Solution of Mixed Convective Boundary Layer Flow Around a Vertical Slender Body with Suction and Blowing" has been approved by the board of examiners for the partial fulfillment of the requirements for the degree of Master of Philosophy (M. Phil.) in the Department of Mathematics, Khulna university of Engineering & Technology, Khulna, Bangladesh in August 2014.

BOARD OF EXAMINARS

1. 12.08.14
Dr. M. M. Touhid Hossain — Chairman (Supervisor)
Professor
Department of Mathematics
Khulna University of Engineering & Technology
Khulna-9203, Bangladesh.

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Dr. A.R.M. Jalal Uddin Jamali — Member
Head
Department of Mathematics
Khulna University of Engineering & Technology
Khulna-9203, Bangladesh.

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Khulna-9203, Bangladesh.

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Department of Mathematics
Khulna University of Engineering & Technology
Khulna-9203, Bangladesh.

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Dr. Wázed Ali Pramanik — Member (External)
Professor
Department of Mathematics
University of Rajshahi
Rajshahi-6205, Bangladesh.

Dedication

To my beloved Daughter

MESHKATUL JANNAT

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The author

ABSTRACT

In this dissertation, numerical solution of mixed convective laminar boundary layer flow around a vertical slender body with suction or blowing has been investigated. Firstly, the governing boundary layer partial differential equations have been made dimensionless and then simplified by using Boussinesq approximation. Secondly, similarity transformations are introduced on the basis of detailed analysis in order to transform the simplified coupled partial differential equations into a set a ordinary differential equations. The transformed complete similarity equations are solved numerically by using computer software. The flow phenomenon has been characterized with the help of obtained flow controlling parameters such as suction parameter, buoyancy parameter, Prandtl number, body-radius parameter and other driving parameters. Finally the effects of involved parameters on the velocity and temperature distributions are presented graphically. It is found that a small suction or blowing can play a significant role on the velocity and temperature fields.

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Nomenclature

x, y	Cartesian Coordinates
t	Time
F_W	Suction parameter
u	Fluid velocity in the *x* direction
v	Fluid velocity in the *y* direction
k	Thermal conductivity
M	Mach number
μ	Dynamic coefficient of viscosity
η	Similarity variable
ν	Kinematic coefficient of viscosity
ψ	Stream function
ρ	Fluid density
θ	Dimensionless temperature
γ	Boundary layer thickness
τ_W	Shearing stress at the wall
ϕ	Velocity potential
p	Pressure
Q	Universal gas constant
m	Molecular weight of the gas
$f(\eta)$	Similarity function
β	Coefficient of thermal expansion

q_W Heat transfer coefficient

u_e External velocity

L_c Characteristics length

T_e Ambient temperature

g_x x Component body force

R_e Reynolds number

C_p Specific heat

G_r Grashop number

Fr Froude number

$\Pr$ Paradtl number

E_c Eckert number

$\frac{U_F^2}{u_e^2}$ Buoyancy parameter

R_0 Body-radius parameter

CHAPTER I

Introduction and Literature Review

Fluid dynamics is a subject of widespread interest to researcher and it become an obvious challenge for the scientists, engineers as well as users to understand more about fluid motion. An important contribution to the fluid dynamics is the concept of boundary layer flow introduced first by L. Prandtl [50]. The concept of the boundary layer is the consequence of the fact that flows at high Reynolds numbers can be divided into unequally spaced regions. A very thin layer (called boundary layer) in the vicinity (of the object) in which the viscous effects dominate, must be taken into account, and for the bulk of the flow region, the viscosity can be neglected and the flow corresponds to the in viscid outer flow. Although the boundary layer is very thin, it plays a vital role in the fluid dynamics. Boundary layer theory has become an essential study now-a-days in analyzing the complex behaviors of real fluids. The concept of boundary layer can be used to simplify the Navier-Stocks' equations to such an extent that the viscous effects of flow parameters are evaluated, and these are useable in many practical problems (viz the drag on ships and missiles, the efficiency of compressors and turbines in jet engines, the effectiveness of air intakes for ram and turbojets and so on).

Further the boundary layer effects caused by free convection are frequently observed in our environmental happenings and engineering devices. We know that if externally induced flow is provided and flows arising naturally solely due to the effect of the differences in density caused by temperature or concentration differences in the body force field (such as gravitational field), this type of flow is called `free convection' or 'natural convection' flow. The density difference causes buoyancy effects and these effects act as 'driving forces' due to which the flow is generated. Hence free convection is the process of heat transfer which occurs due to movement of the fluid particles by density differences associated with temperature differences in a fluid. In such case, the free stream velocity falls away, in deed, no reference velocity does a priori exist. If the density in the vicinity of the object is kept constant, natural convection flow can not be formed. Thus, the natural convection is an effect of variable properties, where there is a mutual coupling between momentum and heat transport. The direct origin of the formation of natural convection flows is a heat transfer via conduction through the fixed surfaces surrounding the fluid. If the surface temperature is greater than that of ambient fluid, heat is transferred from the plate to the fluid leads to an increase in temperature of the fluid close the surfaces and to a change in the density, because it is temperature dependent. If the density decreases with increasing temperature, buoyancy forces arise close to the surface and warmer fluid moves upwards. Such buoyancy forces are proportional to the coefficient of thermal expansion β_T, defined as $\beta_T = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{p=\text{constant}}$,where ρ, T and p are density, temperature and pressure respectively. It is observed that $\beta_T = \frac{1}{T}$ for a perfect gas, and we see that stream is well approximated by the perfect-gas result $\beta_T T = 1$ at low pressure and high temperature. Also $\beta_T < \frac{1}{T}$ for a liquid and may even be negative, and $\beta_T > \frac{1}{T}$ for imperfect gas, particularly at high pressure. β_T is also useful in estimating the dependence of enthalpy 'h' on pressure, from the thermodynamic relation $dh = c_p dT + (1-\beta_T T)\frac{dp}{\rho}$, where T is the absolute temperature. For the perfect gas, the second term vanishes, so that $h = h(T)$ only. The natural convection studies begun in the year 1881 with Lorentz and continued at a relatively constant rate until recently. This mode of heat transfer occurs very commonly, the cooling of transmission lines, electric transformers and rectifiers, the heating of rooms by use of radiators, the heat transfer from hot pipes and ovens surrounded by cooled air, cooling the reactor core (in nuclear power plant) and carry out the heat generated by nuclear fission etc. The Mixed convection flows, combined forced and free convection flows, arise in many transport processes in engineering devices and in nature. This follows are characterized by the buoyancy parameter (measure of the influence of the free convection in comparison with that of forced convection on the fluid flow) which depends on the flow configuration and the surface heating conditions. Bulks of information are now available in literature about the boundary layer form of natural convection flows over bodies of different shapes. The theoretical, experimental and numerical analysis for the natural and the mixed convection boundary layer flow about isothermal, vertical porous flat plates have been carried out widely by many authors (viz. [13, 38, 40, 45, 57, 60, 63, 64]) in view of its applications in many engineering and geographical problems. Ramanaiah et al. [52] considered the problem of mixed convection over a horizontal plate subjected to a temperature or surface heat flux varying as a power of x.

Schmidt [56] was apparently the first researcher who investigated experimentally the behavior of the flow near the leading edge above a flat horizontal surface. The theoretical analysis of the laminar, two-dimensional, steady natural convection boundary layer flow on a semi-infinite horizontal flat plate was first considered by Stewartson [61] (later corrected by Gill, Zeh and Del-Casal [17]). In that analysis he used the Buossinesq approximation to show how the boundary layer analysis could be incorporated with the natural convection on rectangular plates, which are of high plane form aspect ratio.

Rotem and claassen [54] investigated the boundary layer equation over a semi-infinite horizontal surface of uniform temperature and results were presented for some specific values of Prandtl number with its limits from zero to infinity. The effect of buoyancy forces that exist in boundary layer flow, over a horizontal surface, where the surface temperature differs from that of ambient fluid, was studied by Sparrow and Minkowycz [59]. The free convection above a heated and almost horizontal plate has been treated by Jones [31].

The problem of mixed convection due to a heated or cooled vertical flat plate provides one of the most basic scenarios for heat transfer theory and thus is of considerable theoretical and practical interest and has been extensively studied by Sparrow et al. [65], Wilks [68], Afzal and Banthiya [5], Hunt and Wilks [24], Lin and Chen [35], Hussain and Afzal [27], Merkin et al. [42] etc. However, the problem of forced, free and mixed convection flows past a heated or cooled body with porous wall is of interest in relation to the boundary layer control on airfoil, lubrication of ceramic machine parts and food processing. Watanabe [69] has considered the mixed convection boundary layer flow past an isothermal vertical porous flat plate with uniform suction or injection. Satter [62] made analytical studies on the combined forced and free convection flow in a porous medium. Further, a vast literature of similarity solution has appeared in the area of fluid mechanics, heat transfer, and mass transfer, etc. as it is one of the important means for the reduction of a number of independent variable with simplifying assumptions. It is revealed that the similarity solution, which being attained for some suitable values of different parameters, might be thought of being the solution of the convective boundary-layer context either near the leading edge or far away in the downstream. Deswita et al. [13] obtained a similarity solution for the steady laminar free convection boundary layer flow on a horizontal plate with variable wall temperature.

The boundary layer type of the natural convection flow, which occurs on the upper surface of heated horizontal surface has been investigated theoretically and experimentally by among other, Rotem and Claassen [55], Pera and Gebhart [47 , 48] and Goldstein, et al [18]. It is seen from their experiments and also from the flow visualization of Husra and Sparrow [23] that a boundary layer starts from each edge of a plate edge, each boundary layer having its leading at a straight-side plate edge. The boundary layer development occurs normal to the corresponding edge so that collisions between opposing boundary layer flows occur on the plate surface. After collision, the fluid cheeked in the boundary layer forms a rising buoyant plume.

Furthermore, the study of complete similarity solutions of the unsteady laminar natural convection boundary layer flow about a heated horizontal semi-infinite porous plate have been considered by Hossain et al. [25, 26] even the solution of a system of coupled partial differential equations with boundary conditions is often difficult and even impossible to solve with the usual classical method. Thus, it is imperative to reduce the number of variables from the system which reached in a stage of great extent. Similarity solution is one of the important means for the reduction of a number of independent variables with simplifying assumptions and finally the system of partial differential equations reduces to a set of ordinary differential equations successfully. The similarity solutions in the context of mixed convection boundary layer flow of steady viscous incompressible fluid over an impermeable vertical flat plate were discussed by Ishak et al. [29]. Ramanaiah G and Malarvizhi G. [53] studied the similarity solutions of free, mixed and forced convection problems in a saturated porous media.

In 1978, Johnson and Cheng [30] examined the necessary and sufficient conditions under which similarity solutions exist for free convection boundary layers adjacent to flat plates in porous media. The solutions obtained in their work were more general than those appearing in the previous studies. With a parameter associated with the body shapes a similarity solution on the natural convection flow has also been studied by Pop and Takhar [49]. Ferdows et al. [15] have been made a similarity analysis for the forced as well as free convection boundary layer flow of an electrically conducting viscous incompressible fluid past a semi-infinite non-conducting vertical porous plate by introducing a time dependent suction.

Most of the above analysis were based on the Buossinesq approximation and have been concerned with the seeking of similarity solutions in which the plate temperature varies with the distance from plate leading edge. In this approximation thus density, viscosity, thermal conductivity and specific heat variations are ignored except for the necessary inclusion of the density-variation in the body force term.

An analysis is performed by Chen et al. [9] to study the flow and heat transfer characteristic of laminar natural convection in boundary layer flows from horizontal, inclined and vertical plates with power law variation of the wall temperature.

In most of the above analysis the boundary layer of the natural convection flows were considered over heated or uniformly heated horizontal vertical, horizontal or near horizontal, semi-infinite, rectangular porous plates. The surface is impermeable to the fluid, so that there is no transpiration i.e., suction or blowing velocity normal to the surface. This led to the kinematic boundary condition $v_w = 0$.

The problem of boundary layer control has become very important factor; in actual application it is often necessary to prevent separation. The separation of the boundary layer is generally undesirable, since separated flow causes a great increase in the drag experienced by the body. So it is often necessary to prevent separation in order to reduce pressure drag.

Suction (or blowing) is one of the useful means in preventing boundary layer separation. The effect of suction consists in the removal of decelerated particles from the boundary layer before they are given a chance to cause separation. The surface is considered to be permeable to the fluid, so that the surface will allow a non-zero normal velocity and fluid is either sucked or blown through it. In doing this however, no-slip condition $u_w = 0$ at the surface (non-moving) shall continue to remain valid.

In driving the boundary layer equation, it is anticipated that the v-component of the velocity is small quantity of the order of magnitude $O\left(\mathrm{Re}^{\frac{1}{2}}\right)$ and it is assumed that the suction (or blowing) velocity $v_w = 0$ normal to the surface has its magnitude of order (characteristic Reynolds number)$^{-1/2}$. The consequence of this is that outer flow is independent of v_w and the boundary condition at the surface is given by $y = 0\,;\, u = 0,\ v = v_w(x)$.

Suction or blowing causes double effects with respect to the heat transfer. On the one hand, the temperature profile is influenced by the changed velocity field in the boundary-layer, leading to a change in the heat conduction at the surface. On the other hand, convective heat transfer occurs at the surface along with the heat conduction for $v_W \neq 0$. A summary of flow separation and its control can be found in Chang [6, 7].

The study of natural convection on a horizontal plate with suction and blowing is of huge interest in many engineering applications, for instance, transpiration cooling, boundary layer control and other diffusion operations. The effects of blowing and suction on forced or free convection flow over vertical as well as horizontal plates were analyzed in a systematic way by Gortler [19], Sparrow and Cess [58], Koh and Hartnett [32], Gersten and Gross [16], Merkin [37, 39], Vedhanaygarm, Altenkirch and Eichhorn [66], Hasio-Tsung and Wen-Shing [22], Merkin [41] and Acharya, Shing and Dash [1] etc.

Using the usual asymptotic approach, the similar solutions of the steady natural convection boundary layer for a non-similar flow situation on a horizontal plate with large suction approximation has been developed by Afzal and Hussain [3]. A detailed study on similarity solutions for free convection boundary layer flow over a permeable wall in a fluid saturated porous media was carried out by Chaudhary et al. [8]. They have shown that the system depends on the power law exponent and the dimensionless surface mass transfer rate. They also examined the range of exponent under which the solution exists. With constant plate temperature and particular distribution of blowing rate Clarke and Riley [11] obtained a special case of similarity solution, allowing variable fluid density. But there is still a shortage of accurate data for a wide range of both suction and blowing rate. Lin and Yu [34] presented a non-similar solution for the laminar free convection flow over a semi-infinite heated upward-facing horizontal porous plate with suitable transpiration rate as a power-law variation. Emphasis was given for an isothermal plate under the condition of uniform blowing or suction. Lately, using a parameter concerned pseudo-similarity technique of time and position coordinates, Cheng and Huang [10] studied the unsteady laminar boundary layer flow and heat transfer in the presence and absence of heat source or sink on a continuous moving and stretching isothermal surface with suction and blowing. In their analysis they paid attention on the temporal developments of the hydrodynamic and thermal characteristics after the sudden simultaneous changes in momentum and heat transfer. Recently, an analysis is performed by Aydin, O. and Kayato, A. [4] for the laminar boundary layer flow over a porous horizontal flat plate, particularly, to study the effect of uniform suction/injection on the heat transfer. Using the constant surface temperature as thermal boundary condition they also investigated the effect of Prandtl number on heat transfer.

Recently, Hossain and Mojumder [21] presented the similarity solution for the steady laminar free convection boundary layer flow generated above a heated horizontal rectangular surface. They investigated the effect of suction and blowing on fluid flow and heat transfer as well as skin friction coefficients. They also found that suction increased skin-friction and heat transfer coefficients whereas injection caused a decrease in both.

Hossain *et al.* [28] obtained a complete similarity solution of the unsteady laminar combined free and forced convection boundary layer flow about a heated vertical porous plate in viscous incompressible fluid and investigated the effects of several involved parameters on the velocity and temperature fields and other flow parameters like skin friction, heat transfer coefficients across the boundary layer. The combined free and force convective laminar fluid motion caused by a heated (or cooled) vertical slender body moving through a viscous fluid has not so far been considered for a large scale study. Van Dyke [67] successfully analyzed a natural convection flow near a vertical thin needle for the case of a constant surface temperature. Kuiken [33] has studied the axi-symmetric free convective boundary layer along an isothermal vertical cylinder of constant thickness.

The problem of forced laminar flow over thin needles, such that the boundary layer thickness is comparable to the local needle thickness, has been investigated by Lee [36] and Narain and

Uberoi [44]. The combined free and force convective laminar fluid flow for the steady case has been studied also by Narain and Uberoi [43] for slender needles for the cases of isothermal wall and uniform wall heat flux. Furthermore, no attention has been paid to the corresponding unsteady problems of needle flows which may have applications in the field of aeronautics, atomic power, chemical engineering and electrical engineering etc.

The purpose of the present study is, therefore, to find a possible similarity solution of the combined free and force convective laminar fluid motion caused by a heated (or cooled) axi-symmetric slender body of finite axial length immersed vertically in a viscous incompressible fluid. The thermal distributions on the outer surface of the body as well as the motion of the body itself are assumed to be unsteady. Furthermore, throughout the investigation, the effect of suction or injection has been taken into consideration. We are attempted to investigate the effects of several involved parameters on the velocity and temperature fields and other flow parameters like skin friction, heat transfer coefficients across the boundary layer. We are also tried to calculate the role of suction or injection velocity on these parameters as well.

In attacking this problem the equations expressing conservation of mass, momentum and energy will be formulated in a manner which readily admits the variation of thermodynamic and transport properties of fluid with temperature and pressure. The governing non-dimensional boundary layer partial differential equations are simplified first based on the Boussinesqu approximation. The similarity transformations are then introduced on the basis of detailed analysis in order to transform the simplified coupled partial differential equations into a set of ordinary differential equations. The transformed complete similarity equations are then solved numerically by using computer software. The flow phenomenon has been characterized with the help of obtained flow controlling parameters such as suction parameter, buoyancy parameter, Prandtl number and the other driving parameter.

The numerical solutions including the velocity and temperature fields are to be presented for different selected values of the appeared dimensionless parameters. The influences of these various parameters on the velocity and temperature profiles will be exhibited in the present analysis. It may be expected that the effects of suction and blowing can play an important role on the velocity and temperature fields, so that their effects should be taken into account with other useful parameters associated.

Here we adopt the method of classical 'separation of variables' which is of the simplest and most straightforward method of determining similarity solutions. This method was first initiated by Abbott and Kline (1960). In this method, a form of similarity variable is chosen, the given PDE is changed under the selected co-ordinate system. The dependent variables are to be expressed in terms of the product of separable functions of the new independent variables where each function is dependent on the single variable. Substitution of the product from of the dependent variables in to the original PDE generally leads to an equation in which no functions of single variable can be isolated on the two sides of the equation unless certain parameters are to be specified. Usually, these parameters can be specified quite readily and "separation of the variables" is achieved. On this way the separation proceeds until the one side becomes an ODE. Four different similarity cases arise here, viz. Case A, Case B, Case C and Case D, on the basis of our assumptions.

This thesis is composed of Five Chapters. An introduction of basic principles of boundary layer theory, natural convection flows, suction and blowing phenomena with historical review of earlier researches and background of our problem are presented in CHAPTER I.

Basic equations governing the problem, dimensional analysis with simplifying assumptions and similarity transformations with possible similarity case are given in CHAPTER II. In CHAPTER III, a detailed discussion of one of the four similarity case, namely, Case A has given. Under the considered condition, the numerical solutions with graphs and tables have also been given there for some selected values of the appeared parameters. The effects of these parameters on several variables will also be exhibited in the analysis. CHAPTER IV is concerned with the study of another similarity case (Case B). The numerical solutions with the graph and tables for this case are also displayed there. We also have predicted the role of small suction or blowing velocity on these parameters concerned.

In CHAPTER V, the conclusions gained from this work and brief descriptions for further works related to our present researches are discussed.

CHAPTER-II

Mathematical Formulations

2.1 Basic Equations

An axi-symmetric heated (or cooled) slender body of finite axial length is immersed vertically in a viscous fluid of variable properties. The surface temperature ($=T_w$), the velocity and the temperature of the undisturbed fluid (u_e and T_e) close to the body surface but outside the boundary-layer are all general functions of x and t. r_w is the radial distance from the axis of symmetry to the surface of the body, x is the distance measured along the axis of symmetry of the body and t is the time. The physical configuration and the coordinate system of the problem are shown in Fig.2.1.

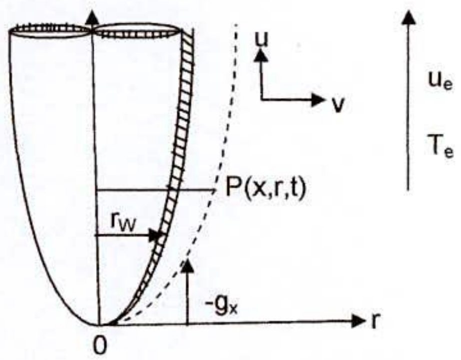

Fig: 2.1 physical configuration and coordinate system

The influence of body force generated by buoyancy effects on the flow field near the surface is significant if the Froude number of such a flow field is of order unity. That is the non dimensional form of the buoyancy force is

$$\frac{T_w - T_e}{T_e}\frac{g_x L_c}{U^2} \cong 0(1) \tag{2.1}$$

where g_x is the gravity component in the x-direction, L_c is a suitable characteristic length and U is a suitable characteristic velocity. In attacking this problem the equations expressing conservation of mass, momentum and energy will be formulated in a manner which readily admits the variation of thermo-dynamic and transport properties of the fluid with temperature and pressure.

If u, v are the components of the velocity in the x and r directions respectively then the equation governing the motion of the fluid as shown in Fig. 2.1 may be written as follows:

$$r\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho ru)+\frac{\partial}{\partial r}(\rho rv)=0 \tag{2.2}$$

$$\begin{aligned}\upsilon\frac{Du}{Dt}=\rho g_x-\frac{\partial p}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}\left[\mu r\left(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial x}\right)\right]\\+\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}+\lambda\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)+\frac{\partial u}{\partial x}\right)\right]\end{aligned} \tag{2.3}$$

$$\begin{aligned}\rho\frac{Dv}{Dt}=-\frac{\partial p}{\partial r}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial r}\right)\right]+\frac{2\mu}{r}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)\\+\frac{\partial}{\partial r}\left[2\mu\frac{\partial v}{\partial r}+\lambda\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)+\frac{\partial v}{\partial x}\right)\right]\end{aligned} \tag{2.4}$$

$$\begin{aligned}\rho C_P\frac{DT}{Dt}=\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)\\+T\beta_T\left\{u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial r}\right\}+\varphi\end{aligned} \tag{2.5}$$

where

$$\varphi=\mu\left[2\left\{\left(\frac{\partial v}{\partial r}\right)^2+\left(\frac{v}{r}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial r}\right)^2\right\}\right]+\lambda\left[\frac{\partial v}{\partial r}+\frac{v}{r}+\frac{\partial u}{\partial x}\right]^2$$

and $$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial r}$$

Also ρ is the density, p is the pressure, μ and λ are the dynamic and second coefficients of viscosity, T the temperature, k the thermal conductivity C_p the specific heat at constant pressure and $\beta_T=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_P$ is the coefficient of thermal expansion of the fluid. As usual in the boundary-layer approximation, the equations (2.2)-(2.5) are reduced to (on the basis of non-dimensionalisation in boundary-layer theory)

$$r\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho r u)+\frac{\partial}{\partial r}(\rho r v)=0 \tag{2.6}$$

$$\rho\frac{Du}{Dt}=\rho g_x-\frac{\partial p}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial u}{\partial r}\right) \tag{2.7}$$

$$0=\frac{\partial p}{\partial r} \tag{2.8}$$

$$\rho C_p\frac{DT}{Dt}=\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) \tag{2.9}$$

In their non-dimensional form, the boundary-layer momentum equations (2.7) and (2.8) ignore terms of order Re^{-1}, whilst the energy equation (2.9) ignores terms of order Re^{-1} and E (for the stress work terms). Here $R_e=\frac{UL_c}{\nu_0}$, $E=\frac{U^2}{C_p(T_w-T_e)}$ are the Reynolds and Eckert numbers of the flow respectively. We are concerned with those types of boundary-layer flows where $Re\rightarrow\infty$ and $E<<1$.

Imposing the boundary conditions at the outer edge of the boundary layer (as $\rho\rightarrow\rho_e$, $u\rightarrow u_e$, $T\rightarrow T_e$, $p\rightarrow p_e$ and $\frac{\partial p}{\partial r}\rightarrow 0$) we get from the equations (2.7)-(2.9)

$$\rho_e\frac{\partial u_e}{\partial t}+\rho_e u_e\frac{\partial u_e}{\partial x}=\rho_e g_x-\frac{\partial p_e}{\partial x} \tag{2.10}$$

$$p=p(x,t) \tag{2.11}$$

$$\frac{\partial T_e}{\partial t}+u_e\frac{\partial T_e}{\partial x}=0 \tag{2.12}$$

Here subscript 'e' refers to conditions in the outer or external flow. The values with the subscript are the values to which the solutions in the boundary-layer must be matched as the boundary-layer co-ordinate normal to the surface tends to infinity. In general u_e , T_e , ρ_e are functions of x and t. However, equation (2.12) has the solution $T_e=$ constant for a given fluid element. In what follows it will be assumed that

T_e = constant throughout the flow field. By virtue of (2.6)-(2.9) and (2.10)-(2.12), the boundary-layer equations which are to be studied in this chapter may be written in the following forms:

$$r\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho r u)+\frac{\partial}{\partial r}(\rho r v)=0 \tag{2.13}$$

$$\rho\frac{Du}{Dt}=(\rho-\rho_e)g_x+\rho_e\left\{\frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial x}\right\}+\frac{1}{r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial u}{\partial r}\right) \tag{2.14}$$

$$\rho C_p\frac{DT}{Dt}=\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) \tag{2.15}$$

2.2. Transformations

The equations (2.13)-(2.15) take the following forms for a Boussinesq fluid:

$$\frac{\partial}{\partial x}(ru)+\frac{\partial}{\partial r}(rv)=0 \tag{2.16}$$

$$\frac{Du}{Dt}=-g_x\beta_T\Delta T\theta+\frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial x}+\frac{v}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{2.17}$$

$$\frac{D\theta}{Dt}=-\left\{u\frac{\partial}{\partial x}(\log\Delta T)+\frac{\partial}{\partial t}(\log\Delta T)\right\}\theta+\frac{1}{\Pr}\frac{v}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right) \tag{2.18}$$

where $v=\dfrac{\mu}{\rho}$, $\Pr=\dfrac{\mu C_p}{k}$, $T-T_0=\Delta T_0$, $\Delta T=T_W-T_0$ and $T_e=T_0$ is treated here as constant temperature for the ambient fluid. Since the Boussinesq from of the state relations is $\rho=\rho(T)$, it follows that $\rho_e=\rho_0$ (constant). T and T_W in general depend on both x and t. A solution of the equations (2.16)-(2.18) is now sought, these equations being valid in the limit $Re\rightarrow\infty$ and $E\rightarrow 0$. Higher order effects are not discussed here, as the present investigation concerns the first order boundary- layer approximations only.

The complexity of the above governing differential equations makes the use of simplifying approximations desirable so that tractable solutions may be obtained. The method of similarity provides a convenient and accurate procedure for computing heat transfer, skin friction and other laminar boundary-layer characteristics. Guided by this idea independent variables *(x,r,t)* are changed to a new set of variables $(\xi, \phi\ , \tau)$ where the relations between the two sets are

$$\xi = x, \qquad \tau = t, \qquad \phi = \frac{r^2}{2\gamma(x,\,t)} \tag{2.19}$$

Here $\gamma(x,t)$is thought to be proportional to the square of the local boundary-layer thickness. This definition of ϕ arises purely for reasons of convenience in using the axi-symmetric from of the equations. From (2.19), we get

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \frac{\phi}{\gamma}\gamma_\tau \frac{\partial}{\partial \phi} \tag{2.20}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{\phi}{\gamma}\gamma_\xi \frac{\partial}{\partial \phi} \tag{2.21}$$

$$\frac{\partial}{\partial r} = \frac{r}{\gamma}\frac{\partial}{\partial \phi} \tag{2.22}$$

Here suffices denote partial differentiation with respect to corresponding arguments. The continuity equation (2.16) is identically satisfied by introducing a stream function $\psi(x,\,r,\,t)$ defined by

$$ru = \frac{\partial \psi}{\partial r} \tag{2.23}$$

$$-rv = \frac{\partial \psi}{\partial x} \tag{2.24}$$

By virtue of (2.22) and (2.23) we have

$$u = 2\frac{\partial \psi}{\partial (r^2)} = \frac{\partial}{\partial \phi}\left\{\frac{\psi}{\gamma(x,t)}\right\} \tag{2.25}$$

Using, for the moment, a non-dimensional scaling factor *U (x, t)* for the velocity component *u* we can write

$$\frac{u}{U} = \frac{\partial}{\partial \phi}\left\{\frac{\psi}{\gamma U}\right\}$$

$$\text{or } \psi(\xi, \phi, \tau) - \psi(\xi, \phi_0, \tau) = \gamma\, U \int_{\phi_0}^{\phi} \frac{u}{U} d\phi$$

$$= \gamma\, U\, F(\xi, \phi, \tau) \quad \text{(say)} \tag{2.26}$$

where $F(\xi, \phi, \tau) = \int_{\phi_0}^{\phi} \frac{u}{U} d\phi$ and ϕ_0 is the value of ϕ on the body. That is

$$\phi_0 = \frac{r_W^2}{2\gamma(x\,,\,t)} \tag{2.27}$$

In view of (2.21), (2.24) and (2.26) we can write

$$\begin{aligned}
-rv &= \left(\frac{\partial}{\partial \xi} - \frac{\phi}{\gamma}\gamma_\xi \frac{\partial}{\partial \phi}\right)\psi(\xi, \phi, \tau) \\
&= \frac{\partial \psi(\xi, \phi, \tau)}{\partial \xi} - \frac{\phi}{\gamma}\gamma_\xi \frac{\partial \psi(\xi, \phi, \tau)}{\partial \phi} \\
&= \frac{\partial}{\partial \xi}\{\gamma\, U\, F(\xi, \phi, \tau) + \psi(\xi, \phi_0, \tau)\} - \frac{\phi}{\gamma}\gamma_\xi \frac{\partial}{\partial \phi}\{\gamma\, U\, F(\xi, \phi, \tau) + \psi(\xi, \phi_0, \tau)\}
\end{aligned}$$

$$\text{or,} -rv = (\gamma U\, F)_\xi - \phi\, \gamma_\xi\, U\, F_\phi - r_W v_W \tag{2.28}$$

where
$$-r_W v_W = \psi_\xi(\xi, \phi_0, \tau) \tag{2.29}$$

We assume that the surface of the body is porous, therefore $v_W \neq 0$ represents the suction or blowing effects and since $r_w = r_w\,(x)$ only we take $\psi_\xi(\xi, \phi_0, \tau) \neq 0$. In this situation the convective operator $\frac{D}{Dt}$ becomes

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial r}$$

$$= \frac{\partial}{\partial \tau} - \frac{\phi}{\gamma}\gamma_\tau \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \phi}\left(\frac{\psi}{\gamma}\right)\left(\frac{\partial}{\partial \xi} - \frac{\phi}{\gamma}\gamma_\xi \frac{\partial}{\partial \phi}\right) - \frac{1}{r}\left\{(\gamma U\, F)_\xi - \phi \gamma_\xi U F_\phi - r_W v_W\right\}\frac{r}{\gamma}\frac{\partial}{\partial \phi}$$

$$= \frac{\partial}{\partial \tau} - \frac{\phi}{\gamma}\gamma_\tau \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \phi}\left\{\frac{\gamma UF - \psi(\xi,\phi_0,\tau)}{\gamma}\right\}\left(\frac{\partial}{\partial \xi} - \frac{\phi}{\gamma}\gamma_\xi \frac{\partial}{\partial \phi}\right) - \frac{(\gamma U F)_\xi}{\gamma}\frac{\partial}{\partial \phi}$$
$$+ \frac{1}{\gamma}\phi\gamma_\xi UF_\phi \frac{\partial}{\partial \phi} + \frac{r_W v_W}{\gamma}\frac{\partial}{\partial \phi}$$

$$= \frac{\partial}{\partial \tau} - \frac{\phi}{\gamma}\gamma_\tau \frac{\partial}{\partial \phi} + UF_\phi \frac{\partial}{\partial \xi} - \frac{1}{\gamma}\phi\gamma_\xi UF_\phi \frac{\partial}{\partial \phi} - \frac{(\gamma U F)_\xi}{\gamma}\frac{\partial}{\partial \phi}$$
$$+ \frac{1}{\gamma}\phi\gamma_\xi UF_\phi \frac{\partial}{\partial \phi} + \frac{r_W v_W}{\gamma}\frac{\partial}{\partial \phi}$$

$$= \frac{\partial}{\partial \tau} - \left\{\frac{\phi}{\gamma}\gamma_\tau + \frac{(\gamma UF)_\xi}{\gamma} - \frac{r_W V_W}{\gamma}\right\}\frac{\partial}{\partial \phi} + UF_\phi \frac{\partial}{\partial \xi} \tag{2.30}$$

In attempting separation of variables for $F(\xi,\phi,\tau)$ and $\theta(\xi,\phi,\tau)$ we write

$$F(\xi,\phi,\tau) = L(\xi,\tau)\tilde{f}(\phi) , \qquad \theta(\xi,\phi,\tau) = m(\xi,\tau)\vartheta(\phi) \tag{2.31}$$

Since $\theta(\xi,\phi_0,\tau) = 1$,

We may put without loss of generality

$$m(\xi,\tau) = 1 \text{ Or } \theta(\phi_0) = 1 \tag{2.32}$$

Now,

$$\frac{v}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{v}{r}.\frac{r}{\gamma}\frac{\partial}{\partial \phi}\left[r\frac{r}{\gamma}\frac{\partial}{\partial \phi}\left\{\frac{\partial}{\partial \phi}\left(\frac{\psi}{\gamma}\right)\right\}\right]$$
$$= \frac{v}{\gamma}\frac{\partial}{\partial \phi}\left[2\phi\frac{\partial}{\partial \phi}\left\{\frac{\partial}{\partial \phi}\left(\frac{\gamma UF}{\gamma}\right)\right\}\right]$$
$$= \frac{v}{\gamma}\frac{\partial}{\partial \phi}\left[2\phi\frac{\partial}{\partial \phi}\left\{\frac{\partial}{\partial \phi}\left(UL\tilde{f}\right)\right\}\right]$$
$$= \frac{2v}{\gamma}UL\frac{\partial}{\partial \phi}\left[\phi\frac{\partial}{\partial \phi}\left\{\frac{\partial \tilde{f}}{\partial \phi}\right\}\right]$$
$$= \frac{2v}{\gamma}UL\frac{\partial}{\partial \phi}\left[\phi\frac{\partial}{\partial \phi}\left(\tilde{f}_\phi\right)\right]$$
$$= \frac{2v}{\gamma}UL\frac{\partial}{\partial \phi}\left[\phi\,\tilde{f}_{\phi\phi}\right]$$

$$=\frac{2\nu}{\gamma}\phi UL\tilde{f}_{\phi\phi\phi}+\frac{2\nu}{\gamma}UL\,\tilde{f}_{\phi\phi} \tag{2.33}$$

$$\frac{\partial u_e}{\partial t}=\left(\frac{\partial}{\partial\tau}-\frac{\phi}{\gamma}\gamma_\tau\frac{\partial}{\partial\phi}\right)u_e$$

$$=\frac{\partial u_e}{\partial\tau}-\frac{\phi}{\gamma}\gamma_\tau\frac{\partial u_e}{\partial\phi} \tag{2.34}$$

$$u_e\frac{\partial u_e}{\partial x}=u_e\left(\frac{\partial}{\partial\xi}-\frac{\phi}{\gamma}\gamma_\xi\frac{\partial}{\partial\phi}\right)u_e$$

$$=u_e\frac{\partial u_e}{\partial\xi}-u_e\frac{\phi}{\gamma}\gamma_\xi\frac{\partial u_e}{\partial\phi} \tag{2.35}$$

By virtue of the above equations from (2.19) to (2.35) the momentum equation (2.17) reduce to

$$\left(\frac{\partial}{\partial\tau}-\left\{\frac{\phi}{\gamma}\gamma_\tau+\frac{(\gamma UF)_\xi}{\gamma}-\frac{r_W V_W}{\gamma}\right\}\frac{\partial}{\partial\phi}+UF_\phi\frac{\partial}{\partial\xi}\right)\frac{\partial}{\partial\phi}\left(\frac{\psi}{\gamma}\right)=-g_x\beta_T\Delta T\theta+\frac{\partial u_e}{\partial\tau}-\frac{\phi}{\gamma}\gamma_\tau\frac{\partial u_e}{\partial\phi}$$
$$+u_e\frac{\partial u_e}{\partial\xi}-u_e\frac{\phi}{\gamma}\gamma_\xi\frac{\partial u_e}{\partial\phi}+\frac{2\nu}{\gamma}\phi UL\tilde{f}_{\phi\phi\phi}+\frac{2\nu}{\gamma}UL\,\tilde{f}_{\phi\phi}$$

$$\Rightarrow\left(\frac{\partial}{\partial\tau}-\left\{\frac{\phi}{\gamma}\gamma_\tau+\frac{(\gamma UF)_\xi}{\gamma}-\frac{r_W V_W}{\gamma}\right\}\frac{\partial}{\partial\phi}+UF_\phi\frac{\partial}{\partial\xi}\right)\frac{\partial}{\partial\phi}\left(\frac{\gamma UF+\psi(\xi,\phi_0,\tau)}{\gamma}\right)=-g_x\beta_T\Delta T\theta$$
$$+\frac{\partial u_e}{\partial\tau}-\frac{\phi}{\gamma}\gamma_\tau\frac{\partial u_e}{\partial\phi}+u_e\frac{\partial u_e}{\partial\xi}-u_e\frac{\phi}{\gamma}\gamma_\xi\frac{\partial u_e}{\partial\phi}+\frac{2\nu}{\gamma}\phi UL\tilde{f}_{\phi\phi\phi}+\frac{2\nu}{\gamma}UL\,\tilde{f}_{\phi\phi}$$

$$\Rightarrow\left(\frac{\partial}{\partial\tau}-\left\{\frac{\phi}{\gamma}\gamma_\tau+\frac{(\gamma UF)_\xi}{\gamma}-\frac{r_W V_W}{\gamma}\right\}\frac{\partial}{\partial\phi}+UF_\phi\frac{\partial}{\partial\xi}\right)\frac{\partial}{\partial\phi}\left(UL\tilde{f}\right)=-g_x\beta_T\Delta T\theta$$
$$+\frac{\partial u_e}{\partial\tau}-\frac{\phi}{\gamma}\gamma_\tau\frac{\partial u_e}{\partial\phi}+u_e\frac{\partial u_e}{\partial\xi}-u_e\frac{\phi}{\gamma}\gamma_\xi\frac{\partial u_e}{\partial\phi}+\frac{2\nu}{\gamma}\phi UL\tilde{f}_{\phi\phi\phi}+\frac{2\nu}{\gamma}UL\,\tilde{f}_{\phi\phi}$$

$$\Rightarrow\left(\frac{\partial}{\partial\tau}-\left\{\frac{\phi}{\gamma}\gamma_\tau+\frac{(\gamma UF)_\xi}{\gamma}-\frac{r_W V_W}{\gamma}\right\}\frac{\partial}{\partial\phi}+UF_\phi\frac{\partial}{\partial\xi}\right)\left(UL\tilde{f}_\phi\right)=-g_x\beta_T\Delta T\theta$$
$$+\frac{\partial u_e}{\partial\tau}-\frac{\phi}{\gamma}\gamma_\tau\frac{\partial u_e}{\partial\phi}+u_e\frac{\partial u_e}{\partial\xi}-u_e\frac{\phi}{\gamma}\gamma_\xi\frac{\partial u_e}{\partial\phi}+\frac{2\nu}{\gamma}\phi UL\tilde{f}_{\phi\phi\phi}+\frac{2\nu}{\gamma}UL\,\tilde{f}_{\phi\phi}$$

$$\Rightarrow \frac{\partial}{\partial \tau}\left(UL\tilde{f}_{\phi}\right)-\left\{\frac{\phi}{\gamma}\gamma_{\tau}+\frac{(\gamma UF)_{\xi}}{\gamma}-\frac{r_W V_W}{\gamma}\right\}\frac{\partial}{\partial \phi}\left(UL\tilde{f}_{\phi}\right)+UF_{\phi}\frac{\partial}{\partial \xi}\left(UL\tilde{f}_{\phi}\right)=-g_x\beta_T\Delta T\theta$$
$$+\frac{\partial u_e}{\partial \tau}-\frac{\phi}{\gamma}\gamma_{\tau}\frac{\partial u_e}{\partial \phi}+u_e\frac{\partial u_e}{\partial \xi}-u_e\frac{\phi}{\gamma}\gamma_{\xi}\frac{\partial u_e}{\partial \phi}+\frac{2\nu}{\gamma}\phi UL\tilde{f}_{\phi\phi\phi}+\frac{2\nu}{\gamma}UL\,\tilde{f}_{\phi\phi}$$

$$\Rightarrow (UL)_{\tau}\tilde{f}_{\phi}-\left\{\frac{\phi}{\gamma}\gamma_{\tau}+\frac{\left(\gamma UL\tilde{f}\right)_{\xi}}{\gamma}-\frac{r_W V_W}{\gamma}\right\}UL\tilde{f}_{\varphi\varphi}+U\left(L\tilde{f}\right)_{\phi}\frac{\partial}{\partial \xi}\left(UL\tilde{f}_{\phi}\right)=-g_x\beta_T\Delta T\theta$$
$$+\frac{\partial u_e}{\partial \tau}-\frac{\phi}{\gamma}\gamma_{\tau}\frac{\partial u_e}{\partial \phi}+u_e\frac{\partial u_e}{\partial \xi}-u_e\frac{\phi}{\gamma}\gamma_{\xi}\frac{\partial u_e}{\partial \phi}+\frac{2\nu}{\gamma}\phi UL\tilde{f}_{\phi\phi\phi}+\frac{2\nu}{\gamma}UL\,\tilde{f}_{\phi\phi}$$

$$\Rightarrow (UL)_{\tau}\tilde{f}_{\phi}-\frac{\phi}{\gamma}\gamma_{\tau}UL\tilde{f}_{\varphi\varphi}-\frac{(\gamma UL)_{\xi}\tilde{f}}{\gamma}UL\tilde{f}_{\varphi\varphi}+\frac{r_W V_W}{\gamma}UL\tilde{f}_{\varphi\varphi}+UL\tilde{f}_{\phi}(UL)_{\xi}\tilde{f}_{\phi}=-g_x\beta_T\Delta T\theta$$
$$+\frac{\partial u_e}{\partial \tau}-\frac{\phi}{\gamma}\gamma_{\tau}\frac{\partial u_e}{\partial \phi}+u_e\frac{\partial u_e}{\partial \xi}-u_e\frac{\phi}{\gamma}\gamma_{\xi}\frac{\partial u_e}{\partial \phi}+\frac{2\nu}{\gamma}\phi UL\tilde{f}_{\phi\phi\phi}+\frac{2\nu}{\gamma}UL\,\tilde{f}_{\phi\phi}$$

$$\Rightarrow \frac{\gamma}{UL}(UL)_{\tau}\tilde{f}_{\phi}-\frac{\gamma}{UL}\frac{\phi}{\gamma}\gamma_{\tau}UL\tilde{f}_{\varphi\varphi}-\frac{\gamma}{UL}\frac{(\gamma UL)_{\xi}\tilde{f}}{\gamma}U^2L\tilde{f}_{\varphi\varphi}+\frac{\gamma}{UL}\frac{r_W V_W}{\gamma}UL\tilde{f}_{\varphi\varphi}+\frac{\gamma}{UL}UL\tilde{f}_{\phi}(UL)_{\xi}\tilde{f}_{\phi}$$
$$=-\frac{\gamma}{UL}g_x\beta_T\Delta T\theta+\frac{\gamma}{UL}\left(\frac{\partial u_e}{\partial \tau}+u_e\frac{\partial u_e}{\partial \xi}\right)+2\nu\phi\tilde{f}_{\phi\phi\phi}+2\nu\,\tilde{f}_{\phi\phi}$$

$$\Rightarrow \frac{\gamma}{UL}(UL)_{\tau}\tilde{f}_{\phi}-\phi\gamma_{\tau}\tilde{f}_{\varphi\varphi}-(\gamma UL)_{\xi}\tilde{f}\,\tilde{f}_{\varphi\varphi}+r_W V_W\tilde{f}_{\varphi\varphi}+\gamma(UL)_{\xi}\tilde{f}_{\phi}^{2}$$
$$=-\frac{\gamma}{UL}g_x\beta_T\Delta T\theta+\frac{\gamma}{UL}\left(\frac{\partial u_e}{\partial \tau}+u_e\frac{\partial u_e}{\partial \xi}\right)+2\nu\phi\tilde{f}_{\phi\phi\phi}+2\nu\,\tilde{f}_{\phi\phi}$$

$$\Rightarrow 2\nu\left\{(\bar{\phi}+a_3)\tilde{f}_{\bar{\phi}\bar{\phi}}\right\}_{\bar{\phi}}+\left\{a_0(\bar{\phi}+a_3)+a_9\right\}\tilde{f}_{\bar{\phi}\bar{\phi}}+(a_1+a_2)\tilde{f}\tilde{f}_{\bar{\phi}\bar{\phi}}$$
$$-a_2\tilde{f}_{\bar{\phi}}^{2}-a_4\tilde{f}_{\bar{\phi}}+a_5\vartheta+a_6=0 \qquad (2.36)$$

and energy equation (2.18) reduce

$$\left[\frac{\partial}{\partial\tau}-\left\{\frac{\phi}{\gamma}\gamma_\tau+\frac{(\gamma UF)_\xi}{\gamma}-\frac{r_W V_W}{\gamma}\right\}\frac{\partial}{\partial\phi}+UF_\phi\frac{\partial}{\partial\xi}\right](m\vartheta)=-\left\{u\frac{\partial}{\partial x}(\log\Delta T)+\frac{\partial}{\partial t}(\log\Delta T)\right\}(m\vartheta)$$
$$+\frac{1}{\Pr}\frac{\nu}{r}\frac{r}{\gamma}\frac{\partial}{\partial\phi}\left\{r\frac{r}{\gamma}\frac{\partial}{\partial\phi}(m\vartheta)\right\}$$

$$\Rightarrow\frac{\partial}{\partial\tau}(m\vartheta)-\frac{\phi}{\gamma}\gamma_\tau\frac{\partial}{\partial\phi}(m\vartheta)-\frac{\left(\gamma UL\tilde{f}\right)_\xi}{\gamma}\frac{\partial}{\partial\phi}(m\vartheta)+\frac{r_W V_W}{\gamma}\frac{\partial}{\partial\phi}(m\vartheta)$$
$$=-\left\{u\frac{\partial}{\partial x}(\log\Delta T)+\frac{\partial}{\partial t}(\log\Delta T)\right\}(m\vartheta)+\frac{1}{\Pr}\frac{\nu}{\gamma}\frac{\partial}{\partial\phi}\left\{2\phi\frac{\partial}{\partial\phi}(m\vartheta)\right\}$$

$$\Rightarrow-\frac{\phi}{\gamma}\gamma_\tau\vartheta_\phi-\frac{(\gamma UL)_\xi}{\gamma}\tilde{f}\vartheta_\phi+\frac{r_W V_W}{\gamma}\vartheta_\phi$$
$$=-\left\{u\frac{\partial}{\partial x}(\log\Delta T)+\frac{\partial}{\partial t}(\log\Delta T)\right\}(\vartheta)+\frac{1}{\Pr}\frac{\nu}{\gamma}\frac{\partial}{\partial\phi}\left\{2\phi\frac{\partial}{\partial\phi}(\vartheta)\right\}$$

$$\Rightarrow-\frac{\phi}{\gamma}\gamma_\tau\vartheta_\phi-\frac{(\gamma UL)_\xi}{\gamma}\tilde{f}\vartheta_\phi+\frac{r_W V_W}{\gamma}\vartheta_\phi$$
$$=-\left\{u\frac{\partial}{\partial x}(\log\Delta T)+\frac{\partial}{\partial t}(\log\Delta T)\right\}(\vartheta)+\frac{1}{\Pr}\frac{\nu}{\gamma}\frac{\partial}{\partial\phi}\{2\phi\vartheta_\phi\}$$

$$\Rightarrow\frac{2\nu}{\Pr}\left\{(\bar{\phi}+a_3)\vartheta_{\bar{\phi}}\right\}_{\bar{\phi}}+\left\{a_0(\bar{\phi}+a_3)+a_9\right\}\vartheta_{\bar{\phi}}+(a_1+a_2)\tilde{f}\vartheta_{\bar{\phi}}$$
$$-\left(a_7+a_8\tilde{f}_{\bar{\phi}}\right)\vartheta=0 \qquad (2.37)$$

where

(*i*) $\gamma_\tau=a_0$

(*ii*) $(\gamma UL)_\xi=\gamma_\xi UL+\gamma(UL)_\xi=a_1+a_2$

(*iii*) $\frac{r_W^2}{2\gamma}=a_3=\phi_0$

(*iv*) $\frac{\gamma(UL)_\tau}{UL}=a_4$

(*v*) $\gamma(UL)_\xi=a_2$

$(vi)\quad \gamma_\xi(UL) = a_1$,

$(vii)\quad -\frac{\gamma}{UL} g_x \beta_T \Delta T = a_5$

$(viii)\quad \frac{\nu}{UL}\{(u_e)_\tau + u_e(u_e)_\xi\} = a_6$

$(ix)\quad \nu(\log\Delta T)_\tau = a_7$

$(x)\quad \nu UL\{\log\Delta T\}_\xi = a_8$

$(xi)\quad -r_W V_W = a_9$

$$\text{and } \overline{\phi} = \frac{r^2 - r_W^2}{2\gamma(x,t)} = \phi - \phi_0,\quad \frac{\partial}{\partial\phi} \equiv \frac{\partial}{\partial\overline{\phi}}.\frac{\partial\overline{\phi}}{\partial\phi} \equiv \frac{\partial}{\partial\overline{\phi}} \tag{2.38}$$

Similar solutions for (2.36) and (2.37) exist only when all a's defined in equations (2.38) are finite and independent of ξ and τ. In such cases, the equations (2.36) and (2.37) will be reduced to ordinary non-linear differential equations (i.e. equations containing one independent variable). If $u_e = 0$, UL is treated then as the non-dimensionalising characteristic velocity (e.g. maximum velocity) induced by buoyancy effects at a particular station *(x.r.t)*. On the other hand if $u_e \neq 0$ without loss of generality we may put $UL = u_e$ to simplify the outer boundary condition. This then asserts that the velocity component u is non-dimensionalised by the external forcing velocity. Thus the equations (2.34) give us the relations for UL (ξ, τ) or consequently $u_e(\xi,\tau)$ and $\gamma(\xi, \tau)$ separately. The latter are the scale factors for the velocity component u and the square of the ordinate r. Hence the contractions of ξ and τ expressed in terms of the a's in equations (2.38) become the conditions to transform the boundary-layer equations into similarity equations.

Integrating (i) of (2.38) (i), we get

$$\gamma = a_0\tau + B(\xi) \tag{2.39}$$

Again integrating (ii) of (2.38), we get

$$\gamma UL = (a_1 + a_2)\xi + A(\tau) \tag{2.40}$$

Differentiating (2.39) with respect to ξ and using (2.38)(vi) we get

$$\gamma_\xi = \frac{d}{d\xi}\{B(\xi)\} = \frac{a_1}{UL} \tag{2.41}$$

Similarly differentiating (2.40) with respect to τ and making use of conditions (ii) and (iv) of (2.38) we get $(\gamma UL)_\tau = \dfrac{d\,A(\tau)}{d\tau}$

$$\Rightarrow \gamma(UL)_\tau + \gamma_\tau UL = \frac{d\,A(\tau)}{d\tau}$$

$$\Rightarrow UL\,a_4 + a_0 UL = \frac{d\,A(\tau)}{d\tau}$$

$$UL(a_0 + a_4) = \frac{d}{d\tau}\{A(\tau)\} \qquad (2.42)$$

As result of (2.41) and (2.42) it may be obtained that

$$\frac{d}{d\tau}\{A(\tau)\}.\frac{d}{d\xi}\{B(\xi)\} = a_1(a_0 + a_4) \qquad (2.43)$$

Since $\gamma(\xi, \tau)$ and $UL\ (\xi, \tau)$ depend solely on the choice of $A\ (\tau)$ and $B\ (\xi)$, the equation (2.43) plays a significant role in determining the possible cases of similarity solutions.

These possible cases are:

A. Both $\dfrac{d}{d\tau}\{A(\tau)\}$ and $\dfrac{d}{d\xi}\{B(\xi)\}$ are constants (2.44)

B. $\dfrac{d}{d\tau}\{A(\tau)\} = 0$ and $\dfrac{d}{d\xi}\{B(\xi)\} \neq 0$ (2.45)

C. Both $\dfrac{d}{d\tau}\{A(\tau)\}$ and $\dfrac{d}{d\xi}\{B(\xi)\}$ are zero (2.46)

D. $\dfrac{d}{d\tau}\{A(\tau)\} \neq 0$ and $\dfrac{d}{d\xi}\{B(\xi)\}$ =0 (2.47)

It is seen that the similarity conditions (2.38) lead to four possible classes of similarity listed in equations (2.44)-(2.47). However in the present case an additional condition for similarity arises because ϕ_0 (equation (2.27) must be constant also. Since r_w is a function of x only, for ϕ_0 to be constant therefore requires that γ is a function of x only. Thus similarity is achieved in the present problem in those circumstances for which $a_0 = 0$. With this restriction the various similarity cases are discussed consecutively in Chapter III and Chapter IV, each one being considered separately either as a combined or a natural convection problem.

CHAPTER: III

Similarity Solution: Study of Case A

In this chapter we will discuss the similarity case, viz., Case A which is obtained in the similarity analysis as given in Chapter II.

When $\frac{dA(\tau)}{d\tau}$ and $\frac{dB(\xi)}{d\xi}$ both are finite constants, then from the equation (2.42) we have

$$UL(a_0 + a_4) = \frac{d}{d\tau}\{A(\tau)\}$$

3.1 Purely Unsteady Case $(UL \propto t^n)$

In this case also it is found that

$$\gamma = a_0\tau + B(\xi)$$

Now,

$$\frac{a_4}{a_0} = \frac{\gamma(UL)_\tau}{UL\gamma_\tau}$$

Or, $\frac{(UL)_\tau}{UL} = \frac{\gamma_\tau}{\gamma}\frac{a_4}{a_0}$

Integrating we get

$$\ln(UL) = \frac{a_4}{a_0}\ln(\gamma) + \ln(K)$$

Or, $UL = K\gamma^{\frac{a_4}{a_0}}$

Or, $UL = K(a_0\tau + B)^{\frac{a_4}{a_0}}$; B and K are constants

So that once more similarity solutions are not possible, However, It is possible in this case for a_o to be zero. This particular situation is discussed next.

3.2 Analysis for Combined Convection in Exponential Unsteady Situation $(UL \propto e^t)$

A case situation is found to be possible in a particular unsteady case if one of the similarity requirements stated in equations (2.38) be set arbitrarily to zero(i.e. $a_o = 0$). This assumption gives us for the unsteady case A (in view of equations (2.38) (i) and (2.38) (iv)

$$\gamma = B \tag{3.1}$$

$$\frac{\gamma(UL)_\tau}{UL} = a_4$$

$$\text{or, } \frac{(UL)_\tau}{UL} = \frac{a_4}{\gamma}$$

Integrating

$$\ln(UL) = \frac{a_4}{\gamma}\tau + \ln(K)$$

$$\text{Or, } UL = Ke^{\frac{a_4}{\gamma}\tau}$$

$$\text{Or, } UL = Ke^{\frac{a_4}{B}\tau} \tag{3.2}$$

where K and B are constants of integration. For the above specified values of γ and UL, and substituting the above equations in turn into the equations (2.38), we obtain the following relations between the constants:

$a_0 = a_1 = a_2 = 0$, $\quad a_3 = \frac{r_W^2}{2B}$, $a_4 =$ arbitrary,

$$\begin{aligned} a_5 &= -\frac{\gamma}{UL} g_x \beta_T \Delta T \\ &= -\frac{g_x \beta_T \Delta T B}{K e^{\frac{a_4}{B}\tau}} \\ &= -\frac{g_x \beta_T \Delta T B}{K} e^{\frac{-a_4}{B}\tau} \end{aligned}$$

$a_6 = a_4$, $a_7 = a_4$, $a_8 = 0$ and $a_9 =$ arbitrary.

Hence the general equations (2.36) and (2.37) take the forms for this case:

$$2\nu\left\{(\bar{\phi} + a_3)\tilde{f}_{\bar{\phi}\bar{\phi}}\right\}_{\bar{\phi}} + a_9 \tilde{f}_{\bar{\phi}\bar{\phi}} - a_4 \tilde{f}_{\bar{\phi}} + a_5 \vartheta + a_4 = 0$$

$$\text{Or, } 2\nu\left\{(\bar{\phi} + a_3)\tilde{f}_{\bar{\phi}\bar{\phi}}\right\}_{\bar{\phi}} + a_9 \tilde{f}_{\bar{\phi}\bar{\phi}} + a_4(1 - \tilde{f}_{\bar{\phi}}) + a_5 \vartheta = 0 \tag{3.4}$$

$$\text{Or, } \frac{2\nu}{P_r}\left\{(\bar{\phi} + a_3)\vartheta_{\bar{\phi}}\right\}_{\bar{\phi}} + a_9 \vartheta_{\bar{\phi}} - a_4 \vartheta = 0 \tag{3.5}$$

Let us now substitute $\tilde{f} = \alpha_1 f$ and $\bar{\phi} = \alpha_2 \eta$ where the arbitrary constants α_1 and α_2 are introduced so as to provide the convenience in simplifications to the above equations.

It is convenient to choose $\alpha_1 = \alpha_2$ and $\dfrac{a_4\alpha_1}{\nu} = 1$ and writing $\dfrac{r_W^2}{2\alpha_1 B} = R_0$ the above two equations are simplified to

Equation (3.4):

$$\frac{2\nu}{\alpha_1}\left\{\left(\alpha_1\eta + \frac{r_W^2}{2B}\right)\frac{1}{\alpha_1}f_{\eta\eta}\right\}_\eta + \frac{a_9}{\alpha_1}f_{\eta\eta} + a_4(1 - f_\eta) + a_5\vartheta = 0$$

$$\text{Or, } 2\left\{\left(\eta + \frac{r_W^2}{2B\alpha_1}\right)f_{\eta\eta}\right\}_\eta + \frac{a_9}{\nu}f_{\eta\eta} + \frac{a_4\alpha_1}{\nu}(1 - f_\eta) + \frac{a_5\alpha_1}{\nu}\vartheta = 0$$

$$\text{Or, } 2\{(\eta + R_0)f_{\eta\eta}\}_\eta + F_W f_{\eta\eta} + 1 - f_\eta + \frac{U_F^2}{u_e^2}\vartheta = 0$$

$$\text{Or, } 2(\eta + R_0)f_{\eta\eta\eta} + (2 + F_W)f_{\eta\eta} + 1 - f_\eta + \frac{U_F^2}{u_e^2}\vartheta = 0 \tag{3.6}$$

Equation (3.5):

$$\frac{2\nu}{P_r}\left\{\left(\bar{\phi} + \frac{r_W^2}{2B}\right)\vartheta_{\bar{\phi}}\right\}_{\bar{\phi}} + a_9\vartheta_{\bar{\phi}} - a_4\vartheta = 0$$

$$\text{Or, } \frac{2\nu}{P_r}\frac{1}{\alpha_1}\left\{\left(\alpha_1\eta + \frac{r_W^2}{2B}\right)\frac{1}{\alpha_1}\vartheta_\eta\right\}_\eta + \frac{1}{\alpha_1}a_9\vartheta_\eta - a_4\vartheta = 0$$

$$\text{Or, } \frac{2}{P_r}\left\{\left(\eta + \frac{r_W^2}{2B\alpha_1}\right)\vartheta_\eta\right\}_\eta + \frac{1}{\nu}a_9\vartheta_\eta - \frac{\alpha_1 a_4}{\nu}\vartheta = 0$$

$$\text{Or, } 2(\eta + R_0)\vartheta_{\eta\eta} + (2 + P_r F_W)\vartheta_\eta - P_r\vartheta = 0 \tag{3.7}$$

where $UL = u_e$ is considered here without loss of generality for combined convection,

$$\frac{U_F^2}{u_e^2} = \frac{a_5}{a_4} = \frac{-g_x\,\beta_T\Delta T(Cu_e)}{u_e^2} \tag{3.8}$$

$$\text{Or, } U_F^2 = -g_x\,\beta_T\Delta T(Cu_e) \tag{3.9}$$

and $\dfrac{a_9}{\nu} = F_W = -\dfrac{r_W\,v_W}{\nu}$ and the characteristic length is Cu_e. The constant 'C' represents here some non-dimensional characteristic time scale and is substituted for $\dfrac{B}{a_4}$, the inverse the coefficient of the external exponential forcing velocity $u_e\,(= u_0 e^{\frac{a_4}{B}\tau})$. Until the appropriate boundary conditions are specified the transformed equations (3.6) and (3.7) remain incomplete. Hence the similarity equations for this case with controlling parameters R_0, Pr, F_W and $\frac{U_F^2}{u_e^2}$ are

$$2(\eta + R_0)f_{\eta\eta\eta} + (2 + F_W)f_{\eta\eta} + 1 - f_\eta + \frac{U_F^2}{u_e^2}\vartheta = 0 \tag{3.10}$$

$$2(\eta + R_0)\vartheta_{\eta\eta} + (2 + P_r F_W)\vartheta_\eta - P_r\vartheta = 0 \tag{3.11}$$

The boundary conditions are

$$f(0) = f_\eta(0) = 0 \quad , \; f_\eta(\infty) = 1 \; , \vartheta(0) = 1, \; \vartheta(\infty) = 0 \tag{3.12}$$

The similarity function $f(\eta)$, the similarity variable η, the body radius parameter R_0, the velocity components u, v and other boundary-layer characteristics (3.12) are

$$\begin{aligned}
\psi &= \gamma U F + \psi(\xi, \phi_0, \tau) \\
&= \gamma U L \tilde{f}(\bar{\phi}) + \psi(\xi, \phi_0, \tau) \\
&= \gamma U L \alpha_1 f(\eta) + \psi(\xi, \phi_0, \tau) \\
&= \gamma U L \frac{v}{a_4} f(\eta) + \psi(\xi, \phi_0, \tau) \\
&= v U L \frac{\gamma}{a_4} f(\eta) + \psi(\xi, \phi_0, \tau) \\
&= v U L \frac{B}{a_4} f(\eta) + \psi(\xi, \phi_0, \tau) \\
&= C v u_e f(\eta) + \psi(\xi, \phi_0, \tau)
\end{aligned} \tag{3.13}$$

$$\alpha_1 \eta = \bar{\phi}$$

Or, $\alpha_1 \eta = \frac{r^2 - r_W^2}{2\gamma}$

$$\eta = \frac{r^2 - r_W^2}{2\alpha_1 \gamma} = \frac{r^2 - r_W^2}{2\frac{v}{a_4}B} = \frac{r^2 - r_W^2}{2\frac{B}{a_4}v}$$

$$= \frac{r^2 - r_W^2}{2C\nu} \tag{3.14}$$

$$R_0 = \frac{r_W^2}{2\alpha_1 B}$$

$$= \frac{r_W^2}{2\frac{\nu}{a_4}B}$$

$$= \frac{r_W^2}{2\frac{B}{a_4}\nu}$$

$$= \frac{r_W^2}{2C\nu} \tag{3.15}$$

$$u = \frac{\partial}{\partial\phi}\left(\frac{\psi}{\gamma}\right)$$

$$= \frac{\partial}{\partial\phi}\left\{\frac{\gamma U F + \psi(\xi,\phi_0,\tau)}{\gamma}\right\}$$

$$= \frac{\partial}{\partial\phi}\left(\frac{\gamma U F}{\gamma}\right) + \frac{\partial}{\partial\phi}\left\{\frac{\psi(\xi,\phi_0,\tau)}{\gamma}\right\}$$

$$= \frac{\partial}{\partial\phi}(U F)$$

$$= \frac{\partial}{\partial\phi}\left\{U L \widetilde{f}(\phi)\right\}$$

$$= U L \frac{\partial \widetilde{f}(\phi)}{\partial\phi}$$

$$= U L \widetilde{f}_{\bar{\phi}}$$

$$= u_e f_\eta(\eta) \tag{3.16}$$

$$-rv = (\gamma U F)_\xi - \phi\gamma_\xi U F_\phi - r_W v_W$$

$$= \frac{\partial}{\partial\xi}(\gamma U F) - \phi\gamma_\xi U F_\phi - r_W v_W$$

$$= \frac{\partial}{\partial\xi}\left\{\gamma U L \widetilde{f}(\phi)\right\} - \phi\gamma_\xi U \frac{\partial}{\partial\phi}\left\{L \widetilde{f}(\phi)\right\} - r_W v_W$$

$$= 0 - \phi\gamma_\xi U L \frac{\partial}{\partial\phi}\left\{\widetilde{f}(\phi)\right\} - r_W v_W$$

$$v = \frac{1}{r} r_W v_W \tag{3.17}$$

$$\tau_W = \mu \frac{\partial u}{\partial r} \quad \text{at} \quad r = r_W$$

$$= \mu \frac{r}{\gamma} \frac{\partial}{\partial \phi} \left\{ \frac{\partial}{\partial \phi} \left(\frac{\psi}{\gamma} \right) \right\}$$

$$= \mu \frac{r}{\gamma} \frac{\partial}{\partial \phi} \left\{ \frac{\partial}{\partial \phi} \left(\frac{\gamma U F + \psi(\xi, \phi_0, \tau)}{\gamma} \right) \right\}$$

$$= \mu \frac{r}{\gamma} \frac{\partial}{\partial \phi} \left\{ \frac{\partial}{\partial \phi} (UF) \right\}$$

$$= \mu \frac{r}{\gamma} \frac{\partial}{\partial \phi} \left\{ \frac{\partial}{\partial \phi} \left(U L \tilde{f} \right) \right\}$$

$$= \mu \frac{r_W U L}{\gamma} \tilde{f}_{\bar{\phi}\bar{\phi}}$$

$$= \mu \frac{r_W u_e}{\gamma} \frac{1}{\alpha_1} f_{\eta\eta}(0)$$

$$= \mu u_e \frac{r_W}{\gamma} \frac{a_4}{\nu} f_{\eta\eta}(0)$$

$$= \mu u_e \frac{r_W}{\frac{B}{a_4} \nu} f_{\eta\eta}(0)$$

$$= \mu u_e \frac{r_W}{C \nu} f_{\eta\eta}(0)$$

$$= \mu u_e \frac{r_W}{\sqrt{C \nu} \sqrt{C \nu}} f_{\eta\eta}(0)$$

$$= \frac{\mu u_e}{\sqrt{C \nu}} \sqrt{2 R_0} f_{\eta\eta}(0) \tag{3.18}$$

$$q_W = -K \frac{\partial T}{\partial r} \quad \text{at} \quad r = r_W$$

$$= -K \frac{r}{\gamma} \frac{\partial T}{\partial \phi}$$

$$= -K \frac{r_W}{\gamma} \Delta T \frac{\partial \vartheta}{\partial \bar{\phi}}$$

$$= -K\frac{r_W}{\gamma}\Delta T\frac{1}{\alpha_1}\vartheta_\eta(0)$$

$$= -K\frac{r_W}{B}\Delta T\frac{a_4}{\nu}\vartheta_\eta(0)$$

$$= -K\frac{r_W}{\frac{B}{a_4}\nu}\Delta T\vartheta_\eta(0)$$

$$= -K\frac{r_W}{C\nu}\Delta T\vartheta_\eta(0)$$

$$= -K\frac{r_W}{\sqrt{C\nu}\sqrt{C\nu}}\Delta T\vartheta_\eta(0)$$

$$q_W = -\frac{K\Delta T}{\sqrt{C\nu}}\sqrt{2R_0}\,\vartheta_\eta(0) \qquad (3.19)$$

where τ_W and q_W are the shearing stress and the heat transfer rate at the wall respectively.

To have this class of similarity solution the following physical conditions must be satisfied:

(i) $u_e \infty e^{\frac{t}{C}}$ (3.20)

(ii) $\Delta T \infty e^{\frac{t}{C}}$ (3.21)

(iii) $r_W = \sqrt{2R_0}\sqrt{C\nu}$ (3.22)

The analytical solution of the energy equation (3.11) may be expressed in closed form satisfying the attached boundary conditions as

$$\vartheta(\eta) \equiv \vartheta(z) = \frac{I_0(z)}{I_0(z_0)}\left\{1 - \frac{\int_{z_0}^{z}\frac{dz_1}{z_1 I_0^2(z_1)}}{\int_{z_0}^{\infty}\frac{dz_1}{z_1 I_0^2(z_1)}}\right\} \qquad (3.23)$$

where $z = \sqrt{2\Pr\eta}$, $z_0 = \sqrt{2\Pr R_0}$, η and R_0 are given by the equations (3.14) and (3.15). $I_0(z)$ is the modified Bessel function in z of order zero. If $Pr = 1$, the solution for the momentum equation (3.10) becomes simpler. Under this restricted value of Pr (= 1) the solution may be written as

$$1 - f_\eta(\eta) \equiv F(z)$$

$$= \frac{I_0(z)}{I_0(z_0)}\left[1 - \frac{\int_{z_0}^{z}\frac{dz_1}{z_1 I_0^2(z_1)}}{\int_{z_0}^{\infty}\frac{dz_1}{z_1 I_0^2(z_1)}} + R_E\left\{M(z) - M(\infty)\frac{\int_{z_0}^{z}\frac{dz_1}{z_1 I_0^2(z_1)}}{\int_{z_0}^{\infty}\frac{dz_1}{z_1 I_0^2(z_1)}}\right\}\right] \tag{3.24}$$

where $R_E = \frac{U_F^2}{u_e^2}$ and

$$M(z) = \int_{z_0}^{z}\left[z_1\, I_0^2(z_1)\int_{z_0}^{z}\frac{dz_2}{z_2 I_0^2(z_2)}\left\{1 - \frac{\int_{z_0}^{z}\frac{dz_1}{z_1 I_0^2(z_1)}}{\int_{z_0}^{\infty}\frac{dz_1}{z_1 I_0^2(z_1)}}\right\}\right]dz_1$$

$$- \int_{z_0}^{z}\left[z_1\, I_0^2(z_1)\int_{z_0}^{z}\frac{dz_2}{z_2 I_0^2(z_2)}\left\{1 - \frac{\int_{z_0}^{z}\frac{dz_1}{z_1 I_0^2(z_1)}}{\int_{z_0}^{\infty}\frac{dz_1}{z_1 I_0^2(z_1)}}\right\}\right]dz_1 \tag{3.25}$$

3.3 Natural Convection Analysis ($\Delta T \propto e^{\frac{t}{C}}$)

Returning to derive the similarity equations for natural convection flows for this possible case we have to set $a_6 = 0$. Hence substituting values of γ and UL expressed by the equations (3.1) and (3.2) in the similarity requirements (2.38). We can obtain the relations between the constants:

$a_0 = a_1 = a_2 = 0$, $\quad a_3 = \frac{r_W^2}{2B}$, a_4 = arbitrary, $a_5 = \frac{-g_x\,\beta_T\,\Delta TB}{K}e^{\frac{-a_4}{B}\tau}$

$a_6 = 0$, $a_7 = a_4$, $a_8 = 0$ and a_9 = arbitrary.

Thus the general equations (2.36) and (2.37) reduce to the following forms in natural convection flows:

$$2\nu\left\{(\bar{\phi} + a_3)\tilde{f}_{\bar{\phi}\bar{\phi}}\right\}_{\bar{\phi}} + a_9\tilde{f}_{\bar{\phi}\bar{\phi}} - a_4\tilde{f}_{\bar{\phi}} + a_5\vartheta = 0 \tag{3.26}$$

$$\frac{2\nu}{\Pr}\left\{(\bar{\phi} + a_3)\vartheta_{\bar{\phi}}\right\}_{\bar{\phi}} + a_9\vartheta_{\bar{\phi}} - a_4\vartheta = 0 \tag{3.27}$$

Substituting $\tilde{f} = \alpha_1 f$ and $\bar{\phi} = \alpha_1 \eta$ choosing, $\frac{a_4 \alpha_1}{\nu} = 1$ and writing $\frac{r_W^2}{2\alpha_1 B} = R_0$, the above equations becomes

$$\frac{2\nu}{\alpha_1}\left\{\left(\alpha_1\eta + \frac{r_W^2}{2B}\right)\frac{1}{\alpha_1}f_{\eta\eta}\right\}_\eta + \frac{a_9}{\alpha_1}f_{\eta\eta} - a_4 f_\eta + a_5\vartheta = 0$$

$$\text{Or, } 2\left\{\left(\eta + \frac{r_W^2}{2B\alpha_1}\right)f_{\eta\eta}\right\}_\eta + \frac{a_9}{\nu}f_{\eta\eta} - \frac{a_4\alpha_1}{\nu}f_\eta + \frac{a_5\alpha_1}{\nu}\vartheta = 0$$

$$\text{Or, } 2\{(\eta + R_0)f_{\eta\eta}\}_\eta + F_W f_{\eta\eta} - f_\eta + \frac{a_5}{a_4}\vartheta = 0 \tag{3.28}$$

$$\frac{2\nu}{\Pr}\left\{\left(\bar{\phi} + \frac{r_W^2}{2B}\right)\vartheta_{\bar{\phi}}\right\}_{\bar{\phi}} + a_9\vartheta_{\bar{\phi}} - a_4\vartheta = 0$$

$$\text{Or, } \frac{2\nu}{\Pr}\frac{1}{\alpha_1}\left\{\left(\alpha_1\eta + \frac{r_W^2}{2B}\right)\frac{1}{\alpha_1}\vartheta_\eta\right\}_\eta + \frac{1}{\alpha_1}a_9\vartheta_\eta - a_4\vartheta = 0$$

$$\text{Or, } \frac{2}{\Pr}\left\{\left(\eta + \frac{r_W^2}{2B\alpha_1}\right)\vartheta_\eta\right\}_\eta + \frac{1}{\nu}a_9\vartheta_\eta - \frac{\alpha_1 a_4}{\nu}\vartheta = 0$$

$$\text{Or, } \frac{2}{\Pr}\{(\eta + R_0)\vartheta_\eta\}_\eta + F_W\vartheta_\eta - \vartheta = 0 \tag{3.29}$$

where $\frac{a_9}{\nu} = F_W = -\frac{r_W\, v_W}{\nu}$

$$\text{Now } \quad \frac{a_5}{a_4} = \frac{-g_x\beta_T\Delta T}{K}.\frac{B}{a_4}e^{-\frac{a_4}{B}\tau} = \frac{U_F^2}{(UL)^2} \tag{3.30}$$

$$U_F^2 = -g_x\beta_T\Delta T\, C(UL)\,, \quad \frac{B}{a_4} = C \tag{3.31}$$

For natural convection, to find the free convection velocity U_F, solely generated by buoyancy effects caused by temperature difference $(=\Delta T)$, we may now choose

$$U_F = UL \tag{3.32}$$

resulting in

$$U_F = -g_x\beta_T\Delta T\, C \tag{3.33}$$

Here 'C' is a non-dimensional characteristic time scale. Hence the simplified forms of the similarity equations are (in view of (3.30) and (3.31))

$$2(\eta+R_0)f_{\eta\eta\eta}+(2+F_W)f_{\eta\eta}-f_\eta+\vartheta=0 \tag{3.34}$$

$$2(\eta+R_0)\vartheta_{\eta\eta}+(2+P_rF_W)\vartheta_\eta-P_r\,\vartheta=0 \tag{3.35}$$

The boundary conditions are

$$f(0)=f_\eta(0)=f_\eta(\infty)=0\,,\qquad \vartheta(0)=1,\;\; \vartheta(\infty)=0 \tag{3.36}$$

The parameters are P_r, F_W and R_0 only. The solution for (3.35) is the equation (3.23) and that of (3.34) is (for $Pr=1$)

$$-f_\eta(\eta)=I_0(z)\left\{M(z)-M(\infty)\frac{\int_{z_0}^{z}\frac{dz_1}{z_1I_0^2(z_1)}}{\int_{z_0}^{\infty}\frac{dz_1}{z_1I_0^2(z_1)}}\right\} \tag{3.37}$$

where $M(z)$ is given by the expression (3.24). The similarity function $f(\eta)$, the similarity variable η, the velocity components u, v are given by

$$\psi=\gamma U\,F+\psi(\xi,\phi_0,\tau)$$

$$=\gamma U\,L\tilde{f}(\bar{\phi})+\psi(\xi,\phi_0,\tau)$$

$$=\gamma U_F\,\alpha_1 f(\eta)+\psi(\xi,\phi_0,\tau)$$

$$=-\gamma\,g_x\,\beta_T\,\Delta T\,C\,\alpha_1 f(\eta)+\psi(\xi,\phi_0,\tau)$$

$$=B\frac{\nu}{a_4}(-g_x\,\beta_T\,\Delta T\,C)f(\eta)+\psi(\xi,\phi_0,\tau)$$

$$=C\,\nu\,(-g_x\,\beta_T\,\Delta T\,C)f(\eta)+\psi(\xi,\phi_0,\tau)$$

or, $$\psi=C\nu(-g_x\beta_T\Delta T\,C)f(\eta)+\psi(\xi,\phi_0,\tau) \tag{3.38}$$

$$\alpha_1\eta=\bar{\phi}$$

$$\alpha_1\eta=\frac{r^2-r_W^2}{2\gamma}$$

$$\eta=\frac{r^2-r_W^2}{2\alpha_1\gamma}$$

$$=\frac{r^2-r_W^2}{2\frac{\nu}{a_4}B}$$

$$= \frac{r^2 - r_W^2}{2\dfrac{B}{a_4}\nu}$$

$$= \frac{r^2 - r_W^2}{2C\nu} \tag{3.39}$$

$$u = \frac{\partial}{\partial \phi}\left(\frac{\psi}{\gamma}\right)$$

$$= \frac{\partial}{\partial \phi}\left\{\frac{\gamma U F + \psi(\xi, \phi_0, \tau)}{\gamma}\right\}$$

$$= \frac{\partial}{\partial \phi}\left(\frac{\gamma U F}{\gamma}\right) + \frac{\partial}{\partial \phi}\left\{\frac{\psi(\xi, \phi_0, \tau)}{\gamma}\right\}$$

$$= \frac{\partial}{\partial \phi}(U F)$$

$$= \frac{\partial}{\partial \phi}\left\{U L \tilde{f}(\phi)\right\}$$

$$= U L \frac{\partial}{\partial \phi}\left\{\tilde{f}(\phi)\right\}$$

$$= U_F \frac{\partial}{\partial \phi}\left\{\tilde{f}(\phi)\right\}$$

$$u = \left(-C g_x \beta_T \Delta T\right) f_\eta(\eta) \tag{3.40}$$

$$-rv = (\gamma U F)_\xi - \phi \gamma_\xi U F_\phi - r_W v_W$$

$$= \frac{\partial}{\partial \xi}(\gamma U F) - \phi \gamma_\xi U F_\phi - r_W v_W$$

$$= \frac{\partial}{\partial \xi}\left\{\gamma U L \tilde{f}(\phi)\right\} - \phi \gamma_\xi U \frac{\partial}{\partial \phi}\left\{L \tilde{f}(\phi)\right\} - r_W v_W$$

$$= 0 - \phi \gamma_\xi U_F \frac{\partial}{\partial \phi}\left\{\tilde{f}(\phi)\right\} - r_W v_W$$

$$-rv = 0 - r_W v_W$$

$$-v = -\frac{1}{r} r_W v_W \tag{3.41}$$

The body radius parameter 'R_0' is given by the equation (3.15). The remaining boundary-layer characteristics of interest associated with the equations (3.34) and (3.35) may be derived from the equations (3.25) and (3.37).

Sometimes, in natural convection flow it is preferable to use a non-dimensional Grashof number throughout the analysis. In doing so the following substitution is necessary

$$2(\eta + R_0) = \overline{\eta}_v^2 \tag{3.42}$$

In view of (3.40) and writing

$$\overline{\eta}_v = R_v + \eta_v \tag{3.43}$$

We have

$$2(\eta + R_0) = (R_v + \eta_v)^2$$

Or, $\eta = \frac{1}{2}(R_v + \eta_v)^2 - R_0$

Or, $\frac{\partial \eta}{\partial \eta} = \frac{\partial}{\partial \eta}\left\{\frac{1}{2}(R_v + \eta_v)^2 - R_0\right\}$

Or, $1 = \frac{\partial}{\partial \eta_v}\left\{\frac{1}{2}(R_v + \eta_v)^2 - R_0\right\}\frac{\partial \eta_v}{\partial \eta}$

Or, $1 = \frac{1}{2}2(R_v + \eta_v)\frac{\partial \eta_v}{\partial \eta}$

Or, $\frac{\partial \eta_v}{\partial \eta} = \frac{1}{(R_v + \eta_v)}$

and

$$f_\eta = \frac{\partial f}{\partial \eta}$$

$$= \frac{\partial f}{\partial \eta_v}\frac{\partial \eta_v}{\partial \eta}$$

$$= \frac{1}{(R_v + \eta_v)} f_{\eta_v}$$

$$f_{\eta\eta} = \frac{\partial}{\partial \eta}\left\{\frac{1}{(R_v + \eta_v)} f_{\eta_v}\right\}$$

$$= \frac{\partial}{\partial \eta_v}\left\{\frac{1}{(R_v + \eta_v)} f_{\eta_v}\right\}\frac{\partial \eta_v}{\partial \eta}$$

$$= \left\{\frac{1}{(R_v + \eta_v)} f_{\eta_v\eta_v} - \frac{1}{(R_v + \eta_v)^2} f_{\eta_v}\right\}\frac{1}{(R_v + \eta_v)}$$

$$= \frac{1}{(R_v + \eta_v)^2} f_{\eta_v\eta_v} - \frac{1}{(R_v + \eta_v)^3} f_{\eta_v}$$

$$f_{\eta\eta\eta} = \frac{\partial}{\partial \eta}\left\{\frac{1}{(R_v + \eta_v)^2} f_{\eta_v\eta_v} - \frac{1}{(R_v + \eta_v)^3} f_{\eta_v}\right\}$$

$$= \frac{\partial}{\partial \eta_v}\left\{\frac{1}{(R_v + \eta_v)^2} f_{\eta_v\eta_v} - \frac{1}{(R_v + \eta_v)^3} f_{\eta_v}\right\}\frac{\partial \eta_v}{\partial \eta}$$

$$= \left\{\frac{1}{(R_v + \eta_v)^2} f_{\eta_v\eta_v\eta_v} - \frac{2}{(R_v + \eta_v)^3} f_{\eta_v\eta_v} - \frac{1}{(R_v + \eta_v)^3} f_{\eta_v\eta_v} + \frac{3}{(R_v + \eta_v)^4} f_{\eta_v}\right\}\frac{1}{(R_v + \eta_v)}$$

$$= \frac{1}{(R_v + \eta_v)^3} f_{\eta_v\eta_v\eta_v} - \frac{3}{(R_v + \eta_v)^4} f_{\eta_v\eta_v} + \frac{3}{(R_v + \eta_v)^5} f_{\eta_v}$$

Again,

$$\vartheta_\eta = \frac{\partial \vartheta}{\partial \eta}$$

$$= \frac{\partial \vartheta}{\partial \eta_v}\frac{\partial \eta_v}{\partial \eta}$$

$$= \frac{1}{(R_v + \eta_v)} \vartheta_{\eta_v}$$

$$\vartheta_{\eta\eta} = \frac{\partial}{\partial \eta}\left\{\frac{1}{(R_v + \eta_v)} \vartheta_{\eta_v}\right\}$$

$$= \frac{\partial}{\partial \eta_v}\left\{\frac{1}{(R_v + \eta_v)} \vartheta_{\eta_v}\right\}\frac{\partial \eta_v}{\partial \eta}$$

$$= \left\{\frac{1}{(R_v + \eta_v)} \vartheta_{\eta_v\eta_v} - \frac{1}{(R_v + \eta_v)^2} \vartheta_{\eta_v}\right\}\frac{1}{(R_v + \eta_v)}$$

$$= \frac{1}{(R_v + \eta_v)^2} \vartheta_{\eta_v\eta_v} - \frac{1}{(R_v + \eta_v)^3} \vartheta_{\eta_v}$$

By using the above relations, the equations (3.33) and (3.34) will be transformed to

$$f_{\eta_v\eta_v\eta_v} - (1 - F_W)\frac{f_{\eta_v\eta_v}}{(R_v + \eta_v)} - (1 - F_W)\frac{f_{\eta_v}}{(R_v + \eta_v)^2} + (R_v + \eta_v)\vartheta = 0 \tag{3.44}$$

$$P_r^{-1}\{(R_v + \eta_v)\vartheta_{\eta_v}\}_{\eta_v} + F_W \vartheta_{\eta_v} (R_v + \eta_v)\vartheta = 0 \tag{3.45}$$

with the boundary conditions

$$f(0) = f_{\eta_v}(0) = f_{\eta_v}(\infty) = \vartheta(\infty) = 0, \quad \vartheta(0) = 1 \tag{3.46}$$

The similarity function $f(\eta_v)$, the similarity variable η, the body radius parameter R_v, the velocity components u, v and other essential boundary-layer characteristics τ_W and q_W associated with the equations (3.44) and (3.45) are given by the following relations:

$$\psi = \gamma U F + \psi(\xi, \phi_0, \tau)$$
$$= \gamma U L \tilde{f}(\bar{\phi}) + \psi(\xi, \phi_0, \tau)$$
$$= \gamma U L \alpha_1 f(\eta) + \psi(\xi, \phi_0, \tau)$$
$$= \gamma U L \frac{\nu}{a_4} f(\eta) + \psi(\xi, \phi_0, \tau)$$
$$= \nu U L \frac{\gamma}{a_4} f(\eta) + \psi(\xi, \phi_0, \tau)$$
$$= \nu U_F \frac{B}{a_4} f(\eta) + \psi(\xi, \phi_0, \tau)$$
$$= \nu(-g_x \beta_T \Delta T C) C f(\eta) + \psi(\xi, \phi_0, \tau)$$
$$= \nu(-g_x \beta_T \Delta T C) C f(\eta) + \psi(\xi, \phi_0, \tau)$$

or, $\psi = \nu \sqrt{C \nu}\, G_r f(\eta_v) + \psi(\xi, \phi_0, \tau)$ (3.47)

$$\eta_v = \frac{r - r_W}{\sqrt{C \nu}} \tag{3.48}$$

$$R_v = \frac{r_W}{\sqrt{C \nu}} \tag{3.49}$$

$$u_r = \frac{\partial}{\partial \phi}\left(\frac{\psi}{\gamma}\right)$$
$$= \frac{\partial}{\partial \phi}\left\{\frac{\gamma U F + \psi(\xi, \phi_0, \tau)}{\gamma}\right\}$$
$$= \frac{\partial}{\partial \phi}\left(\frac{\gamma U F}{\gamma}\right) + \frac{\partial}{\partial \phi}\left\{\frac{\psi(\xi, \phi_0, \tau)}{\gamma}\right\}$$
$$= \frac{\partial}{\partial \phi}(U F)$$
$$= \frac{\partial}{\partial \phi}\{U L \tilde{f}(\phi)\}$$
$$= U L \frac{\partial}{\partial \phi}\{\tilde{f}(\phi)\}$$

$$= U_F \frac{\partial}{\partial \phi}\{\tilde{f}(\phi)\}$$

$$u_r = G_r f_{\eta_v}(\eta_v), \tag{3.50}$$

$$-rv = (\gamma U F)_\xi - \phi \gamma_\xi U F_\phi - r_W v_W$$

$$= \frac{\partial}{\partial \xi}(\gamma U F) - \phi \gamma_\xi U F_\phi - r_W v_W$$

$$= \frac{\partial}{\partial \xi}\{\gamma U L\tilde{f}(\phi)\} - \phi \gamma_\xi U \frac{\partial}{\partial \phi}\{L\tilde{f}(\phi)\} - r_W v_W$$

$$= 0 - \phi \gamma_\xi U_F \frac{\partial}{\partial \phi}\{\tilde{f}(\phi)\} - r_W v_W$$

$$-rv = 0 - r_W v_W$$

$$-v = -\frac{1}{r} r_W v_W \tag{3.51}$$

$$R_v \tau_W = G_r \frac{\mu_W}{C\nu} f_{\eta_v \eta_v}(0) \tag{3.52}$$

$$R_v q_W = -\frac{k_W}{C\nu} \vartheta_{\eta_v}(0) \tag{3.53}$$

where $G_r = \left[-\frac{g_x \beta_T \Delta T}{\nu^2}(C\nu)^{\frac{3}{2}}\right]$, the local Grashof number based on constant characteristic length $\left(= \sqrt{C\nu}\right)$ and *'C'* denotes here some non-dimensional characteristic time.

3.4 Numerical Scheme and Procedure

The set of differential equations (3.10) – (3.11) with the boundary conditions (3.12) are solved numerically by using computer software. Here the velocity f_η and temperature ϑ are determined as a function of coordinate η. The values proportional to the coefficient skin friction $f_{\eta\eta}(0)$ and the rate heat transfer $-\vartheta_\eta(0)$ are also evaluated for this case and numerical results thus obtained in terms of the similarity variables are displayed in graphs and tables for several selected values of the appeared parameters F_W, $\frac{U_F^2}{u_e^2}$, R_0 and Pr below.

3.5 Numerical Results and Discussion

The effects of F_W on the velocity and temperature fields are plotted in Figure 3.1 and Figure 3.2, respectively. From Figures it is observed that, in all cases the velocity is starting at zero, and then velocity increases with the increase of η near the leading edge and finally moves towards 1.0 asymptotically but temperature is starting at 1.0, then it decreases asymptotically and finally leads to zero with the increase of η. From Figure 3.1 we see that for the case of suction $(F_W > 0)$, the velocity increases with decreasing F_W but for blowing case $(F_W < 0)$, velocity increases with the increase of the magnitude of blowing. The usual stabilizing effect of the suction parameter on the boundary layer growth is also evident from this figure.

From Figure 3.2 it is observed that for both the cases of suction and blowing, temperature decreases quickly close to the leading edge and away from it temperature decreases asymptotically and finally leads to zero with the increase of η. For the case of suction $(F_W > 0)$, temperature decreases with decreasing suction. But for the case of blowing $(F_W < 0)$, temperature decreases more with the increase of the magnitude of blowing.

Figure 3.3 and Figure 3.4 show the effect of bouncy parameter $\frac{U_F^2}{u_e^2}$ on the velocity and temperature fields respectively. Physically, the flow is said to be aided when $\frac{U_F^2}{u_e^2} > 0$ and is called an opposing flow when $\frac{U_F^2}{u_e^2} < 0$. Further, when $U_F^2 << u_e^2$, the flow becomes forced flow, whereas for $u_e^2 << U_F^2$ the flow becomes a free convection flow. We see from the Figure 3.3 that with the increase in $\frac{U_F^2}{u_e^2}$ from negative to positive values, in all cases the velocity is starting at zero and increasing asymptotically to 1.0. But with the increase in $\frac{U_F^2}{u_e^2}$ the rate of change of velocity increases slightly. Thus before being asymptotically goes to 1.0 far away; velocity is higher for higher values of $\frac{U_F^2}{u_e^2}$ within the boundary layer. Since the energy equation given by (2.28) is independent of the buoyancy parameter $\frac{U_F^2}{u_e^2}$, no effect of this parameter on the temperature field is observed as shown in Figure 3.4.

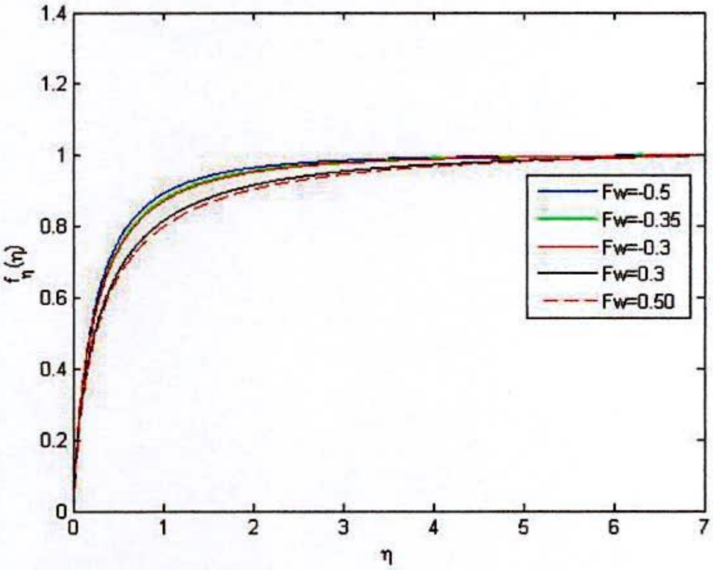

Figure 3.1: Velocity profiles for different values of F_W (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $R_0 = 0.1$ and $\Pr = 0.71$).

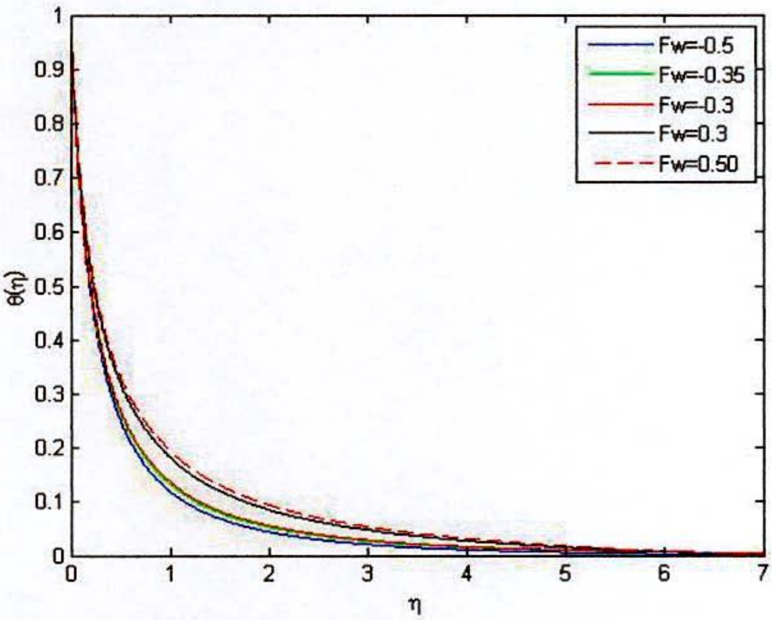

Figure 3.2: Temperature profiles for different values of F_W (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $R_0 = 0.1$ and $\Pr = 0.71$).

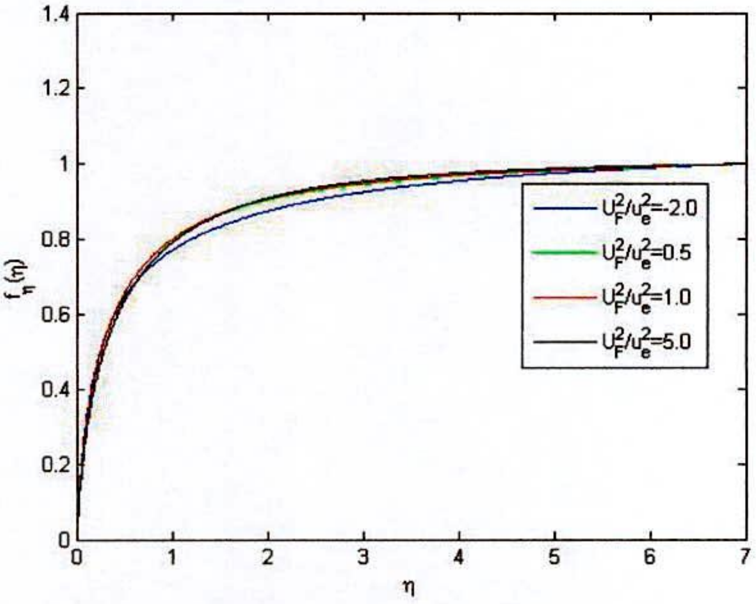

Figure 3.3: Velocity profiles for different values of $\frac{U_F^2}{u_e^2}$ (with fixed values of $F_W = 0.5$ $R_0 = 0.1$ and $\Pr = 0.71$).

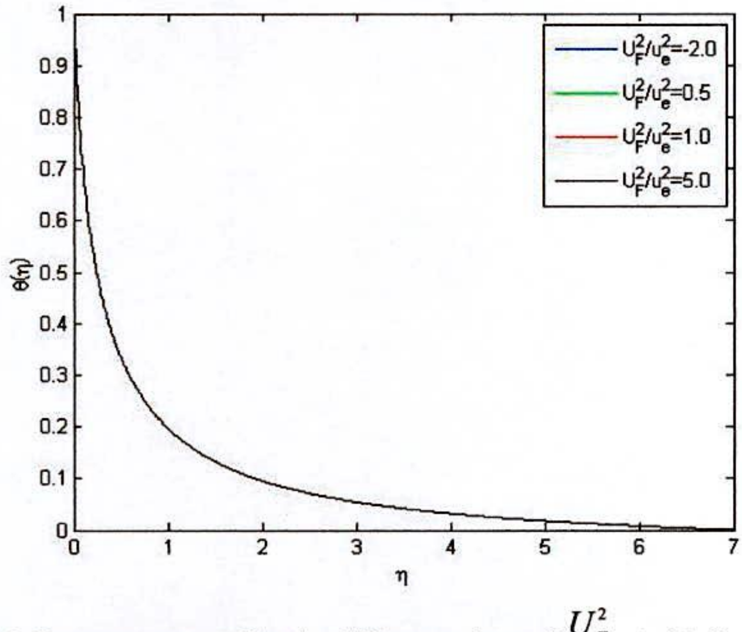

Figure 3.4: Temperature profiles for different values of $\frac{U_F^2}{u_e^2}$ (with fixed values of $F_W = 0.5$, $R_0 = 0.1$ and $\Pr = 0.71$).

The body radius parameter R_0 depends on the shape of the slender body. There is a remarkable consequence of the variation of R_0 on the velocity and temperature fields as are observed in Figure 3.5 and Figure 3.6. Like before, for all values of R_0, velocity starts from zero and then increases with the increase of η and finally tends to 1.0, whereas temperature launches from 1.0, then decreases with increasing η and finally asymptotically leads to zero, for large value of η. We see from Figure 3.5 that within the boundary layer, velocity highly increases with the increase of R_0, before asymptotically being 1.0 for large value of η. From Figure 3.6 it is observed that temperature decreases sharply with the increase in R_0 before being zero asymptotically for higher value of η.

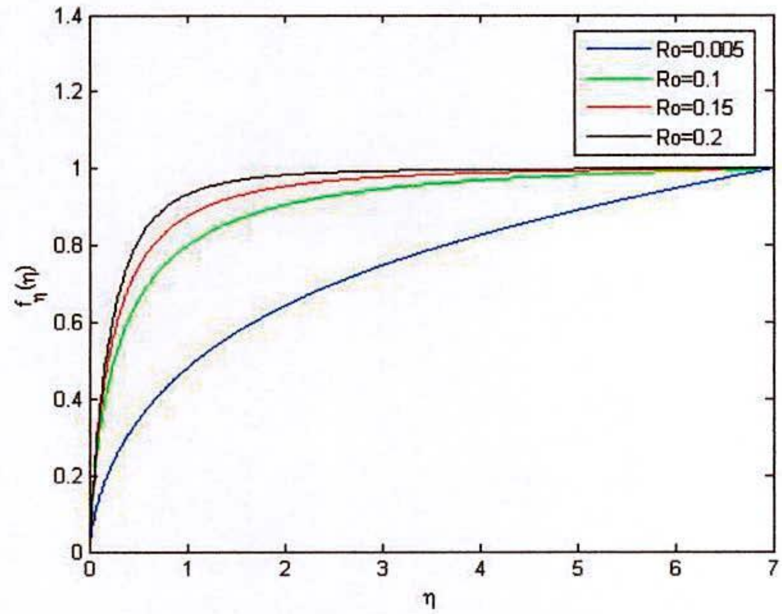

Figure 3.5: Velocity profiles for different values of R_0 (with fixed values of $F_W = 0.5$, $\frac{U_F^2}{u_e^2} = 0.5$ and Pr=0.71)

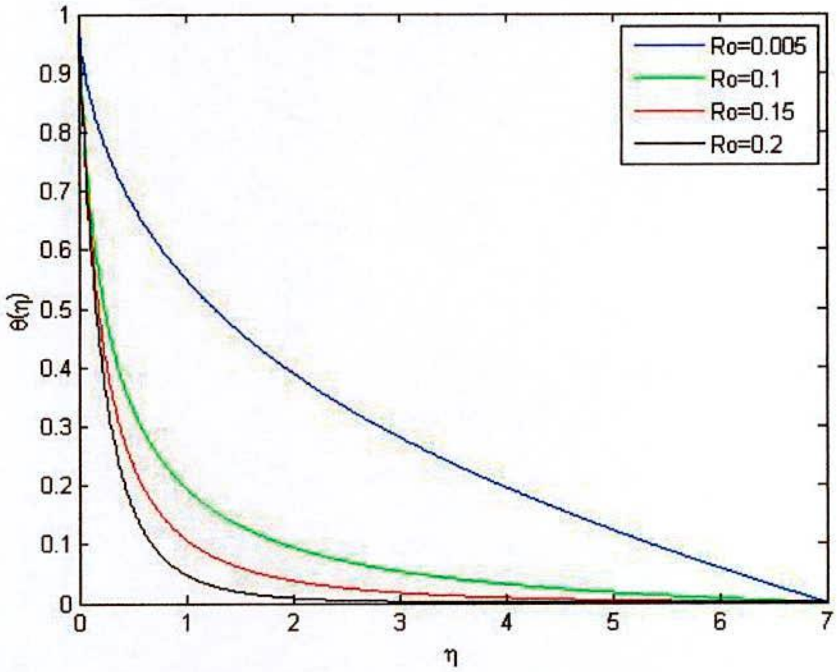

Figure 3.6: Temperature profiles for different values of R_0 (with fixed values of $F_W = 0.5$, $\frac{U_F^2}{u_e^2} = 0.5$ and $\Pr = 0.71$).

The last controlling parameter is the Prandtl number $\Pr\left(=\frac{\mu C_p}{k}\right)$ which depends on the properties of the medium. Here velocity exhibits minor changes while temperature exhibits significant changes with the variation of Prandtl number $\Pr$. As observed from Figure 3.7 that, velocity increases negligibly with the increase of $\Pr$ before being 1.0 asymptotically in all cases for large value of η. From Figure 3.8, we see that, temperature decreases momentously with the decrease in $\Pr$ and asymptotically leads to zero for higher η. Thus before being asymptotically zero temperature is lower for lower values of $\Pr$ within the boundary layer.

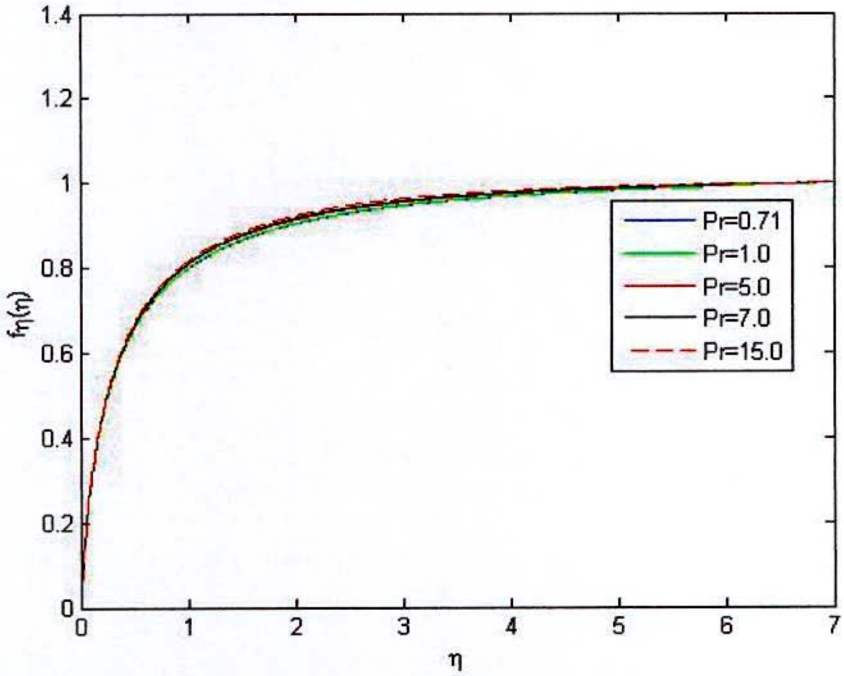

Figure 3.7: Velocity profiles for different values of Pr (with fixed values of $F_W = 0.5$, $\frac{U_F^2}{u_e^2} = 0.5$ and $R_0 = 0.1$).

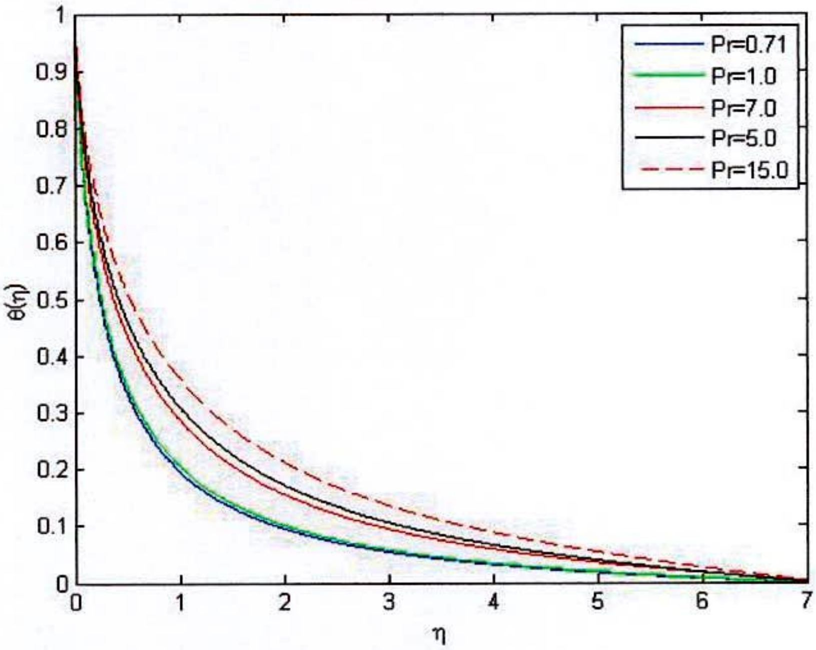

Figure 3.8: Temperature profiles for different values of Pr (with fixed values of $F_W = 0.5$, $\frac{U_F^2}{u_e^2} = 0.5$ and $R_0 = 0.1$).

The values proportional to the coefficients of skin friction $f^{//}(0)$ and heat transfer $-\vartheta^{/}(0)$ are tabulated in Table (3.1) – (3.4).

Table 3.1: Values proportional to the coefficients of skin-friction ($f^{//}(0)$) and heat transfer ($-\vartheta^{/}(0)$) with the variation of suction parameter F_W (for fixed $U_F^2/u_e^2 = 0.5$, $R_0 = 0.1$ and $\Pr = 0.71$).

F_W	$f^{//}(0)$	$-\vartheta^{/}(0)$
0.50	0.52341	-1.52677
0.30	0.84083	-2.83987
-0.30	-0.49478	3.80024
-0.50	-0.38129	3.02430

Table 3.2: Values proportional to the coefficients of skin-friction ($f^{//}(0)$) and heat transfer ($-\vartheta^{/}(0)$) with the variation of buoyancy parameter U_F^2/u_e^2 (for fixed $F_W = 0.5$, $R_0 = 0.1$ and $\Pr = 0.71$).

U_F^2 / u_0^2	$f^{//}(0)$	$-\vartheta^{/}(0)$
-2.0	0.69423	-1.52677
0.5	0.52341	-1.52677
1.0	0.49312	-1.52677
5.0	0.30218	-1.52677

Table 3.3: Values proportional to the coefficients of skin-friction $f^{//}(0)$ and heat transfer $-\vartheta^{/}(0)$ with the variation of driving parameter R_0 (for fixed $F_W = 0.5$, $U_F^2/u_e^2 = 0.5$ and Pr=0.71).

R_0	$f^{//}(0)$	$-\vartheta^{/}(0)$
0.005	0.52340	-1.52677
0.10	0.52341	-1.62166
0.15	0.51021	-1.63965
0.20	0.50385	-1.70361

Table 3.4: Values proportional to the coefficients of skin-friction ($f^{//}(0)$) and heat Transfer ($-\vartheta^{/}(0)$) with the variation of Prandtl number Pr (for fixed $F_W = 0.5$, $U_F^2/u_e^2 = 0.5$ and $R_0 = 0.1$).

Pr	$f^{//}(0)$	$-\vartheta^{/}(0)$
0.71	0.52341	-1.52677
1.0	0.39925	-1.40267
5.0	0.07591	1.50934
7.0	0.04762	1.75093

CHAPTER IV

Similarity Solution: Study of Case B

4.1 Analysis for combined convection $\left(u_e \infty \left(x+x_0\right)^{\beta}\right)$

We shall now discuss in detail the restrictions on the general equations for the case B. These restrictions are illustrated in equations (2.45). In view of equations (2.41) and (2.42), the equation (2.45) may be written as

$\frac{dA(\tau)}{d\tau} = UL(a_4 + a_0) = 0 \Rightarrow a_0 = -a_4$, provided $UL \neq 0$

and

$\frac{dB(\xi)}{d\xi} = \frac{a_1}{UL} \neq 0 \Rightarrow UL$ is a function of ξ only. As a result of the above two statements the equations (2.39) and (2.40) may now be put in the following forms:

$$\gamma = a_0\tau + B(\xi)$$

$$\gamma UL = (a_1 + a_2)\xi + A(\tau)$$

Since UL is a function of ξ only γ must also be a function of ξ only, so it is necessary to set $a_0 = a_4 = 0$. Consequently the expressions for B(ξ) are

$$\gamma = B(\xi) = \frac{(a_1 + a_2)\xi + A}{UL} \tag{4.1}$$

and

$$\frac{dB(\xi)}{d\xi} = \frac{a_1}{UL} \tag{4.2}$$

Equations (4.1) and (4.2) jointly must specify γ and UL and these are found to be,

$$\frac{a_1}{a_1 + a_2} = \frac{\gamma_\xi UL}{(\gamma UL)_\xi}$$

Or, $\gamma_\xi UL = \left(\gamma UL\right)_\xi \frac{a_1}{a_1 + a_2}$

Or, $\frac{\gamma_\xi}{\gamma} = \frac{\left(\gamma UL\right)_\xi}{\gamma UL} \frac{a_1}{a_1 + a_2}$

Integrating

$$\ln\gamma = \ln K_1 + \frac{a_1}{a_1+a_2}\ln(\gamma UL)$$

Or, $\ln\gamma = \ln\left\{K_1(\gamma UL`)^{\frac{a_1}{a_1+a_2}}\right\}$

Or, $\gamma = K_1(\gamma UL`)^{\frac{a_1}{a_1+a_2}}$

Or, $\gamma = K_1\{(a_1+a_2)\xi + A\}^{\frac{a_1}{a_1+a_2}}$ (4.3)

Again,

$$\frac{a_2}{a_1+a_2} = \frac{\gamma(UL)_\xi}{(\gamma UL)_\xi}$$

Or, $\gamma(UL)_\xi = \frac{a_2}{a_1+a_2}(\gamma UL)_\xi$

Or, $\frac{\gamma UL(UL)_\xi}{UL} = \frac{a_2}{a_1+a_2}(\gamma UL)_\xi$

Or, $\frac{(UL)_\xi}{UL} = \frac{a_2}{a_1+a_2}\frac{(\gamma UL)_\xi}{\gamma UL}$

Integrating,

$$\ln(UL) + \ln K_1 = \frac{a_2}{a_1+a_2}\ln(\gamma UL)$$

Or, $\ln(ULK_1) = \ln(\gamma UL)^{\frac{a_2}{a_1+a_2}}$

Or, $ULK_1 = (\gamma UL)^{\frac{a_2}{a_1+a_2}}$

Or, $UL = \frac{1}{K_1}(\gamma UL)^{\frac{a_2}{a_1+a_2}}$

Or, $UL = \frac{1}{K_1}\{(a_1+a_2)\xi + A\}^{\frac{a_2}{a_1+a_2}}$ (4.4)

respectively, where K_1 is a constant of integration. Without loss of generality we may write $UL = u_e$ for combined convection flows. By virtue of equations (4.3) – (4.4) the general conditions for similarity requirements (2.38) yield relations between the constants (i.e. a's). These relations are

$a_0 = 0, \quad a_1, a_2$ arbitrary,

$$a_3 = \frac{r_W^2}{2\gamma}$$

$$\text{Or, } a_3 = \frac{r_W^2}{2K_1\{(a_1+a_2)\xi + A\}^{\frac{a_1}{a_1+a_2}}},$$

$$a_4 = 0,$$

$$a_5 = -\frac{\gamma}{UL} g_x \beta_T \Delta T$$

$$\text{Or, } a_5 = -\frac{K_1\{(a_1+a_2)\xi + A\}^{\frac{a_1}{a_1+a_2}}}{K_1^{-1}\{(a_1+a_2)\xi + A\}^{\frac{a_2}{a_1+a_2}}} g_x \beta_T \Delta T$$

$$\text{Or, } a_5 = -g_x \beta_T \Delta T\, K_1^2 \{(a_1+a_2)\xi + A\}^{\frac{a_1-a_2}{a_1+a_2}},$$

$$a_6 = \frac{\gamma}{UL}\{(u_e)_\tau + u_e (u_e)_\xi\}$$

$$\text{Or, } a_6 = \frac{\gamma}{UL}\{(UL)_\tau + UL(UL)_\xi\}$$

$$\text{Or, } a_6 = \frac{\gamma}{UL}\{UL(UL)_\xi\} \quad \because (UL)_\tau = 0$$

$$\text{Or, } a_6 = \gamma (UL)_\xi$$

$$\text{Or, } a_6 = a_2,$$

$$a_7 = \gamma(\log \Delta T)_\tau$$

$$\text{Or, } a_7 = 0,$$

$$a_8 = \gamma UL\{\log \Delta T\}_\xi \qquad \left(\text{Since } \Delta T = \frac{-a_5 UL}{g_x \beta_T \gamma} = \frac{-a_5(a_1\xi + A)}{g_x \beta_T (a_0 \tau + B)}\right)$$

$$\text{Or, } a_8 = \gamma UL \frac{\partial}{\partial \xi}\left\{\log\left(\frac{-a_5 UL}{g_x \beta_T \gamma}\right)\right\}$$

$$\text{Or, } a_8 = \gamma UL \frac{\partial}{\partial \xi}\{\log(-a_5) + \log(UL) - \log g_x - \log \beta_T - \log(\gamma)\}$$

$$\text{Or, } a_8 = \gamma UL\left\{\frac{(UL)_\xi}{UL} - \frac{\gamma_\xi}{\gamma}\right\}$$

Or, $a_8 = \gamma UL \frac{(UL)_\xi}{UL} - \gamma UL \frac{\gamma_\xi}{\gamma}$

Or, $a_8 = \gamma (UL)_\xi - UL\gamma_\xi$

Or, $a_8 = a_2 - a_1$, and a_9 is arbitrary.

In view of the above relations, the general equations (2.36) and (2.37) take the following forms in this case:

$$2\nu\left[\{\bar{\phi} + a_3\}\tilde{f}_{\bar{\phi}\bar{\phi}}\right]_{\bar{\phi}} + a_9\tilde{f}_{\bar{\phi}\bar{\phi}} + (a_1 + a_2)\tilde{f}\,\tilde{f}_{\bar{\phi}\bar{\phi}} - a_2\tilde{f}_{\bar{\phi}}^2 + a_5\vartheta + a_2 = 0$$

$$\text{Or, } 2\nu\left[\{\bar{\phi} + a_3\}\tilde{f}_{\bar{\phi}\bar{\phi}}\right]_{\bar{\phi}} + a_9\tilde{f}_{\bar{\phi}\bar{\phi}} + (a_1 + a_2)\tilde{f}\,\tilde{f}_{\bar{\phi}\bar{\phi}} + a_2\left(1 - \tilde{f}_{\bar{\phi}}^2\right) + a_5\vartheta = 0 \tag{4.5}$$

and

$$\frac{2\nu}{P_r}\left[\{\bar{\phi} + a_3\}\vartheta_{\bar{\phi}}\right]_{\bar{\phi}} + a_9\vartheta_{\bar{\phi}} + (a_1 + a_2)\tilde{f}\,\vartheta_{\bar{\phi}} + (a_1 - a_2)\tilde{f}_{\bar{\phi}}\vartheta = 0 \tag{4.6}$$

As in sub-section Case (A) ((3.2) and (3.3)), substituting $\tilde{f} = \alpha_1 f$, $\bar{\phi} = \alpha_1\eta$ choosing $\frac{a_1 + a_2}{\nu}\alpha_1 = 1$, finally writing $\frac{r_W^2 u_e}{2\nu(\xi + \xi_0)} = R_0$ and $\frac{a_2}{a_1 + a_2} = \beta$ the above equations (4.5) and (4.6) are simplified to

$$\frac{2\nu}{\alpha_1^2}\{(\alpha_1\eta + a_3)f_{\eta\eta}\}_\eta + \frac{a_9}{\alpha_1}f_{\eta\eta} + (a_1 + a_2)f\,f_{\eta\eta} + a_2\left(1 - f_\eta^2\right) + a_5\vartheta = 0$$

$$\text{Or, } \frac{2\nu}{\alpha_1}\left\{\left(\eta + \frac{a_3}{\alpha_1}\right)f_{\eta\eta}\right\}_\eta + \frac{a_9}{\alpha_1}f_{\eta\eta} + (a_1 + a_2)f\,f_{\eta\eta} + a_2\left(1 - f_\eta^2\right) + a_5\vartheta = 0$$

$$\text{Or, } 2\left\{\left(\eta + \frac{a_3}{\alpha_1}\right)f_{\eta\eta}\right\}_\eta + \frac{a_9}{\nu}f_{\eta\eta} + \frac{a_1 + a_2}{\nu}\alpha_1 f\,f_{\eta\eta} + \frac{a_2\alpha_1}{\nu}\left(1 - f_\eta^2\right) + \frac{a_5\alpha_1}{\nu}\vartheta = 0$$

$$\text{Or, } 2(\eta + R_0)f_{\eta\eta\eta} + 2f_{\eta\eta} + F_W f_{\eta\eta} + f\,f_{\eta\eta} + \beta\left(1 - f_\eta^2\right) + \frac{U_F^2}{u_e^2}\vartheta = 0$$

$$\text{Or, } 2(\eta + R_0)f_{\eta\eta\eta} + (2 + F_W + f)f_{\eta\eta} + \beta\left(1 - f_\eta^2\right) + \frac{U_F^2}{u_e^2}\vartheta = 0 \tag{4.7}$$

and

$$\frac{2\nu}{\Pr\alpha_1^2}\left\{(\alpha_1\eta + a_3)\vartheta_\eta\right\}_\eta + \frac{a_9}{\alpha_1}\vartheta_\eta + \frac{a_1 + a_2}{\alpha_1}f\,\vartheta_\eta + (a_1 - a_2)f_\eta\vartheta = 0$$

$$\text{Or, } \frac{2\nu}{\Pr \alpha_1}\left\{\left(\eta+\frac{a_3}{\alpha_1}\right)\vartheta_\eta\right\}_\eta + \frac{a_9}{\alpha_1}\vartheta_\eta + \frac{a_1+a_2}{\alpha_1} f\,\vartheta_\eta + (a_1-a_2) f_\eta \vartheta = 0$$

$$\text{Or, } \frac{2}{\Pr}\left\{(\eta+R_0)\vartheta_\eta\right\}_\eta + \frac{a_9}{\nu}\vartheta_\eta + \frac{a_1+a_2}{\nu} f\,\vartheta_\eta + \frac{(a_1-a_2)\alpha_1}{\nu} f_\eta \vartheta = 0$$

$$\text{Or, } 2(\eta+R_0)\vartheta_{\eta\eta} + 2\vartheta_\eta + \Pr F_W \vartheta_\eta + \Pr f\,\vartheta_\eta + \Pr\left\{\frac{(a_1+a_2)\alpha_1}{\nu} - 2\frac{a_2}{\nu}\alpha_1\right\} f_\eta \vartheta = 0$$

$$\text{Or, } 2(\eta+R_0)\vartheta_{\eta\eta} + \Pr\left[\left(\frac{2}{\Pr}+F_W+f\right)\vartheta_\eta + (1-2\beta) f_\eta \vartheta\right] = 0 \tag{4.8}$$

The boundary conditions are $f(0)=f_\eta(0)=\vartheta(\infty)=0,\ f_\eta(\infty)=\vartheta(0)=1$ (4.9)

Here the free convection velocity U_F, caused by the temperature difference $(T_w - T_0)$, is connected with by the following equation:

$$U_F^2 = -g_x \beta_T \Delta T(\xi+\xi_0) \tag{4.10}$$

where $(\xi+\xi_0)$, i.e. $(x+x_0)$, is a local characteristic length. The controlling parameters established here are Pr, the Prandtl number of the fluid, R_0, the body radius parameter, $\frac{U_F^2}{u_e^2}$, the square of the ratio between the free convection velocity U_F (Ostrach [46]) and the forcing velocity u_e and β, the exponent of the external velocity $\left(= u_e \propto (x+x_0)^\beta\right)$ variations. The latter exponent is consequently related to the exponent of the surface temperature $\left(= \Delta T \propto (x+x_0)^{2\beta-1}\right)$ variations. For the above class of similarity solution, the following restrictions must be satisfied:

(i) $u_e \propto (x+x_0)^\beta$

(ii) $(T_w - T_0) \propto (x+x_0)^{2\beta-1}$

(iii) $r_W \propto (x+x_0)^{\frac{1-\beta}{2}}$ $\left[\because R_0 = \frac{r_W^2 u_e}{2\nu(x+x_0)}\right]$

The case $\beta = 0$ results in the outer flow moving with constant velocity around a vertical body which is a paraboloid of revolution with latusrectum of length $\frac{2\nu R_0}{u_0}$ and the surface temperature varies inversely as the distance along the axis measured from the stagnation point. The restrictions (i) and (iii) are basically for forced flow, originally established by Probstein and Elliott [51]. The similarity function, similarity variable, the velocity components and the body-radius parameter $(= R_0)$ related to the equations (4.7) and (4.8) are

$$\psi = \gamma U F + \psi(\xi, \phi_0, \tau)$$
$$= \gamma U L \tilde{f}(\bar{\phi}) + \psi(\xi, \phi_0, \tau)$$
$$= \gamma U L \alpha_1 f(\eta) + \psi(\xi, \phi_0, \tau)$$
$$= \gamma U L \frac{\nu}{a_1 + a_2} f(\eta) + \psi(\xi, \phi_0, \tau)$$
$$= \gamma U L \frac{\nu \beta}{a_2} f(\eta) + \psi(\xi, \phi_0, \tau)$$
$$= \gamma U L \frac{\nu \beta}{\gamma (UL)_\xi} f(\eta) + \psi(\xi, \phi_0, \tau)$$
$$= \nu \frac{U L \beta}{(UL)_\xi} f(\eta) + \psi(\xi, \phi_0, \tau)$$
$$= \nu \frac{\beta}{\log(UL)} f(\eta) + \psi(\xi, \phi_0, \tau)$$
$$= \nu \log(UL)^{-\beta} f(\eta) + \psi(\xi, \phi_0, \tau)$$
$$= \nu (x + x_0) f(\eta) + \psi(\xi, \phi_0, \tau)$$

Hence $f(\eta) = \{\nu(x + x_0)\}^{-1} \{\psi(\xi, \phi, \tau) - \psi(\xi, \phi_0, \tau)\}$ (4.10)

$$\alpha_1 \eta = \bar{\phi}$$

Or, $\alpha_1 \eta = \dfrac{r^2 - r_W^2}{2\gamma}$

Or, $\eta = \dfrac{r^2 - r_W^2}{2\alpha_1 \gamma}$

$$= \frac{r^2 - r_W^2}{2 \dfrac{\nu}{a_1 + a_2} \gamma}$$
$$= \frac{(r^2 - r_W^2) a_2}{2 \nu \gamma \beta}$$
$$= \frac{(r^2 - r_W^2) \gamma (UL)_\xi}{2 \nu \gamma \beta}$$

$$= \frac{(r^2 - r_W^2)(UL)_\xi \, UL}{2\nu\beta UL}$$

$$= \frac{(r^2 - r_W^2)\frac{\partial}{\partial\xi}\{\log(UL)\}u_e}{2\nu\beta}$$

$$= \frac{(r^2 - r_W^2)u_e}{2\nu(x + x_0)^\beta}$$

Hence $\eta = \frac{\left(r^2 - r_W^2\right)u_e}{2\nu\left(x + x_0\right)}$ (4.11)

$$u = \frac{\partial}{\partial\phi}\left(\frac{\psi}{\gamma}\right)$$

$$= \frac{\partial}{\partial\phi}\left\{\frac{\gamma U F + \psi(\xi, \phi_0, \tau)}{\gamma}\right\}$$

$$= \frac{\partial}{\partial\phi}\left(\frac{\gamma U F}{\gamma}\right) + \frac{\partial}{\partial\phi}\left\{\frac{\psi(\xi, \phi_0, \tau)}{\gamma}\right\}$$

$$= \frac{\partial}{\partial\phi}(U F)$$

$$= \frac{\partial}{\partial\phi}\{U L \tilde{f}(\phi)\}$$

$$= U L \frac{\partial \tilde{f}(\phi)}{\partial\phi}$$

$$= U L \tilde{f}_{\bar{\phi}}$$

Hence $u = u_e f_\eta(\eta)$ (4.12)

$$-v = \frac{\nu}{r}\left[f(\eta) - (1-\beta)(\eta + R_0) f_\eta(\eta) - r_W v_W\right] \quad (4.13)$$

$$R_0 = \frac{r_W^2}{2\alpha_1 B}$$

$$= \frac{r_W^2}{2\frac{\nu}{a_1 + a_2}}$$

$$= \frac{r_W^2(a_1 + a_2)}{2\nu}$$

$$= \frac{r_W^2 \gamma (UL)_\xi}{2\nu \beta}$$

Hence $R_0 = \dfrac{r_W^2 u_e}{2\nu(x + x_0)}$ (4.14)

It is interesting to mention here that in the absence of buoyancy effects $\left(e.g.\ U_F^2 = 0\right)$ the similarity solution of the momentum equation given by Glauert and Lighthill [20] may be obtained for β=0 in present case with a minor change in similarity variable $(= \eta \text{ to } 2\eta_a)$.

4.2 Flow Parameters

The viscous drag of a body moving through a fluid is obtained from the shearing stress distribution at the wall. This is one of the boundary-layer characteristics of interest. The shearing stress at the wall is

$$\tau_W = \mu \left.\frac{\partial u}{\partial r}\right]_{r=r_W} \tag{4.15}$$

and in terms of the similarity variable,

$$\tau_W = \mu \frac{\partial u}{\partial r} \quad \text{at} \quad r = r_W$$

$$= \mu \frac{r}{\gamma} \frac{\partial}{\partial \phi} \left\{ \frac{\partial}{\partial \phi} \left(\frac{\psi}{\gamma} \right) \right\}$$

$$= \mu \frac{r}{\gamma} \frac{\partial}{\partial \phi} \left\{ \frac{\partial}{\partial \phi} \left(\frac{\gamma U F + \psi(\xi, \phi_0, \tau)}{\gamma} \right) \right\}$$

$$= \mu \frac{r}{\gamma} \frac{\partial}{\partial \phi} \left\{ \frac{\partial}{\partial \phi} (UF) \right\}$$

$$= \mu \frac{r}{\gamma} \frac{\partial}{\partial \phi} \left\{ \frac{\partial}{\partial \phi} \left(U L \tilde{f} \right) \right\}$$

$$= \mu \frac{r_W U L}{\gamma} \tilde{f}_{\bar{\phi}\bar{\phi}}$$

$$= \mu \frac{r_W u_e}{\gamma} \frac{1}{\alpha_1} f_{\eta\eta}(0)$$

$$= \mu u_e \frac{r_W}{\gamma} \frac{a_1 + a_2}{\nu} f_{\eta\eta}(0)$$

$$= \mu u_e \frac{r_W}{\gamma} \frac{a_2}{\nu \beta} f_{\eta\eta}(0)$$

$$= \mu u_e \frac{r_W}{\gamma} \frac{\gamma (UL)_\xi}{\nu \beta} f_{\eta\eta}(0)$$

$$= \mu u_e r_W \frac{(UL)_\xi}{\nu \beta} f_{\eta\eta}(0)$$

Hence $\tau_W = \frac{\mu r_W}{\nu} \frac{u_e^2}{(\xi + \xi_0)} f_{\eta\eta}(0)$ (4.16)

Usually, the skin friction and Reynolds number are combined to given the boundary-layer shear stress at the wall in the form,

$$C_f \sqrt{R_{e_x}} = 2\sqrt{2R_0} f_{\eta\eta}(0) \tag{4.17}$$

where

$C_f = \frac{2\lambda_w}{\rho u_e^2}$ (local skin friction)

$R_{e_x} = \frac{u_e (x + x_0)}{\nu}$ (local Reynolds number)

$R_0 = \frac{r_W^2 u_e}{2\nu (x + x_0)}$ (local body radius parameter)

and $(x + x_0)$ is the local characteristic length.

It is necessary to know the heat transfer rate between the boundary-layer and the wall. The heat transfer rate at the wall is given by Fourier's law and is related to the non-dimensional temperature function by

$$q_W = -K \frac{\partial T}{\partial r}, \quad r = r_W$$

$$= -K \frac{r}{\gamma} \frac{\partial T}{\partial \phi}$$

$$= -K \frac{r_W}{\gamma} \Delta T \frac{\partial \vartheta}{\partial \bar{\phi}}$$

$$= -K \frac{r_W}{\gamma} \Delta T \frac{1}{\alpha_1} \vartheta_\eta(0)$$

$$= -K \frac{r_W}{\gamma} \Delta T \frac{a_1 + a_2}{\nu} \vartheta_\eta(0)$$

$$= -K\frac{r_W}{\gamma}\Delta T\frac{a_2}{\nu\beta}\vartheta_\eta(0)$$

$$= -K\frac{r_W}{\gamma}\Delta T\frac{\gamma(UL)_\xi}{\nu\beta}\vartheta_\eta(0)$$

$$= -K\frac{r_W}{\nu}\Delta T\frac{(UL)_\xi}{\beta}\vartheta_\eta(0)$$

Hence $q_W = -\dfrac{K\,\Delta T\, r_w u_e}{\nu(x+x_0)}\vartheta_\eta(0)$ (4.18)

Hence the counterpart of the equation (4.17) is

$$\frac{N_{u_x}}{\sqrt{R_{e_x}}} = -\sqrt{2R_0}\,\vartheta_\eta(0) \tag{4.19}$$

Due to the non-linearity of the ordinary differential equation it is not possible to obtain analytical solutions of the equations (4.7) – (4.8) as in the Case (A) ((3.2) and (3.3)). Therefore numerical solutions are obtained for specified values of the controlling parameters F_W, Pr, R_0, β and $\dfrac{U_F^2}{u_e^2}$.

4.3 Analysis for Natural Convection $\left(\Delta T \propto (x+x_0)^{2\beta-1}\right)$

For natural convection $u_e = 0$ yields $a_6 = 0$. Here, UL (obtained in equation (4.4)) should be considered as the non-dimensionalsing characteristic velocity (maximum velocity within the boundary-layer generated by the buoyancy effects). Without loss of generality by setting $\dfrac{U_F^2}{(UL)^2} = 1$, we get UL or $U_F = \left[-g_x\beta_T\,\Delta T(x+x_0)\right]^{\frac{1}{2}}$. Replacing u_e by U_F in the combined convection analysis and setting $a_6 = 0$, the governing simplified equations for this case take the following forms:

$$\frac{2\nu}{\alpha_1^2}\left\{(\alpha_1\eta + a_3)f_{\eta\eta}\right\}_\eta + \frac{a_9}{\alpha_1}f_{\eta\eta} + (a_1+a_2)f\,f_{\eta\eta} - a_2 f_\eta^2 + a_5\vartheta = 0$$

$$\text{Or, } \frac{2\nu}{\alpha_1}\left\{\left(\eta + \frac{a_3}{\alpha_1}\right)f_{\eta\eta}\right\}_\eta + \frac{a_9}{\alpha_1}f_{\eta\eta} + (a_1+a_2)f\,f_{\eta\eta} - a_2 f_\eta^2 + a_5\vartheta = 0$$

$$\text{Or, } 2\left\{\left(\eta + \frac{a_3}{\alpha_1}\right)f_{\eta\eta}\right\}_\eta + \frac{a_9}{\nu}f_{\eta\eta} + \frac{a_1+a_2}{\nu}\alpha_1 f\,f_{\eta\eta} - \frac{a_2\,\alpha_1}{\nu}f_\eta^2 + \frac{a_5\alpha_1}{\nu}\vartheta = 0$$

$$\text{Or, } 2(\eta+R_0)f_{\eta\eta\eta}+2f_{\eta\eta}+F_W f_{\eta\eta}+f\,f_{\eta\eta}-\beta f_\eta^2+\frac{U_F^2}{u_e^2}\vartheta=0$$

$$\text{Or, } 2(\eta+R_0)f_{\eta\eta\eta}+(2+F_W+f)f_{\eta\eta}-\beta f_\eta^2+\vartheta=0 \tag{4.20}$$

and

$$\frac{2\nu}{\Pr\alpha_1^2}\{(\alpha_1\eta+a_3)\vartheta_\eta\}_\eta+\frac{a_9}{\alpha_1}\vartheta_\eta+(a_1+a_2)f\,\vartheta_\eta+(a_1-a_2)f_\eta\vartheta=0$$

$$\text{Or, } \frac{2\nu}{\Pr\alpha_1}\left\{\left(\eta+\frac{a_3}{\alpha_1}\right)\vartheta_\eta\right\}_\eta+\frac{a_9}{\alpha_1}\vartheta_\eta+(a_1+a_2)f\,\vartheta_\eta+(a_1-a_2)f_\eta\vartheta=0$$

$$\text{Or, } \frac{2}{\Pr}\{(\eta+R_0)\vartheta_\eta\}_\eta+\frac{a_9}{\nu}\vartheta_\eta+\frac{a_1+a_2}{\nu}\alpha_1 f\,\vartheta_\eta+\frac{(a_1-a_2)\alpha_1}{\nu}f_\eta\vartheta=0$$

$$\text{Or, } 2(\eta+R_0)\vartheta_{\eta\eta}+2\vartheta_\eta+\Pr F_W\vartheta_\eta+\Pr f\,\vartheta_\eta+\Pr\left\{\frac{(a_1+a_2)\alpha_1}{\nu}-2\frac{a_2}{\nu}\alpha_1\right\}f_\eta\vartheta=0$$

$$\text{Or, } 2(\eta+R_0)\vartheta_{\eta\eta}+\Pr\left[\left(\frac{2}{\Pr}+F_W+f\right)\vartheta_\eta+(1-2\beta)f_\eta\vartheta\right]=0 \tag{4.21}$$

The boundary conditions are $f(0)=f_\eta(0)=f_\eta(\infty)=0,\ \vartheta(0)=1,\ \vartheta(\infty)=0$ (4.22)

The similarity function, the similarity variable and the velocity components associated with the above equations are

$$f(\eta)=\{\nu(x+x_0)\}^{-1}[\psi(\xi,\phi,\tau)-\psi(\xi,\phi_0,\tau)] \tag{4.23}$$

$$\eta=\left[-\frac{g_x\beta_T\,\Delta T(x+x_0)^3}{4\nu^2}\right]\frac{(r^2-r_W^2)}{2\nu(x+x_0)^2} \tag{4.24}$$

$$u=U_F f_\eta(\eta), \tag{4.25}$$

$$-v=\frac{\nu}{r}\left[f(\eta)-(1-\beta)(\eta+R_0)f_\eta(\eta)-r_W v_W\right] \tag{4.26}$$

respectively. The parameters are F_W Pr, β and R_0, where β is related to the exponent of the ΔT -variations $\left(=Q(x+x_0)^{2\beta-1}\right)$ and R_0 is the body radius parameter given by

$$R_0=\left[-\frac{g_x\beta_T\Delta T(x+x_0)^3}{4\nu^2}\right]^{\frac{1}{2}}\frac{r_W^2}{(x+x_0)^2} \tag{4.27}$$

The parameter β establishes a bridge between the exponents of ΔT, r_W and u_e-variations for combined convection flows. For this situation (natural convection flows) similarity solutions exist only when the natures of ΔT and r_W are

(i) $\Delta T \propto x^{2\beta-1}$

(ii) $r_W \propto x^{\frac{1-\beta}{2}}$

It will be recalled that Van Dyke [67] has also studied this type of flow and the above statement agrees with that of Van Dyke [67]. The body shear stress and heat transfer rate are given by

$$\lambda_w = \sqrt{2R_0}\,\frac{\mu_r \nu}{(x+x_0)^2}\left[-\frac{g_x\beta_T\Delta T(x+x_0)^3}{\nu^2}\right]^{\frac{3}{4}} f_{\eta\eta}(0) \tag{4.28}$$

$$q_w = -\sqrt{2R_0}\,\frac{k_W\Delta T}{(x+x_0)}\left[-\frac{g_x\beta_T\Delta T(x+x_0)^3}{\nu^2}\right]^{\frac{1}{4}} \vartheta_\eta(0) \tag{4.29}$$

The expressions ((4.28) – (4.29)) replacing u_e by U_F in the definitions of C_f, R_e and R_0, where

$$U_F^2 = -g_x\beta_T\Delta T(x+x_0) \tag{4.30}$$

Substituting $2(\eta+R_0)=(R_v+\eta_v)^2$ (where R_v is constant) gives

$$\eta = \frac{1}{2}(R_v+\eta_v)^2 - R_0$$

Or, $1=\frac{1}{2}.2(R_v+\eta_v)\frac{\partial\eta_v}{\partial\eta}$

Or, $\frac{\partial\eta_v}{\partial\eta}=\frac{1}{(R_v+\eta_v)}$

Now,

$$f_\eta = \frac{\partial f}{\partial\eta} = \frac{\partial f}{\partial\eta_v}\frac{\partial\eta_v}{\partial\eta} = \frac{1}{(R_v+n_v)}f_{\eta_v}$$

$$f_{\eta\eta} = \frac{\partial}{\partial\eta}\left(\frac{\partial f}{\partial\eta}\right) = \frac{\partial}{\partial\eta_v}\left(\frac{\partial f}{\partial\eta}\right)\frac{\partial\eta_v}{\partial\eta} = \frac{\partial}{\partial\eta_v}\left(\frac{1}{R_v+n_v}f_{\eta_v}\right)\frac{1}{(R_v+n_v)}$$
$$= \left[\frac{1}{R_v+n_v}f_{\eta_v\eta_v} - \frac{1}{(R_v+n_v)^2}f_{\eta_v}\right]\frac{1}{R_v+\eta_v}$$

$$f_{\eta\eta\eta} = \frac{\partial}{\partial\eta}(f_{\eta\eta}) = \frac{\partial}{\partial\eta_v}(f_{\eta\eta})\frac{\partial\eta_v}{\partial\eta}$$

$$=\frac{\partial}{\partial\eta_v}\left[\frac{1}{(R_v+\eta_v)^2}f_{\eta_v\eta_v}-\frac{1}{(R_v+\eta_v)^3}f_{\eta_v}\right]\frac{1}{(R_v+\eta_v)}$$

$$=\frac{1}{(R_v+\eta_v)^3}f_{\eta_v\eta_v\eta_v}-\frac{3}{(R_v+\eta_v)^4}f_{\eta_v\eta_v}+\frac{3}{(R_v+\eta_v)^5}f_{\eta_v}$$

Again,

$$\vartheta_\eta=\frac{\partial\vartheta}{\partial\eta}=\frac{\partial\vartheta}{\partial\eta_v}.\frac{\partial\eta_v}{\partial\eta}=\frac{1}{R_v+\eta_v}\vartheta_{\eta_v}$$

$$\vartheta_{\eta\eta}=\frac{\partial}{\partial\eta}\left(\frac{\partial\vartheta_\eta}{\partial\eta}\right)=\frac{\partial}{\partial\eta_v}\left(\frac{\partial\vartheta_\eta}{\partial\eta}\right).\frac{\partial\eta_v}{\partial\eta}$$

$$=\frac{1}{(R_v+\eta_v)^2}\vartheta_{\eta_v\eta_v}-\frac{1}{(R_v+\eta_v)^3}\vartheta_{\eta_v}$$

The equation (4.20) is transformed to

$$(R_v+\eta_v)^2\left[\frac{1}{(R_v+\eta_v)^3}f_{\eta_v\eta_v\eta_v}-\frac{3}{(R_v+\eta_v)^4}f_{\eta_v\eta_v}+\frac{3}{(R_v+\eta_v)^5}f_{\eta_v}\right]$$
$$+(2+F_W+f)\left[\frac{1}{(R_v+\eta_v)^2}f_{\eta_v\eta_v}-\frac{1}{(R_v+\eta_v)^3}f_{\eta_v}\right]-\beta\frac{1}{(R_v+\eta_v)^2}f_{\eta_v}^2+\vartheta=0$$

Or, $$\frac{1}{(R_v+\eta_v)}f_{\eta_v\eta_v\eta_v}-\frac{3}{(R_v+\eta_v)^2}f_{\eta_v\eta_v}+\frac{3}{(R_v+n_v)^3}f_{\eta_v}+(2+F_W+f)\left[\frac{1}{(R_v+\eta_v)^2}f_{\eta_v\eta_v}\right.$$
$$\left.-\frac{1}{(R_v+\eta_v)^3}f_{\eta_v}\right]-\beta\frac{1}{(R_v+\eta_v)^2}f_{\eta_v}^2+\vartheta=0$$

Or, $$f_{\eta_v\eta_v\eta_v}-\frac{3}{(R_v+\eta_v)}f_{\eta_v\eta_v}+\frac{3}{(R_v+\eta_v)^2}f_{\eta_v}+(2+F_W+f)\left[\frac{1}{(R_v+\eta_v)}f_{\eta_v\eta_v}\right.$$
$$\left.-\frac{1}{(R_v+\eta_v)^2}f_{\eta_v}\right]-\beta\frac{1}{(R_v+\eta_v)}f_{\eta_v}^2+\vartheta(R_v+\eta_v)=0$$

Or, $$f_{\eta_v\eta_v\eta_v}-\frac{(1-F_W-f)}{(R_v+\eta_v)}f_{\eta_v\eta_v}+\frac{(1-F_W-f)}{(R_v+\eta_v)^2}f_{\eta_v}-\beta\frac{1}{(R_v+\eta_v)}f_{\eta_v}^2+\vartheta(R_v+n_v)=0 \quad (4.31)$$

The equation (4.21) is transformed to

$$(R_v+\eta_v)^2\left[\frac{1}{(R_v+\eta_v)^2}\vartheta_{\eta_v\eta_v}-\frac{1}{(R_v+\eta_v)^3}\vartheta_{\eta_v}\right]$$

$$+\Pr\left[\left(\frac{2}{p_r}+F_W+f\right)\frac{1}{R_v+\eta_v}\vartheta_{\eta_v}+(1-2\beta)\frac{1}{R_v+\eta_v}f_{\eta_v}\vartheta\right]=0$$

Or, $\vartheta_{\eta_v\eta_v}-\frac{1}{(R_v+\eta_v)}\vartheta_{\eta_v}+\Pr\left[\left(\frac{2}{p_r}+F_W+f\right)\frac{1}{R_v+\eta_v}\vartheta_{\eta_v}+(1-2\beta)\frac{1}{R_v+\eta_v}f_{\eta_v}\vartheta\right]=0$

Or, $\Pr^{-1}\left[\vartheta_{\eta_v\eta_v}-\frac{1}{(R_v+\eta_v)}\vartheta_{\eta_v}+\frac{2}{R_v+\eta_v}\vartheta_{\eta_v}\right]+(F_W+f)\frac{1}{R_v+\eta_v}\vartheta_{\eta_v}+(1-2\beta)\frac{1}{R_v+\eta_v}f_{\eta_v}\vartheta=0$

Or, $\Pr^{-1}\left\{(R_v+\eta_v)\vartheta_{\eta_v\eta_v}+\vartheta_{\eta_v}\right\}+(F_W+f)\vartheta_{\eta_v}+(1-2\beta)f_{\eta_v}\vartheta=0$

Or, $\Pr^{-1}\left\{(R_v+\eta_v)\vartheta_{\eta_v}\right\}_{\eta_v}+(F_W+f)\vartheta_{\eta_v}+(1-2\beta)f_{\eta_v}\vartheta=0$ (4.32)

The above equations with the boundary conditions attached to (4.22) are the similarity equations derived by Van Dyke [67]. The parameter β in the present form of the equations has been changed to m by a relation $m=$ (1-2 β) for Van Dyke's [67] equations. Van Dyke [67] displayed velocity and temperature profiles for $\Pr=0.72$, $\beta=0.5$ and $R_v=k=0.05$ although no numerical data were included. He remarked that his numerical predictions did not agree well with the experimental results. This may, in part, be due to the use of the approximate Boussinesq form of the boundary-Layer equations in their complete variable properties form, the flow equations will be shown later to be controlled by another parameter $\left(\frac{\Delta T}{T_r}\right)$ for constant surface temperature.

The skin friction $(\infty f_{\eta\eta}(0))$ and heat transfer rate $(\infty\vartheta_\eta(0))$ are related to the corresponding factors $(=f_{\eta_v\eta_v}(0)$ and $=\vartheta_{\eta_v}(0))$ of the equations (4.31) and (4.32) by the following equations:

$$f_{\eta\eta}(0)=\frac{1}{R_v^2}f_{\eta_v\eta_v}(0)$$

and

$$\vartheta_\eta(0)=\frac{1}{R_v}\vartheta_{\eta_v}(0)$$

where $2R_0=R_v^2$ holds good for the body radius parameters used in the two system of equations (4.20) and (4.21).

Changing the similarity variable η to η_c where $2(\eta + R_0) = \eta_c$ the equations (4.20) and (4.21) may be transformed to those dealt by Cebeci and Na [12] with a particular value of the parameter $\beta\left(=\frac{1}{2}\right)$. The Nusselt number and the skin friction coefficient for a given value of the local Reynolds number (based on free convection velocity U_F) increase with decreasing values of the body radius parameter R_0 for the specified values of Pr and β. Similar effects were also shown graphically by Cebeci and Na [12]. Furthermore, the present investigation shows that, for fixed values of Pr and R_0 the Nusselt number increases with increasing values of the parameter β, whereas the skin friction coefficient correspondingly decreases (rather slowly). This is due to the fact that, at least for β>1, with the increasing values of β, the solid body becomes more and more flat at the leading edge and ultimately the body shape takes the form of a horizontal circular disc having its maximum thickness at the centre with the heated surface facing upwards.

4.4 Analysis for Combined Convection in Exponential Steady Situation $\left(u_e \propto e^{\frac{x}{d}}\right)$

Finally in this research for possible similarity cases, we look for another possible situation, when any one of the constants considered arbitrary in case (4.1) be set equal to zero, If we set either a_1 or $a_2 = 0$, this leads to the special cases in which β is either 1 or zero in the case (4.1). On the other hand if we set $(a_1 + a_2) = 0$, we obtain, different equation for UL and for this case we obtain,

$$\gamma = B(\xi) \tag{4.35}$$

$$\gamma UL = A \tag{4.36}$$

By virtue of (2.38) one finds, from the equations (4.35) and (4.36), γ and UL in the following forms:

$$\gamma = B(\xi) = e^{-\frac{a_2(\xi+\xi_0)}{A}} \tag{4.37}$$

$$UL = Ae^{\frac{a_2(\xi+\xi_0)}{A}} \tag{4.38}$$

In view (4.37) and (4.38) the relations between the constants are easy to obtain from the similarity requirements (2.38). These are

$$a_0 = (a_1 + a_2) = a_4 = 0,\ a_2 = \text{arbitrary},\ a_3 = \frac{r_w^2}{2e^{\frac{-a_2(\xi+\xi_0)}{A}}}$$

$$a_5 = \frac{-g_x \beta_T \Delta T}{A} e^{\frac{-2a_2(\xi+\xi_0)}{A}},\ a_6 = a_2,\ a_7 = 0 \text{ and } a_8 = 2a_2$$

As before, after simplifications, the general equations (2.36) and (2.37) can be reduced to the following forms in combined convection flows:

$$\{2(\eta + R_0) f_{\eta\eta}\}_\eta + 1 - f_\eta^2 + \frac{U_F^2}{u_e^2}\vartheta = 0 \tag{4.39}$$

$$P_r^{-1}\{2(\eta + R_0)\vartheta_\eta\}_\eta - f_\eta \vartheta = 0 \tag{4.40}$$

The boundary conditions are

$$f(0) = f_\eta(0) = 0,\ f_\eta(\infty) = \vartheta(0) = 1,\ \vartheta(\infty) = 0 \tag{4.41}$$

In this case, the similarity function, the similarity variable, the velocity components, the body radius parameter and the free convection velocity $(= U_F)$ are given by

$$\psi = \nu d f(n),\ \eta = \frac{r^2 - r_w^2}{2\nu d} u_e \tag{4.42}$$

$$u = u_e f_\eta(\eta), -v = \frac{\nu}{r}(\eta + R_0) f_\eta(\eta), \tag{4.43}$$

$$R_0 = \frac{r_W^2}{2\nu d} u_e,\quad U_F^2 = -g_x \beta_T \Delta T d \tag{4.44}$$

respectively. Here 'd' is a non-dimensional constant characteristic length. It is substituted for the inverse of the co-efficient of the exponential term in the external (=forcing) velocity $u_2\left(= Ae\frac{a_2(x+x_0)}{A}, \text{ so we put } \frac{A}{a_2} = d\right)$. It can be easily shown that the following restrictions on the variations in u_e, ΔT and r_W are necessary for the above type of similarity solution to exist:

(i) $u_e \propto e^{\frac{x+x_0}{d}}$

(ii) $\Delta T \propto e^{-\frac{2(x+x_0)}{d}}$

(iii) $r_W^2 \propto e^{-\frac{(x+x_0)}{d}}$

In this case the governing differential equations are controlled by only three parameters, namely Pr, R_0 and $\frac{U_F^2}{u_e^2}$. Other essential flow parameters like the skin friction coefficient and the heat transfer coefficient may easily be obtained from the equations (4.43) and Fourier's law. These are in the usual forms of

$$C_f\sqrt{\mathrm{Re}_d} = 2\sqrt{2R_0}\, f_{\eta\eta}(0) \tag{4.45}$$

where suffix $'d'$ denotes that the above coefficients are based on the constant characteristic length d.

In the absence of an external forcing velocity u_e, the motion of the fluid around the object is determined entirely by the ΔT-variations. Such a flow in this case will be governed by the following similarity equations:

$$\{2(\eta + R_0) f_{\eta\eta}\}_\eta - f_\eta^2 + \vartheta = 0 \tag{4.47}$$

$$P_r^{-1}\{(\eta + R_0)\vartheta_\eta\}_\eta - f_\eta \vartheta = 0 \tag{4.48}$$

The boundary conditions attached to (4.47) and (4.48) are

$$f(0) = f_\eta(0) = f_\eta(\infty) = 0, \vartheta(0) = 1, \vartheta(\infty) = 0 \tag{4.49}$$

The similarity function $f(\eta)$ the similarity variable η the velocity components u, v and the body radius parameter R_0 associated with the equations to (4.47) and (4.48) are

$$f(\eta) = \frac{\psi}{\nu d} \tag{4.50}$$

$$\eta = \left[-\frac{g_x \beta_T \Delta T d^3}{4\nu^2}\right]^{\frac{1}{2}} \frac{r^2 - r_w^2}{d^2} \tag{4.51}$$

$$u = \left[-g_x \beta_T \Delta T d\right]^{\frac{1}{2}} f_\eta(\eta), \tag{4.52}$$

$$-v = \frac{\nu}{r}(\eta + R_0) f_\eta(\eta) - \frac{r_W v_W}{r} \tag{4.53}$$

$$R_0 = \left[-\frac{g_x \beta_T \Delta T d^3}{4\nu^2}\right]^{\frac{1}{2}} \frac{r_w^2}{d^2} \tag{4.54}$$

respectively. The skin friction τ_w and the heat transfer rate q_w at the body surface may be expressed in the forms of the equations (4.28) and (4.29) where $(x + x_0)$ there should be replaced by `d' in the present case.

4.5 Numerical Scheme and Procedure

The set of differential equations (4.7)–(4.8) with the boundary conditions (4.9) are solved numerically by using computer software. Like Case A, here the velocity f_η and temperature ϑ are determined as a function of coordinate η. The skin friction coefficient $f_{\eta\eta}(0)$ and the heat transfer rate $-\vartheta_\eta(0)$ are also evaluated for this case and numerical results thus obtained in terms of the similarity variables are displayed in graphs and tables for several selected values of the established parameters F_w, β, $\frac{U_F^2}{u_e^2}$, R_0 and Pr below.

4.6 Numerical Results and Discussion

To obtain the solution of differential equations (4.7) – (4.8) with the boundary conditions (4.9), we have adopted a numerical procedure based on computer software. The effects of suction parameter F_w, driving parameter β (the ratio between the changes of local boundary-layer thickness with regard to position and time), the buoyancy parameter $\frac{U_F^2}{u_e^2}$ (the square of the ratio between the fluid velocity caused by buoyancy effects and external velocity for the forced flow), the body radius parameter R_0 and the Prandtl number Pr are plotted in the Figure (4.1) – (4.10). To observe the effect of F_w, the other four parameters β, $\frac{U_F^2}{u_e^2}$, R_0 and Pr are taken constants. Similarly, we observe the effect of the parameters β, $\frac{U_F^2}{u_e^2}$, R_0 and Pr by taking the rest four parameters constant respectively.

The effects of F_w on the velocity and temperature fields are plotted in the Figure 4.1 and Figure 4.2. From Figure 4.1, we observe that the velocity is increasing for increasing values of F_w in the region $\eta < 1$. The velocity becomes negative when $\eta > 1.66$ (except for $F_w = 0.3$). Then the velocity again start increasing with increasing η and further become positive after $\eta > 6.9$. The maximum velocity appears at $\eta \approx 8.75$ and then finally converges to 1.0. Further we conclude that for the case of suction, the velocity increases with decreasing values of F_w and it decreases with increasing values of the magnitude of blowing in the region $0 \le \eta < 0.45$ and after that no sequential effect of F_w on the velocity fields are observed.

From Figure 4.2 we see that the temperature falls quickly close to $\eta = 0$ (leading edge) and away from it temperature decrease asymptotically and finally become zero with the increase of η. For the case of suction ($F_w > 0$), temperature decreases with decreasing suction. For the case of blowing ($F_w < 0$), temperature decreases swiftly close to the leading edge as the magnitude of blowing increases and away from it temperature decreases asymptotically and finally become zero for large value of η.

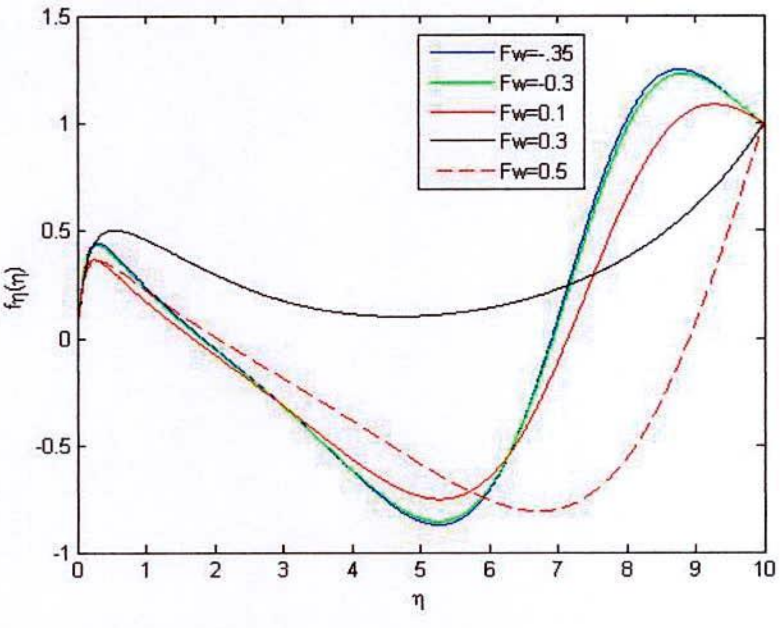

Figure 4.1: Velocity profiles for different values of F_w (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $R_0 = 0.1$ and Pr=0.71, β=0.5).

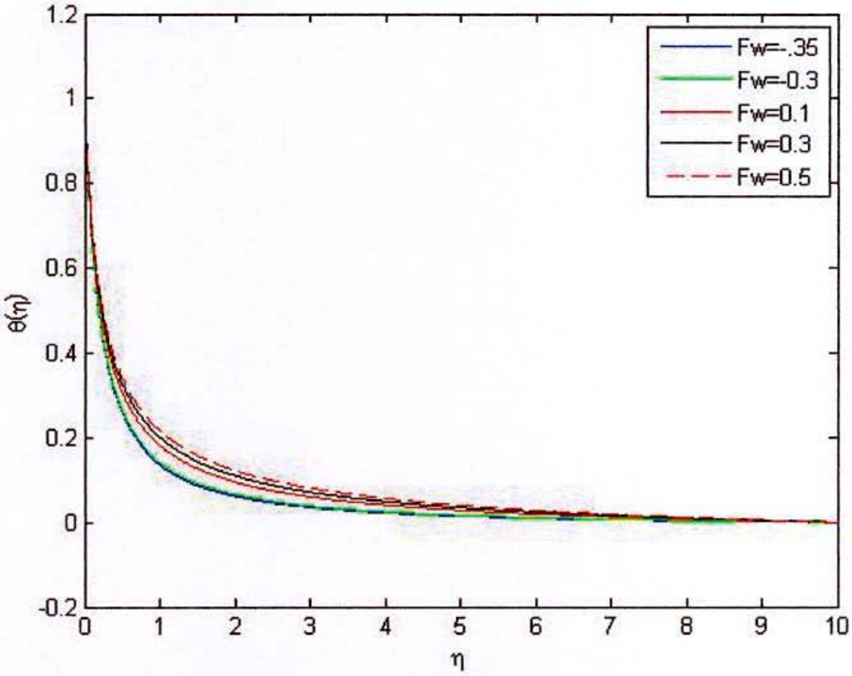

Figure 4.2: Temperature profiles for different values of F_w (with fixed values of $\frac{U_F^2}{u_e^2}=0.5$, $R_0=0.1$ and Pr=0.71, $\beta=0.5$).

The body radius parameter R_0 depends on the shape of the slender body. The velocity and temperature fields exhibit remarkable changes with the variation of R_0 as observed from Figure 4.3 and Figure 4.4. It is observed from Figure 4.3 that with the increase in η for different values of R_0, velocity increases with the increase of R_0 when $\eta \le 0.4$ and then velocity decrease with increasing η and after a certain stage velocity becomes negative when $\eta < 0.72$ (except for $F_w = 0.3$) for a large period and then velocity again become positive and finally leads to one asymptotically. It is observed from Figure. 4.4 that with the increase in η for different values of R_0 the temperature falls quickly with the increase in R_0 and finally leads to zero asymptotically.

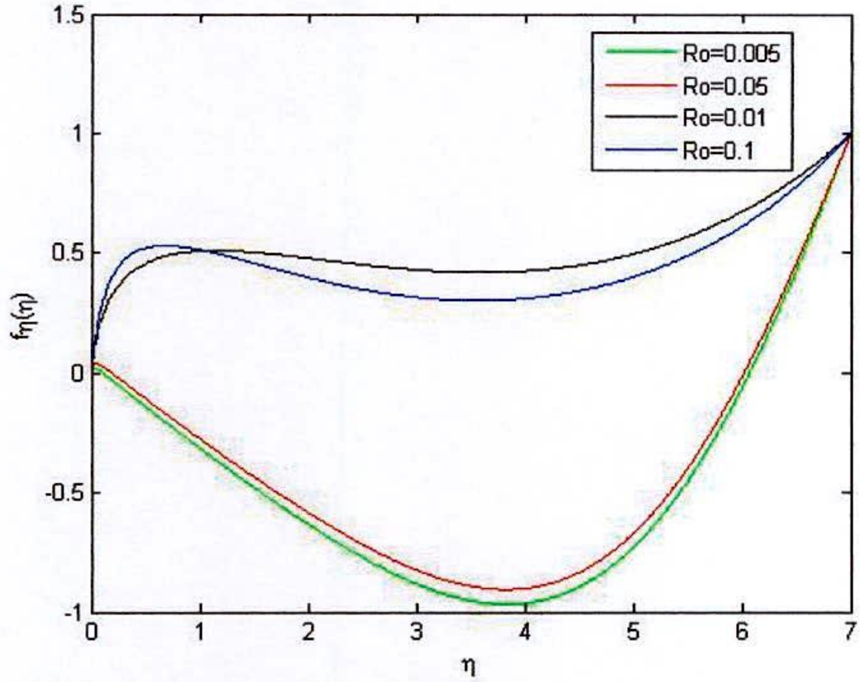

Figure 4.3: Velocity profiles for different values of R_0 (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $F_w = 0.5$ and $\Pr = 0.71$, $\beta = 0.5$).

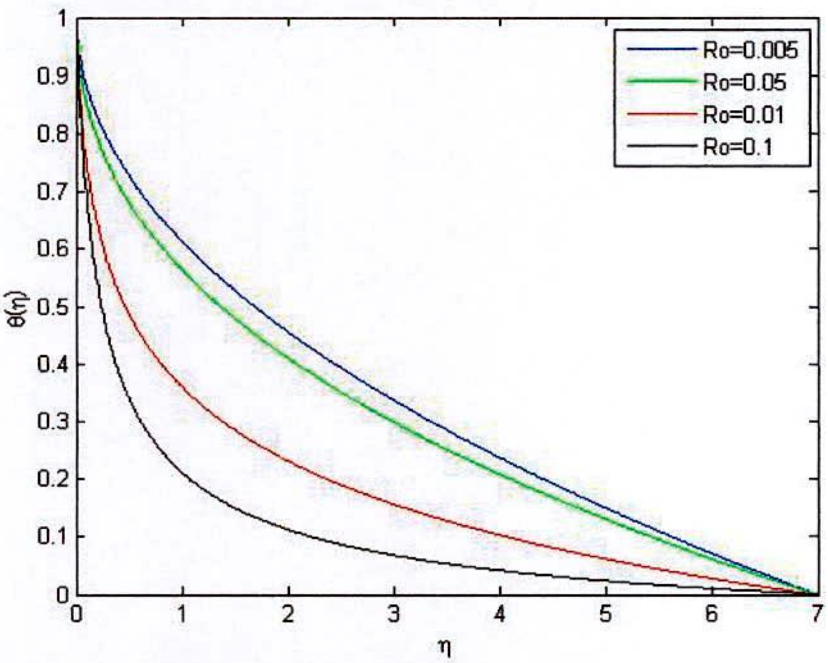

Figure 4.4: Temperature profiles for different values of R_0 (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $F_w = 0.5$ and $\Pr = 0.71$, $\beta = 0.5$).

The other controlling parameter is the Prandtl number $\Pr\left(=\frac{\mu C_p}{k}\right)$ which depends on the properties of the medium. The velocity and temperature fields exhibit considerable changes with the variation of Pr as observed from Figure 4.5 and Figure 4.6. It is observed from Figure 4.5 that with the increase in η for different values of Pr, the velocity increases slightly with increasing Pr and finally leads to one asymptotically all together. Like before, temperature decrease quickly with the decreasing Pr close to the leading edge and finally leads to zero smoothly for large η as is seen in Figure 4.6.

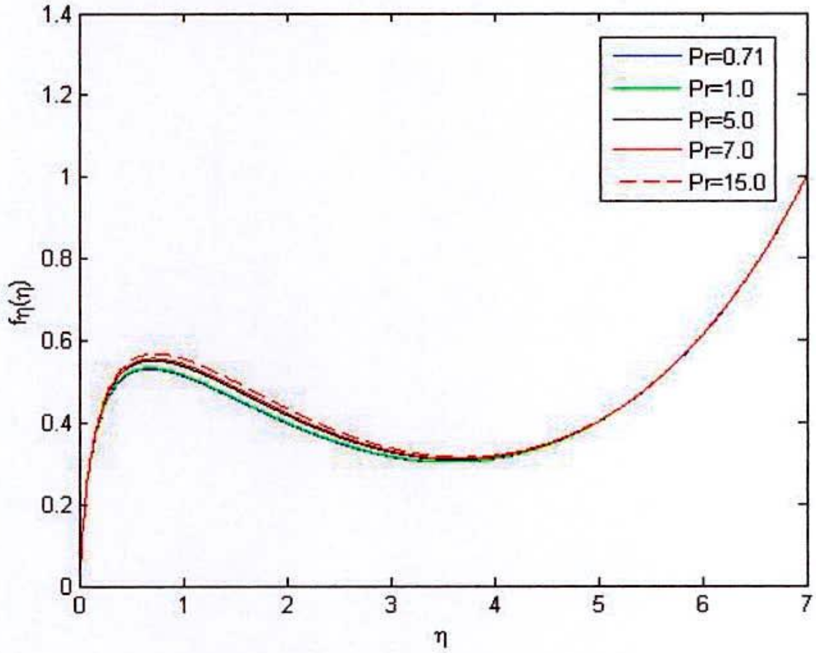

Figure 4.5: Velocity profiles for different values of Pr (with fixed values of $\frac{U_F^2}{u_e^2}=0.5$, $\beta=0.5$, $F_w=0.5$ and $R_0=0.1$).

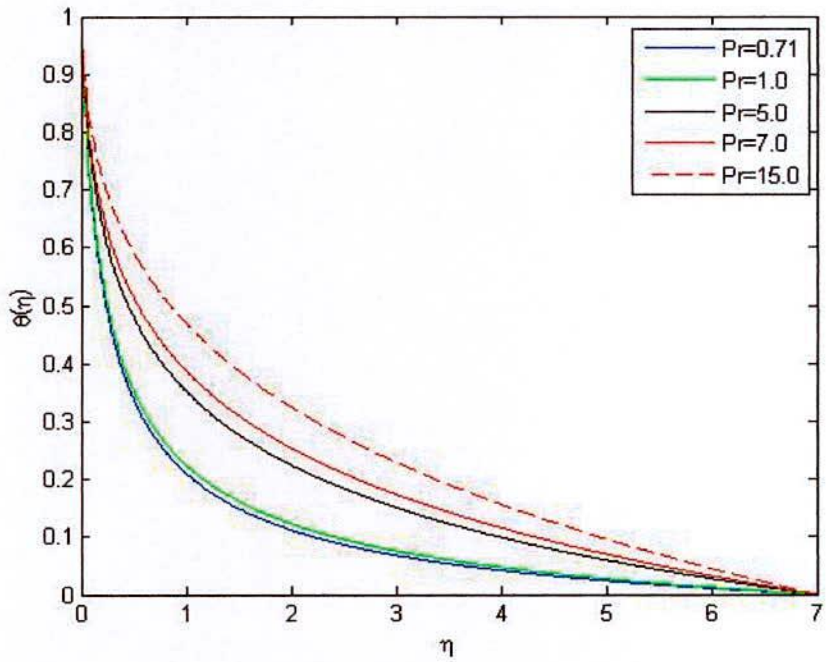

Figure 4.6: Temperature profiles for different values of Pr (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $\beta = 0.5$, $F_w = 0.5$ and $R_0 = 0.1$).

Figure 4.7 and Figure 4.8 exhibit the effects of the driving parameter β on the velocity and temperature fields, respectively. The velocity and temperature fields exhibit remarkable changes with the variation of β as observed from Figure 4.7 and Figure 4.8. It is observed from Figure 4.7 that with the increase in η for different values of β, the velocity increases with increasing β when $\eta < 0.8$. The velocity profiles change their directions (for those values of $\beta > 1.5$) and become negative after $\eta > 1.8$. The magnitude of velocity increase with the increase of β in the negative flow region (for $\beta > 1.5$) where $\eta > 4.0$ and velocity again become positive and finally leads to one asymptotically. Here velocity again increases with the decrease of β in the positive flow region where $\eta > 6.8$.

Like before, temperature decreases faster with the increasing β and finally leads to zero asymptotically as is seen in Figure 4.8.

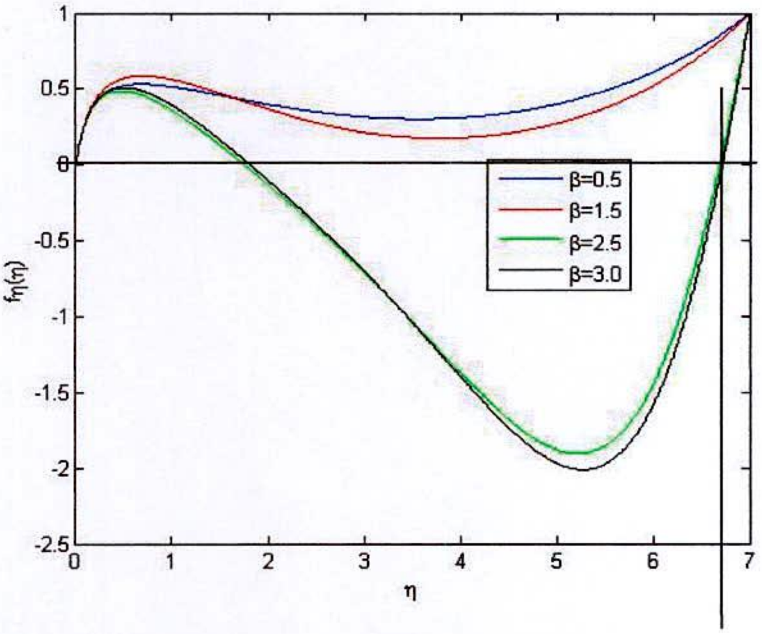

Figure 4.7: Velocity profiles for different values of β (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $\Pr = 0.71$, $F_w = 0.5$ and $R_0 = 0.1$).

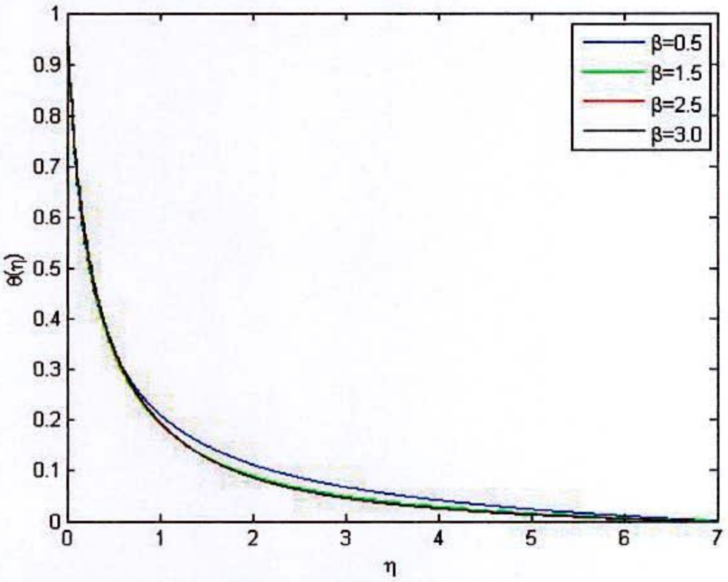

Figure 4.8: Temperature profiles for different values of β (with fixed values of $\frac{U_F^2}{u_e^2} = 0.5$, $\Pr = 0.71$, $F_w = 0.5$ and $R_0 = 0.1$).

From Figure 4.9 it is observed that, with the increase in the buoyancy parameter $\frac{U_F^2}{u_e^2}$ from negative to positive values, the velocity increases and thus goes to 1.0 asymptotically without being negative within a short range for all values of $\frac{U_F^2}{u_e^2}\left(\text{except for } \frac{U_F^2}{u_e^2}=5.0\right)$. But a chaotic behaviour is observed for large value of the buoyancy parameter $\left(\frac{U_F^2}{u_e^2}=5.0\right)$. Here the velocity increases close to the leading edge and then decreases with increasing η, become negative when $\eta > 0.8$ and finally leads to 1.0 after being positive when $\eta > 6.45$.

No significant effect of $\frac{U_F^2}{u_e^2}$ on the temperature field is found as is seen in Figure 4.10.

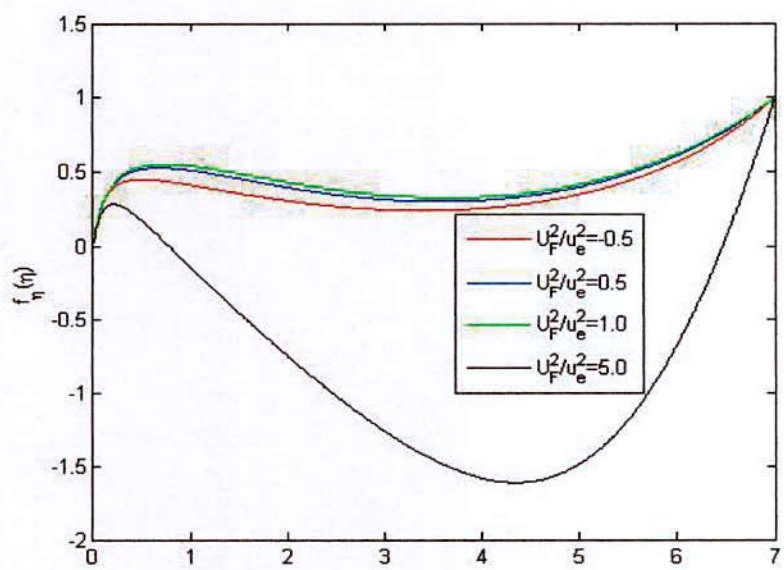

Figure 4.9: Velocity profiles for different values of $\frac{U_F^2}{u_e^2}$ (with fixed values of $\beta = 0.5$, $\Pr = 0.71$, $F_w = 0.5$ and $R_0 = 0.1$).

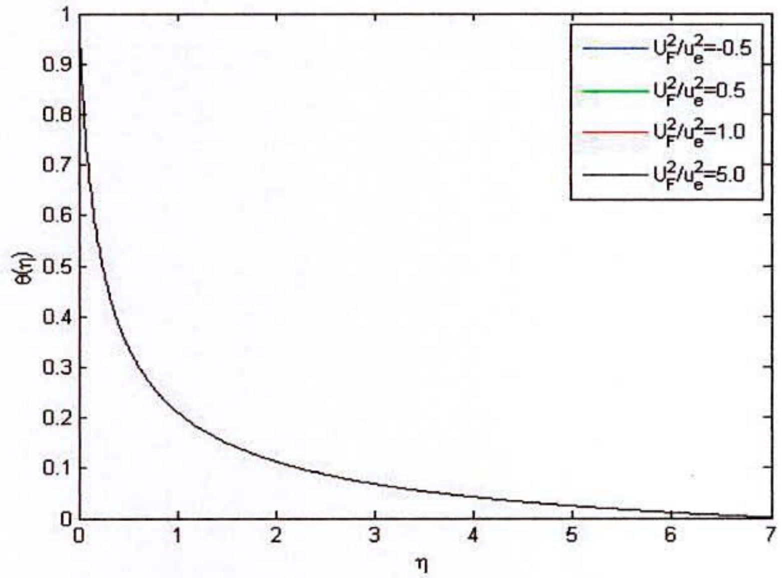

Figure 4.10: Temperature profiles for different values of $\frac{U_F^2}{u_e^2}$ (with fixed values of $\beta = 0.5$, $\Pr = 0.71$, $F_w = 0.5$ and $R_0 = 0.1$).

The values proportional to the coefficients of skin friction $f^{//}(0)$ and heat transfer $-\vartheta^{/}(0)$ are tabulated in Table (4.1) – (4.5).

Table 4.1: Values proportional to the coefficients of skin-friction ($f^{//}(0)$) and heat transfer ($-\vartheta^{/}(0)$) with the variation of suction parameter F_w (for fixed $U_F^2/u_e^2 = 0.5$, $R_0 = 0.1$, $\beta = 0.5$ and $\Pr = 0.71$)

F_w	$f^{//}(0)$	$-\vartheta^{/}(0)$
0.50	0.088367	-6.02010
0.30	0.088918	-5.43997
-0.30	0.087382	-3.84120
-0.50	0.086666	-3.59908

Table 4.2: Values proportional to the coefficients of skin-friction $f^{//}(0)$ and heat transfer $-\vartheta^{/}(0)$ with the variation of body radius parameter R_0 (for fixed $F_w = 0.5$, $U_F^2/u_e^2 = 0.5$, $\beta = 0.5$ and $\Pr = 0.71$.

R_0	$f^{//}(0)$	$-\vartheta^{/}(0)$
0.005	0.478168	-56.42110
0.01	0.312535	-46.75050
0.05	0.132539	-11.59130
0.1	0.088367	-6.02010

Table 4.3: Values proportional to the coefficients of skin-friction ($f''(0)$) and heat Transfer ($-\vartheta'(0)$) with the variation of Prandtl number Pr (for fixed $\beta = 0.5$, $U_F^2/u_e^2 = 0.5$ and $R_0 = 0.1$)

Pr	$f''(0)$	$-\vartheta'(0)$
0.71	0.088367	-6.02010
1.00	0.077284	-7.882919
5.00	0.029811	-33.61870
7.0	0.010929	-52.283413

Table 4.4: Values proportional to the coefficients of skin-friction ($f''(0)$) and heat Transfer ($-\vartheta'(0)$) with the variation of driving parameter β (for fixed $F_w = 0.5$, Pr=0.71, $U_F^2/u_e^2 = 0.5$ and $R_0 = 0.1$)

β	$f''(0)$	$-\vartheta'(0)$
0.5	0.088367	-6.02010
1.5	0.088367	-6.02017
2.5	0.074113	-5.20063
3.0	0.062708	-4.54997

Table 4.5: Values proportional to the coefficients of skin-friction ($f''(0)$) and heat Transfer ($-\vartheta'(0)$) with the variation of buoyancy parameter U_F^2/u_e^2 (for fixed $F_w = 0.5$, $\beta = 0.5$, $R_0 = 0.1$ and $\mathrm{Pr} = 0.71$)

U_F^2/u_e^2	$f''(0)$	$-\vartheta'(0)$
0.5	0.088367	-6.02010
1.0	0.176796	-6.02010
2.0	0.265931	-6.02010
5.0	0.695437	-6.02010

CHAPTER V

Conclusions and Recommendations

By using the technique of similarity solutions, the governing boundary layer equations for the two-dimensional mixed convective laminar boundary layer flow around a vertical slender body have been solved in this present research works. It is considered that the surface of the slender is porous so that the effects of suction and blowing are taken into account. Different similarity cases arise with the choice of $\frac{dA}{d\tau}$ and $\frac{dB}{d\xi}$ either zero or constant. Similarity solutions for two of the cases, namely, Case A and Case B have been studied in two different Chapters of this thesis. On the basis of the findings, it is observed from Case A that:

(a) Velocity increases with the decrease of suction but for the case of blowing the velocity increases with the increase of the magnitude of blowing.

(b) Temperature decreases with decreasing suction but it decreases more with the increase of the magnitude of blowing.

(c) With the increase in the buoyancy parameter $\frac{U_F^2}{u_e^2}$ from negative to positive values, the velocity enhanced slightly and thus the velocity goes to 1.0 asymptotically within a short range.

(d) No effect of buoyancy parameter $\frac{U_F^2}{u_e^2}$ on the temperature field is observed.

(e) Velocity highly increases with the increase of R_0 and finally leads to 1.0 asymptotically for large η.

(f) Temperature decreases sharply with the increase of R_0 and finally approaches zero asymptotically for large η.

(g) With the increase in $\Pr$, the velocity increases negligibly and finally leads to the value 1.0 asymptotically for large value of η.

(h) Temperature decreases momentously with decreasing Pr and finally approaches zero asymptotically.

But in the study of case B it is observed that this case is very case sensitive relative to the values of controlling parameters and no symmetric relationships of controlling parameters on the flow variables are observed here.

On the basis of the findings, it is observed from Case B that:

(a) The possible values of F_W are restricted to $-0.35 \le F_W \le 0.5$.

(b) For the case of suction, the velocity increases with decreasing values of F_w and it decreases with increasing values of the magnitude of blowing in the region $0 \le \eta < 0.55$. After that no sequential effect of F_w on the velocity fields are observed.

(c) For the case of suction, temperature profiles decrease with decreasing suction and for the case of blowing, temperature decreases swiftly close to the leading edge as the magnitude of blowing increases and away from it temperature decreases asymptotically and finally become one for large value of η.

(d) With the increase in R_0, the velocity increases when $\eta \le 0.4$ and then velocity decrease with increasing η and after a certain stage velocity becomes negative when $\eta < 0.72$ (except for $F_w = 0.3$) for a large period and then velocity again become positive and finally leads to 1.0 asymptotically.

(e) With the increase in R_0 the temperature decreases quickly and finally leads to zero asymptotically.

(f) Velocity increases slightly with increasing Pr and finally leads to one asymptotically all together.

(g) Temperature decreases quickly with the decreasing Pr close to the leading edge and finally leads to zero smoothly for large η

(h) No sequential effect of β on the velocity field is seen.

(i) Temperature decreases faster with the increasing β and finally leads to zero asymptotically.

(j) With the increase in the buoyancy parameter $\frac{U_F^2}{u_e^2}$ from negative to positive values, the velocity increases and thus goes to 1.0 asymptotically without being negative within a short range for all values of $\frac{U_F^2}{u_e^2}\left(\text{except for } \frac{U_F^2}{u_e^2}=5.0\right)$.

(k) A chaotic behavior is observed for large value of the buoyancy parameter $\left(\frac{U_F^2}{u_e^2}=5.0\right)$.

(l) No significant effect of $\frac{U_F^2}{u_e^2}$ on the temperature field is found.

Therefore, our conclusion is that, study of Case A is more suitable than Case B.

Further study is necessary to solve rest of the cases.

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