Approximate Solution Techniques for Fourth Order Nonlinear Ordinary Differential Systems

by

Md. Asraful Alom
Roll No. 1051506

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Master of Philosophy
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Dedicated to My

Beloved Parents

and

Affectionate Son

Declaration

This is to certify that the thesis work entitled "**Approximate Solution Techniques for Fourth Order Nonlinear Ordinary Differential Systems**" has been carried out by Md. Asraful Alom in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

Signature of Supervisor

Dr. M. Alhaz uddin
Assistant Professor
Department of Mathematics
KUET, Khulna-9203

Signature of Student

Alom/28.11.12

Md. Asraful Alom
Roll No. 1051506
Session: January, 2010

Approval

This is to certify that the thesis work submitted by Md. Asraful Alom entitled "**Approximate Solution Techniques for Fourth Order Nonlinear Ordinary Differential Systems**" has been approved by the board of examiners for the partial fulfillment of the requirements for the degree of Master of Philosophy in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh in December 2012.

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The author

Abstract

A perturbation method known as "the asymptotic averaging method" in the theory of nonlinear oscillations was first presented by Krylov and Bogoliubov (KB) in 1947. Primarily, the method was developed only to obtain the periodic solutions of second order weakly nonlinear differential systems. Later, the method of KB has been improved and justified by Bogoliubov and Mitropolskii in 1967. In literature, this method is known as the Krylov-Bogoliubov-Mitropolskii (KBM) method. Now a days, this method is used for obtaining the solutions of second, third and fourth order nonlinear differential systems for oscillatory, damped oscillatory, over damped, critically damped and more critically damped cases by imposing some restrictions. Ji-Huan He has developed a homotopy perturbation method for second order strongly nonlinear differential systems without damping. Recently, Uddin *et al.* have developed approximate analytical technique for second order strongly nonlinear differential systems with damping combining He's homotopy perturbation technique and the extended form of the KBM method. In this thesis, an analytical approximate technique will be presented by combining the He's homotopy perturbation technique and the extended form of the KBM method for solving certain type of fourth order strongly nonlinear differential systems with small damping and cubic nonlinearity. Also, the KBM method will be modified and elaborated to find out the solutions of fourth order weakly and near critically damped nonlinear differential systems by imposing some restrictions on the eigen values. To justify the presented methods, the approximate solutions have been compared to those solutions obtained by the fourth order ***Runge-Kutta*** method.

Publications

The following papers have been extracted from this thesis:

1. M. Alhaz Uddin, Md. Asraful Alom and M. Wali Ullah, "An analytical approximate technique for solving a certain type of fourth order strongly nonlinear oscillatory differential system with small damping", Far East Journal of Mathematical Sciences (FJMS), Vol. 67, No. 1, pp. 59-72, 2012.

2. Md. Asraful Alom and M. Alhaz Uddin, "Approximate Solution of Fourth Order Near Critically Damped Nonlinear Systems with Special Conditions", J. Bangladesh Academy of Sciences (accepted for publication and it will be appeared), Vol. 36, No. 2, December 2012.

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CHAPTER 1

Introduction

Differential equation is a mathematical tool, which has its application in many branches of knowledge of mankind. Numerous physical, mechanical, chemical, biological, biochemical, and many other relations appear mathematically in the form of differential equations that are linear or nonlinear, autonomous or non-autonomous. Generally, in many physical phenomena, such as spring-mass systems, resistor-capacitor-inductor circuits, bending of beams, chemical reactions, the motion of pendulums, the motion of the rotating mass around another body, etc, the differential equations occur. Also, in ecology and economics the differential equations are vastly used. Basically, many differential equations involving physical phenomena are nonlinear. Differential equations, which are linear, are comparatively easy to solve and nonlinear are laborious and in some cases it is impossible to solve them analytically. In such situations, mathematicians, physicists and engineers convert the nonlinear equations into linear equations by imposing some conditions. The method of small oscillations is a well-known example of the linearization. But, such a linearization is not always possible and when it is not, then the original nonlinear equation itself must be used. The study of nonlinear equations is generally confined to a variety of rather special cases, and one must resort to various methods of approximation.

At first Van der Pol [1] paid attention to the new (self-excitations) oscillations and indicated that their existence is inherent in the nonlinearity of the differential equations characterizing the process. This nonlinearity appears, thus, as the very essence of these phenomena and by linearizing the differential equation in the sense of the method of small oscillations, one simply eliminates the possibility of investigating such problems. Thus, it is necessary to deal with the nonlinear problems directly instead of evading them by dropping the nonlinear terms. To solve nonlinear differential equations there exist some methods. Among the methods, the method of perturbations, i.e., asymptotic expansions in terms of a small parameter are foremost. Perturbation methods have recently received much attention as methods for accurately and quickly computing numerical solutions of dynamic stochastic economic equilibrium models, both single-agent or rational expectations models and multi-agent or game theoretic models. A perturbation method is based on the following aspects: The equations to be solved are sufficiently "smooth" or sufficiently differentiable a number of times in the required regions of variables and parameters.

The KBM [2-4] method was developed for the systems only to obtain the periodic solutions of second order nonlinear differential equations. Now, the method is used to obtain oscillatory as well as damped, critically damped, over damped, near critically damped, more critically damped oscillatory and non-oscillatory solutions of second, third, fourth etc, order nonlinear differential equations by imposing some restrictions to obtain the uniform solutions. Ji-Huan He [5-7] has developed a homotopy perturbation technique for solving second order strongly nonlinear differential systems without damping effects. Belendez *et al.* [8] have applied He's homotopy perturbation method to duffing harmonic oscillator. Later, Uddin *et al.* [9-11] have presented an approximate technique for solving second order strongly nonlinear oscillatory differential systems with damping effects combing by the He's [5-8] homotopy perturbation and the KBM [2-4] methods. The method of KB [2] is an asymptotic method in the sense that $\varepsilon \to 0$. An asymptotic series itself may not be convergent, but for a fixed number of terms, the approximate solution tends to the exact solution as $\varepsilon \to 0$. It may be noted that the term asymptotic is frequently used in the theory of oscillations in the sense that $\varepsilon \to \infty$. But, in this case, the mathematical method is quite different. It is an important approach to the study of such nonlinear oscillations in the small parameter expansion. Two widely spread methods in this theory are mainly used in the literature; one is averaging asymptotic KBM method and the other is multitime scale method. The KBM method is particularly convenient and is the extensively used technique to obtain the approximate solutions among the methods used to study the nonlinear differential systems with small nonlinearity. The KBM method starts with the solution of linear equation (sometimes called the generating solution of the linear equation), assuming that in the nonlinear case, the amplitude and phase in the solution of the linear differential equation are time dependent functions instead of constants. This method introduces an additional condition on the first derivative of the assumed solution for determining the solution of a second order equation. The KBM method is demanded that the asymptotic solutions are free from secular terms. These assumptions are mainly valid for second and third order equations. But, for the fourth order differential equation, the correction terms sometimes contain secular terms, although the solution is generated by the classical KBM asymptotic method. For this reason, the traditional solutions fail to explain the proper situation of the systems. To remove the presence of secular terms and obtain the desired results, we need to impose some conditions. The KBM method with some special conditions for fourth order weakly nonlinear differential systems and the homotopy perturbation technique for fourth order strongly nonlinear differential systems with damping almost remain untouched. The main objective of this thesis is to find out these limitations and to fill these gaps and to determine the proper solutions under some special conditions. The results may be used in mechanics, physics, chemistry, plasma physics, circuit and control theory, population dynamics, etc.

In this thesis, we have chosen fourth order nonlinear autonomous differential equations that describes strongly nonlinear oscillatory differential system with small damping and cubic nonlinearity to solve by combining the He's [5-11] homotopy perturbation technique and the extended form of the KBM [2-4] method. Also the modified KBM method has been extended to solve weakly and near critically damped nonlinear differential system.

In Chapter 2, the review of literature is presented. **In Chapter 3**, an approximate analytical technique has been developed for obtaining the solution of certain type of fourth order strongly nonlinear oscillatory differential system with small damping and cubic nonlinearity. The KBM method has been extended to solve weakly and near critically damped nonlinear non-oscillatory differential system with special conditions **in Chapter 4**. Finally, **in Chapter 5**, the conclusions are discussed.

CHAPTER 2

Literature Review

Nonlinear differential equations show varieties characters. But, mathematical formulations of many physical problems often result in differential equations are nonlinear. In many situations, linear differential equation is substituted for a nonlinear differential equation, which approximates the former equation closely enough to give expected result. In many cases such linearization is not possible, and, when it is not, the original nonlinear differential equation must be tackled directly. During last several decades in the 20^{th} century, some Russian scientists like Mandelstam and Papalexi [12], Krylov and Bogoliubov [2], Bogoliubov and Mitropolskii [3] unitedly investigated the nonlinear dynamics. To solve nonlinear differential equations there exist some methods. Among the methods, the method of perturbations, i.e., an asymptotic expansion in terms of small parameter is foremost. Firstly, Krylov and Bogoliubov [2] considered equations of the form

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}, t, \varepsilon), \tag{2.1}$$

where $\dot{x}$ denotes ordinary derivative with respect to t, ε is a small positive parameter and f is a power series in ε, whose coefficients are polynomials in x, $\dot{x}$, $\sin t$ and $\cos t$ and their proposed solution procedure is known as KB method. In general, f does not contain either ε or t. To describe the behavior of nonlinear oscillations by the solutions obtained by perturbation method, Lindstedt [13], Glyden [14], Liapounoff [15], Poincare [16] discussed only periodic solutions, transient were not considered. Most probably, Poisson initiated approximate solutions of nonlinear differential equations around 1830 and the technique was established by Liouville. The KBM [2-4] method started with the solution of the linear equation, assuming that in the nonlinear systems, the amplitude and phase variables in the solution of the linear equation are time dependent functions rather than constants. This procedure introduces an additional condition on the first derivative of the assumed solution for determining the desired results. Some meritful works are done and elaborative uses have been made by Stoker [17], McLachlan [18], Minorsky [19], Nayfeh [20] and Bellman [21]. Duffing [22] has investigated many significant results about the periodic solutions of the following nonlinear damped differential equation

$$\ddot{x} + 2k\,\dot{x} + \omega^2 x = -\varepsilon\, x^3. \tag{2.2}$$

Sometimes different types of nonlinear phenomena occur, when the amplitude of the dependent variable of a dynamic system is less than or greater than unity. The damping is negative when the amplitude is less than unity and the damping is positive when the amplitude is greater than unity. The governing equation having these phenomena is

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0. \tag{2.3}$$

This equation is known as Van der Pol equation. Kruskal [23] has extended the KB method to solve the fully nonlinear differential equation of the following form

$$\ddot{x} = F(x, \dot{x}, \varepsilon). \tag{2.4a}$$

Cap [24] has studied nonlinear systems of the form

$$\ddot{x} + \omega^2 f(x) = \varepsilon\, F(x, \dot{x}). \tag{2.4b}$$

Generally, since f does not contain either ε or t, thus the equation (2.1) becomes

$$\ddot{x} + \omega^2 x = \varepsilon\, f(x, \dot{x}). \tag{2.5}$$

As pointed certain that, in the treatment of nonlinear oscillations by perturbation method, only periodic solutions were discussed, transients were not considered by different investigators, where as Krylov and Bogoliubov have discussed transient response firstly. When $\varepsilon = 0$, the equation (2.5) reduces to linear equation and its solution is

$$x = a\cos(\omega t + \varphi), \tag{2.6}$$

where a and φ are arbitrary constants to be determined by using the given initial conditions.

When $\varepsilon \neq 0$, but is sufficiently small, then Krylov and Bogoliubov assumed that the solution of equation (2.5) is still given by equation (2.6) together with the derivative of the form

$$\dot{x} = -a\omega\sin(\omega t + \varphi), \tag{2.7}$$

where a and φ are functions of t, rather than being constants. In this case, the solution of equation (2.5) is

$$x = a(t)\cos(\omega t + \varphi(t)) \tag{2.8}$$

and the derivative of the solution is

$$\dot{x} = -a(t)\omega\sin(\omega t + \varphi(t)). \tag{2.9}$$

Differentiating the assumed solution equation (2.8) with respect to time t, we obtain

$$\dot{x} = \dot{a}\cos\psi - a\omega\sin\psi - a\dot{\varphi}\sin\psi,\ \ \psi = \omega t + \varphi(t). \tag{2.10}$$

Using the equations (2.7) and (2.10), we get

$$\dot{a}\cos\psi = a\dot{\varphi}\sin\psi. \tag{2.11}$$

Again, differentiating equation (2.9) with respect to *t*, we have

$$\ddot{x} = -\dot{a}\omega\sin\psi - a\omega^2\cos\psi - a\omega\dot{\varphi}\cos\psi. \tag{2.12}$$

Putting the value of $\ddot{x}$ from equation (2.12) into the equation (2.5) and using equations (2.8) and (2.9), we obtain

$$\dot{a}\,\omega\sin\psi + a\omega\dot{\varphi}\cos\psi = -\varepsilon\, f(a\cos\psi, -a\omega\sin\psi). \tag{2.13}$$

Solving equations (2.11) and (2.13), we have

$$\dot{a} = -\frac{\varepsilon}{\omega}\sin\psi\, f(a\cos\psi, -a\omega\sin\psi), \tag{2.14}$$

$$\dot{\varphi} = -\frac{\varepsilon}{a\omega}\cos\psi\, f(a\cos\psi, -a\omega\sin\psi). \tag{2.15}$$

It is observed that, a basic differential equation (2.5) of the second order in the unknown x, reduces to two first order differential equations (2.14) and (2.15) in the unknowns a and φ.

Moreover, $\dot{a}$ and $\dot{\varphi}$ are proportional to ε; a and φ are slowly varying functions of the time period $T = \frac{2\pi}{\omega}$. It is noted that these first order equations are now written in terms of the amplitude a and phase φ as dependent variables. Therefore, the right sides of equations (2.14) and (2.15) show that both $\dot{a}$ and $\dot{\varphi}$ are periodic functions of time T. In this case, the right-hand terms of these equations contain a small parameter ε and also contain both a and φ, which are slowly varying functions of the time t.with period $T = \frac{2\pi}{\omega}$. We can transform the equations (2.14) and (2.15) into more convenient form.

Now, expanding

$\sin\psi\, f(a\cos\psi, -a\omega\sin\psi)$ and $\cos\psi\, f(a\cos\psi, -a\omega\sin\psi)$ in Fourier series in the total phase ψ, the first approximate solution of equation (2.5), by averaging equations (2.14) and (2.15) with period $T = \frac{2\pi}{\omega}$, is

$$\begin{aligned} \langle\dot{a}\rangle &= -\frac{\varepsilon}{2\pi\,\omega}\int_0^{2\pi}\sin\psi\, f(a\cos\psi, -a\omega\,\sin\psi)\,d\psi, \\ \langle\dot{\varphi}\rangle &= -\frac{\varepsilon}{2\pi\,\omega\, a}\int_0^{2\pi}\cos\psi\, f(a\cos\psi, -a\,\omega\,\sin\psi)\,d\psi, \end{aligned} \tag{2.16}$$

where a and φ are independent of time t under the integrals. Krylov and Bogoliubov [2] called their method asymptotic in the sense that $\varepsilon \to 0$. An asymptotic series itself is not convergent, but for a fixed number of terms the approximate solution tends to the exact solution as $\varepsilon \to 0$. Later, this technique has been extended mathematically by Bogoliubov and Mitropolskii, and has extended to non-stationary vibrations by Mitropolskii [4]. They have assumed the solution of the nonlinear differential equation (2.5) of the form

$$x = a\cos\psi + \varepsilon u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \dots\dots\dots\dots + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}), \tag{2.17}$$

where u_k, $(k = 1, 2, \dots\dots, n)$ are periodic functions of ψ with a period 2π, and the quantities a and ψ are functions of time t and defined by the following first order ordinary differential equations

$$\begin{aligned} \dot{a} &= \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \dots\dots\dots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}), \\ \dot{\psi} &= \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \dots\dots + \varepsilon^n B_n(a) + O(\varepsilon^{n+1}). \end{aligned} \tag{2.18}$$

The functions $u_{k,}$ A_k and B_k, $(k = 1, 2, \dots\dots, n)$ are to be chosen in such a way that the equation (2.17), after replacing a and ψ by the functions defined in equation (2.18), is a solution of equation (2.5). Since there are no restrictions in choosing functions A_k and B_k, it generates the arbitrariness in the definitions of the functions u_k (Bogoliubov and Mitropolskii). To remove this arbitrariness, the following additional conditions are imposed

$$\begin{aligned} &\int_0^{2\pi} u_k(b,\psi)\cos\psi \, d\psi = 0, \\ &\int_0^{2\pi} u_k(a,\psi)\sin\psi \, d\psi = 0. \end{aligned} \tag{2.19}$$

Absences of secular terms in all successive approximations are guaranteed by these conditions. Differentiating equation (2.17) two times with respect to t, substituting the values of x, $\dot{x}$ and $\ddot{x}$ into equation (2.5), using these relations in equation (2.18) and equating the coefficients of ε^k, $(k = 1, 2, \dots\dots, n)$ result a recursive system leads to

$$\omega^2((u_k)_{\psi\psi} + u_k) = f^{(k-1)}(a,\psi) + 2\omega(a\,B_k\cos\psi + A_k\sin\psi), \tag{2.20}$$

where $(u_k)_\psi$ denotes partial derivatives with respect to ψ,

$f^{(0)}(a,\psi) = f(a\cos\psi, -a\omega\sin\psi)$ and

$$f^{(1)}(a,\psi) = u_1 f_x(a\cos\psi, -a\omega\sin\psi) + (A_1\cos\psi - aB_1\sin\psi + \omega(u_1)_\psi)\times$$
$$f_{\dot{x}}(\cos\psi, -a\omega\sin\psi) + (aB_1^2 - A_1\frac{dA_1}{da})\cos\psi + (2A_1B_1 - aA_1\frac{dB_1}{da})\sin\psi - \tag{2.21}$$
$$2\omega(A_1(u_1)_{a\psi} + B_1(u_1)_{\psi\psi}).$$

Here $f^{(k-1)}$ is a periodic function of ψ with period 2π which depends also on the amplitude a. Therefore, $f^{(k-1)}$ and u_k can be expanded in a Fourier series as

$$\begin{aligned} f^{(k-1)}(a,\psi) &= g_0^{(k-1)}(a) + \sum_{n=1}^{\alpha}(g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi), \\ u_k(a,\psi) &= v_0^{(k-1)}(a) + \sum_{n=1}^{\alpha}(v_n^{(k-1)}(a)\cos n\psi + \omega_n^{(k-1)}(a)\sin n\psi), \end{aligned} \tag{2.22}$$

where

$$g_0^{(k-1)} = \frac{1}{2\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\,d\psi. \tag{2.23}$$

Here, $v_1^{(k-1)} = \omega_1^{(k-1)} = 0$ for all values of k, because both integrals of equation (2.19) are vanished. Substituting these values into the equation (2.20), we obtain

$$\begin{aligned} &\omega^2 v_0^{(k-1)}(a) + \sum_{n=2}^{\alpha}\omega^2(1-n^2)[v_n^{(k-1)}(a)\cos n\psi + \omega_n^{(k-1)}(a)\sin n\psi] \\ &= g_0^{(k-1)}(a) + (g_1^{(k-1)}(a) + 2\omega a B_k)\cos n\psi + (h_1^{(k-1)}(a) + 2\omega A_k)\sin\psi \\ &+ \sum_{n=2}^{\alpha}[g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi]. \end{aligned} \tag{2.24}$$

Now, equating the coefficients of the harmonics of the same order, yield

$$\begin{aligned} &g_1^{(k-1)}(a) + 2\omega a B_k = 0, \quad h_1^{(k-1)}(a) + 2\omega A_k = 0, \quad v_0^{(k-1)}(a) = \frac{g_0^{(k-1)}(a)}{\omega^2}, \\ &v_n^{(k-1)}(a) = \frac{g_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad \omega_n^{(k-1)}(a) = \frac{h_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad n \geq 1. \end{aligned} \tag{2.25}$$

These are the sufficient conditions to obtain the desired order of approximation. For the first order approximation, we have

$$\begin{aligned} A_1 &= -\frac{h_1^{(0)}(a)}{2\omega} = -\frac{1}{2\pi\omega}\int_0^{2\pi} f(a\cos t\psi, -a\omega\sin\psi)\sin\psi\,d\psi, \\ B_1 &= -\frac{g_1^{(0)}}{2a\omega} = -\frac{1}{2\pi a\omega}\int_0^{2\pi} f(a\cos t\psi, -a\omega\sin\psi)\cos\psi\,d\psi. \end{aligned} \tag{2.26}$$

Thus, the variational equations in equation (2.18) become

$$\dot{a} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\, d\psi,$$
$$\dot{\psi} = \omega - \frac{\varepsilon}{2\pi a\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi\, d\psi. \tag{2.27}$$

It is seen that, the equation (2.27) are similar to the equation (2.16). Thus, the first order solution obtained by Bogoliubov and Mitropolskii [3] is identical to the original solution obtained by Krylov and Bogoliubov [2]. In literature, this method is well known as Krylov-Bogoliubov-Mitropolskii (KBM) [2-4] method. Secondly, higher order solutions can be found easily. The correction term u_1 is obtained by equation (2.22) on using equation (2.25) as

$$u_1 = \frac{g_0^{(0)}(a)}{\omega^2} + \sum_{n=2}^{\alpha} \frac{g_n^{(0)}(a)\cos n\psi + h_n^{(0)}(a)\sin n\psi}{\omega^2(1-n^2)} \tag{2.28}$$

The solution equation (2.17) together with u_1 is known as the first order improved solution in which a and ψ are obtained from equation (2.27). If the values of the functions A_1 and B_1 are substituted from equation (2.26) into the second relation of equation (2.21), the function $f^{(1)}$ and in the similar way, the functions A_2, B_2 and u_2 can be found. Therefore, the determination of the higher order approximation is completed. The KB [2] method is very similar to that of Van der Pol [1] and related to it. Van der Pol has applied the method of variation of constants to the basic solution $x = a\cos\omega t + b\sin\omega t$ of $\ddot{x} + \omega^2 x = 0$, on the other hand Krylov-Bogoliubov have applied the same method to the basic solution $x = a\cos(\omega t + \varphi)$ of the same equation. Thus, in the KB method the varied constants are a and φ, while in the Van der Pol's method the constants are a and b. The method of KB seems more interesting from the point of view of applications, since it deals directly with the amplitude and phase of the quasi-harmonic oscillation.

The solution of the equation (2.4a) is based on recurrent relations and is given as the power series of the small parameter. Cap [24] has solved the equation (2.4b) by using elliptical functions in the sense of Krylov and Bogoliubov. The method of KB has been extended by Popov [25] to damped nonlinear differential systems represented by the following equation

$$\ddot{x} + 2k\dot{x} + \omega^2 x = \varepsilon f(\dot{x}, x), \tag{2.29}$$

where $-2k\dot{x}$ is the linear damping force and $0 < k < \omega$. It is noteworthy that, because of the importance of the Popov's method in the physical systems, involving damping force, Mendelson [26] and Bojadziev [27] have retrieved Popov's results. In case of damped nonlinear differential systems, the first equation of equation (2.18) has been replaced by

$$\dot{a} = -k\,a + \varepsilon\,A_1(a) + \varepsilon^2 A_2(a) + \dots\dots\dots\dots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}). \tag{2.18a}$$

Murty and Deekshatulu [28] have developed a simple analytical method to obtain the time response of second order nonlinear over damped systems with small nonlinearity represented by the equation (2.29), based on the KB method of variation of parameters. Alam [29] has extended the KBM method to find solutions of over damped nonlinear differential systems, when one root of the auxiliary equation becomes much smaller than the other root. According to the KBM method, Murty *et al.* [30] have found a hyperbolic type asymptotic solution of an over damped system represented by the nonlinear differential equation (2.29), i.e., in the case $k > \omega$. They have used hyperbolic functions, $\cosh\varphi$ and $\sinh\varphi$ instead of their circular counterpart, which are used by Krylov, Bogoliubov, Mitropolskii, Popov and Mendelson. In case of oscillatory or damped oscillatory process may be used arbitrarily for all kinds of initial conditions. But, in case of non-oscillatory systems $\cosh\varphi$ or $\sinh\varphi$ should be used depending on the given set of initial conditions (Murty *et al.* [30], Bojadziev and Edwards [31], Murty [32]). Murty [32] has presented a unified KBM method for solving the nonlinear systems represented by the equation (2.29), which cover the undamped, damped and over-damped cases. Bojadziev and Edwards [31] have investigated solutions of oscillatory and non-oscillatory systems represented by equation (2.29) when k and ω are slowly varying functions of time t. Arya and Bojadziev [33-34] have examined damped oscillatory systems and time dependent oscillating systems with slowly varying parameters and delay. Sattar [35] has developed an asymptotic method to solve a second order critically damped nonlinear system represented by equation (2.29). He has found the asymptotic solution of the equation (2.29) in the following form

$$x = a(1+\psi) + \varepsilon\,u_1(a,\psi) + \dots\dots\dots\dots + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}), \tag{2.30}$$

where a is defined by the equation (2.18a) and ψ is defined by

$$\dot{\psi} = 1 + \varepsilon C_1(a) + \varepsilon^2 C_2(a) + \dots\dots\dots\dots + \varepsilon^n C_n(a) + O(\varepsilon^{n+1}) \tag{2.18b}$$

Osiniskii [36] has extended the KBM method to the following third order nonlinear differential equation

$$\dddot{x} + c_1 \ddot{x} + c_2 \dot{x} + c_3 x = \varepsilon f(\ddot{x}, \dot{x}, x), \tag{2.31}$$

where ε is a small positive parameter and f is a given nonlinear function. He has assumed the asymptotic solution of equation (2.31) in the form

$$x = a + b\cos\psi + \varepsilon u_1(a,b,\psi) + \dots\dots\dots + \varepsilon^n u_n(a,b,\psi) + O(\varepsilon^{n+1}), \tag{2.32}$$

where each u_k $(k = 1, 2, \dots\dots, n)$ is a periodic function of ψ with period 2π and a, b and ψ are functions of time t, and they are given by

$$\begin{aligned} \dot{a} &= -\lambda a + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \dots\dots\dots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}), \\ \dot{b} &= -\mu a + \varepsilon B_1(b) + \varepsilon^2 B_2(b) + \dots\dots\dots + \varepsilon^n B_n(b) + O(\varepsilon^{n+1}), \\ \dot{\psi} &= \omega + \varepsilon C_1(b) + \varepsilon^2 C_2(b) + \dots\dots\dots + \varepsilon^n C_n(b) + O(\varepsilon^{n+1}), \end{aligned} \tag{2.33}$$

where $-\lambda$, $-\mu \pm \omega$ are the eigen values of the equation (2.31) when $\varepsilon = 0$.

By using the KBM method, Bojadziev [37] has investigated solutions of nonlinear damped oscillatory systems with small time lag. Bojadziev [38] has also found solutions of damped forced nonlinear vibrations with small time delay. Bojadziev [39], Bojadziev and Chan [40] have applied the KBM method to solve problems of population dynamics. Bojadziev [27] has used the KBM method to investigate solutions of nonlinear systems arised from biological and biochemical fields. Lin and Khan [41] have also used the KBM method to some biological problems. Proskurjakov [42] and Bojadziev *et al.* [43] have investigated periodic solutions of nonlinear systems by the KBM and Poincare method, and have compared the two solutions. Bojadziev and Lardner [44-45] have investigated monofrequent oscillations in mechanical systems including the case of internal resonance, governed by hyperbolic differential equations with small nonlinearities. Bojadziev and Lardner [46] have also investigated solution for a certain hyperbolic partial differential equation with small nonlinearity and large time delay included into both unperturbed and perturbed parts of the equation. Rauch [47] has studied oscillations of a third order nonlinear autonomous system. Bojadziev [48] and Bojadziev and Hung [49] have developed a technique by using the method of KBM to investigate a weakly nonlinear differential system with strong damping. Osiniskii [50] has also extended the KBM method to a third order nonlinear partial differential equation with initial friction and relaxation. Mulholland [51] has studied nonlinear oscillations governed by a third order differential equation. Lardner and Bojadziev [52] have investigated nonlinear damped oscillations governed by a third order partial differential equation. They have introduced the concept of "couple amplitude" where the unknown functions A_k, B_k and C_k depend on both the amplitudes a and b. Bojadziev [48] and Bojadziev and Hung [49] have used at least two trial solutions to investigate time dependent differential systems; one is for resonant case and the other is for non-resonant case. But, Alam [53] has used only one set of variational equations, arbitrarily for both resonant and non-resonant cases. Alam *et al.* [54] have presented a general form of the KBM method for solving nonlinear partial differential equations. Raymond and Cabak [55] have examined the effects of internal resonance on impulsive forced nonlinear systems with two-degree-of-freedom. Later, Alam [56-57] has extended the method to *nth*, $n \geq 2,3$ order nonlinear differential systems. Alam [58] has presented a new perturbation method based on the KBM method to find approximate solutions of second order nonlinear systems with large damping. Alam *et al.* [59] have investigated perturbation solution of a second order time dependent nonlinear system based on the modified KBM method. Sattar [60] has extended the KBM asymptotic method for three dimensional over damped nonlinear systems. In recent years, He [5] has developed some new approaches to Duffing equation with strongly and high order nonlinearity (I) linearized perturbation method. In another paper, He [6] has obtained the approximate solution of nonlinear differential equation with convolution product nonlinearities. Also, He [7] has presented a new interpretation of homotopy perturbation method. Belendez *et al.* [8] have presented the application of He's homotopy perturbation method to Duffing harmonic oscillator. Lim *et al.* [61] have also presented a new analytical approach to the Duffing harmonic oscillator. Later, Uddin *et al.* [9-11] have presented an approximate technique for solving second order strongly nonlinear oscillatory differential systems with damping effects combing by the He's [5-8] homotopy perturbation and the KBM [2-4] methods. Alam and Sattar [62] have studied time dependent third order oscillating systems with damping based on the extended form of KBM method. Alam [29] and Alam *et al.* [63] have developed a simple method to obtain the time response of second order over damped nonlinear systems together with slowly varying coefficients under some special conditions. Later, Alam [57] and Alam and Hossain [64] have extended the method presented in [29] to obtain the time response of n-th order $(n \geq 2)$, over damped systems. Alam and Sattar [65] have presented a unified KBM method for solving third order nonlinear systems. Later, Alam [66] has extended the method to nonlinear over damped systems. Also, Alam *et al.* [67] have extended the method to certain non-oscillatory nonlinear systems with varying coefficients. Alam [68] has also presented a unified KBM method, which is not the formal form of the original KBM method, for solving *nth*, $n \geq 2,3$ order nonlinear systems. The solution contains some unusual variables, yet this solution is very important. Alam [69] has also presented a modified and compact form of the KBM unified method for solving an *nth*, $n \geq 2,3$ order nonlinear differential equation. The formula presented in [69] is compact, systematic and practical, and easier than that of [68]. Alam and Sattar [70] have developed a method to solve third order critically damped autonomous nonlinear systems. Alam [71] has redeveloped the method presented in [70] to find approximate solutions of critically damped nonlinear systems in presence of different damping forces by considering different sets of variational equations. Later, he has unified the KBM method for solving critically damped nonlinear system whose unequal eigen values are in integral multiple. Alam [72] has also extended the method to a third order over damped system when two of the eigen values are almost equal (i.e., the system is near to the critically damped) and the rest is small. Alam [73] has presented an asymptotic method for certain third order non-oscillatory nonlinear system, which gives desired results when the damping force is near to the critically damping force. Akbar *et al.* [74] have presented an asymptotic method based on the KBM method to solve the fourth order over damped nonlinear systems. Akbar *et al.* [75] have also developed a simple technique for obtaining certain over damped solution of an *nth* order nonlinear differential equation. Akbar *et al.* [76] have presented the KBM unified method for solving n-th order nonlinear systems under some special conditions including the case of internal resonance. Later, Akbar *et al.* [77] have extended the KBM method for solving fourth order more critically damped nonlinear systems. Akbar *et al.* [78] have also developed perturbation theory for fourth order nonlinear systems with large damping. Haque *et al.* [79] have investigated the solution of fourth order critically damped oscillatory nonlinear systems when two of the eigen values are real and equal and the other two are complex conjugate. Uddin and Sattar [80] have presented an asymptotic method for solving fourth order weakly nonlinear differential system with strong damping and slowly varying coefficients. Recently, Rahman *et al.* [81] have developed a technique for solving of fourth order near critically damped nonlinear systems.

CHAPTER 3

An Analytical Approximate Technique for Solving a Certain Type of Fourth Order Strongly Nonlinear Oscillatory Differential System with Small Damping

3.1 Introduction

The study of nonlinear problems is of crucial importance in all areas of applied mathematics, physics, engineering, medical science and other disciplines, since most of the phenomena in the real world are essentially nonlinear and described by nonlinear differential systems. It is too much difficult to handle nonlinear problems and in general, it is often very difficult to get an analytical approximate solution than a numerical one. The most common methods for constructing the analytical approximate solutions to the nonlinear oscillator equations are the perturbation techniques. Some well known perturbation methods are the KBM method [2-4], the Lindstedt-Poincare (LP) method [13, 16], and the method of multiple time scales [82]. Almost all perturbation methods are based on an assumption that small parameters must exist in the equations, which is too strict to find wide application of the classical perturbation methods. It determines not only the accuracy of the perturbation approximations, but also the validity of the perturbation methods itself. However, in science and engineering, there exist many nonlinear problems which do not contain any small parameter; especially those appear in nature with strong nonlinearity. Therefore, many new techniques have been proposed to eliminate the "small parameter" assumption, such as the homotopy perturbation method (HPM). In recent years, He [5] has developed some new approaches to Duffing equation with strongly and high order nonlinearity (I) linearized perturbation method. In another paper, He [6] has obtained the approximate solution of nonlinear differential equation with convolution product nonlinearities. Recently, He [7] has presented a new interpretation of homotopy perturbation method. Belendez *et al.* [8] have presented the application of He's homotopy perturbation method to Duffing harmonic oscillator. Lim *et al.* [61] have also presented a new analytical approach to the Duffing- harmonic oscillator. Alam [68] has investigated a unified KBM method for solving $nth, n \geq 2, 3$ order nonlinear systems. Also, Alam [83] has investigated a unified KBM method for solving $nth, n \geq 3$ order nonlinear differential system with slowly varying coefficients. Later, Alam *et al.* [84] have presented an asymptotic method for third order nonlinear system with varying coefficients. Akbar *et al.* [76] have presented the KBM unified method for solving an *nth* order nonlinear differential equation under some special conditions including the internal resonance for weakly nonlinear system. Recently, Uddin *et al.* [9-11] have presented an approximate technique for solving second order strongly nonlinear oscillatory differential systems with damping effects combing by the He's homotopy perturbation and the KBM methods. Many physical and engineering problems occur in nature which does not contain a small parameter, *i. e.,* appear with damping and strong nonlinearity. The more difficult and no less important case, the fourth order strongly nonlinear oscillatory differential systems with damping and cubic nonlinearity has remained almost untouched. The aim of this chapter is to fill this gap. So in this chapter, we are interested to develop a coupling analytical approximate technique based on the He's homotopy perturbation and the KBM methods to solve a certain type of fourth order strongly nonlinear oscillatory differential systems with damping and cubic nonlinearity. This method transforms a difficult problem under simplification into a simple problem which is easy to solve and understand and there is no any complexity for handling this method. The presented method has been successfully applied to solve fourth order strongly nonlinear damped oscillatory differential system with an example. The advantage of this method is that the first order analytical approximate solutions show a good agreement with the corresponding numerical solutions.

3.2 The Proposed Method

Let us consider the fourth order nonlinear ordinary differential equation (ODE) with damping in the following form

$$x^{(4)} + (\omega_1^2 + \omega_2^2)\ddot{x} + \omega_1^2\omega_2^2 x = -(p\dddot{x} + q\dot{x}) + \varepsilon f(x), \tag{3.1}$$

where over dots represent the derivatives with respect to time t, ω_1 and ω_2 are the frequencies for double mode of vibrations of the systems, ε is a positive parameter which is not necessarily small, p, q are unknown constants and $f(x)$ is a given nonlinear function. If we choose $p = 4k$ and $q = 4k^3 + 2k(\omega_1^2 + \omega_2^2) - 12\,k^3$ then the eigen values of equation (3.1) becomes $-k \pm i\omega_1$, $-k \pm i\omega_2$, where $k << 1$ represents any positive constant. Now, we are going to consider [9-11] the following substitution

$$x = y(t)e^{-kt}. \tag{3.2}$$

Differentiating equation (3.2) four times with respect to time t, and then substituting the derivatives $x^{(4)}, \dddot{x}, \ddot{x}, \dot{x}$ together with x and the values of p and q into equation (3.1) and simplifying them, we get

$$y^{(4)} + (\omega_1^2 + \omega_2^2 - 6k^2)\ddot{y} + \{5k^4 - k^2(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2\}y = \varepsilon e^{kt} f(ye^{-kt}). \tag{3.3}$$

According to the homotopy perturbation method [5-11] equation (3.3) reduces to

$$\begin{aligned} &y^{(4)} + (\omega_1^2 + \omega_2^2 - 6k^2)\ddot{y} + \left(5k^4 - k^2(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2 + \lambda\right)y \\ &= \lambda y + \varepsilon e^{kt} f(ye^{-kt}). \end{aligned} \tag{3.4}$$

Equation (3.4) can be rewritten as

$$y^{(4)} + (\omega_1^2 + \omega_2^2 - 6k^2)\ddot{y} + \omega^2 y = \lambda y + \varepsilon e^{kt} f(ye^{-kt}), \tag{3.5}$$

where $\omega^2 = 5k^4 - k^2(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2 + \lambda.$ (3.6)

Herein ω is a constant for undamped nonlinear oscillator and known as the frequency in literature and λ is an unknown function which can be determined by eliminating the secular terms. But, for damped nonlinear oscillatory differential systems, ω is a time dependent function and it varies slowly with time t. To handle this situation, we can use the extended form of the KBM [2-3] method by Mitropolskii [4]. According to this method, we choose the solution of equation (3.5) (for a single mode of vibrations) in the following form

$$y = a\cos\varphi, \tag{3.7}$$

where the amplitude a and the phase φ vary slowly with time t and they satisfy the following first order ordinary differential equations

$$\begin{aligned} \dot{a} &= \varepsilon A_1(a,\tau) + \varepsilon^2 A_2(a,\tau) + \cdots, \\ \dot{\varphi} &= \mu_1(\tau) + \varepsilon B_1(a,\tau) + \varepsilon^2 B_2(a,\tau) + \cdots, \end{aligned} \tag{3.8}$$

where ε is a small positive parameter and $\tau = \varepsilon t$ is the slowly varying time. Now, differentiating equation (3.7) four times with respect to time t and utilizing equation (3.8) and then taking the terms up to $O(\varepsilon)$, we obtain

$$\begin{aligned} &y^{(4)} + \mu_1^2\ddot{y} + \mu_2^2(\ddot{y} + \mu_1^2 y) \\ &= 5\varepsilon a\mu_1^2\mu_1'\sin\varphi + 2\varepsilon\mu_1^3(A_1\sin\varphi + aB_1\sin\varphi) \\ &+ \mu_2^2(-\varepsilon a\mu_1'\sin\varphi - 2\varepsilon\mu_1(A_1\sin\varphi + aB_1\cos\varphi)). \end{aligned} \tag{3.9}$$

Equation (3.9) can be re-written as

$$y^{(4)} + (\mu_1^2 + \mu_2^2)\ddot{y} + \mu_1^2 \mu_2^2 y$$
$$= \varepsilon \mu_1'(5\mu_1^2 - \mu_2^2) a \sin\varphi + 2\varepsilon \mu_1(\mu_1^2 - \mu_2^2)(A_1 \sin\varphi + a B_1 \cos\varphi). \tag{3.10}$$

Now,

$$y^{(4)} + (\mu_1^2 + \mu_2^2)\ddot{y} + \mu_1^2 \mu_2^2 y = 0, \tag{3.11}$$

which leads to

$$\varepsilon \mu_1'(5\mu_1^2 - \mu_2^2) a \sin\varphi + 2\varepsilon \mu_1(\mu_1^2 - \mu_2^2)(A_1 \sin\varphi + a B_1 \cos\varphi) = 0. \tag{3.12}$$

Equating the coefficients of $\sin\varphi$ and $\cos\varphi$ from equation (3.12), we obtain

$$A_1 = -\frac{\mu_1'(5\mu_1^2 - \mu_2^2) a}{2\mu_1(\mu_1^2 - \mu_2^2)}, \qquad B_1 = 0. \tag{3.13}$$

Comparing equation (3.10) with equation (3.5)

$$\mu_1^2 + \mu_2^2 = \omega_1^2 + \omega_2^2 - 6k^2, \qquad \mu_1^2 \mu_2^2 = 5k^4 - k^2(\omega_1^2 + \omega_2^2) + \omega_1^2 \omega_2^2 + \lambda. \tag{3.14}$$

Now,

$$\mu_1^2 - \mu_2^2 = \sqrt{(\omega_1^2 - \omega_2^2)^2 - 8k^2(\omega_1^2 + \omega_2^2) - 4\lambda}. \tag{3.15}$$

By solving equation (3.14) and equation (3.15), we get

$$\mu_1 = \sqrt{(\omega_1^2 + \omega_2^2 - 6k^2 + \sqrt{(\omega_1^2 - \omega_2^2)^2 - 8k^2(\omega_1^2 + \omega_2^2) - 4\lambda})/2}, \tag{3.16}$$

$$\mu_2 = \sqrt{(\omega_1^2 + \omega_2^2 - 6k^2 - \sqrt{(\omega_1^2 - \omega_2^2)^2 - 8k^2(\omega_1^2 + \omega_2^2) - 4\lambda})/2}. \tag{3.17}$$

Putting the value of y from equation (3.7) into equation (3.2) and the values of A_1 and B_1 from equation (3.13) into equation (3.8), we obtain

$$x = a e^{-kt} \cos\varphi, \tag{3.18}$$

$$\dot{a} = -\frac{\varepsilon \mu_1'(5\mu_1^2 - \mu_2^2) a}{2\mu_1(\mu_1^2 - \mu_2^2)}, \qquad \dot{\varphi} = \mu_1(\tau). \tag{3.19}$$

Equation (3.18) represents the first order analytical approximate solution of equation (3.1) by the proposed coupling technique. Usually, the integration of equation (3.19) is accomplished by the well-known techniques of calculus [82, 85], but sometimes they are calculated by a numerical procedure [9-11, 34, 48, 68, 76, 83, 84, 86] with the help of equations (3.16) - (3.17). Thus, the determination of first order analytical approximate solutions of equation (3.1) is completed by the proposed method.

3.3 Example

As an example of the above procedure, we are going to consider the fourth order strongly nonlinear oscillatory differential system with small damping and cubic nonlinearity in the following form

$$x^{(4)} + (\omega_1{}^2 + \omega_2{}^2)\ddot{x} + \omega_1{}^2\omega_2{}^2 x = -(p\ddot{x} + q\dot{x}) + \varepsilon x^3, \tag{3.20}$$

where $f(x) = x^3$ is the given nonlinear function. Now, by using the transformation equation (3.2) and then simplifying them and according to the homotopy perturbation method [5-11], we obtain

$$y^{(4)} + (\mu_1^2 + \mu_2^2)\ddot{y} + \mu_1^2\mu_2^2 y = \lambda y + \varepsilon y^3 e^{-2kt}. \tag{3.21}$$

According to the extended form of the KBM method [2-4], the solution of equation (3.21) is assumed as

$$y = a\cos\varphi, \tag{3.22}$$

where a and φ are given by

$$\dot{a} = -\frac{\varepsilon\,\mu_1'(5\mu_1^2 - \mu_2^2)a}{2\mu_1(\mu_1^2 - \mu_2^2)}, \qquad \dot{\varphi} = \mu_1(\tau). \tag{3.23}$$

Now, putting the value of y from equation (3.22) into the right hand side of equation (3.21), we have

$$y^{(4)} + (\mu_1^2 + \mu_2^2)\ddot{y} + \mu_1^2\,\mu_2^2\, y = \left(\lambda\, a + \frac{3\varepsilon a^3 e^{-2kt}}{4}\right)\cos\varphi + \frac{\varepsilon a^3 e^{-2kt}}{4}\cos 3\varphi. \tag{3.24}$$

The requirement of no secular terms in particular solution of equation (3.24) implies that the coefficient of the $\cos\varphi$ term is zero. Setting this term to zero, we obtain

$$\lambda\, a + \frac{3\varepsilon a^3 e^{-2kt}}{4} = 0. \tag{3.25}$$

For the nontrivial solution $(i.e., a \neq 0)$, equation (3.25) leads to

$$\lambda = -\frac{3\varepsilon a^2 e^{-2kt}}{4}. \tag{3.26}$$

Now, putting the value of λ from equation (3.26) into equations (3.16) – (3.17) we obtain

$$\mu_1 = \sqrt{(\omega_1^2 + \omega_2^2 - 6k^2 + \sqrt{(\omega_1^2 - \omega_2^2)^2 - 8k^2(\omega_1^2 + \omega_2^2) + 3\varepsilon a^2 e^{-2kt}})/2}, \tag{3.27}$$

$$\mu_2 = \sqrt{(\omega_1^2 + \omega_2^2 - 6k^2 - \sqrt{(\omega_1^2 - \omega_2^2)^2 - 8k^2(\omega_1^2 + \omega_2^2) + 3\varepsilon a^2 e^{-2kt}})/2}. \tag{3.28}$$

Squaring equation (3.27) and expanding according to the binomial theorem and then simplifying, we obtain

$$\mu_1^2 = \omega_1^2 - 3k^2 - \frac{2k^2(\omega_1^2 + \omega_2^2)}{\omega_1^2 - \omega_2^2} - \frac{3\varepsilon a^2 e^{-2kt}}{4(\omega_1^2 - \omega_2^2)}. \quad (3.29)$$

Differentiating equation (3.29) with respect to t and treating a as constant, we obtain

$$2\mu_1\mu_1' = \frac{3k\,a_0^2 e^{-2kt}}{2(\omega_1^2 - \omega_2^2)}. \quad (3.30)$$

Dividing equation (3.30) by equation (3.29) we have

$$\frac{\mu_1'}{\mu_1} = \frac{3ka_0^2 e^{-2kt}}{4(\omega_1^2 - \omega_2^2)}\left(\frac{(\omega_1^2 - \omega_2^2)(\omega_1^2 - 5k^2) - 4k^2\omega_2^2}{(\omega_1^2 - \omega_2^2)} - \frac{3\varepsilon a_0^2 e^{-2kt}}{4(\omega_1^2 - \omega_2^2)}\right)^{-1}, \quad (3.31)$$

which can be written as

$$\frac{\mu_1'}{2\mu_1} = \frac{3ka_0^2 e^{-2kt}}{8\{(\omega_1^2 - \omega_2^2)(\omega_1^2 - 5k^2) - 4k^2\omega_2^2\}}\left(1 + \frac{3\varepsilon a_0^2 e^{-2kt}}{4\{(\omega_1^2 - \omega_2^2)(\omega_1^2 - 5k^2) - 4k^2\omega_2^2\}}\right). \quad (3.32)$$

Now, putting equation (3.32) into equation (3.13) and taking the terms for $O(\varepsilon)$, we get

$$A_1 = r\,a\,e^{-2kt}, \quad B_1 = 0, \quad (3.33)$$

where

$$r = -\frac{3ka_0^2(5\mu_1^2 - \mu_2^2)}{8(\mu_1^2 - \mu_2^2)\{(\omega_1^2 - \omega_2^2)(\omega_1^2 - 5k^2) - 4k^2\omega_2^2\}}. \quad (3.34)$$

Thus, equation (3.23) becomes

$$\dot{a} = \varepsilon r\,a e^{-2kt}, \quad \dot{\varphi} = \mu_1(\tau). \quad (3.35)$$

Integrating equation (3.35), we obtain

$$a = a_0 \exp\left(\frac{\varepsilon r}{2k}(1 - e^{-2kt})\right), \qquad \varphi = \varphi_0 + \int_0^t \mu_1(\tau)dt, \quad (3.36)$$

where a_0 and φ_0 are the initial amplitude and phase variables for the nonlinear differential systems respectively.

Thus, the first order analytical approximate solution of equation (3.20) is given by

$$x = a\,e^{-kt}\cos\varphi, \quad (3.37)$$

where a and φ are calculated from equation (3.36) with help of equations (3.27) – (3.28).

3.4 Results and Discussion

In this chapter, a new coupling analytical approximate technique has been presented to obtain the first order analytical approximate solutions for a certain type of fourth order strongly nonlinear oscillatory differential systems with damping and the method has been successfully implemented to illustrate the effectiveness and convenience of the proposed method. The first order analytical approximate solutions of equation (3.20) are computed by equation (3.37) with the help of equations (3.27) – (3.28) and equation (3.36) and the corresponding numerical solutions are obtained by the well known fourth order ***Runge-Kutta*** method.

Furthermore, the presented method is simple and the advantage of this method is that the first order approximate solutions show good agreement (see also **Figs. 3.1-3.4**) with the corresponding numerical solutions for several damping effects. The initial approximation can be freely chosen, which is identified via various methods in the references. The approximations obtained by the presented method are valid not only for strongly nonlinear oscillatory differential systems, but also for weak one with small damping effects. **Figs. 3.1-3.2** are provided to compare the solutions obtained by the presented method to the corresponding numerical solutions with small damping for strongly nonlinear oscillatory differential systems. Also, **Figs. 3.3-3.4** are cited to compare the solutions obtained by the proposed method to the corresponding numerical solutions for weakly nonlinear oscillatory differential systems with small damping effects. From the **Figs. 3.1-3.4** it is seen that, the obtained analytical approximate solutions for both strongly and weakly nonlinear differential systems show good agreement with those solutions obtained by the fourth order ***Runge-Kutta*** method.

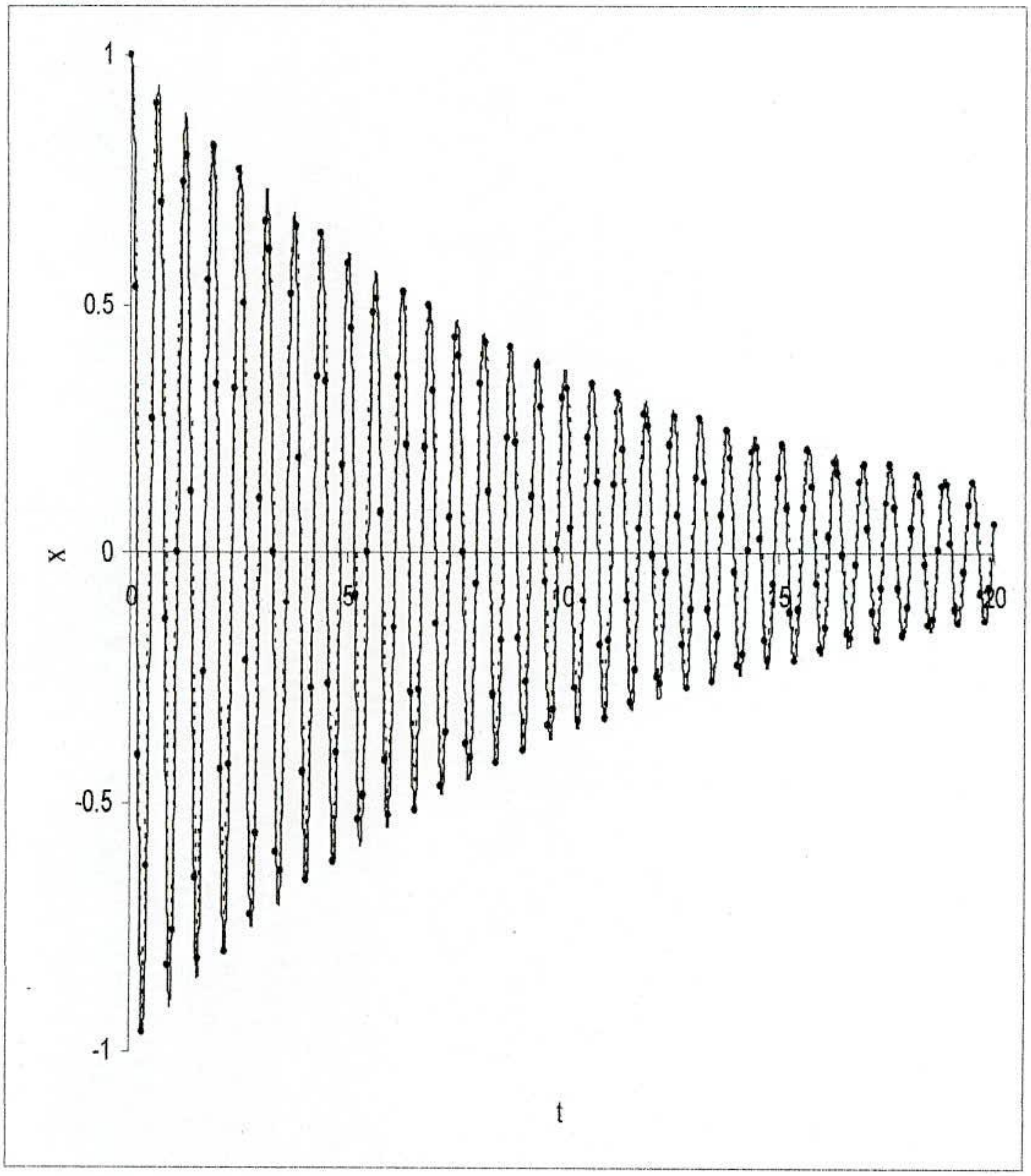

Fig. 3.1 First approximate solution of equation (3.20) is denoted by $-\bullet-$ (dashed lines) by the proposed coupling technique with the initial conditions $a_0 = 1.0,\ \varphi_0 = 0$ or $[x(0) = 1.00000,\ \dot{x}(0) = -0.10003,\ \ddot{x}(0) = -99.92663,\ \dddot{x}(0) = 30.08070]$ when $\omega_1 = 10.0$, $\omega_2 = 5.0$, $k = 0.1$, $\varepsilon = 1.0$ and $f = x^3$. Corresponding numerical solution is denoted by - (solid line).

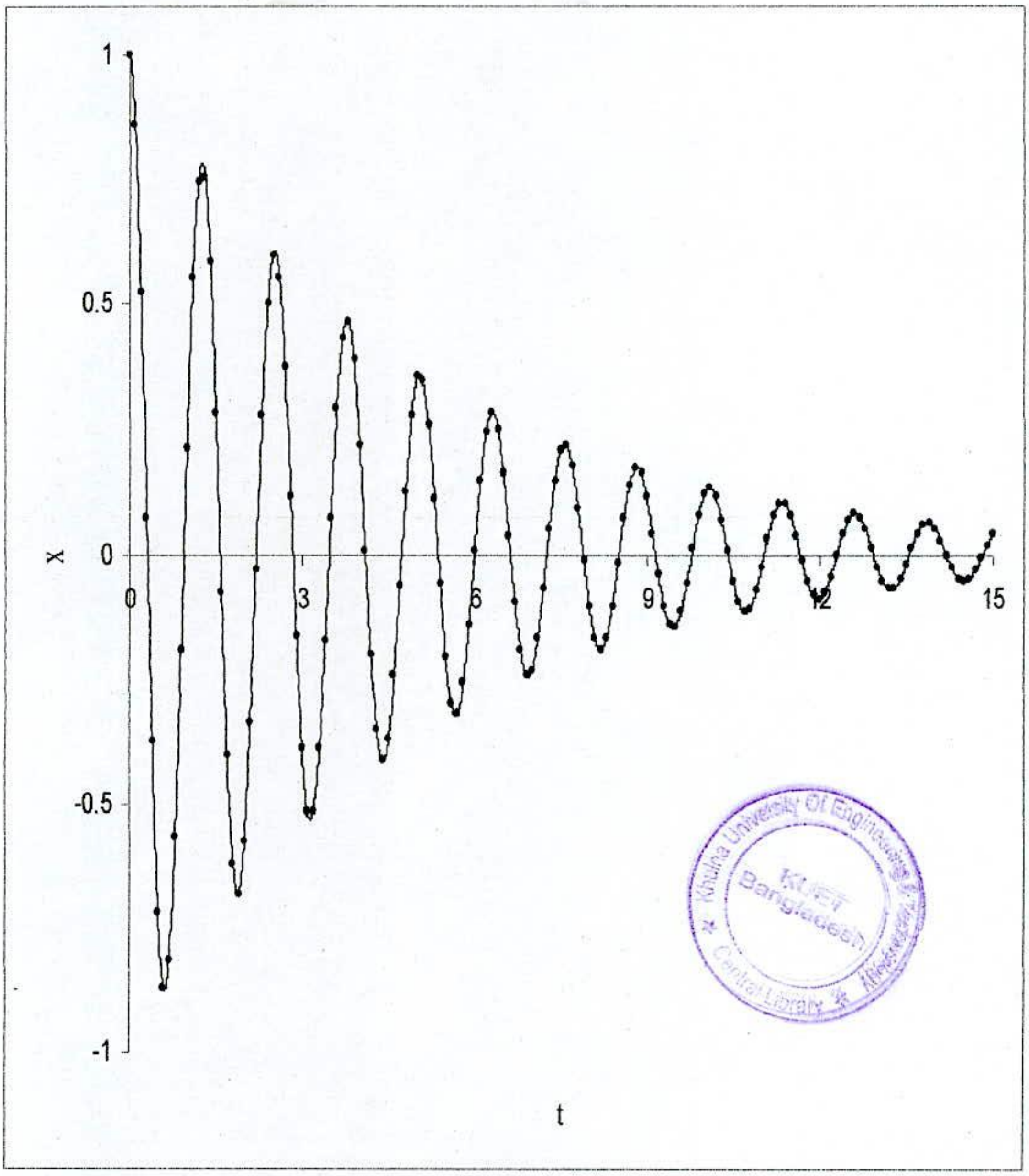

Fig. 3.2 First approximate solution of equation (3.20) is denoted by $-\bullet-$ (dashed lines) by the proposed coupling technique with the initial conditions $a_0 = 1.0,\ \varphi_0 = 0$ or $[x(0) = 1.00000, \dot{x}(0) = -0.20065,\ \ddot{x}(0) = -24.76098,\ \dddot{x}(0) = 15.06427]$ when $\omega_1 = 5.0$, $\omega_2 = 1.0$, $k = 0.2$, $\varepsilon = 1.0$ and $f = x^3$. Corresponding numerical solution is denoted by - (solid line).

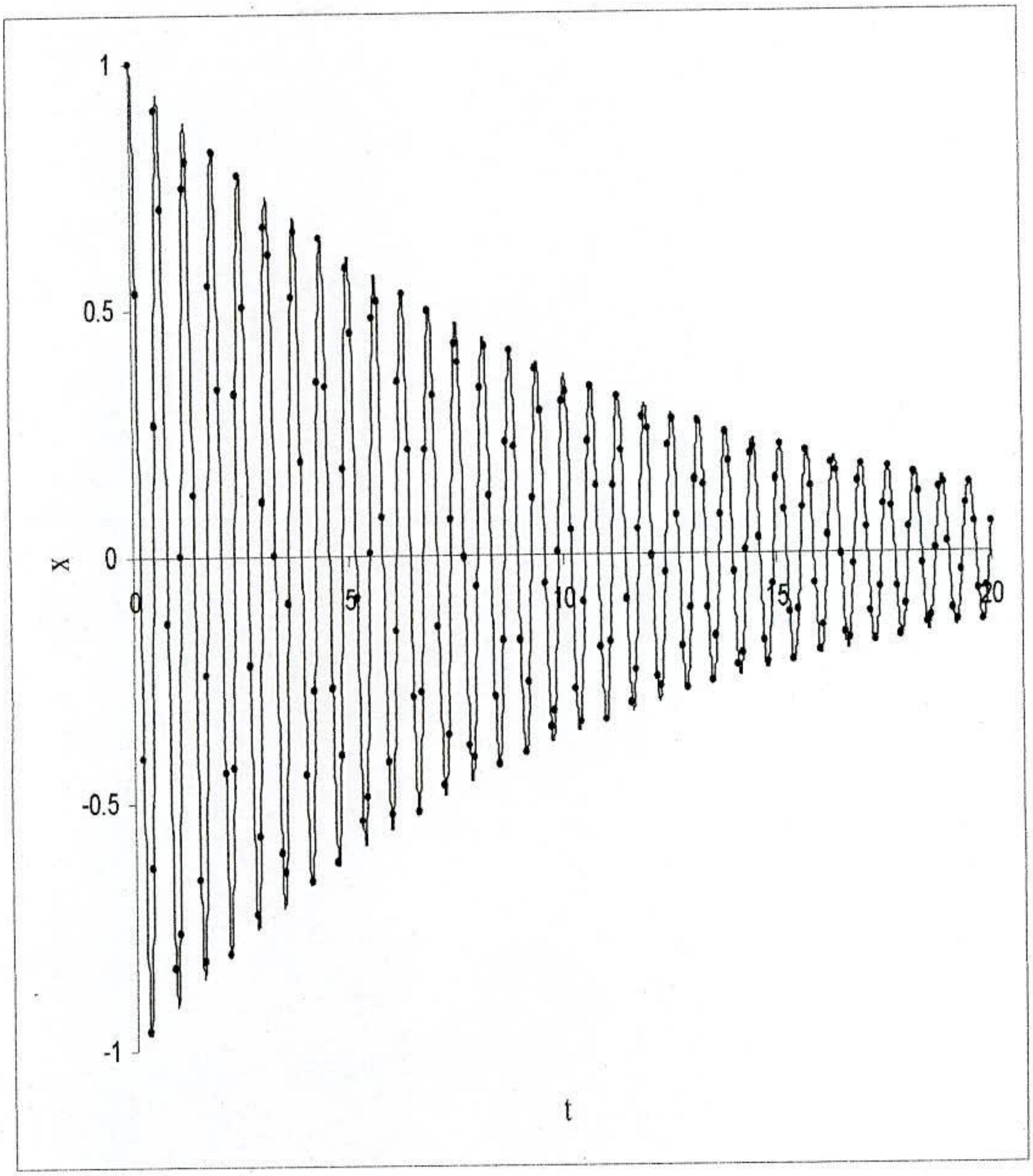

Fig. 3.3 First approximate solution of equation (3.20) is denoted by $-\bullet-$ (dashed lines) by the proposed coupling technique with the initial conditions $a_0 = 1.0,\ \varphi_0 = 0$ or $[x(0) = 1.00000,\ \dot{x}(0) = -0.1000,\ \ddot{x}(0) = -99.93565,\ \dddot{x}(0) = 30.07827]$ when $\omega_1 = 10.0$, $\omega_2 = 5.0$, $k = 0.1$, $\varepsilon = 0.1$ and $f = x^3$. Corresponding numerical solution is denoted by - (solid line).

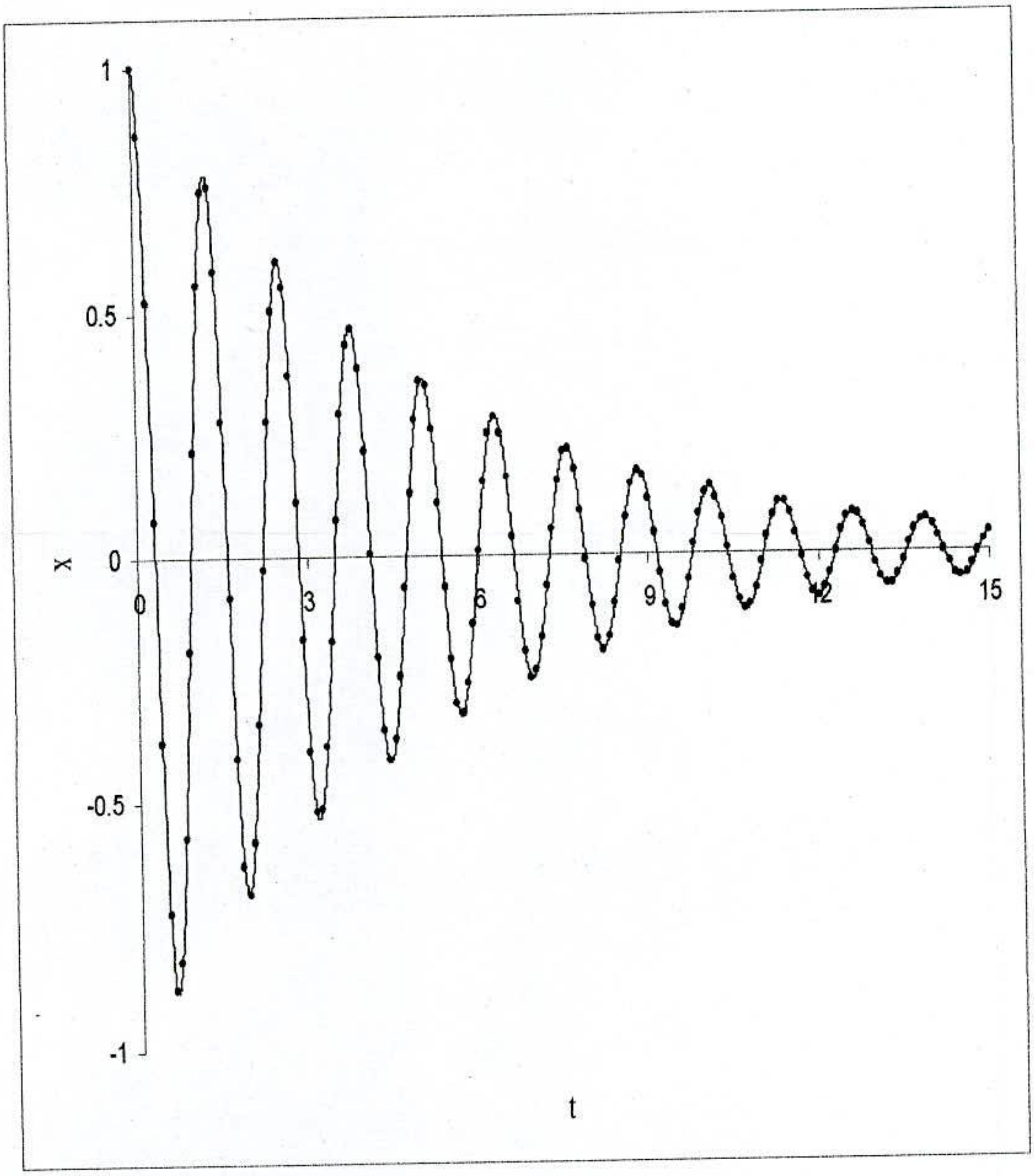

Fig. 3.4 First approximate solution of equation (3.20) is denoted by $-\bullet-$ (dashed lines) by the proposed coupling technique with the initial conditions $a_0 = 1.0$, $\varphi_0 = 0$ or $[x(0) = 1.00000,\ \dot{x}(0) = -0.20007,\ \ddot{x}(0) = -24.78982,\ \dddot{x}(0) = 15.05779]$ when $\omega_1 = 5.0$, $\omega_2 = 1.0$, $k = 0.2$, $\varepsilon = 0.1$ and $f = x^3$. Corresponding numerical solution is denoted by - (solid line).

CHAPTER 4

Approximate Solution of Fourth Order Near Critically Damped Nonlinear Systems with Special Conditions

4.1 Introduction

The KBM [2-4] method is one of the widely used techniques to obtain analytical approximate solution of weakly nonlinear systems and this method was originally developed for finding periodic solution of nonlinear systems with small nonlinearities. The method was extended by Popov to damped oscillatory systems. Murty *et al.* [30] have investigated an over-damped nonlinear system using Bogoliubov's method. Murty [32] has presented a unified KBM method for solving second order nonlinear systems which cover the un-damped, damped and over-damped cases. Alam and Sattar [70] have extended the KBM method for third order critically damped nonlinear systems. Alam [72] has also investigated the solution of third order nonlinear systems when two of the eigen values are almost equal and the other is small. Akbar *et al.* [77] have extended the KBM method for solving fourth order more critically damped nonlinear systems. Haque *et al.* [79] have investigated the solution of fourth order critically damped oscillatory nonlinear systems when two of the eigen values are real and equal and the other two are complex conjugate. Uddin and Sattar [80] have presented an asymptotic method for solving fourth order weakly nonlinear differential system with strong damping and slowly varying coefficients. Recently, Rahman *et al.* [81] have developed a technique for solving of fourth order near critically damped nonlinear systems.

For the relation $\lambda_4 \approx \lambda_3 + 2\lambda_1$, the solution obtained in Rahman *et al.* [81] break-down. The aim in this chapter is to fill this gap, that is, we are interested to investigate the solution when the relation $\lambda_4 \approx \lambda_3 + 2\lambda_1$ exists among the eigen values $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. The solutions obtained by this technique show good coincidence with those obtained by numerical method.

4.2 The Method

Let us consider the fourth order weakly nonlinear ordinary differential equation in the following form

$$x^{(4)} + c_1\dddot{x} + c_2\ddot{x} + c_3\dot{x} + c_4 x = -\varepsilon f(x), \tag{4.1}$$

where over dots represent the derivatives with respect to time t, ε is a positive small parameter, c_1, c_2, c_3, c_4 are constants and $f(x)$ is the given nonlinear function. The constants are defined in terms of the eigen values $-\lambda_i$, $i = 1, 2, 3, 4$ of the unperturbed equation of equation (4.1) as $c_1 = \sum_{i=1}^{4}\lambda_i$, $c_2 = \sum_{\substack{i,j=1\\ i\neq j}}^{4}\lambda_i\lambda_j$, $c_3 = \sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{4}\lambda_i\lambda_j\lambda_k$ and $c_4 = \prod_{i=1}^{4}\lambda_i$.

The equation (4.1) becomes linear when $\varepsilon = 0$, and suppose the eigen values $-\lambda_1$ and $-\lambda_2$ are almost equal ($\lambda_1 \approx \lambda_2$) and other two eigen values $-\lambda_3$ and $-\lambda_4$ are distinct. Therefore, the unperturbed solution is

$$x(t,0) = \frac{1}{2}a_{1,0}(e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_{2,0}\left(\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}\right) + a_{3,0}e^{-\lambda_3 t} + a_{4,0}e^{-\lambda_4 t}, \tag{4.2}$$

where $a_{i,0}\,(i = 1, 2, 3, 4)$ are arbitrary constants.

When $\varepsilon \neq 0$, following Alam [72] technique we choose the solution of equation (4.1) in the form

$$\begin{aligned} x(t,\varepsilon) = {} & \frac{1}{2}a_1(t)(e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2(t)\left(\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}\right) + a_3(t)e^{-\lambda_3 t} \\ & + a_4(t)e^{-\lambda_4 t} + \varepsilon u_1(a_1, a_2, a_3, a_4, t) + \varepsilon^2 \ldots, \end{aligned} \tag{4.3}$$

where $a_i\,(i = 1, 2, 3, 4)$ satisfy the following first order differential equations:

$$\dot{a}_i(t) = \varepsilon A_1(a_1, a_2, a_3, a_4, t) + \varepsilon^2 \cdots, \qquad i = 1, 2, 3, 4. \tag{4.4}$$

Confining only to a first few terms $1, 2, 3, \ldots, n$ in the series expansions equations (4.3) and (4.4), we calculate the functions u_1 and A_i, $i = 1, 2, 3, 4$ such that $a_i(t)$, $i = 1, 2, 3, 4$ appearing in equations (4.3) and (4.4) satisfy the given differential equation (4.1) with an accuracy of order ε^{n+1}. To determine the unknown functions u_1, A_1, A_2, A_3, A_4, it is assumed (as customary in the KBM method) that the correction term u_1 does not contain secular-type term $te^{-\lambda_1 t}$, which make them large. Differentiating equation (4.3) four times with respect to t, substituting the derivatives

$x^{(4)}, \dddot{x}, \ddot{x}, \dot{x}$ and x in the original equation (4.1), utilizing the relations presented in equation (4.4) and finally equating the coefficients of ε, we obtain

$$\begin{aligned}
&\frac{1}{2}(e^{-\lambda_1 t}(D-\lambda_1+\lambda_2)(D-\lambda_1+\lambda_3)(D-\lambda_1+\lambda_4)+e^{-\lambda_2 t}(D-\lambda_2+\lambda_1)\times\\
&(D-\lambda_2+\lambda_3)(D-\lambda_2+\lambda_4))A_1+(D+\lambda_4)(e^{-\lambda_1 t}(\lambda_1-\lambda_3-\frac{3}{2}D)+e^{-\lambda_2 t}\times\\
&(\lambda_2-\lambda_3-\frac{3}{2}D))A_2+e^{-\lambda_3 t}(D-\lambda_3+\lambda_1)(D-\lambda_3+\lambda_2)(D-\lambda_3+\lambda_4)A_3+\\
&e^{-\lambda_4 t}(D-\lambda_4+\lambda_1)(D-\lambda_4+\lambda_2)(D-\lambda_4+\lambda_3)A_4+(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2})(D+\lambda_4)\times\\
&D(D+\lambda_3-\frac{\lambda_1+\lambda}{2})A_2-(\frac{\lambda_1 e^{-\lambda_1 t}-\lambda_2 e^{-\lambda_2 t}}{\lambda_1-\lambda_2})D(D+\lambda_3-\frac{\lambda_1+\lambda}{2})A_2+\\
&(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)(D+\lambda_4)u_1=-f^{(0)},
\end{aligned} \tag{4.5}$$

where

$$f^{(0)}=f(x_0) \text{ and } x_0=\frac{1}{2}a_1(t)(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2(t)\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)+a_3(t)e^{-\lambda_3 t}+a_4(t)e^{-\lambda_4 t}.$$

It is assumed that the function $f^{(0)}$ can be expanded in power series (Taylor's series) in the form (see also [3] for details)

$$f^{(0)}=\sum_{r=0}^{n}F_r(a_3 e^{-\lambda_3 t},a_4 e^{-\lambda_4 t})\left\{\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}^r, \tag{4.6}$$

where n is the order of polynomial of the nonlinear function f. This assumption is certainly valid when f is a polynomial function of x. Such polynomial functions cover some special and important systems in mechanics. Following Alam's [72], in this chapter we assume that u_1 does not contain the terms F_0 and F_1 of $f^{(0)}$, since the system is considered to near critically damped. Substituting the value of $f^{(0)}$ from equation (4.6) into equation (4.5) and equating the coefficients of like powers of $\left(\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right)$, we obtain

$$\begin{aligned}
&e^{-\lambda_3 t}(D-\lambda_3+\lambda_1)(D-\lambda_3+\lambda_2)(D-\lambda_3+\lambda_4)A_3+e^{-\lambda_4 t}(D-\lambda_4+\lambda_1)\times\\
&(D-\lambda_4+\lambda_2)(D-\lambda_4+\lambda_3)A_4+\frac{1}{2}\{e^{-\lambda_1 t}(D-\lambda_1+\lambda_2)(D-\lambda_1+\lambda_3)\times\\
&(D-\lambda_1+\lambda_4)+e^{-\lambda_2 t}(D-\lambda_2+\lambda_1)(D-\lambda_2+\lambda_3)(D-\lambda_2+\lambda_4)\}A_1+\\
&(D+\lambda_4)\{e^{-\lambda_1 t}(\lambda_1-\lambda_3-\frac{3}{2}D)+e^{-\lambda_2 t}(\lambda_2-\lambda_3-\frac{3}{2}D)\}A_2-(\frac{\lambda_1 e^{-\lambda_1 t}-\lambda_2 e^{-\lambda_2 t}}{\lambda_1-\lambda_2})\\
&\times D\left(D+\lambda_3-\frac{\lambda_1+\lambda_2}{2}\right)A_2=-F_0-\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})F_1,
\end{aligned} \tag{4.7}$$

$$(D+\lambda_4)\times D\left(D+\lambda_3-\frac{\lambda_1+\lambda_2}{2}\right)A_2=-a_2F_1, \tag{4.8}$$

and

$$(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)(D+\lambda_4)u_1$$
$$=-\sum_{r=2}^{n}F_r(a_3e^{-\lambda_3 t},a_4e^{-\lambda_4 t})\times\left\{\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}^r. \tag{4.9}$$

KBM [2-4], Alam [70, 72] have imposed the condition that u_1 does not contain the fundamental terms (the solution presented in equation (4.2) is called generating solution and its terms are called fundamental terms) of $f^{(0)}$. The solution of equation (4.8) gives value of the unknown function A_2. If the nonlinear function f and the eigen values $-\lambda_1,-\lambda_2,-\lambda_3,-\lambda_4$ of the corresponding linear equation of equation (4.1) are not specified then it is not easy to solve the equation (4.7) for the unknown functions A_1, A_3 and A_4. When these are specified, the values of A_1, A_3 and A_4 can be found subject to the condition that the coefficients in the solutions of A_1, A_3 and A_4 do not become large (see Alam [72], Akbar *et al.* [77] for details), as if A_1, A_3 and A_4 do not contain terms involving te^{-t}. In this chapter, we have imposed the conditions that the relation $\lambda_4\approx\lambda_3+2\lambda_1$ but $\lambda_4<2\lambda_3$ exists among the eigen values $\lambda_1,\lambda_3,\lambda_4$ (also $\lambda_1\to\lambda_2$ since the system is near critically damped). These relations are important, because under these relations the coefficients in the solutions of A_1, A_3 and A_4 do not become large. Under these imposed conditions, we obtain the values of A_1, A_3 and A_4 from equation (4.7). Substituting the values of A_1, A_2, A_3 and A_4 in the equation (4.4), we obtain the solutions of $\dot{a}_i(t)$ $(i=1,2,3,4)$, which are proportional to the small parameter ε. So they are slowly varying functions of time t, that is, they are almost constants and by integrating we obtain the values of a_i $(i=1,2,3,4)$. It is laborious work to solve the equation (4.9) for u_1. However, as $\lambda_1\to\lambda_2$ it takes simple form

$$(D+\lambda_1)^2(D+\lambda_3)(D+\lambda_4)u_1=-\sum_{r=2}^{n}F_r(a_3e^{-\lambda_3 t},a_4e^{-\lambda_4 t})\left\{e^{-\lambda_1 t}(a_1-a_2t)\right\}^r. \tag{4.10}$$

Solving equation (4.10), we obtain the value of u_1. Finally, substituting the values of a_i $(i=1,2,3,4)$ and u_1 in the equation (4.3), we obtain the complete solution of equation (4.1).

4.3 Example

For an example of the above method, we consider the following fourth order nonlinear differential equation,

$$x^{(4)} + c_1 \dddot{x} + c_2 \ddot{x} + c_3 \dot{x} + c_4 x = -\varepsilon x^3, \tag{4.11}$$

Here $f(x) = x^3$ and $x_0 = \frac{1}{2} a_1 (e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2 \left(\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right) + a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t}$.

Thus, $f^{(0)} = \left\{ \frac{1}{2} a_1 (e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2 \left(\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right) + a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t} \right\}^3$,

$$F_0 = (a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t})^3,\ F_1 = 3(a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t})^2. \tag{4.12}$$

According to the equations (4.7)-(4.9), we obtain

$$\begin{aligned} &e^{-\lambda_3 t}(D-\lambda_3+\lambda_1)(D-\lambda_3+\lambda_2)(D-\lambda_3+\lambda_4)A_3 + e^{-\lambda_4 t}(D-\lambda_4+\lambda_1)\times \\ &(D-\lambda_4+\lambda_2)(D-\lambda_4+\lambda_3)A_4 + \frac{1}{2}\{e^{-\lambda_1 t}(D-\lambda_1+\lambda_2)(D-\lambda_1+\lambda_3)\times \\ &(D-\lambda_1+\lambda_4) + e^{-\lambda_2 t}(D-\lambda_2+\lambda_1)(D-\lambda_2+\lambda_3)(D-\lambda_2+\lambda_3)\}A_1 + (D+\lambda_4)\times \\ &\{e^{-\lambda_1 t}(\lambda_1-\lambda_3-\frac{3}{2}D) + e^{-\lambda_2 t}(\lambda_2-\lambda_3-\frac{3}{2}D)\}A_2 - (\frac{(\lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t})}{(\lambda_1-\lambda_2)})\times \\ &D(D+\lambda_3-\frac{\lambda_1+\lambda_2}{2})A_2 = -[(a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t})^3 + 3(a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t})^2 \times \\ &\frac{1}{2} a_1 (e^{-\lambda_1 t} + e^{-\lambda_2 t})], \end{aligned} \tag{4.13}$$

$$(D+\lambda_4)\times D(D+\lambda_3-\frac{\lambda_1+\lambda_2}{2})A_2 = -3a_2(a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t})^2, \tag{4.14}$$

and

$$\begin{aligned} &(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)(D+\lambda_4)u_1 \\ &= -\sum_{r=2}^{3} F_r(a_3 e^{-\lambda_3 t}, a_4 e^{-\lambda_4 t}) \left\{ \frac{1}{2} a_1 (e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2 \left(\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \right\}^r. \end{aligned} \tag{4.15}$$

Solving equation (4.14), we obtain

$$A_2 = a_2[n_1 a_3^2 e^{-2\lambda_3 t} + n_2 a_3 a_4 e^{-(\lambda_3+\lambda_4)t} + n_3 a_4^2 e^{-2\lambda_4 t}], \tag{4.16}$$

where

$$n_1 = \frac{3}{\lambda_3(\lambda_1 + \lambda_2 + 2\lambda_3)(2\lambda_3 - \lambda_4)}, \qquad n_2 = \frac{12}{\lambda_3(\lambda_3 + \lambda_4)(\lambda_1 + \lambda_2 + 2\lambda_4)}, \tag{4.17}$$
$$n_3 = \frac{3}{\lambda_4^2(\lambda_1 + \lambda_2 - 2\lambda_3 + 4\lambda_4)}.$$

Substituting the value of A_2 from equation (4.16) into equation (4.13). In order to separate the equation (4.13) for determining the unknown functions A_1, A_3 and A_4, we use the conditions as discussed in the method (see also Alam [72], Akbar *et al.* [77]). It is to note that our solution approaches toward critically damped solution (see Alam [72]) if $\lambda_1 \to \lambda_2$. However, equation (4.13) has not an exact solution unless $\lambda_1 \to \lambda_2$. Now, we consider $\lambda_4 \approx \lambda_3 + 2\lambda_1$ but $\lambda_4 \langle 2\lambda_3$. Under these imposed conditions and by equating like terms on both sides of the equation (4.13), we obtain

$$\begin{aligned} & e^{-\lambda_1 t}(D - \lambda_1 + \lambda_2)(D - \lambda_1 + \lambda_3)(D - \lambda_1 + \lambda_4)A_1 \\ & = -a_2 a_3^2 n_1 \lambda_2 \lambda_3(\lambda_1 + \lambda_2 + 2\lambda_3) t e^{-(\lambda_1 + 2\lambda_3)t} - \frac{1}{2} a_2 a_3 a_4 n_2 \lambda_2 (2\lambda_4^2 + 2\lambda_3\lambda_4 + \\ & \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_4) t e^{-(\lambda_1 + \lambda_3 + \lambda_4)t} - a_2 a_4^2 n_3 \lambda_2 \lambda_4 (\lambda_1 + \lambda_2 - 2\lambda_3 + \\ & 4\lambda_4) t e^{-(\lambda_1 + 2\lambda_4)t}, \end{aligned} \tag{4.18}$$

$$\begin{aligned} & e^{-\lambda_3 t}(D - \lambda_3 + \lambda_1)(D - \lambda_3 + \lambda_2)(D - \lambda_3 + \lambda_4)A_3 \\ & = [a_2 n_1\{(\lambda_1 + 2\lambda_3)(\lambda_1 + 2\lambda_3 - \lambda_4) + \lambda_3(\lambda_1 + \lambda_2 + 2\lambda_3)\} - \frac{3}{2} a_1] a_3^2 e^{-(\lambda_1 + 2\lambda_3)t} \\ & + [a_2 n_1 (\lambda_2 + 2\lambda_3)(\lambda_2 + 2\lambda_3 - \lambda_4) - \frac{3}{2} a_1] a_3^2 e^{-(\lambda_2 + 2\lambda_3)t} - a_3^3 e^{-3\lambda_3 t} \\ & - 3a_3^2 a_4 e^{-(2\lambda_3 + \lambda_4)t}, \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} & e^{-\lambda_4 t}(D - \lambda_4 + \lambda_1)(D - \lambda_4 + \lambda_2)(D - \lambda_4 + \lambda_3)A_4 \\ & = [\frac{1}{2} a_2 n_2 \{(\lambda_1 + \lambda_3)(2\lambda_1 + \lambda_3 + 3\lambda_4) + (2\lambda_4^2 + 2\lambda_3\lambda_4 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \\ & \lambda_1\lambda_4 + \lambda_2\lambda_4)\} - 3a_1] a_3 a_4 e^{-(\lambda_1 + \lambda_3 + \lambda_4)t} + [a_2 n_3 \{(\lambda_1 + \lambda_4)(\lambda_1 - \lambda_3 + 3\lambda_4) \\ & + \lambda_4(\lambda_1 + \lambda_2 - 2\lambda_3 + 4\lambda_4)\} - \frac{3}{2} a_1] a_4^2 e^{-(\lambda_1 + 2\lambda_4)t} + [\frac{1}{2} a_2 n_2 (\lambda_2 + \lambda_3) \times \\ & (2\lambda_2 + \lambda_3 + 3\lambda_4) - 3a_1] a_3 a_4 e^{-(\lambda_2 + \lambda_3 + \lambda_4)t} + [a_2 n_3 (\lambda_2 + \lambda_4)(\lambda_2 - \lambda_3 + 3\lambda_4) \\ & - \frac{3}{2} a_1] a_4^2 e^{-(\lambda_2 + 2\lambda_4)t} - [3a_3 a_4^2 e^{-(\lambda_3 + 2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t}]. \end{aligned} \tag{4.20}$$

The particular solutions of equations (4.18)-(4.20) yield respectively

$$\begin{aligned} A_1 = {} & i_1 a_2 a_3^2 t e^{-(\lambda_1 - \lambda_2 + 2\lambda_3)t} + i_2 a_2 a_3^2 t e^{-(\lambda_1 - \lambda_2 + 2\lambda_3)t} + i_3 a_2 a_3 a_4 t e^{-(\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4)t} \\ & + i_4 a_2 a_3 a_4 t e^{-(\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4)t} + i_5 a_2 a_4^2 t e^{-(\lambda_1 - \lambda_2 + 2\lambda_4)t} + i_6 a_2 a_4^2 t e^{-(\lambda_1 - \lambda_2 + 2\lambda_4)t}, \end{aligned} \tag{4.21}$$

$$A_3 = (M_1 a_2 + M_2 a_1) a_3^2 e^{-(\lambda_1+\lambda_3)t} + (M_3 a_2 + M_4 a_1) a_3^2 e^{-(\lambda_2+\lambda_3)t} + M_5 a_3^3 e^{-2\lambda_3 t} + M_6 a_3^2 a_4 e^{-(\lambda_3+\lambda_4)t}, \tag{4.22}$$

and

$$A_4 = (S_1 a_2 + S_2 a_1) a_3 a_4 e^{-(\lambda_1+\lambda_3)t} + (S_3 a_2 + S_4 a_1) a_4^2 e^{-(\lambda_1+\lambda_4)t} + (S_5 a_2 + S_6 a_1) \times a_3 a_4 e^{-(\lambda_2+\lambda_3)t} + (S_7 a_2 + S_8 a_1) a_4^2 e^{-(\lambda_2+\lambda_4)t} + S_9 a_3 a_4^2 e^{-(\lambda_3+\lambda_4)t} + S_{10} a_4^3 e^{-2\lambda_4 t}, \tag{4.23}$$

where

$$r_1 = -n_1 \lambda_2 \lambda_3 (\lambda_1 + \lambda_2 + 2\lambda_3),$$

$$r_2 = -\frac{1}{2} n_2 \lambda_2 (2\lambda_4^2 + 2\lambda_3\lambda_4 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_4),$$

$$r_3 = -n_3 \lambda_2 \lambda_4 (\lambda_1 + \lambda_2 - 2\lambda_3 + 4\lambda_4),$$

$$i_1 = -\frac{r_1}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_4)},$$

$$i_2 = -\frac{r_1}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_4)}\left(\frac{1}{2\lambda_3} + \frac{1}{(\lambda_1+\lambda_3)} + \frac{1}{(\lambda_1+2\lambda_3-\lambda_4)}\right),$$

$$i_3 = -\frac{r_2}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_4)(\lambda_3+\lambda_4)},$$

$$i_4 = -\frac{r_2}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_4)(\lambda_3+\lambda_4)}\left(\frac{1}{(\lambda_1+\lambda_3)} + \frac{1}{(\lambda_1+\lambda_4)} + \frac{1}{(\lambda_3+\lambda_4)}\right),$$

$$i_5 = -\frac{r_3}{2\lambda_4(\lambda_1+\lambda_4)(\lambda_1+2\lambda_4-\lambda_3)},$$

$$i_6 = -\frac{r_3}{2\lambda_4(\lambda_1+\lambda_4)(\lambda_1+2\lambda_4-\lambda_3)}\left(\frac{1}{2\lambda_4} + \frac{1}{(\lambda_1+\lambda_4)} + \frac{1}{(\lambda_1+2\lambda_4-\lambda_3)}\right),$$

$$m_1 = n_1\{(\lambda_1+2\lambda_3)(\lambda_1+2\lambda_3-\lambda_4) + \lambda_3(\lambda_1+\lambda_2+2\lambda_3)\}, \quad m_2 = -\frac{3}{2},$$

$$m_3 = n_1(2\lambda_3+\lambda_2)(2\lambda_3+\lambda_2-\lambda_4), \quad m_4 = -\frac{3}{2},$$

$$s_1 = \frac{1}{2} n_2 \{(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3+3\lambda_4) + (2\lambda_4^2 + 2\lambda_3\lambda_4 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_4)\},$$

$$s_2 = -3, \quad s_3 = n_3\{(\lambda_1+\lambda_4)(\lambda_1-\lambda_3+3\lambda_4) + \lambda_4(\lambda_1+\lambda_2-2\lambda_3+4\lambda_4)\},$$

$$s_4 = -\frac{3}{2}, \quad s_5 = \frac{1}{2} n_2 (\lambda_2+\lambda_3)(2\lambda_2+\lambda_3+3\lambda_4), \quad s_6 = -3,$$

$$s_7 = n_3(\lambda_2+\lambda_4)(\lambda_2-\lambda_3+3\lambda_4), \quad s_8 = -\frac{3}{2},$$

$$M_1 = -\frac{m_1}{2\lambda_3(\lambda_1 + 2\lambda_3 - \lambda_2)(\lambda_1 + 2\lambda_3 - \lambda_4)},$$
$$M_2 = -\frac{m_2}{2\lambda_3(\lambda_1 + 2\lambda_3 - \lambda_2)(\lambda_1 + 2\lambda_3 - \lambda_4)},$$
$$M_3 = -\frac{m_3}{2\lambda_3(\lambda_2 + 2\lambda_3 - \lambda_1)(\lambda_2 + 2\lambda_3 - \lambda_4)},$$
$$M_4 = -\frac{m_4}{2\lambda_3(\lambda_2 + 2\lambda_3 - \lambda_1)(\lambda_2 + 2\lambda_3 - \lambda_4)},$$
$$M_5 = \frac{1}{(\lambda_1 - 3\lambda_3)(\lambda_2 - 3\lambda_3)(3\lambda_3 - \lambda_4)},$$
$$M_6 = \frac{3}{2\lambda_3(\lambda_1 - 2\lambda_3 - \lambda_4)(\lambda_2 - 2\lambda_3 - \lambda_4)},$$
$$S_1 = -\frac{s_1}{(\lambda_1 + \lambda_4)(\lambda_3 + \lambda_4)(-\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)},$$
$$S_2 = -\frac{s_2}{(\lambda_1 + \lambda_4)(\lambda_3 + \lambda_4)(-\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)},$$
$$S_3 = \frac{s_3}{2\lambda_4(\lambda_1 - \lambda_2 + 2\lambda_4)(-\lambda_1 + \lambda_3 - 2\lambda_4)},$$
$$S_4 = \frac{s_4}{2\lambda_4(\lambda_1 - \lambda_2 + 2\lambda_4)(-\lambda_1 + \lambda_3 - 2\lambda_4)},$$
$$S_5 = -\frac{s_5}{(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)},$$
$$S_6 = -\frac{s_6}{(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)},$$
$$S_7 = -\frac{s_7}{2\lambda_4(\lambda_1 - \lambda_2 - 2\lambda_4)(\lambda_2 - \lambda_3 + 2\lambda_4)},$$
$$S_8 = -\frac{s_8}{2\lambda_4(\lambda_1 - \lambda_2 - 2\lambda_4)(\lambda_2 - \lambda_3 + 2\lambda_4)},$$
$$S_9 = \frac{3}{2\lambda_4(\lambda_1 - \lambda_3 - 2\lambda_4)(\lambda_2 - \lambda_3 - 2\lambda_4)}, \tag{4.24}$$
$$S_{10} = -\frac{1}{(\lambda_1 - 3\lambda_4)(\lambda_2 - 3\lambda_4)(\lambda_3 - 3\lambda_4)}.$$

Here u_1 is a correction term and has also very small contribution in the solution. However, it is laborious work to solve the equation (4.15) for u_1. So, we can ignore the calculation of u_1. Putting the values of A_1, A_2, A_3 and A_4 from equations (4.21), (4.16), (4.22), (4.23) into equation (4.4) and integrating, we obtain

$$a_1(t) = a_{1,0} + \varepsilon\Big[a_{2,0}\, a_{3,0}^2 \times \begin{Bmatrix} i_2\left(1-e^{(-\lambda_1+\lambda_2-2\lambda_3)t}\right) \\ -i_1\left(t\,e^{(-\lambda_1+\lambda_2-2\lambda_3)t} + \dfrac{e^{(-\lambda_1+\lambda_2-2\lambda_3)t}-1}{\lambda_1-\lambda_2+2\lambda_3}\right)\end{Bmatrix} / (\lambda_1-\lambda_2+2\lambda_3)$$

$$+ a_{2,0}\, a_{3,0}\, a_{4,0} \times \begin{Bmatrix} i_4\left(1-e^{(-\lambda_1+\lambda_2-\lambda_3-\lambda_4)t}\right) \\ -i_3\left(t\,e^{(-\lambda_1+\lambda_2-\lambda_3-\lambda_4)t} + \dfrac{e^{(-\lambda_1+\lambda_2-\lambda_3-\lambda_4)t}-1}{\lambda_1-\lambda_2+\lambda_3+\lambda_4}\right)\end{Bmatrix} / (\lambda_1-\lambda_2+\lambda_3+\lambda_4)$$

$$+ a_{2,0}\, a_{4,0}^2 \times \begin{Bmatrix} i_6\left(1-e^{(-\lambda_1+\lambda_2-2\lambda_4)t}\right) \\ -i_5\left(t\,e^{(-\lambda_1+\lambda_2-2\lambda_4)t} + \dfrac{e^{(-\lambda_1+\lambda_2-2\lambda_4)t}-1}{\lambda_1-\lambda_2+2\lambda_4}\right)\end{Bmatrix} / (\lambda_1-\lambda_2+2\lambda_4)\Big],$$

$$a_2(t) = a_{2,0} + \varepsilon\, a_{2,0}\left[n_1 a_{3,0}^2\left(\frac{1-e^{-2\lambda_3 t}}{2\lambda_3}\right) + n_2\, a_{3,0}\, a_{4,0}\left(\frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4}\right) + n_3 a_{4,0}^2\left(\frac{1-e^{-2\lambda_4 t}}{2\lambda_4}\right)\right],$$

$$a_3(t) = a_{3,0} + \varepsilon\Big[a_{3,0}^2\{M_1 a_{2,0} + M_2\, a_{1,0}\}\left(\frac{1-e^{-(\lambda_1+\lambda_3)t}}{\lambda_1+\lambda_3}\right) + a_{3,0}^2\{M_3 a_{2,0} + M_4 a_{1,0}\}\left(\frac{1-e^{-(\lambda_2+\lambda_3)t}}{\lambda_2+\lambda_3}\right)$$

$$+ a_{3,0}^3 M_5\left(\frac{1-e^{-2\lambda_3 t}}{2\lambda_3}\right) + a_{3,0}^2 a_4 M_6\left(\frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4}\right)\Big],$$

$$a_4(t) = a_{4,0} + \varepsilon\Big[a_{3,0}\, a_{4,0}\{S_1 a_{2,0} + S_2\, a_{1,0}\}\left(\frac{1-e^{-(\lambda_1+\lambda_3)t}}{\lambda_1+\lambda_3}\right) + a_{4,0}^2\{S_3 a_{2,0} + S_4 a_{1,0}\}\left(\frac{1-e^{-(\lambda_1+\lambda_4)t}}{\lambda_1+\lambda_4}\right)$$

$$+ a_{3,0}\, a_{4,0}\{S_5\, a_{2,0} + S_6\, a_{1,0}\}\left(\frac{1-e^{-(\lambda_2+\lambda_3)t}}{\lambda_2+\lambda_3}\right) + a_{4,0}^2\{S_7\, a_{2,0} + S_8\, a_{1,0}\}\left(\frac{1-e^{-(\lambda_2+\lambda_4)t}}{\lambda_2+\lambda_4}\right)$$

$$+ a_{3,0}\, a_{4,0}^2 S_9\left(\frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4}\right) + a_{4,0}^3 S_{10}\left(\frac{1-e^{-2\lambda_4 t}}{2\lambda_4}\right)\Big].$$

(4.25)

Thus, we obtain the first approximate solution of the equation (4.11) is

$$x(t,\varepsilon) = \frac{1}{2}a_1(e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2\left(\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}\right) + a_3\, e^{-\lambda_3 t} + a_4\, e^{-\lambda_4 t} \qquad (4.26)$$

where a_1, a_2, a_3, a_4 are given by the equation (4.25).

4.4 Results and Discussion

To test the accuracy of the approximate analytical solutions obtained by the presented technique have been compared to the numerical solutions. Firstly, $x(t,\varepsilon)$ is calculated by the equation (4.26) by using the imposed conditions $\lambda_1 \to \lambda_2$, $\lambda_4 \approx \lambda_3 + 2\lambda_1$ but $\lambda_4 < 2\lambda_3$ in which a_1, a_2, a_3, a_4 are calculated by the equation (4.25). The corresponding numerical solution of equation (4.11) is computed by fourth order ***Runge-Kutta*** method. The approximate analytical solutions and numerical solutions are plotted in the **Figs. 4.1-4.4** for different initial conditions.

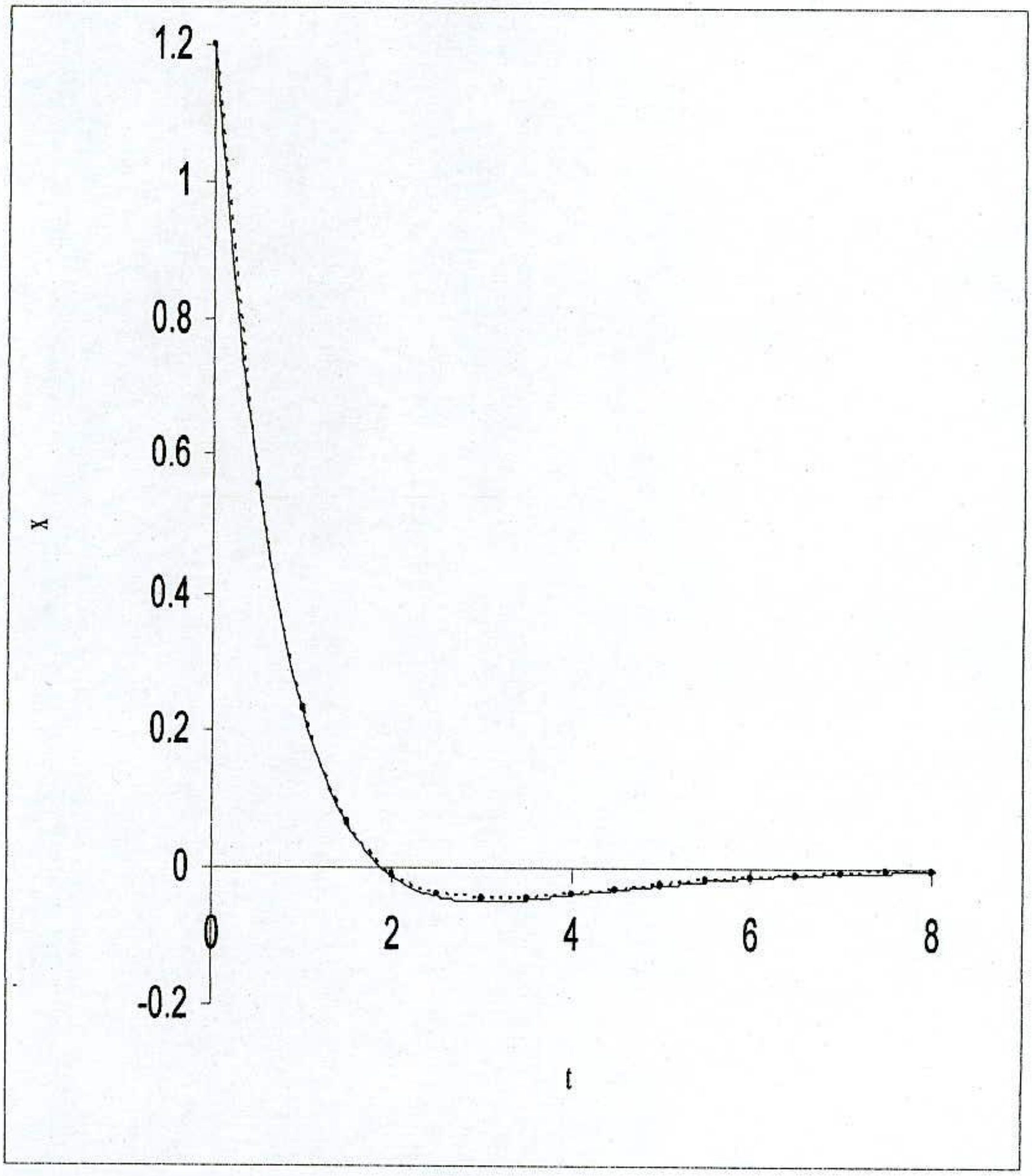

Fig. 4.1 First approximate solution of equation (4.11) is denoted by $-\bullet-$ (dashed lines) by the presented method with the initial conditions $a_{1,0} = 0.6,\ a_{2,0} = 0.6,\ a_{3,0} = 0.6,\ a_{4,0} = 0.6$ or $[x(0) = 1.80000,\ \dot{x}(0) = -2.61561,$ $\ddot{x}(0) = 3.33514,\ \dddot{x}(0) = -3.88353]$ when $\lambda_1 = 0.7, \lambda_2 = 0.95,\ \lambda_3 = 1.18, \lambda_4 = 1.35,\ \varepsilon = 0.1$ and $f = x^3$. Corresponding numerical solution is denoted by - (solid line).

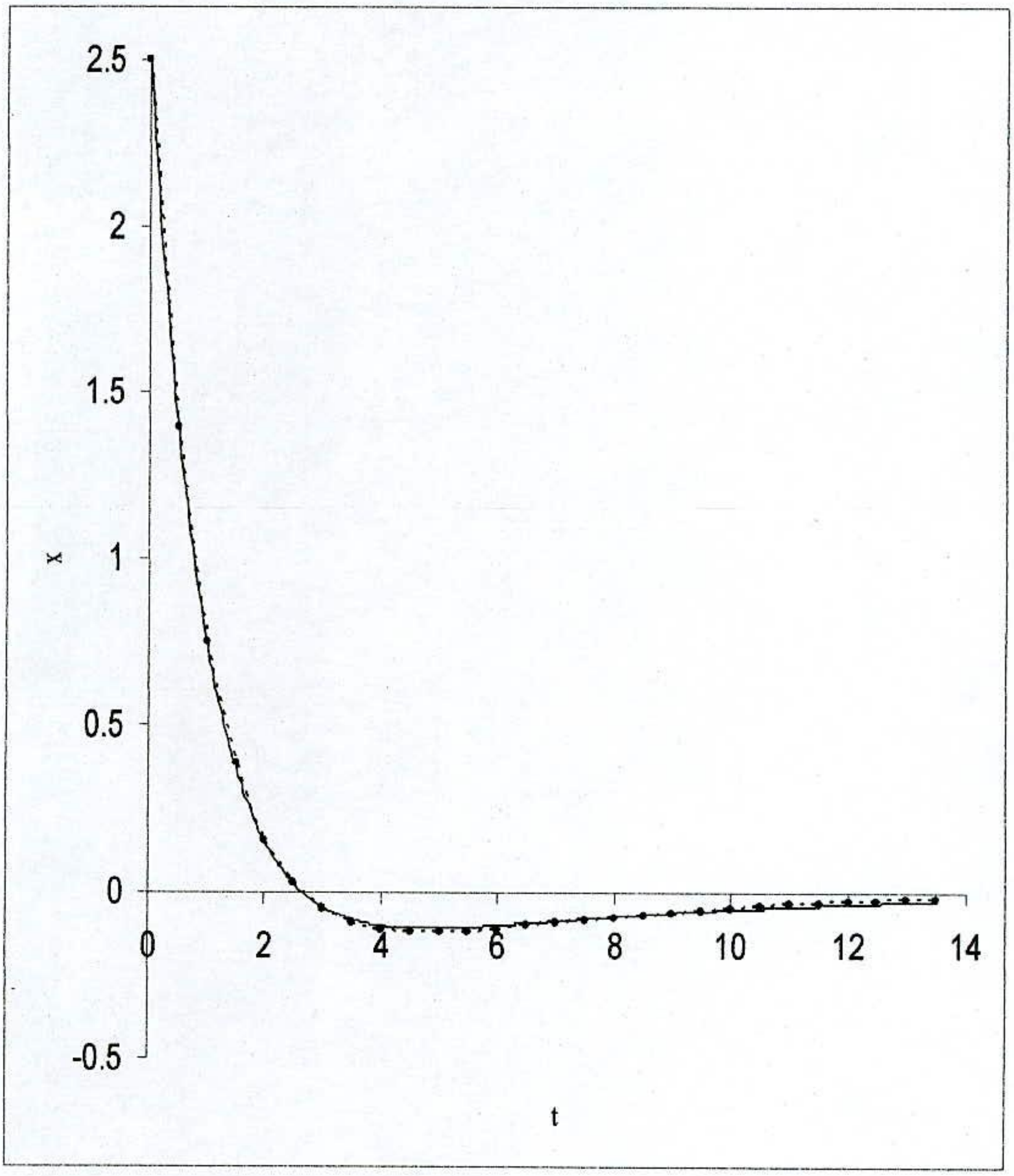

Fig. 4.2 First approximate solution of equation (4.11) is denoted by $-\bullet-$ (dashed lines) by the presented method with the initial conditions $a_{1,0}=1.0,\ a_{2,0}=0.5,\ a_{3,0}=1.0,\ a_{4,0}=0.5$ or $[x(0)=2.50000,\ \dot{x}(0)=-2.88042,$ $\ddot{x}(0)=3.13488,\ \dddot{x}(0)=-3.07011]$ when $\lambda_1=0.25,\quad \lambda_2=0.8,\ \lambda_3=1.2,\ \lambda_4=1.43,\ \varepsilon=0.1$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

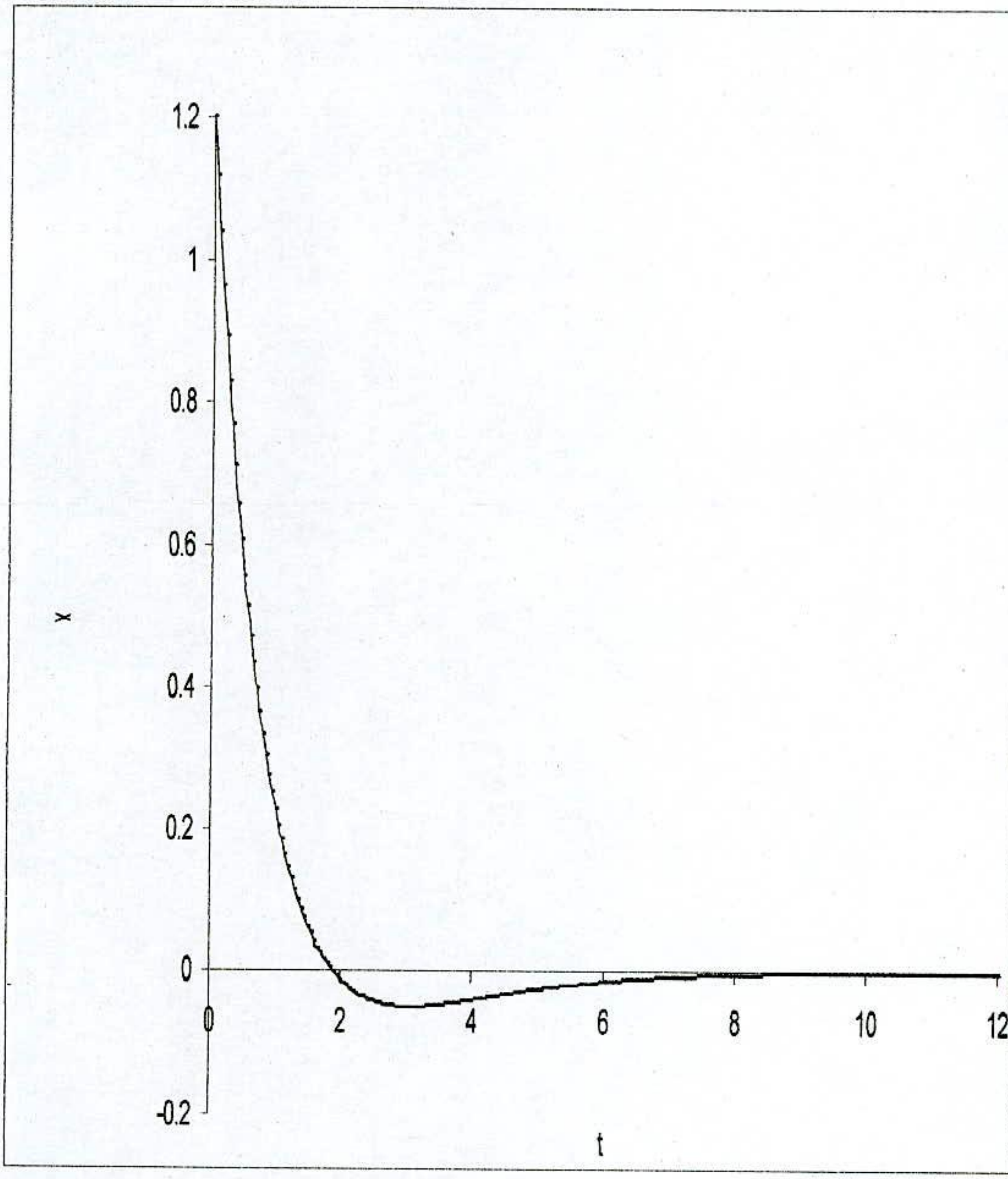

Fig. 4.3 First approximate solution of equation (4.11) is denoted by $-\bullet-$ (dashed lines) by the presented method with the initial conditions $a_{1,0} = 0.4,\ a_{2,0} = 0.4,\ a_{3,0} = 0.4,\ a_{4,0} = 0.4$ or $[x(0) = 1.20000,\ \dot{x}(0) = -1.74277,\ \ddot{x}(0) = 2.22474,\ \dddot{x}(0) = -2.65437]$ when $\lambda_1 = 0.7, \lambda_2 = 0.95,\ \lambda_3 = 1.18, \lambda_4 = 1.35, \varepsilon = 0.1$ and $f = x^3$. Corresponding numerical solution is denoted by - (solid line).

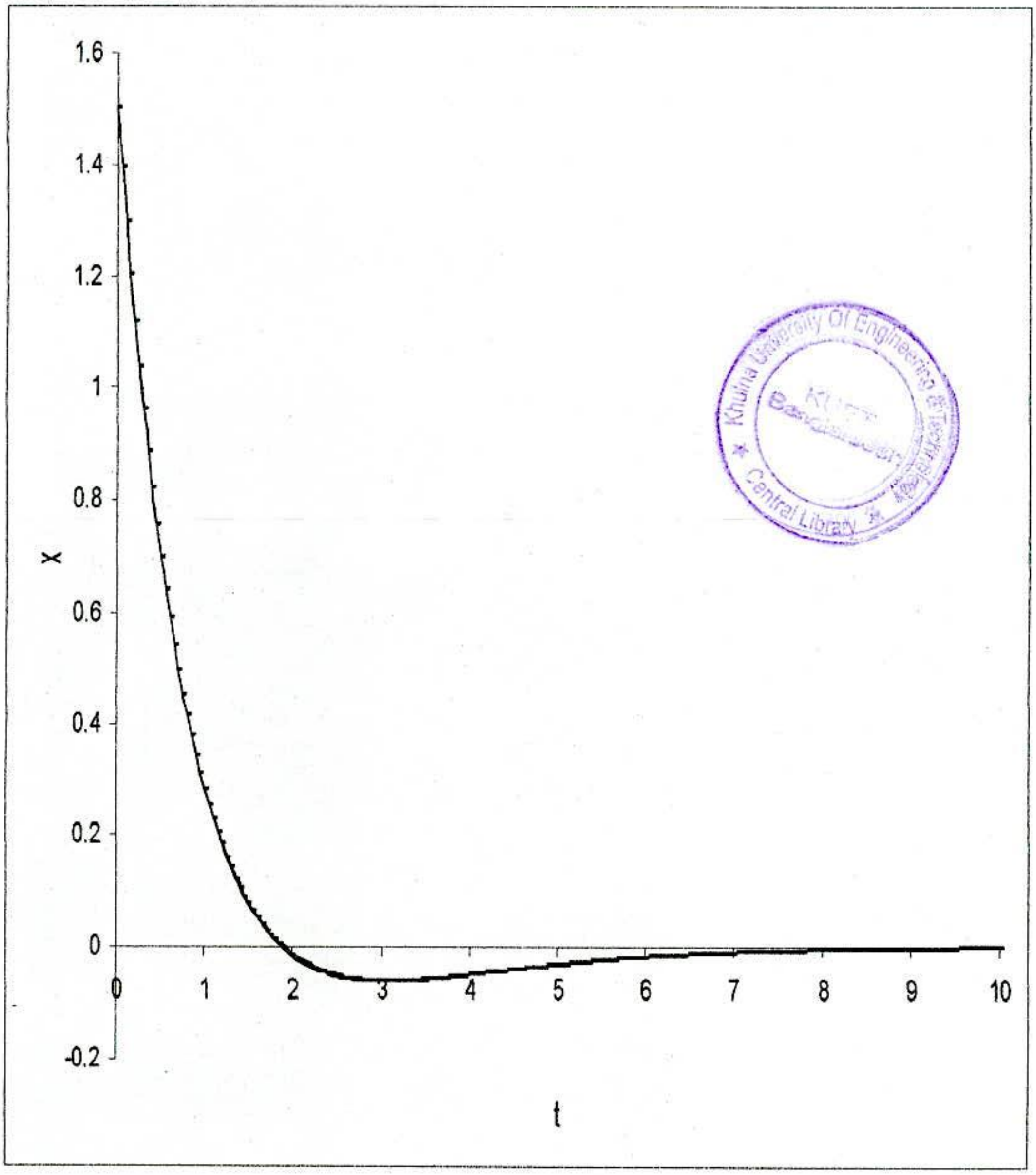

Fig. 4.4 First approximate solution of equation (4.11) is denoted by $-\bullet-$ (dashed lines) by the presented method with the initial conditions $a_{1,0}=0.5,\ a_{2,0}=0.5,\ a_{3,0}=0.5,\ a_{4,0}=0.5$ or $[x(0)=1.50000,\ \dot{x}(0)=-2.17901,$ $\ddot{x}(0)=2.78047,\ \dddot{x}(0)=-3.28167]$ when $\lambda_1=0.7, \lambda_2=0.95,\ \lambda_3=1.18,\ \lambda_4=1.35,\ \varepsilon=0.1$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

CHAPTER 5

Conclusions

The determination of amplitude and phase variables is crucial question in strongly nonlinear damped oscillatory differential systems and they play very important role for any physical problem. The amplitude and phase variables characterize the oscillatory processes. In presence of damping, amplitude $a \to 0$ as $t \to \infty$ (i.e., for large time t). It is also noticed that the some limitations of He's homotopy perturbation (without damping) technique and the KBM method (weak nonlinearity) have been overcome by the proposed method in chapter 3. Moreover, the proposed technique in chapter 3, is able to give the desired results for both strongly and weakly damped nonlinear oscillatory differential systems with small damping effects.

Also, in chapter 4, the KBM method has been extended for solving the fourth order near critically damped nonlinear systems under some special conditions with small nonlinearities, when the four eigen values of the corresponding linear equation are real and negative numbers. From the **Figs. 4.1-4.4**, it is noticed that the solutions obtained by the presented method show good agreement with those obtained by the numerical method.

REFERENCES

[1] Van der Pol, B., On Relaxation Oscillations, Philosophical Magazine, 7-th series, Vol. 2, 1926.

[2] Krylov, N. N. and Bogoliubov, N. N., Introduction to Nonlinear Mechanics, Princeton University Press, New Jersey, 1947.

[3] Bogoliubov, N. N. and Mitropolskii, Yu. A., Asymptotic Methods in the Theory of Nonlinear Oscillation, Gordan and Breach, New York, 1961.

[4] Mitropolskii, Yu. A., Problems on Asymptotic Methods of Non-stationary Oscillations (in Russian), Izdat, Nauka, Moscow, 1964.

[5] He, J. H., Some New Approaches to Duffing Equation with Strongly and High Order Nonlinearity (I) Linearized Perturbation Method, J. Communications in Nonlinear Science & Numerical Simulation,Vol. 4, No. 1, pp. 78-80, 1999.

[6] He, J. H., Approximate Solution of Nonlinear Differential Equations with Convolution Product Nonlinearities, Journal of Computer methods in applied mechanics and engineering, Vol. 167, pp. 69-73, 1998.

[7] He, J. H., New Interpretation of Homotopy Perturbation Method, International Journal of Modern Physics B, Vol.20, No. 18, pp. 2561-2568, 2006.

[8] Belendez, A., Hernandez, A., Belendez, T., Fernandez, E., Alvarez, M. L. and Neipp, C., Application of He's Homotopy Perturbation Method to Duffing Harmonic Oscillator, International Journal of Nonlinear Science and Numerical Simulation, Vol. 8(1), pp. 78-88, 2007.

[9] Uddin, M. A., Sattar, M. A. and Alam, M. S., An Approximate Technique for Solving Strongly Nonlinear Differential Systems with Damping Effects, Indian Journal of Mathematics, Vol. 53 (1), pp. 83-98, 2011.

[10] Uddin, M. A. and Sattar, M. A., An approximate technique for solving strongly nonlinear biological systems with small damping effects, Journal of the Calcutta Mathematical Society, Vol. 7 (1), pp. 51-62, 2011.

[11] Uddin, M. A. and Sattar, M. A., An Approximate Technique to Duffing Equation With Small Damping And Slowly Varying Coefficients, J. Mechanics of continua and Mathematical Sciences, Vol. 5(2), pp. 627-642, 2011.

[12] Mandelstam, L. and Papalexi N., Expose des Recherches Recentes sur les Oscillations Non-linearities (Outline of Recent Research on Non-linear Oscillations), Journal of Technical physics, USSR, 1934.

[13] Lindstedt, A., Memoires de I, Ac. Imper, des Science de st. Petersburg, 31, 1883.

[14] Gylden, Differentialgleichungen der Storungs Theorie (Differential Equations of the Theory of perturbation), Petersbourg, Vol. 31, 1883.

[15] Liapounoff, M. A., Probleme General de la Stabilite du Mouvement (General Problems of Stability of Motion), Annales de la Faculte des Sciences de Toulouse, Paris, Vol. 9, 1907.

[16] Poincare, H., Les Methods Nouvelles de la Mecanique Celeste, Paris, 1892.

[17] Stoker, J. J., Nonlinear Vibrations in Mechanical and Electrical Systems, Interscience, New York, 1950.

[18] McLachlan, N. W., Ordinary Nonlinear differential Equations in Engineering and Physical Science, Clarendon Press, Oxford, 1950.

[19] Minorsky, N., Nonlinear oscillations, Van Nostrand, Princeton, N. J., pp. 375, 1962.

[20] Nayfeh, A. H., Perturbation Methods, John Wiley and Sons, New York, 1973.

[21] Bellman, R., Perturbation Techniques in Mathematics, Physics and Engineering, Holt, Rinehart and Winston, New York, 1966.

[22] Duffing, G., Erzwungene Schwingungen bei Veranderlicher Eigen Frequenz und Ihre Technische Bedeutung, Ph. D. Thesis (Sammlung Vieweg, Braunchweig) 1918.

[23] Kruskal, M., Asymptotic Theory of Hamiltonian and Other Systems with all Situations Nearly Periodic, J. Math. Phys., Vol. 3, pp. 806-828, 1962.

[24] Cap, F. F., Averaging Method for the Solution of Nonlinear Differential Equations with Periodic Non-harmonic Solutions, Int. J. Nonlinear Mech., Vol. 9, pp. 441-450, 1974.

[25] Popov, I. P., A Generalization of the Bogoliubov Asymptotic Method in the Theory of Nonlinear Oscillations (in Russian), Dokl. Akad. USSR Vol. 3, pp. 308-310, 1956.

[26] Mendelson, K. S., Perturbation Yheory for Damped Nonlinear Oscillations, J. Math. Physics, Vol. 2, pp. 413-415, 1970.

[27] Bojadziev, G. N., Damped Oscillating Processes in Biological and Biochemical Systems, Bull. Math. Biol., Vol. 42, pp. 701-717, 1980.

[28] Murty, I. S. N., Deekshatulu, B. L., Method of Variation of Parameters for Over-damped Nonlinear Systems, J. Control, Vol. 9, no. 3, pp. 259-266, 1969.

[29] Alam, M. S., On Some Special Conditions of Over-damped Nonlinear Systems, Soochow J. Math., Vol. 29, pp. 181-190, 2003.

[30] Murty, I. S. N., Deekshatulu, B. L. and Krishna G., On an Asymptotic Method of Krylov-Bogoliubov for Over-damped Nonlinear Systems, J. Frank. Inst., Vol. **288**, pp. 49-65, 1969.

[31] Bojadziev, G. N. and Edwards, J., On Some Method for N0n-oscillatory and Oscillatory Processes, J. Nonlinear Vibration Probs., Vol. 20, pp. 69-79, 1981.

[32] Murty, I. S. N., A Unified Krylov-Bogoliubov method for Solving Second Order Nonlinear Systems, Int. J. Nonlinear Mech. Vol. **6**, pp. 45-53, 1971.

[33] Arya, J. C. and Bojadziev, G. N., Damped Oscillating Systems Modeled by Hyperbolic Differential Equations with Slowly Varying Coefficients, Acta Mechanica, Vol. 35, pp. 215-221, 1980.

[34] Arya, J. C. and Bojadziev, G. N., Time Dependent Oscillating Systems with Damping, Slowly Varying Parameters, and Delay, Acta Mechanica, Vol. 41, pp. 109-119, 1981.

[35] Sattar, M. A., An Asymptotic Method for Second Order Critically Damped Nonlinear Equations, J. Frank. Inst., Vol. 321, pp. 109-113, 1986.

[36] Osiniskii, Z., Longitudinal, Torsional and Bending Vibrations of a Uniform Bar with Nonlinear Internal Friction and Relaxation, Nonlinear Vibration Problems, Vol. 4, pp. 159-166, 1962.

[37] Bojadziev, G. N., On Asymptotic Solutions of Nonlinear Differential Equations with Time Lag, Delay and Functional Differential Equations and Their Applications (edited by K. Schmit), 299-307, New York and London: Academic Press, 1972.

[38] Bojadziev, G. N., Damped Forced Nonlinear Vibrations of Systems with Delay, J. Sound and Vibrations, Vol. 46, pp. 113-120, 1976.

[39] Bojadziev, G. N., The Krylov-Bogoliubov-Mitropolskii Method Applied to Models of Population Dynamics, Bulletin of Mathematical Biology, Vol. 40, pp. 335-345, 1978.

[40] Bojadziev, G. N. and Chan, S., Asymptotic Solutions of Differential Equations with Time Delay in Population Dynamics, Bull. Math. Biol., Vol. 41, pp. 325-342, 1979.

[41] Lin, J. and Khan, P. B., Averaging Methods in Prey-Pridictor Systems and Related Biological models, J. Theor. Biol. Vol. 57, pp. 73-102, 1974.

[42] Proskurjakov, A. P., Comparison of the Periodic Solutions of Quassi-linear Systems Constructed by the Method of Poincare and Krylov-Bogoliubov (in Russian), Applied Math. And Mech., 28, 1964.

[43] Bojadziev, G. N. and Lardner, R. W. and Arya J. C., On the Periodic Solutions of Differential Equations Obtained by the Method of Poincare and Krylov-Bogoliubov, J. Utilitas Mathematica, Vol. 3, pp. 49-64, 1973.

[44] Bojadziev, G. N. and Lardner, R. W., Monofrequent Oscillations in Mechanical Systems Governed by Hyperbolic Differential Equation with Small Nonlinearities, Int. J. Nonlinear Mech., Vol. 8, pp. 289-302, 1973.

[45] Bojadziev, G. N. and Lardner, R. W., Second Order Hyperbolic Equations with Small Nonlinearities in the Case of Internal Resonance, Int. J. Nonlinear Mech., Vol. 9, pp. 397-407, 1974.

[46] Bojadziev, G. N. and Lardner, R. W., Asymptotic Solution of a Nonlinear Second Order Hyperbolic Differential Equation with Large Time Delay, J. Inst. Math. Applies, Vol. 14, pp. 203-210, 1974.

[47] Rauch, L. L., Oscillations of a Third Order Nonlinear Autonomous System, in Contribution to the Theory of Nonlinear Oscillations, pp. 39-88, New Jersey, 1950.

[48] Bojadziev, G. N., Damped Nonlinear Oscillations Modeled by a 3-Dimensional Differential System, Acta Mechanica, Vol. 48, pp. 193-201, 1983.

[49] Bojadziev, G. N. and Hung C. K., Damped Oscillations Modeled by a 3-Dimensional Time Dependent Differential Systems, Acta Mechanica, Vol. 53, pp. 101-114, 1984.

[50] Osiniskii, Z., Vibration of a One Degree Freedom System with Nonlinear Internal Friction and Relaxation, Proceedings of International Symposium of Nonlinear Vibrations, Vol. 111, pp. 314-325, Kiev, Izadt, Akad, Nauk USSR, 1963.

[51] Mulholland, R. J., Nonlinear Oscillations of Third Order Differential Equation, Int. J. Nonlinear Mechanics, Vol. 6, pp. 279-294, 1971.

[52] Lardner, R. W. and Bojadziev, G. N., Asymptotic Solutions for Third Order Partial Differential Equations with Small Nonlinearities, Meccanica, Vol. 14, pp. 249-256, 1979.

[53] Alam, M. S., Damped Oscillations Modeled by an n-th Order Time Dependent Quasi-linear Differential System, Acta Mechanica, Vol. 169, pp. 111-122, 2004.

[54] Alam, M. S., Akbar, M. A. and Islam, M. Z., A General Form of Krylov-Bogoliubov-Mitropolskii Method for Solving Non-linear Partial Differential Equations, Journal of Sound and Vibration, Vol. 285, pp. 173-185, 2005.

[55] Raymond, P. Vitro and Cabak G., The Effects of Internal Resonance on Impulsively Forced Nonlinear Systems with Two Degree of Freedom, Int. J. Nonlinear Mechanics, Vol. 14, pp. 93-99, 1979.

[56] Alam, M. S., Perturbation Theory for n-th Order Nonlinear Systems with Large damping, Indian J. pure appl. Math., Vol. 33, pp. 1677-1684, 2002.

[57] Alam, M. S., Method of Solution to the n-th Order Over-damped Nonlinear Systems Under Some Special Conditions, Bull. Cal. Math. Soc., Vol. 94, pp. 437-440, 2002.

[58] Alam, M. S., Perturbation Theory for Nonlinear Systems with Large Damping, Indian J. pure appl. Math.,Vol. 32, pp. 453-461, 2001.

[59] Alam, M. S., Hossain, B. and Shanta, S. S., Krylov-Bogoliubov-Mitropolskii Method for Time Dependent Nonlinear Systems with Damping, Mathematical Forum, Vol. 14, pp. 53-59, 2001.

[60] Sattar, M. A., An Asymptotic Method for Three Dimensional Over-damped Nonlinear Systems, Ganit, J. Bangladesh Math. Soc., Vol. 13, pp. 1-8, 1993.

[61] Lim, C. W. and Wu, B. S., A New Analytical Approach to the Duffing- Harmonic Oscillator, Physics Letters A Vol.311, pp.365-373, 2003.

[62] Alam, M. S. and Sattar, M. A., Time Dependent Third-order Oscillating Systems with Damping, Acta Ciencia Indica, Vol. 27, pp. 463-466, 2001.

[63] Alam, M. S., Hossain, M. B. and Akbar, M. A., On a Special Condition of Over-damped Nonlinear Systems with Slowly Varying Coefficients, Journal of Interdisciplinary Mathematics, Vol. 7, pp. 255-260, 2004.

[64] Alam, M. S. and Hossain, M. B., On Some Special Conditions of n-th Order Non-oscillatory Nonlinear Systems, Communication of Korean Math. Soc., Vol. 18, pp. 755-765, 2003.

[65] Alam, M. S. and Sattar, M. A., A Unified Krylov-Bogoliubov-Mitropolskii Method for Solving Third Order Nonlinear Systems, Indian J. pure appl. Math., Vol. 28, pp. 151-167, 1997.

[66] Alam, M. S., Asymptotic Methods for Second Order Over-Damped and Critically Damped Nonlinear Systems, Soochow Journal of Math., Vol. 27, pp. 187-200, 2001.

[67] Alam, M. S., Alam, M. F. and Shanta, S. S., Approximate Solution of Non-oscillatory systems with Slowly Varying Coefficients, Ganit-J, Bangladesh Math. Soc., Vol. 21, pp. 55-59, 2001.

[68] Alam, M. S., A Unified Krylov-Bogoliubov-Mitropolskii Method for Solving n-th Order Nonlinear Systems, J. Frank. Inst., Vol. 339, pp. 239-248, 2002.

[69] Alam, M. S., A Modified and Compact Form of Krylov-Bogoliubov-Mitropolskii Unified Method for an n-th Order Nonlinear Differential Equation, Int. J. Nonlinear Mechanics, Vol. 39, pp. 1343-1357, 2004.

[70] Alam, M. S. and Sattar, M. A., An Asymptotic Method for Third Order Critically Damped Nonlinear Equations, J. Mathematical and Physical Sciences, Vol. 30, pp. 291-298, 1996.

[71] Alam, M. S., Bogoliubov's Method for Third Order Critically Damped Nonlinear Systems, Soochow J. Math., Vol. 28, pp. 65-80, 2002.

[72] Alam, M. S., Asymptotic Method for Non-oscillatory Nonlinear Systems, Far East J. Appl. Math., Vol. 7, pp. 119-128, 2002.

[73] Alam, M. S., Asymptotic Method for Certain Third Order Non-oscillatory Nonlinear Systems, J. Bangladesh Academy of Sciences, Vol. 27, pp. 141-148, 2003.

[74] Akbar, M. A., Paul A. C. and Sattar, M. A., An Asymptotic Method of Krylov-Bogoliubov for Fourth Order Over-damped Nonlinear Systems, Ganit-J. Bangladesh Math. Soc., Vol. 22, pp. 83-96, 2002.

[75] Akbar, M. A., Alam, M. S. and Sattar, M. A., A Simple Technique for Obtaining Certain Over-damped Solutions of an n-th Order Nonlinear Differential Equation, Soochow Journal of Mathematics, Vol. 31(2), pp. 291-299, 2005.

[76] Akbar, M. A., Alam, M. S. and Sattar, M. A., Krylov-Bogoliubov-Mitropolskii Unified Method for Solving n-th Order Nonlinear Differential Equations Under Some Special Conditions including the Case of Internal Resonance, Int. J. Non-linear Mech, Vol. 41, pp. 26-42, 2006.

[77] Akbar, M. A., Uddin, M. S., Islam, M. R. and Soma, A. A., Krylov-Bogoliubov-Mitropolskii (KBM) Method for Fourth Order More Critically Damped Nonlinear Systems, J. Mech. of Continua and Math. Sciences Vol. **2(1)**, pp. 91-107, 2007.

[78] Akbar, M. A., Alam, M. S., Shanta, S. S., Uddin, M. S. and Shamsuzzoha, M., "Perturbation Method for Fourth Order Nonlinear Systems with Large Damping", Bulletin of Calcutta Math. Soc., Vol. 100(1), pp. 85-92, 2008.

[79] Haque, B. M. I., Alom, M. A. and Rahman, M. H., Perturbation Solution of Fourth Order Critically Damped Oscillatory Nonlinear Systems, International Journal of Basic & Applied Sciences, Vol. 11, No. 3, pp. 68-77, 2011.

[80] Uddin, M. A. and Sattar, M. A., Approximate Solution of a Fourth Order Weakly Nonlinear Differential System with Strong Damping and Slowly Varying Coefficients By Unified KBM Method, J. Bangladesh Academy of Sciences, Vol. 34(1), pp. 71-82, 2010.

[81] Rahman, M. H., Haque, B. M. I. and Akbar, M. A., Asymptotic Solutions of Fourth Order Near Critically Damped Nonlinear Systems, Journal of Informatics and Mathematical Sciences, Vol. 1, no. 1, pp. 61-73, 2009.

[82] Nayfeh, A. H., Introduction to Perturbation Techniques, Wiley, New York, 1981.

[83] Alam, M. S., A Unified KBM Method for Solving n-th Order Nonlinear Differential Equation with Varying Coefficients, J. Sound and Vibration, Vol. 265, pp. 987-1002, 2003.

[84] Alam, M. S. and Sattar, M. A., Asymptotic Method for Third Order Nonlinear Systems with Slowly Varying Coefficients, Journal of Southeast Asian Bulletin of Mathematics, Vol. 28, pp. 979-987, 2004.

[85] Murdock, J. A., Perturbations: Theory and Methods, Wiley, New York, 1991.

[86] Sachs, R. K., Hlatky, L. R. and Hahnfeldt, P., Simple ODE models of tumor growth and anti-angiogenic or radiation treatment, J. Mathematical and Computer Modeling, Vol. 33, pp. 1297-1305, 2001.