Approximate Solutions of Second and Fourth Order Ordinary Differential Systems with Strong Generalized Nonlinearity

by

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Dedicated to My

Beloved Parents

and

Affectionate Son

Declaration

This is to certify that the thesis work entitled "**Approximate Solutions of Second and Fourth Order Ordinary Differential Systems with Strong Generalized Nonlinearity**" has been carried out by **Md. Wali Ullah** in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

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The author

Abstract

A perturbation method known as "the asymptotic averaging method" in the theory of nonlinear oscillations was first presented by Krylov and Bogoliubov (KB) in 1947. Primarily, the method was developed only to obtain the periodic solutions of second order weakly nonlinear differential systems. Later, the method of KB has been improved and justified by Bogoliubov and Mitropolskii in 1967. In literature, this method is known as the Krylov-Bogoliubov-Mitropolskii (KBM) method. Now a days, this method is used for obtaining the solutions of second, third and fourth order weakly nonlinear differential systems for oscillatory, damped oscillatory, over damped, critically damped and more critically damped cases by imposing some special restrictions. Ji-Huan He has developed a homotopy perturbation method for solving second order strongly nonlinear differential systems without damping. Uddin *et al.* have presented an approximate analytical technique for second order strongly nonlinear differential systems with damping by combining He's homotopy perturbation technique and the extended form of the KBM method. Recently, Uddin *et al.* have developed an analytical approximate technique for solving a certain type of fourth order strongly nonlinear oscillatory differential systems with small damping and cubic nonlinearity by combining He's homotopy perturbation and the extended form of the KBM methods. In this thesis, approximate analytical techniques shall be presented by combining the He's homotopy perturbation technique and the extended form of the KBM method for solving the second and fourth order nonlinear ordinary differential systems with strong generalized nonlinearity. To justify the presented methods, the approximate solutions have been compared to those solutions obtained by the fourth order **Runge-Kutta** method.

Publications:

1. M. Alhaz Uddin and M. Wali Ullah, 2013, "A coupling approximate analytical technique for solving certain type of fourth order strongly generalized nonlinear damped oscillatory differential system", Indian Journal of Theoretical Physics, Vol.61(3), pp.179-194.

2. M. Alhaz Uddin and M. Wali Ullah, "An approximate analytical technique for solving second order strongly nonlinear generalized Duffing oscillator with small damping" (Submitted, 2013).

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CHAPTER I

Introduction

Differential equation is a mathematical tool, which has its application in many branches of knowledge of mankind. Numerous physical, mechanical, chemical, biological, biochemical and many other relations appear mathematically in the form of differential equations that are linear or nonlinear, autonomous or non-autonomous. Generally, in many physical phenomena, such as spring-mass systems, resistor-capacitor-inductor circuits, bending of beams, chemical reactions, the motion of pendulum, the motion of the rotating mass around another body, etc, the differential equations occur. Also, in ecology and economics the differential equations are vastly used. Basically, many differential equations involving physical phenomena are nonlinear. Differential equations, which are linear, are comparatively easy to solve and nonlinear are laborious and in some cases it is impossible to solve them analytically. In such situations, mathematicians, physicists and engineers convert the nonlinear equations into linear equations i.e., they linearize them by imposing some special conditions. The method of small oscillations is a well-known example of the linearization. But, such a linearization is not always possible and when it is not, then the original nonlinear equation itself must be used. The study of nonlinear equations is generally confined to a variety of rather special cases, and one must resort to various methods of approximation.

At first Van der Pol [1] paid attention to the new (self-excitations) oscillations and indicated that their existence is inherent in the nonlinearity of the differential equations characterizing the process. Thus, this nonlinearity appears as the very essence of these phenomena and by linearizing the differential equations in the sense of the method of small oscillations, one simply eliminates the possibility of investigating such problems. Thus, it is necessary to deal with the nonlinear problems directly instead of evading them by dropping the nonlinear terms. To solve nonlinear differential equations there exist some methods. Among the methods, the method of perturbations, i.e., asymptotic expansions in terms of a small parameter are foremost. Perturbation methods have recently received much attention as methods for accurately and quickly computing numerical solutions of dynamic, stochastic and economic equilibrium models, both single-agent or rational expectations models and multi-agent or game theoretic models. A perturbation method is based on the following aspects: the equations to be solved are sufficiently "smooth" or sufficiently differentiable a number of times in the required regions of variables and parameters.

The KBM [2-3] method was developed for the systems only to obtain the periodic solutions of second order nonlinear differential equations. Now, the method is used to obtain oscillatory as well as damped, critically damped, over damped, near critically damped, more critically damped oscillatory and non-oscillatory solutions of second, third, fourth etc, order nonlinear differential equations by imposing some suitable restrictions to obtain the uniform solutions. Several authors [5-73] have investigated and developed many significant results concerning the solutions of the weakly nonlinear differential systems. Extensive uses have been made and some important works are done by several authors [5-73] based on the **KBM** method.

Ji-Huan He [74-76] has developed a homotopy perturbation technique for solving second order strongly nonlinear differential systems without damping effects. Belendez *et al.* [77] have applied He's homotopy perturbation method to Duffing harmonic oscillator. Uddin *et al.* [78-79] have presented an approximate technique for solving second order strongly nonlinear oscillatory differential systems with damping effects by combing the He's [74-76] homotopy perturbation and the KBM [2-3] methods. Recently, Uddin *et al.* [82] have developed an analytical approximate technique for solving a certain type of fourth order strongly nonlinear oscillatory differential system with small damping and cubic nonlinearity by combining He's [74-76] homotopy perturbation and the extended form of the KBM [2-3] methods. The method of KB [2] is an asymptotic method in the sense that $\varepsilon \to 0$. An asymptotic series itself may not be convergent, but for a fixed number of terms, the approximate solution tends to the exact solution as $\varepsilon \to 0$. It may be noted that the term asymptotic is frequently used in the theory of oscillations in the sense that $\varepsilon \to \infty$. But, in this case, the mathematical method is quite different. It is an important approach to the study of such nonlinear oscillations in the small parameter expansion. Two widely spread methods in this theory are mainly used in the literature; one is averaging asymptotic KBM method and the other is multiple-time scale method. The KBM method is particularly convenient and extensively used technique to obtain the approximate solutions among the methods used to study the nonlinear differential systems with small nonlinearity. The KBM method starts with the solution of linear equations (sometimes called the generating solution of the linear equation), assuming that in the nonlinear case, the amplitude and phase variables in the solution of the linear differential equation are time dependent functions instead of constants. This method introduces an additional condition on the first derivative of the assumed solution for determining the solution of a second order equation. The KBM method demands that the asymptotic solutions are free from secular terms. These assumptions are mainly valid for second and third order equations. But, for the fourth order differential equation, the correction terms sometimes contain secular terms, although the solution is generated by the classical KBM asymptotic method. For this reason, the traditional solutions fail to explain the proper situation of the systems. To remove the presence of secular terms and to obtain the desired results, we need to impose some special conditions. The KBM method is fail to tackle the second and fourth order ordinary differential systems with strong generalized nonlinearity. Also the homotopy perturbation technique for second and fourth order nonlinear ordinary differential systems with strong generalized nonlinearity in presence of damping almost remains untouched. The main objective of this thesis is to overcome these limitations of KBM and He's homotopy perturbation methods. The results may be used in mechanics, physics, chemistry, plasma physics, circuit and control theory, population dynamics, economics, etc.

In this thesis, He's homotopy perturbation method has been extended for obtaining the analytical approximate solutions of second and fourth order strongly generalized nonlinear differential systems with small damping based on the extended form of the KBM method.

In Chapter II, the review of literature is presented. **In Chapter III**, an approximate analytical technique has been developed for solving second order strongly generalized nonlinear differential system with small damping. Also a coupled analytical approximate technique has been extended for obtaining the solutions of certain type of fourth order strongly generalized nonlinear oscillatory differential systems with small damping based on the He's homotopy perturbation and the extended form of the famous KBM methods **in Chapter IV.** Finally, **in Chapter V**, the conclusions are drown.

CHAPTER II

Literature Review

Nonlinear differential equations show varieties characteristics. But, mathematical formulations of many physical problems often result in differential equations that are nonlinear. In many situations, linear differential equation is substituted for a nonlinear differential equation, which approximates the former equation closely enough to give expected result. In many cases such a linearization is not possible and when it is not, the original nonlinear differential equation must be tackled directly. During last several decades in the *20th* century, some famous Russian scientists like Krylov and Bogoliubov [2], Bogoliubov and Mitropolskii [3], Mitropolskii [4], Mandelstam and Papalexi [5] have investigated the nonlinear dynamics. To solve nonlinear differential equations there exist some methods. Among the methods, the method of perturbations, i.e., an asymptotic expansion in terms of small parameter is foremost. Firstly, Krylov and Bogoliubov (KB) [2] considered the equation of the form

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}, t, \varepsilon), \tag{2.1}$$

where $\ddot{x}$ denotes the second derivative with respect to t, ε is a small positive parameter and f is a power series in ε, whose coefficients are polynomials in $x, \dot{x}, \sin t$ and $\cos t$ and their proposed solution procedure is known as KB method. In general, *f does* not contain either ε or t. In literature, the method presented [2-3] is known as Krylov-Bogoliubov-Mitropolskii (KBM) method. To describe the behavior of nonlinear oscillations by the solutions obtained by the perturbation method, Lindstedt [6], Glyden [7], Liapounoff [8], Poincare [9] discussed only periodic solutions, transient were not considered. Most probably, Poisson initiated approximate solutions of nonlinear differential equations around 1830 and the technique was established by Liouville. The KBM [2-3] method started with the solution of the linear equation, assuming that in the nonlinear systems, the amplitude and phase variables in the solution of the linear equations are time dependent functions rather than constants. This procedure introduces an additional condition on the first derivative of the assumed solution for determining the desired results. Some mirthful works are done and elaborative uses have been made by Stoker [10], McLachlan [11], Minorsky [12] and Bellman [13]. Duffing [14] has investigated many significant results for the periodic solutions of the following damped nonlinear differential system

$$\ddot{x} + 2k\,\dot{x} + \omega^2 x = -\varepsilon\, x^3. \tag{2.2}$$

Sometimes, different types of nonlinear phenomena occur, when the amplitude of the dependent variable of a dynamic system is less than or greater than unity. The damping is negative when the amplitude is less than unity and the damping is positive when the amplitude is greater than unity. The governing equation having these phenomena is

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0. \tag{2.3}$$

In literature, this equation is known as Van der Pol [1] equation and is used in electrical circuit theory. Kruskal [15] has extended the KB [2] method to solve the fully nonlinear differential equation of the following form

$$\ddot{x} = F(x, \dot{x}, \varepsilon). \tag{2.4a}$$

Cap [16] has studied nonlinear system of the form

$$\ddot{x} + \omega^2 x = \varepsilon\, F(x, \dot{x}). \tag{2.4b}$$

Generally, *f* does not contain either ε or *t,* thus the equation (2.1) becomes

$$\ddot{x} + \omega^2 x = \varepsilon\, f(x, \dot{x}). \tag{2.5}$$

In the treatment of nonlinear oscillations by the perturbation method, only periodic solutions are discussed, transients are not considered by different investigators, where as KB [2] have discussed transient response firstly.

When $\varepsilon = 0,$ the equation (2.5) reduces to linear equation and its solution can be obtained as

$$x = a\cos(\omega\, t + \varphi). \tag{2.6}$$

where a and φ are arbitrary constants and the values of a and φ are determined by using the given initial conditions.

When $\varepsilon \neq 0,$ but is sufficiently small, then KB [2] have assumed that the solution of equation (2.5) is still given by equation (2.6) together with the derivative of the form

$$\dot{x} = -a\omega \sin(\omega\, t + \varphi). \tag{2.7}$$

where a and φ are functions of *t,* rather than being constants. In this case, the solution of equation (2.5) is

$$x = a(t)\cos(\omega\, t + \varphi(t)) \tag{2.8}$$

and the derivative of the solution is

$$\dot{x} = -a(t)\omega \sin(\omega t + \varphi(t)). \tag{2.9}$$

Differentiating the assumed solution equation (2.8) with respect to time *t*, we obtain

$$\dot{x} = \dot{a}\cos\psi - a\omega\sin\psi - a\dot{\varphi}\sin\psi, \ \ \psi = \omega t + \varphi(t). \tag{2.10}$$

Using the equations (2.7) and (2.10), we get

$$\dot{a}\cos\psi = a\dot{\varphi}\sin\psi. \tag{2.11}$$

Again, differentiating equation (2.9) with respect to *t*, we have

$$\ddot{x} = -\dot{a}\omega\sin\psi - a\omega^2\cos\psi - a\omega\dot{\varphi}\cos\psi. \tag{2.12}$$

Putting the value of $\ddot{x}$ from equation (2.12) into the equation (2.5) and using equations (2.8) and (2.9), we obtain

$$\dot{a}\omega\sin\psi + a\omega\dot{\varphi}\cos\psi = -\varepsilon f(a\cos\psi, -a\omega\sin\psi). \tag{2.13}$$

Solving equations (2.11) and (2.13), we have

$$\dot{a} = -\frac{\varepsilon}{\omega}\sin\psi\, f(a\cos\psi, -a\omega\sin\psi), \tag{2.14}$$

$$\dot{\varphi} = -\frac{\varepsilon}{a\omega}\cos\psi\, f(a\cos\psi, -a\omega\sin\psi). \tag{2.15}$$

It is observed that, a basic differential equation (2.5) of the second order in the unknown x, reduces to two first order differential equations (2.14) and (2.15) in the unknowns a and φ.

Moreover, $\dot{a}$ and $\dot{\varphi}$ are proportional to ε; a and φ are slowly varying functions of the time period $T = \frac{2\pi}{\omega}$. It is noted that these first order equations are now written in terms of the amplitude a and phase φ as dependent variables. Therefore, the right sides of equations (2.14) and (2.15) show that both a and $\dot{\varphi}$ are periodic functions of period T. In this case, the right-hand terms of these equations contain a small parameter ε and also contain both a and φ, which are slowly varying functions of the time t with period $T = \frac{2\pi}{\omega}$. We can transform the equations (2.14) and (2.15) into more convenient form. Now, expanding $\sin\psi\, f(a\cos\psi, -a\omega\sin\psi)$ and $\cos\psi\, f(a\cos\psi, -a\omega\sin\psi)$ in Fourier series with phase ψ, the first approximate solution of equation (2.5) by averaging equations (2.14) and (2.15) with period $T = \frac{2\pi}{\omega}$, is

$$\langle \dot{a} \rangle = -\frac{\varepsilon}{2\pi\,\omega} \int_0^{2\pi} \sin\psi\, f(a\cos\psi, -a\omega\, \sin\psi)\, d\psi,$$
$$\langle \dot{\varphi} \rangle = -\frac{\varepsilon}{2\pi\,\omega\, a} \int_0^{2\pi} \cos\psi\, f(a\cos\psi, -a\,\omega\, \sin\psi)\, d\psi, \qquad (2.16)$$

where a and φ are independent of time t under the integrals. KB [2] have called their method asymptotic in the sense that $\varepsilon \to 0$. An asymptotic series itself is not convergent, but for a fixed number of terms the approximate solution tends to the exact solution as $\varepsilon \to 0$. Later, this technique has been extended mathematically by Bogoliubov and Mitropolskii [3], and has extended to non-stationary vibrations by Mitropolskii [4]. They have assumed the solution of the nonlinear differential equation (2.5) of the form

$$x = a\cos\psi + \varepsilon\, u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \ldots\ldots\ldots\ldots + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}), \qquad (2.17)$$

where u_k, $(k = 1, 2, \ldots\ldots., n)$ are periodic functions of ψ with a period 2π, and the terms a and ψ are functions of time t and defined by the following first order ordinary differential equations

$$\dot{a} = \varepsilon\, A_1(a) + \varepsilon^2 A_2(a) + \ldots\ldots\ldots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}),$$
$$\dot{\psi} = \omega + \varepsilon\, B_1(a) + \varepsilon^2 B_2(a) + \ldots\ldots. + \varepsilon^n B_n(a) + O(\varepsilon^{n+1}). \qquad (2.18)$$

The functions $u_{k,}$ A_k and B_k, $(k = 1, 2, \ldots\ldots., n)$ are to be chosen in such a way that the equation (2.17), after replacing a and ψ by the functions defined in equation (2.18), is a solution of equation (2.5). Since there are no restrictions in choosing functions A_k and B_k, it generates the arbitrariness in the definitions of the functions u_k (Bogoliubov and Mitropolskii [3]). To remove this arbitrariness, the following additional conditions are imposed

$$\int_0^{2\pi} u_k(b,\psi)\cos\psi\, d\psi = 0,$$
$$\int_0^{2\pi} u_k(a,\psi)\sin\psi\, d\psi = 0. \qquad (2.19)$$

Absences of secular terms in all successive approximations are guaranteed by these conditions. Differentiating equation (2.17) two times with respect to *t*, substituting the values of x, $\dot{x}$ and $\ddot{x}$ into equation (2.5), using these relations in equation (2.18) and equating the coefficients of ε^k, $(k = 1, 2, ...,n)$, leads to

$$\omega^2((u_k)_{\psi\psi} + u_k) = f^{(k-1)}(a,\psi) + 2\omega\,(a\,B_k \cos\psi + A_k \sin\psi), \tag{2.20}$$

where $(u_k)_\psi$ denotes partial derivatives with respect to ψ,

$$\begin{aligned} &f^{(0)}(a,\psi) = f(a\cos\psi, -a\omega\sin\psi) \text{ and} \\ &f^{(1)}(a,\psi) = u_1 f_x(a\cos\psi, -a\omega\sin\psi) + (A_1\cos\psi - aB_1\sin\psi + \omega(u_1)_\psi)\times \\ &f_{\dot{x}}(\cos\psi, -a\omega\sin\psi) + (aB_1^2 - A_1\frac{dA_1}{da})\cos\psi + (2A_1B_1 - aA_1\frac{dB_1}{da})\sin\psi - \\ &2\omega(A_1(u_1)_{a\psi} + B_1(u_1)_{\psi\psi}). \end{aligned} \tag{2.21}$$

Here $f^{(k-1)}$ is a periodic function of ψ with period 2π which depends also on the amplitude a. Therefore, $f^{(k-1)}$ and u_k can be expanded in a Fourier series as

$$\begin{aligned} &f^{(k-1)}(a,\psi) = g_0^{(k-1)}(a) + \sum_{n=1}^{\alpha}(g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi), \\ &u_k(a,\psi) = v_0^{(k-1)}(a) + \sum_{n=1}^{\alpha}(v_n^{(k-1)}(a)\cos n\psi + \omega_n^{(k-1)}(a)\sin n\psi), \end{aligned} \tag{2.22}$$

where

$$g_0^{(k-1)}(a) = \frac{1}{2\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\,d\psi. \tag{2.23}$$

Here, $v_1^{(k-1)} = \omega_1^{(k-1)} = 0$ for all values of k, because both integrals of equation (2.19) are vanished. Substituting these values into the equation (2.20), we obtain

$$\begin{aligned} &\omega^2 v_0^{(k-1)}(a) + \sum_{n=2}^{\alpha}\omega^2(1-n^2)[v_n^{(k-1)}(a)\cos n\psi + \omega_n^{(k-1)}(a)\sin n\psi] \\ &= g_0^{(k-1)}(a) + (g_1^{(k-1)}(a) + 2\omega\,a\,B_k)\cos n\psi + (h_1^{(k-1)}(a) + 2\omega\,A_k)\sin\psi \\ &+ \sum_{n=2}^{\alpha}[g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi]. \end{aligned} \tag{2.24}$$

Now, equating the coefficients of the harmonics of the same order, yield

$$\begin{aligned} &g_1^{(k-1)}(a) + 2\omega\,a\,B_k = 0, \quad h_1^{(k-1)}(a) + 2\omega\,A_k = 0, \quad v_0^{(k-1)}(a) = \frac{g_0^{(k-1)}(a)}{\omega^2}, \\ &v_n^{(k-1)}(a) = \frac{g_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad \omega_n^{(k-1)}(a) = \frac{h_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad n \ge 1. \end{aligned} \tag{2.25}$$

These are the sufficient conditions to obtain the desired order of approximation. For the first order approximation, we have

$$A_1 = -\frac{h_1^{(0)}(a)}{2\omega} = -\frac{1}{2\pi\omega}\int_0^{2\pi} f(a\cos t\psi, -a\omega\sin\psi)\sin\psi\, d\psi,$$
$$B_1 = -\frac{g_1^{(0)}(a)}{2a\omega} = -\frac{1}{2\pi\, a\omega}\int_0^{2\pi} f(a\cos t\psi, -a\omega\sin\psi)\cos\psi\, d\psi. \tag{2.26}$$

Thus, the variational equations in equation (2.18) become

$$\dot{a} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\, d\psi,$$
$$\dot{\psi} = \omega - \frac{\varepsilon}{2\pi\, a\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi\, d\psi. \tag{2.27}$$

It is seen that, the equation (2.27) are similar to the equation (2.16). Thus, the first order solution obtained by Bogoliubov and Mitropolskii [3] is identical to the original solution obtained by KB [2]. In literature, this method is well known as Krylov-Bogoliubov-Mitropolskii (KBM) [2-3] method. The correction term u_1 is obtained from equation (2.22) by using equation (2.25) as

$$u_1 = \frac{g_0^{(0)}(a)}{\omega^2} + \sum_{n=2}^{\alpha} \frac{g_n^{(0)}(a)\cos n\psi + h_n^{(0)}(a)\sin n\psi}{\omega^2(1-n^2)} \tag{2.28}$$

The solution equation (2.17) together with u_1 is known as the first order improved solution in which a and ψ are obtained from equation (2.27). If the values of the functions A_1 and B_1 are substituted from equation (2.26) into the second relation of equation (2.21), the function $f^{(1)}$ and in the similar way, the functions A_2, B_2 and u_2 can be found. Therefore, the determination of the second order approximation is completed. The KB [2] method is very similar to that of Van der Pol [1] and related to it. Van der Pol has applied the method of variation of constants to the basic solution $x = a\cos\omega t + b\sin\omega t$ of $\ddot{x} + \omega^2 x = 0$, on the other hand KB [2] has applied the same method to the basic solution $x = a\cos(\omega t + \varphi)$ of the same equation. Thus, in the KB [2] method the varied constants are a and φ, while in the Van der Pol's method the constants are a and b. The method of KB [2] seems more interesting from the point of view of applications, since it deals directly with the amplitude and phase of the quasi-harmonic oscillation.

The solution of the equation (2.4a) is based on recurrent relations and is given as the power series of the small parameter. Cap [16] has solved the equation (2.4b) by using elliptical functions in the sense of KB [2]. The KB [2] method has been extended by Popov [17] to damped nonlinear differential systems represented by the following equation

$$\ddot{x} + 2k\,\dot{x} + \omega^2 x = \varepsilon\, f(\dot{x}, x), \tag{2.29}$$

where $2k\,\dot{x}$ is the linear damping force and $0 < k < \omega$. It is noteworthy that, because of the importance of the Popov's method in the physical systems, involving damping force, Mendelson [18] and Bojadziev [19] have retrieved Popov's [17] results. In case of damped nonlinear differential systems, the first equation of equation (2.18) has been replaced by

$$\dot{a} = -k\,a + \varepsilon\, A_1(a) + \varepsilon^2 A_2(a) + \ldots\ldots\ldots\ldots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}). \tag{2.18a}$$

Murty and Deekshatulu [20] have developed a simple analytical method to obtain the time response of second order nonlinear over damped systems with small nonlinearity represented by the equation (2.29), based on the KB [2] method of variation of parameters. In accordance to the KBM [2-3] method, Murty *et al.* [21] have found a hyperbolic type asymptotic solution of an over damped system represented by the nonlinear differential equation (2.29), i.e., in the case $k > \omega$. They have used hyperbolic functions, $\cosh\varphi$ and $\sinh\varphi$ instead of their circular counterpart, which are used by KBM [2-3], Popov [17] and Mendelson [18]. Murty [22] has presented a unified KBM method for solving the nonlinear systems represented by the equation (2.29), which cover the undamped, damped and over-damped cases. Bojadziev and Edwards [23] have investigated solutions of oscillatory and non-oscillatory systems represented by equation (2.29) when k and ω are slowly varying functions of time t. Initial conditions may be used arbitrarily for the case of oscillatory or damped oscillatory process. But, in case of non-oscillatory systems $\cosh\varphi$ or $\sinh\varphi$ should be used depending on the given set of initial conditions (Murty *et al.* [21], Murty [22], Bojadziev and Edwards [23]). Arya and Bojadziev [24-25] have examined damped oscillatory systems and time dependent oscillating systems with slowly varying parameters and delay. Sattar [26] has developed an asymptotic method to solve a second order critically damped nonlinear system represented by equation (2.29). He has found the asymptotic solution of the equation (2.29) in the following form

$$x = a(1+\psi) + \varepsilon\, u_1(a,\psi) + \ldots\ldots\ldots\ldots + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}), \tag{2.30}$$

where a is defined by the equation (2.18a) and ψ is defined by

$$\dot{\psi} = 1 + \varepsilon C_1(a) + \varepsilon^2 C_2(a) + \dots\dots\dots + \varepsilon^n C_n(a) + O(\varepsilon^{n+1}) \tag{2.18b}$$

Also Sattar [27] has extended the KBM asymptotic method for three dimensional over damped nonlinear systems

Osiniskii [28] has extended the KBM method to the following third order nonlinear differential equation

$$\dddot{x} + c_1 \ddot{x} + c_2 \dot{x} + c_3 x = \varepsilon f(\ddot{x}, \dot{x}, x), \tag{2.31}$$

where ε is a small positive parameter and f is a given nonlinear function. He has assumed the asymptotic solution of equation (2.31) in the form

$$x = a + b\cos\psi + \varepsilon u_1(a, b, \psi) + \dots\dots\dots + \varepsilon^n u_n(a, b, \psi) + O(\varepsilon^{n+1}), \tag{2.32}$$

where each u_k $(k = 1, 2, \dots\dots, n)$ is a periodic function of ψ with period 2π and $a,\ b$ and ψ are functions of time t, and they are given by

$$\begin{aligned} \dot{a} &= -\lambda a + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \dots\dots\dots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}), \\ \dot{b} &= -\mu a + \varepsilon B_1(b) + \varepsilon^2 B_2(b) + \dots\dots\dots + \varepsilon^n B_n(b) + O(\varepsilon^{n+1}), \\ \dot{\psi} &= \omega + \varepsilon C_1(b) + \varepsilon^2 C_2(b) + \dots\dots\dots + \varepsilon^n C_n(b) + O(\varepsilon^{n+1}), \end{aligned} \tag{2.33}$$

where $-\lambda, -\mu \pm \omega$ are the eigen values of the equation (2.31) when $\varepsilon = 0$.

By using the KBM [2-3] method, Osiniskii [29] has investigated to solve a third order nonlinear partial differential equation with initial friction and relaxation. Lin and Khan [30] have also used the KBM [2-3] method to some biological problems. Proskurjakov [31] has investigated periodic solutions of nonlinear systems by using the Poincare and KBM methods, and has compared the two solutions. Bojadziev [32] has investigated the solution of nonlinear damped oscillatory systems with small time lag. Bojadziev and Lardner [33] have investigated periodic solution of nonlinear systems by using the methods of Poincare and KBM. Bojadziev and Lardner [34] have investigated monofrequent oscillations in mechanical systems including the case of internal resonance, governed by the hyperbolic differential equations with small nonlinearities. Bojadziev and Lardner [35-36] have also investigated solution for certain type of hyperbolic differential equations with small nonlinearities in the case internal resonance and large time delay. Bojadziev [37] has also found solutions of damped forced nonlinear vibrations with small time delay. Bojadziev [38] and Bojadziev and Chan [39] have applied the KBM method for solving the problems in population dynamics. Bojadziev [40] has used the KBM method to investigate the solutions of nonlinear systems that arise from biological and biochemical fields. Bojadziev [41] and Bojadziev and Hung [42] have developed a technique to investigate the solutions of damped nonlinear oscillations modeled by three-dimensional differential system. Rauch [43] has studied oscillations of a third order nonlinear autonomous system. Mulholland [44] has studied nonlinear oscillations governed by a third order differential equation. Lardner and Bojadziev [45] have investigated the solutions of nonlinear damped oscillations governed by a third order partial differential equation. They have introduced the concept of "couple amplitude" where the unknown functions A_k, B_k and C_k depend on both the amplitudes a and b. Raymond and Cabak [46] have examined the effects of internal resonance on impulsive forced nonlinear systems with two-degree-of-freedom. Alam and Sattar [47] have developed a method to solve third order critically damped autonomous nonlinear systems. Alam and Sattar [48] have presented a unified KBM method for solving third order nonlinear systems. Later, Alam [49] has extended the KBM method to over damped nonlinear differential systems. Also, Alam *et al.* [50] have extended the KBM method to certain non-oscillatory nonlinear systems with slowly varying coefficients. Alam and Sattar [51] have studied time dependent third order oscillating systems with damping based on the extended form of the KBM method. Alam *et al.* [52] have investigated the solution of time dependent nonlinear system based on the KBM method. Alam [53] has presented perturbation theory based on the KBM method to find the approximate solutions of second order nonlinear systems with large damping. Later, Alam [54-55] has extended the KBM method to *nth*, $(n \geq 2,3)$ order nonlinear differential systems. Alam [56] has also presented a unified KBM method, which is not the formal form of the original KBM method for solving *nth*, $(n \geq 2,3)$ order nonlinear systems. The solution contains some unusual variables, yet this solution is very important. Alam [57] has extended the KBM method presented in [47] to find the approximate solutions of critically damped nonlinear systems in presence of different damping forces by considering different sets of variational equations. Alam [58] has also extended the KBM method to a third order over damped system when two of the eigen values are almost equal (i.e., the system is near to the critically damped) and the rest is small. Alam [59] has presented an asymptotic method for certain third order non-oscillatory nonlinear system, which gives desired results when the damping force is near to the critically damping force. Alam [60] has developed a simple method to obtain the time response of second order over damped nonlinear systems under some special conditions. Alam [61] has investigated a unified KBM method for solving *nth* order nonlinear differential equation with varying coefficients. Alam and Hossain [62] have extended the method presented in [60] to obtain the time response of *nth* order $(n \geq 2)$, over damped systems. Alam and Sattar [63] have developed an asymptotic method for third order nonlinear systems with slowly varying coefficients. Alam [64] has also developed a modified and compact form of the KBM unified method for solving an *nth*, $(n \geq 2)$ order nonlinear differential equation. The formula presented in [64] is compact, systematic and practical, and easier than that of [56]. Alam *et al.* [65] have developed the KBM method for solving over-damped nonlinear systems with slowly varying coefficients on a special condition. Alam [66] has developed damped oscillations modeled by an *nth* order time dependent quasi-linear differential system. Alam *et al.* [67] have presented a general form of the KBM method for solving nonlinear partial differential equations. Alam *et al.* [68] have presented a general Struble's technique for solving an *nth* order weakly nonlinear differential system with damping. Nayfeh [69-70] and Murdock [71] have developed perturbation methods and theory for obtaining the solutions of weakly nonlinear differential systems. Sachs *et al.* [72] have developed a simple ODE model of tumor growth and anti-angiogenic or radiation treatment.

Lim and Wu [73] have also presented a new analytical approach to the Duffing harmonic oscillator. He [74] has obtained the approximate solution of nonlinear differential equation with convolution product nonlinearities. He [75] has developed some new approaches to Duffing equation with strongly and high order nonlinearity. Also, He [76] has presented a new interpretation of homotopy perturbation method. Belendez *et al.* [77] have presented the application of He's homotopy perturbation method to Duffing harmonic oscillator. Uddin *et al.* [78], Uddin and Sattar [79-80] and Uddin [81] have presented an approximate technique for solving second order strongly nonlinear differential systems with damping by combing the He's [74-76] homotopy perturbation and the extended form of the KBM [2-3] methods. Recently, Uddin *et al.* [82] have also developed an analytical approximate technique for solving a certain type of fourth order strongly nonlinear oscillatory differential system with small damping and cubic nonlinearity by combining He's homotopy perturbation [74-76] and the extended form of the KBM [2-3] methods. Younesian *et al.* [83] have presented frequency analysis of strongly nonlinear generalized Duffing oscillators using He's frequency- amplitude formulation and He's energy balance method.

CHAPTER III

An Approximate Analytical Technique for Solving Second Order Strongly Generalized Nonlinear Differential Systems with Small Damping

3.1 Introduction

The study of nonlinear problems arise in the fields of applied mathematics, physics, engineering, medical science, economics and other disciplines is of crucial importance, since most of the phenomena in the real world are essentially nonlinear and described by nonlinear differential systems with damping. It is too much difficult to handle nonlinear problems and in general, it is often very difficult to get an approximate analytical solution for strongly generalized nonlinear differential systems with damping than a numerical one. The most well-known common methods for constructing the approximate analytical solutions to the nonlinear oscillators are the perturbation techniques. Among of these techniques are the KBM [2-3] method, the Lindstedt-Poincare (LP) method [6, 9] and the method of multiple time scales [69, 70] for this category. Perturbation methods are based on an assumption that small parameters must exist in the equations, which is the major restriction to find wide application of the classical perturbation techniques. It determines not only the accuracy of the perturbation approximations, but also the validity of the perturbation methods itself. However, in science and engineering, there exist many nonlinear oscillatory problems which do not contain any small parameter, especially those appear with strong nonlinearities. Therefore, many new techniques have been proposed to eliminate the "small parameter" assumption. Among of them, the homotopy perturbation method (HPM) is a popular one. Lim and Wu [73] have presented a new analytical approach to the Duffing- harmonic oscillator. He [74] has obtained the approximate solution of nonlinear differential equation with convolution product nonlinearities. In another paper, He [75] has developed some new approaches to Duffing equation with strongly and high order nonlinearity. Recently, He [76] has presented a new interpretation of homotopy perturbation method. Belendez *et al.* [77] have presented the application of He's homotopy perturbation method to Duffing harmonic oscillator. Uddin *et al.* [78] have presented an approximate technique for solving strongly nonlinear differential systems with damping effects and cubic nonlinearity. Uddin and Sattar [79] have developed an approximate technique for solving strongly nonlinear biological systems with small damping effects. Uddin and Sattar [80] have also presented an approximate technique to Duffing equation with small damping and slowly varying coefficients. Uddin [81] has applied He's homotopy perturbation method to Duffing equation with small damping and high order strong nonlinearities. Recently Uddin *et al.* [82] have developed an approximate analytical technique for solving a certain type of fourth order strongly nonlinear oscillatory differential system with small damping and cubic nonlinearity. Younesian *et al.* [83] have presented the frequency analysis of strongly generalized nonlinear Duffing oscillators using He's frequency- amplitude formulation and He's energy balance method. From the early discussion, it has been seen that the most of the authors [74-77, 83] have studied nonlinear differential systems without considering any damping effects. But most of the physical and engineering problems occur in nature as nonlinear differential systems with small damping and the damping term plays important role to the systems. It is mentioned that from our early discussion, the second order strongly generalized nonlinear differential system with small damping has remained almost untouched. The main goal of this chapter is to fill this gap. The advantage of the presented coupling technique is that the first order analytical approximate solutions show a good agreement with the corresponding numerical solutions and the strongly generalized nonlinear differential equation is handled easily while the classical perturbation methods are failed to solve such generalized nonlinear differential systems.

3.2 The Method

We are interested to consider the strongly generalized nonlinear differential equation with small damping modeling in the following form

$$\ddot{x} + 2k\dot{x} + \upsilon^2 x = -\varepsilon(\alpha_3 f_3(x, \dot{x}) + \alpha_5 f_5(x, \dot{x}) + \alpha_7 f_7(x, \dot{x}) + \cdots + \alpha_n f_n(x, \dot{x})), \quad (3.1)$$

under the initial conditions

$$x(0) = b_0, \qquad \dot{x}(0) = 0, \quad (3.2)$$

where over dots denote differentiations with respect to time t, υ is a constant, ε is a parameter not necessarily small, $2k$ is the linear damping coefficient, α_i are constants, b_0 is an initial amplitude, and $f_i(x, \dot{x})$, $i = 3,5,7\cdots n$ are given nonlinear functions and they satisfy the following condition

$$f_i(-x,-\dot{x}) = -f_i(x, \dot{x}). \tag{3.3}$$

To solve the equation (3.1), we are interested to assume the following transformation

$$x = y(t)e^{-kt}. \tag{3.4}$$

Now differentiating equation (3.4) twice with respect to time t and substituting $\ddot{x}$, $\dot{x}$ together with x into equation (3.1), we obtain

$$\ddot{y} + (\upsilon^2 - k^2)y = -\varepsilon e^{kt} \sum_{i=1}^{n} \alpha_i \, f_i\left(ye^{-kt}, (\dot{y} - ky)e^{-kt}\right). \tag{3.5}$$

In accordance to the homotopy perturbation method, equation (3.5) can be re-written as

$$\ddot{y} + \omega^2 y = \lambda y - \varepsilon e^{kt} \sum_{i=1}^{n} \alpha_i \, f_i\left(ye^{-kt}, (\dot{y} - ky)e^{-kt}\right), \tag{3.6}$$

where

$$\omega^2 = \upsilon^2 - k^2 + \lambda. \tag{3.7}$$

Herein ω is a constant for undamped nonlinear oscillators and known as the angular frequency of the nonlinear systems and λ is an unknown function which can be determined by eliminating the secular terms. But for the damped nonlinear differential systems ω is a time dependent function and it varies slowly with time t. To handle this situation, we are going to use the extended form of the KBM [2-3] method. In accordance to this method, we choose the solution of equation (3.6) in the following form

$$y = b\cos\varphi, \tag{3.8}$$

where b and φ vary slowly with respect to time t. In literature b and φ are known as the amplitude and phase variables respectively and they play an important role to nonlinear physical systems. The amplitude b and phase variable φ satisfy the following first order ordinary differential equations

$$\begin{aligned} \dot{b} &= \varepsilon B_1(b,\tau) + \varepsilon^2 B_2(b,\tau) + \cdots, \\ \dot{\varphi} &= \omega(\tau) + \varepsilon C_1(b,\tau) + \varepsilon^2 C_2(b,\tau) + \cdots, \end{aligned} \tag{3.9}$$

where ε is a small positive parameter and $\tau = \varepsilon t$ is the slowly varying time. Now differentiating equation (3.8) twice with respect to time t, utilizing the relations equation (3.9) and substituting $\ddot{y}, y$ into equation (3.6) and then equating the coefficients of $\sin\varphi$ and $\cos\varphi$, we obtain

$$B_1 = -\omega' b/(2\omega),\ C_1 = 0, \tag{3.10}$$

where prime denotes differentiation with respect to slowly varying time τ. Now putting equation (3.8) into equation (3.4) and equation (3.10) into equation (3.9), we obtain the following equations

$$x = b e^{-kt} \cos\varphi, \tag{3.11}$$

$$\dot{b} = -\varepsilon\omega' b/(2\omega), \quad \dot{\varphi} = \omega(\tau). \tag{3.12}$$

Thus, the first order analytical approximate solution of equation (3.1) is obtained by the presented coupling technique by equation (3.11) with the help of equations (3.7) and (3.12). Usually the integration of equation (3.12) is computed by well-known techniques of calculus [69-70], but sometimes they are calculated by a numerical procedure [17-18, 24, 28, 34, 44, 47-68, 77-82].

3.3 Examples

3.3.1 To apply the above procedure for the practical problems, let us consider the strongly generalized nonlinear differential equation with a linear damping effects for $n = 3, 5$ [83] in the following form

$$\ddot{x} + 2k\dot{x} + \upsilon^2 x = -\varepsilon(\alpha_3 x^3 + \alpha_5 x^5), \tag{3.13}$$

where $f_3(x, \dot{x}) = x^3$, $f_5(x, \dot{x}) = x^5$. To solve the equation (3.13), we are interested to assume the following transformation

$$x = y(t) e^{-kt}. \tag{3.14}$$

Now using the transformation equation (3.14) into equation (3.13) and then simplifying them, we obtain

$$\ddot{y} + (\upsilon^2 - k^2) y = -\varepsilon(\alpha_3 y^3 e^{-2kt} + \alpha_5 y^5 e^{-4kt}). \tag{3.15}$$

In accordance to the homotopy perturbation [74-82] technique, equation (3.15) can be written as

$$\ddot{y} + \omega^2 y = \lambda y - \varepsilon(\alpha_3 y^3 e^{-2kt} + \alpha_5 y^5 e^{-4kt}), \tag{3.16}$$

where

$$\omega^2 = \upsilon^2 - k^2 + \lambda. \tag{3.17}$$

Here λ is an unknown function which can be determined by eliminating the secular terms. In accordance to the extended form of the KBM [2-3] method, the solution of equation (3.16) is obtained in the form

$$y = b\cos\varphi, \tag{3.18}$$

where

$$\begin{aligned} \dot{b} &= -\varepsilon\omega' b/(2\omega), \\ \dot{\varphi} &= \omega(\tau). \end{aligned} \tag{3.19}$$

According to the trigonometric identity, we know

$$\cos^n\varphi = \frac{1}{2^{n-1}}\left[\begin{aligned} &\cos n\varphi + n\cos(n-2)\varphi + \frac{n(n-1)}{2!}\cos(n-4)\varphi \\ &+\frac{n(n-1)(n-2)}{3!}\cos(n-6)\varphi + \cdots \end{aligned}\right], \tag{3.20}$$

for all odd n. Now using the value of y from equation (3.18) into the right hand side of equation (3.16) and using the trigonometric identity equation (3.20) and rearranging, we obtain

$$\begin{aligned} \ddot{y} + \omega^2 y &= \left(\lambda\, b - \frac{3\varepsilon\alpha_3 b^3 e^{-2kt}}{4} - \frac{5\varepsilon\alpha_5 b^5 e^{-4kt}}{8}\right)\cos\varphi \\ &-\varepsilon\left(\frac{\alpha_3 b^3 e^{-2kt}}{4} + \frac{5\alpha_5 b^5 e^{-4kt}}{16}\right)\cos 3\varphi + \cdots. \end{aligned} \tag{3.21}$$

The requirement of no secular terms in particular solution of equation (3.16) implies that the coefficient of the $\cos\varphi$ term is zero. Setting this term to zero, we obtain

$$\lambda b - \frac{3\varepsilon\alpha_3 b^3 e^{-2kt}}{4} - \frac{5\varepsilon\alpha_5 b^5 e^{-4kt}}{8} = 0, \tag{3.22}$$

which leads to

$$\lambda = \frac{3\varepsilon\alpha_3 b^2 e^{-2kt}}{4} + \frac{5\varepsilon\alpha_5 b^4 e^{-4kt}}{8}. \tag{3.23}$$

Putting the value of λ from equation (3.23) into equation (3.17), then it leads to

$$\omega^2 = \upsilon^2 - k^2 + \frac{3\varepsilon\alpha_3 b^2 e^{-2kt}}{4} + \frac{5\varepsilon\alpha_5 b^4 e^{-4kt}}{8}. \tag{3.24}$$

From equation (3.24) it is clear that, the frequency of the damped nonlinear differential systems depends on both amplitude b and time t. When $t \to 0$ then equation (3.24) yields

$$\omega_0 = \omega(0) = \sqrt{\upsilon^2 - k^2 + \frac{3\varepsilon\alpha_3\, b_0^2}{4} + \frac{5\varepsilon\alpha_5\, b_0^4}{8}}. \tag{3.25}$$

By integrating the equation (3.19), we get

$$b = b_0\sqrt{\frac{\omega_0}{\omega}}, \qquad \varphi = \varphi_0 + \int_0^t \omega(\tau)dt, \tag{3.26}$$

where $\tau = \varepsilon t$ is the slowly varying time and b_0 and φ_0 are constants of integration and is known as the initial amplitude and phase of the systems. Now putting equation (3.26) into equation (3.24), we obtain a biquadratic algebraic equation in ω in the following form

$$\omega^4 + p\omega^2 + q\omega + r = 0, \tag{3.27}$$

where

$$p = k^2 - \upsilon^2,\ q = -\frac{3\varepsilon\,\alpha_3\,\omega_0\, b_0^2 e^{-2kt}}{4},\ r = -\frac{5\varepsilon\,\alpha_5\,\omega_0^{\,2} b_0^4 e^{-4kt}}{8}. \tag{3.28}$$

The solution of equation (3.27) is computed by using the well-known **Newton-Raphson** method. Thus, the first order analytical approximate solution of equation (3.13) is given by

$$x = b\, e^{-kt}\cos\varphi, \tag{3.29}$$

$$b = b_0\sqrt{\frac{\omega_0}{\omega}}, \qquad \varphi = \varphi_0 + \int_0^t \omega(\tau)dt, \tag{3.30}$$

where $\tau = \varepsilon t$, ω_0 is obtained by equation (3.25), ω is calculated from equation (3.27), b and φ are carried out by equation (3.30).

3.3.2 As a second example, we are going to consider the strongly generalized nonlinear differential system with a linear damping effects [74, 83] modeling in the following form

$$\ddot{x} + 2k\dot{x} + \upsilon^2 x = -\varepsilon(\alpha_3 x^3 + \alpha_5 x^5 + \alpha_7 x^7). \tag{3.31}$$

For solving the equation (3.31), we are going to assume the following transformation

$$x = y(t)\, e^{-kt}. \tag{3.32}$$

Now using the transformation equation (3.32) into equation (3.31) and then simplifying them, we obtain

$$\ddot{y} + (\upsilon^2 - k^2)y = -\varepsilon(\alpha_3 y^3 e^{-2kt} + \alpha_5 y^5 e^{-4kt} + \alpha_7 y^7 e^{-6kt}). \tag{3.33}$$

In accordance to the homotopy perturbation [74-82] method, equation (3.32) yields

$$\ddot{y}+\omega^2 y=\lambda\, y-\varepsilon\,(\alpha_3\, y^3 e^{-2kt}+\alpha_5\, y^5 e^{-4kt}+\alpha_7\, y^7 e^{-6kt}), \tag{3.34}$$

where ω is calculated by the following equation

$$\omega^2=\upsilon^2-k^2+\lambda. \tag{3.35}$$

In accordance to the extended form of the KBM [2-3] method, the solution of equation (3.34) is considered as the following form

$$y=b\cos\varphi, \tag{3.36}$$

where

$$\begin{aligned}\dot{b}&=-\varepsilon\omega' b/(2\omega),\\ \dot{\varphi}&=\omega(\tau).\end{aligned} \tag{3.37}$$

Now using the value of y from equation (3.36) into the right hand side of equation (3.34) and using the trigonometric identity equation (3.20) and rearranging, we obtain

$$\begin{aligned}\ddot{y}+\omega^2 y&=\left(\lambda\, a-\frac{3\varepsilon\alpha_3\, a^3 e^{-2kt}}{4}-\frac{5\varepsilon\alpha_5\, a^5 e^{-4kt}}{8}-\frac{35\varepsilon\alpha_7\, a^7 e^{-6kt}}{64}\right)\cos\varphi\\ &-\varepsilon\left(\frac{\alpha_3 a^3 e^{-2kt}}{4}+\frac{5\alpha_5\, a^5 e^{-4kt}}{16}+\frac{21\alpha_7\, a^7 e^{-6kt}}{64}\right)\cos 3\varphi+\cdots.\end{aligned} \tag{3.38}$$

The requirement of no secular terms in particular solution of equation (3.38) implies that the coefficient of the $\cos\varphi$ term is zero. Setting this term to zero, we obtain

$$\lambda\, b-\frac{3\varepsilon\alpha_3\, b^3 e^{-2kt}}{4}-\frac{5\varepsilon\alpha_5\, b^5 e^{-4kt}}{8}-\frac{35\varepsilon\alpha_7 b^7 e^{-6kt}}{64}=0, \tag{3.39}$$

which leads to

$$\lambda=\frac{3\varepsilon\,\alpha_3\, b^2 e^{-2kt}}{4}+\frac{5\varepsilon\alpha_5\, b^4 e^{-4kt}}{8}+\frac{35\varepsilon\,\alpha_7\, b^6 e^{-6kt}}{64}. \tag{3.40}$$

Putting the value of λ from equation (3.40) into equation (3.35), yields

$$\omega^2=\upsilon^2-k^2+\frac{3\varepsilon\,\alpha_3\, b^2 e^{-2kt}}{4}+\frac{5\varepsilon\,\alpha_5\, b^4 e^{-4kt}}{8}+\frac{35\varepsilon\,\alpha_7\, b^6 e^{-6kt}}{64}. \tag{3.41}$$

From equation (3.41), we obtain (as $t\to 0$)

$$\omega_0=\omega(0)=\sqrt{\upsilon^2-k^2+\frac{3\varepsilon\alpha_3\, b_0^2}{4}+\frac{5\varepsilon\,\alpha_5\, b_0^4}{8}+\frac{35\varepsilon\,\alpha_7\, b_0^6}{64}}. \tag{3.42}$$

By integrating the equation (3.37)

$$b=b_0\sqrt{\frac{\omega_0}{\omega}},\qquad \varphi=\varphi_0+\int_0^t\omega(\tau)dt \tag{3.43}$$

where $\tau = \varepsilon t$. Using the equation (3.43) into equation (3.41), we obtain a fifth degree polynomial in ω in the following form

$$\omega^5 + p\omega^3 + q\omega^2 + r\omega + s = 0, \tag{3.44}$$

where

$$\begin{gathered} p = k^2 - \upsilon^2, \quad q = -\frac{3\varepsilon\alpha_3\omega_0 b_0^2 e^{-2kt}}{4}, \\ r = -\frac{5\varepsilon\alpha_5\omega_0^{\ 2} b_0^4 e^{-4kt}}{8}, s = -\frac{35\varepsilon\alpha_7\omega_0^{\ 3} b_0^{\ 6} e^{-6kt}}{64}. \end{gathered} \tag{3.45}$$

The solution of equation (3.44) is obtained by using the well-known **Newton-Raphson** method.

Thus, the first order analytical approximate solution of equation (3.31) is obtained by

$$x = b e^{-kt}\cos\varphi, \tag{3.46}$$

$$b = b_0\sqrt{\frac{\omega_0}{\omega}}, \qquad \varphi = \varphi_0 + \int_0^t \omega(\tau)dt, \tag{3.47}$$

where $\tau = \varepsilon t$, ω_0 is obtained by equation (3.42), ω is calculated from equation (3.44), b and φ are given by equation (3.47).

3.4 Results and Discussion

In this chapter, we have extended He's homotopy perturbation method to solve second order strongly generalized nonlinear differential system [83] with small damping. It is too much difficult to solve the strongly generalized nonlinear Duffing type problems, especially with small damping and high order nonlinearities by the classical perturbation methods [2-4, 17-68]. But the suggested method has been successfully applied to solve second order strongly generalized nonlinear differential systems with small damping and high order nonlinearities. The first order approximate solutions of equation (3.13) and equation (3.31) are computed with small damping and high order nonlinearities by equations (3.29) and (3.46) respectively and the corresponding numerical solutions are obtained by using fourth order **Runge-Kutta** method. The variational equations of the amplitude and phase variables appeared in a set of first order nonlinear differential equations. The integration of these variational equations is carried out by the well-known techniques of calculus [69-70]. In lack of analytical solutions, they are solved by numerical procedure

[2-4, 17-83]. The amplitude and phase variable change slowly with time t. The behavior of amplitude and phase variable characterizes the oscillating processes and amplitude tends to zero in presence of small damping for large time t ($i.e.$, $t \to \infty$). On the other hand, our proposed technique can take full advantage of the classical perturbation method. The solutions obtained by the presented method show a good agreement with those obtained by the numerical procedure [2-4, 17-83] with several damping effects. It is also noticed that the presented method is also capable to handle the second order weakly generalized nonlinear differential system with damping effects and high order nonlinearities. Comparison is made between the solutions obtained by the presented technique and those obtained by the numerical procedure in **Figs. 3.1-3.2** for both strongly ($\varepsilon = 1.0$) and weakly ($\varepsilon = 0.1$) generalized nonlinear differential systems with small damping effects. Also the solution of the Duffing equation for cubic nonlinearity is obtained from equation (3.13) and equation (3.31) by setting $\alpha_5 = 0, \alpha_7 = 0$ with small damping **(Fig. 3.3)**. Also the average percentage errors have been calculated between numerical and approximate solutions in table 3.1. From the table 3.1, it is clear to us that the average percentage errors except few cases are negligible.

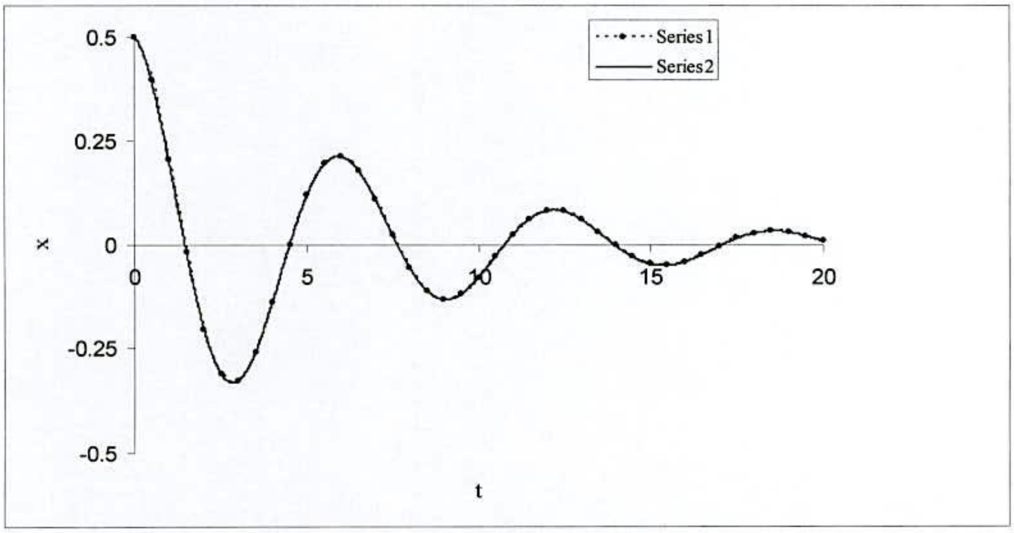

Fig. 3.1 (a) First approximate solution of equation (3.13) is denoted by $-\bullet-$ (dashed lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.07194]$ when $k = 0.15,\ \varepsilon = 1.0,$ $\alpha_3 = 1.0,\ \alpha_5 = 1.0$ and $f_3 = x^3,\ f_5 = x^5$. Corresponding numerical solution is denoted by - (solid line).

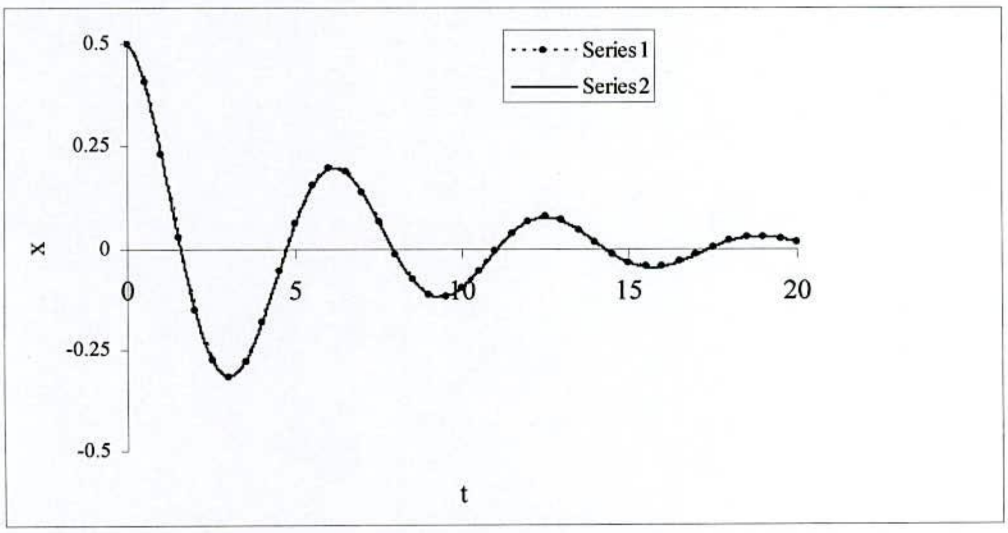

Fig. 3.1 (b) First approximate solution of equation (3.13) is denoted by $-\bullet-$ (dashed lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.07466]$ when $k = 0.15,\ \varepsilon = 0.1,$ $\alpha_3 = 1.0, \alpha_5 = 1.0$ and $f_3 = x^3,\ f_5 = x^5$. Corresponding numerical solution is denoted by - (solid line).

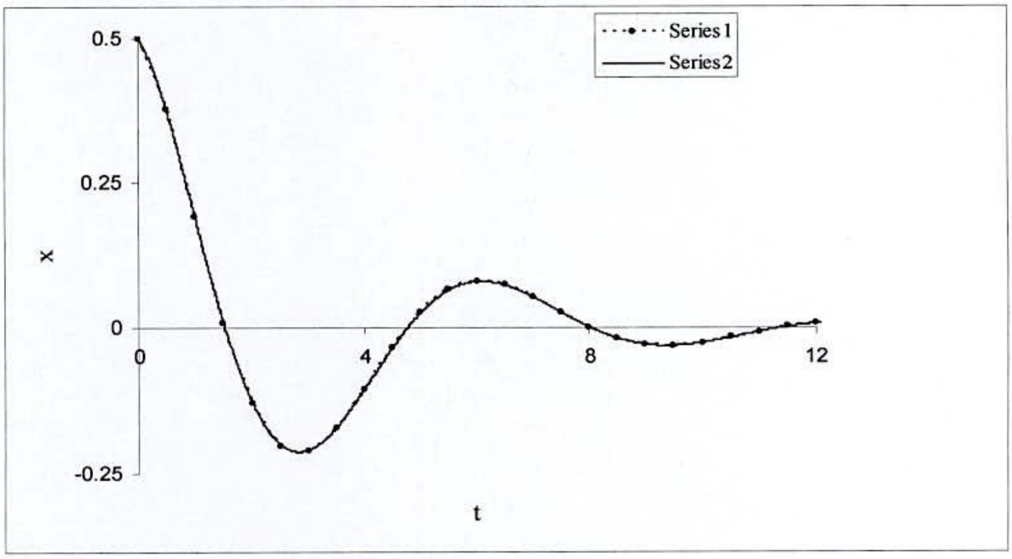

Fig. 3.2 (a) First approximate solution of equation (3.31) is denoted by $-\bullet-$ (dashed lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.13401]$ when $k = 0.3,\ \varepsilon = 1.0$, $\alpha_3 = 1.0, \alpha_5 = 1.0, \alpha_7 = 1.0$ and $f_3 = x^3, f_5 = x^5, f_7 = x^7$. Corresponding numerical solution is denoted by - (solid line).

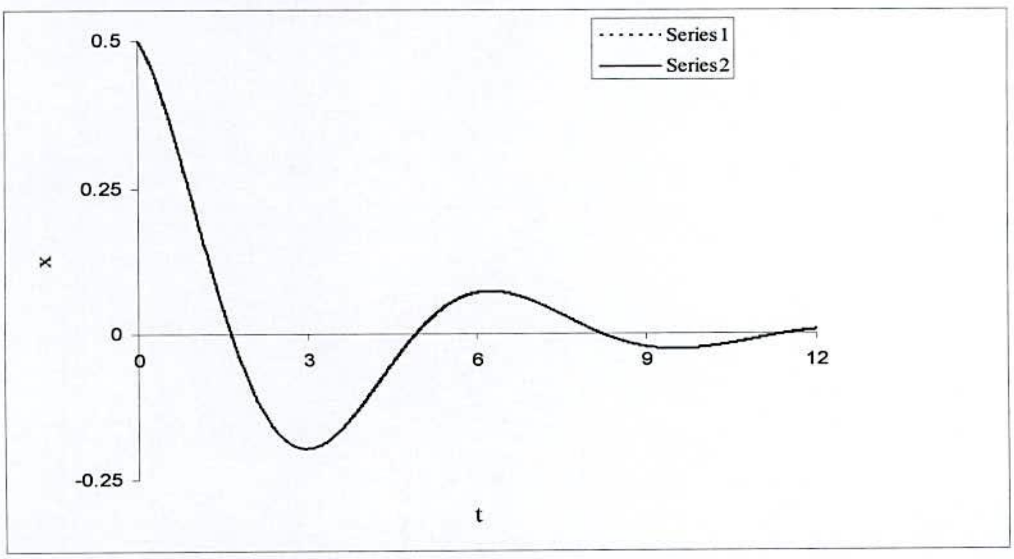

Fig. 3.2 (b) First approximate solution of equation (3.31) is denoted by $-\bullet-$ (dashed lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.14781]$ when $k = 0.3,\ \varepsilon = 0.1$, $\alpha_3 = 1.0, \alpha_5 = 1.0, \alpha_7 = 1.0$ and $f_3 = x^3, f_5 = x^5, f_7 = x^7$. Corresponding numerical solution is denoted by - (solid line).

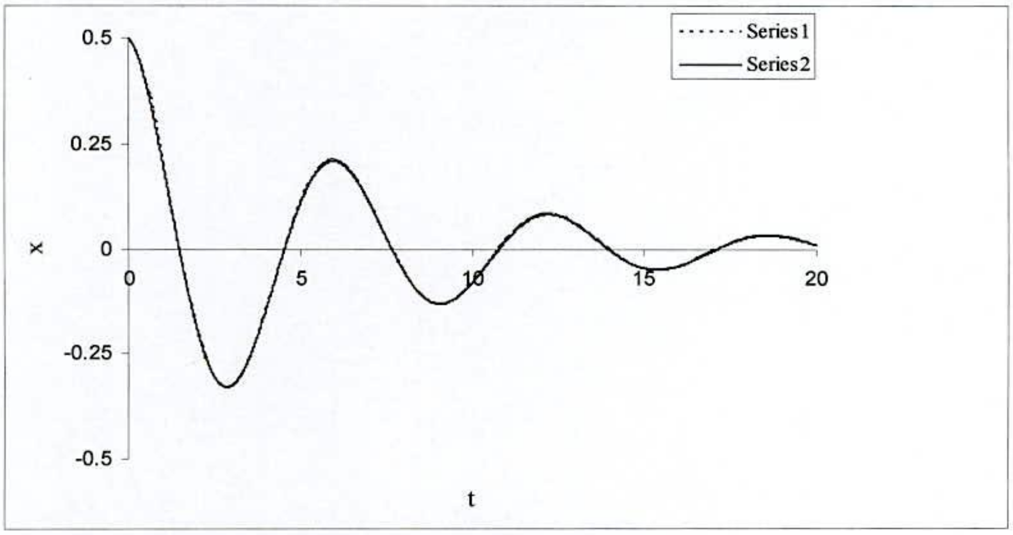

Fig. 3.3 (a) First approximate solution of equation (3.13) is denoted by --- (dashed lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.07281]$ when $k = 0.15,\ \varepsilon = 1.0,\ \alpha_3 = 1.0,\ \alpha_5 = 0.0$ and $f_3 = x^3,\ f_5 = x^5$. Corresponding numerical solution is denoted by - (solid line).

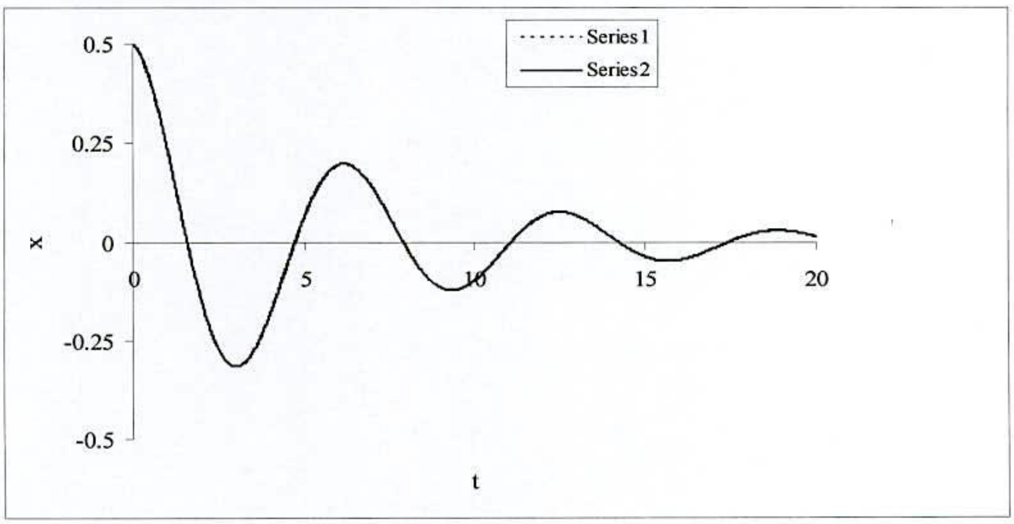

Fig. 3.3 (b) First approximate solution of equation (3.13) is denoted by --- (dashed lines) by the presented analytical technique with the initial conditions $b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.07476]$ when $k = 0.15,\ \varepsilon = 0.1,\ \alpha_3 = 1.0,\ \alpha_5 = 0.0$ and $f_3 = x^3,\ f_5 = x^5$. Corresponding numerical solution is denoted by - (solid line).

Table 3.1: Average percentage error between **numerical** and **approximate** solutions:

Figures No.	**Initial conditions for different values of the parameters.**	**Average Percentage Error**
Fig-3.1(a)	$b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.07194]$ when $k = 0.15,\ \varepsilon = 1.0,\ \alpha_3 = 1.0,\ \alpha_5 = 1.0$ and $f_3 = x^3$, $f_5 = x^5$.	3.083016%
Fig-3.1(b)	$b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.07466]$ when $k = 0.15,\ \varepsilon = 0.1,\ \alpha_3 = 1.0, \alpha_5 = 1.0$ and $f_3 = x^3$, $f_5 = x^5$.	-0.08613%
Fig-3.2(a)	$b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.13401]$ when $k = 0.3,\ \varepsilon = 1.0,\ \alpha_3 = 1.0, \alpha_5 = 1.0, \alpha_7 = 1.0$ and $f_3 = x^3$, $f_5 = x^5$, $f_7 = x^7$.	0.093699%
Fig-3.2(b)	$b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.14781]$ when $k = 0.3,\ \varepsilon = 0.1,\ \alpha_3 = 1.0,\ \alpha_5 = 1.0,\ \alpha_7 = 1.0$ and $f_3 = x^3$, $f_5 = x^5$, $f_7 = x^7$.	4.62758%
Fig-3.3(a)	$b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.07281]$ when $k = 0.15, \varepsilon = 1.0, \alpha_3 = 1.0, \alpha_5 = 0.0$ and $f_3 = x^3$, $f_5 = x^5$.	-1.4236%
Fig-3.3(b)	$b_0 = 0.5,\ \varphi_0 = 0$ or $[x(0) = 0.5,\ \dot{x}(0) = -0.07476]$ when $k = 0.15,\ \varepsilon = 0.1, \alpha_3 = 1.0,\ \alpha_5 = 0.0$ and $f_3 = x^3$, $f_5 = x^5$.	-0.19684%

CHAPTER IV

A Coupling Approximate Analytical Technique for Solving Certain Type of Fourth Order Strongly Generalized Nonlinear Damped Oscillatory Differential System

4.1 Introduction

The most common methods for constructing the approximate analytical solutions to the nonlinear oscillator equations are the perturbation techniques. Some well known perturbation methods are the KBM [2-3] method, the Lindstedt-Poincare (LP) method [6, 9], and the method of multiple time scales [69-70]. Almost all perturbation methods are based on an assumption that small parameters must exist in the equations, which is too strict to find wide application of the classical perturbation methods. It determines not only the accuracy of the perturbation approximations, but also the validity of the perturbation methods itself. However, in science and engineering, there exist many nonlinear problems which do not contain any small parameter; especially those appear in nature with strong nonlinearity. Therefore, many new techniques have been proposed to eliminate the "small parameter" assumption, such as the homotopy perturbation method (HPM), harmonic balance method (HBM), iteration method. Alam [56] has investigated a unified KBM method for solving $nth, (n \geq 2)$ order weakly nonlinear differential systems. Alam and Sattar [63] have presented an asymptotic method for third order nonlinear differential system with slowly varying coefficients. In another paper, Alam [64] has investigated a modified and compact form of KBM unified method for an *nth* order nonlinear differential equation. Alam *et al.* [68] have developed a general Struble's technique for solving an *nth* order weakly nonlinear differential system with damping. Sachs et al. [72] have presented simple ODE models of tumor growth and anti-angiogenic or radiation treatment. Lim and Wu [73] have also presented a new analytical approach to the Duffing- harmonic oscillator. He [74] has obtained the approximate solution of nonlinear differential equation with convolution product nonlinearities without damping. In another paper, He [75] has developed some new approaches to Duffing equation with strong and high order nonlinearity without damping. Recently He [76] has presented a new interpretation of homotopy perturbation method without damping. Belendez *et al.* [77] have presented the application of He's homotopy perturbation method to Duffing harmonic oscillator without damping. Combing He's homotopy perturbation and the extended form of the KBM methods, Uddin *et al.* [78], Uddin and Sattar [79-80] have presented approximate techniques for solving second order strongly nonlinear damped oscillatory differential systems for both cubic and quadratic nonlinearities. Recently Uddin *et al.* [82] have developed an approximate analytical technique for solving a certain type of fourth order strongly nonlinear oscillatory differential system with small damping and cubic nonlinearity. Younesian *et al.* [83] have developed frequency analysis of strongly nonlinear generalized Duffing oscillators using He's frequency- amplitude formulation and He's energy balance method. But many physical and engineering problems occur with fourth order strongly generalized nonlinear damped oscillatory differential systems and they do not contain small parameters, *i. e.,* those appear with small damping and strong nonlinearity. The more difficult and no less important cases, the fourth order strongly generalized nonlinear damped oscillatory differential systems has remained almost untouched. The aim of this chapter is to fill this gap. So in this chapter, we are interested to extend an approximate analytical technique based on the He's homotopy [74-76] perturbation and the extended form of the KBM [2-3] methods to solve certain type of fourth order strongly generalized nonlinear damped oscillatory differential systems. This method transforms a difficult problem under simplification into a simple problem which is easy to solve but the classical perturbation techniques are almost unable to handle the fourth order strongly and weakly generalized nonlinear damped oscillatory differential systems. The presented method has been successfully implemented to solve the fourth order strongly generalized nonlinear damped oscillatory differential systems with example. The advantage of this method is that the first order approximate analytical solutions show a good agreement with the corresponding numerical solutions. Moreover, the presented method is also able to give the desired results for the fourth order weakly generalized nonlinear damped oscillatory differential systems. It is also noted that the presented method is able to handle the fourth order strongly and weakly nonlinear oscillatory differential systems with cubic nonlinearity for several damping effects.

4.2 The Method

We are going to assume the fourth order strongly generalized nonlinear damped oscillatory differential system in the following form

$$x^{(4)} + (\omega_1^2 + \omega_2^2)\ddot{x} + \omega_1^2\omega_2^2 x = -(p\dddot{x} + q\dot{x}) - \varepsilon \sum_j^n \alpha_j f_j(x, \dot{x}), \tag{4.1}$$

under the initial conditions

$$x(0) = a_0, \qquad \dot{x}(0) = \ddot{x}(0) = \dddot{x}(0) = 0, \tag{4.2}$$

where over dots represent the derivatives with respect to time t, ω_1 and ω_2 are the angular frequencies for double mode of vibration of the systems, ε is a positive parameter which is not necessarily small, p, q are unknown constants and $f_j(x, \dot{x}),\ j = 3,5\cdots n$ are given nonlinear functions satisfying the conditions $f_j(-x, -\dot{x}) = -f_j(x, \dot{x})$ and α_j are given positive constants. If we consider $p = 4k$ and $q = 4k^4 + 2k(\omega_1^2 + \omega_2^2) - 12\,k^3$ then the eigen values of equation (4.1) reduces to $-k \pm i\omega_1$ and $-k \pm i\omega_2$, where $k << 1$ represents the significant positive damping effects. In accordance with the earlier works [78-82] the dependent variable is changed by the following transformation

$$x = y(t)e^{-kt}. \tag{4.3}$$

Differentiating equation (4.3) four times with respect to time t, and then substituting the derivatives $x^{(4)}, \dddot{x}, \ddot{x}, \dot{x}$ together with x and the values of p and q into equation (4.1) and after simplifying them, we obtain the following equation

$$\begin{aligned} &y^{(4)} + (\omega_1^2 + \omega_2^2 - 6\,k^2)\ddot{y} + \{5k^4 - k^2(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2\}y \\ &= -\varepsilon e^{kt} \sum_j^n \alpha_j f_j(ye^{-kt}, (\dot{y} - k\,y)e^{-kt}). \end{aligned} \tag{4.4}$$

In accordance to the homotopy perturbation [74-82] method, equation (4.4) can be written as

$$\begin{aligned} &y^{(4)} + (\omega_1^2 + \omega_2^2 - 6k^2)\ddot{y} + \left(5k^4 - k^2(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2 + \lambda\right)y \\ &= \lambda y - \varepsilon e^{kt} \sum_j^n \alpha_j f_j(ye^{-kt}, (\dot{y} - k\,y)e^{-kt}). \end{aligned} \tag{4.5}$$

Equation (4.5) can be re-written as

$$y^{(4)} + (\omega_1^2 + \omega_2^2 - 6k^2)\ddot{y} + \omega^2 y = \lambda\,y - \varepsilon e^{kt} \sum_j^n \alpha_j f_j(ye^{-kt}, (\dot{y} - k\,y)e^{-kt}), \tag{4.6}$$

where

$$\omega^2 = 5k^4 - k^2(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2 + \lambda. \tag{4.7}$$

Herein ω is a constant for undamped nonlinear oscillator and known as the angular frequency in literature and λ is an unknown function which can be determined by eliminating the secular terms. But ω is a time dependent function and it varies slowly with time t for nonlinear damped oscillatory differential system. To tackle this situation, the extended form of the KBM [2-3] method by Mitropolskii [4] is applied. In accordance to this method, we consider the solution (for a single mode of vibration) of equation (4.6) in the following form

$$y = a\cos\varphi, \tag{4.8}$$

where the amplitude a and the phase φ vary slowly with time t and they satisfy the following first order ordinary differential equations

$$\begin{aligned} \dot{a} &= \varepsilon B_1(a,\tau) + \varepsilon^2 B_2(a,\tau) + \cdots, \\ \dot{\varphi} &= \mu_1(\tau) + \varepsilon C_1(a,\tau) + \varepsilon^2 C_2(a,\tau) + \cdots, \end{aligned} \tag{4.9}$$

where ε is a positive parameter which is not necessarily small and $\tau = \varepsilon t$ is the slowly varying time, B_j and C_j are unknown functions and μ_1 is reduced angular frequency of the nonlinear differential systems. Now differentiating equation (4.8) four times with respect to time t and utilizing equation (4.9) and then collecting the terms up to $O(\varepsilon)$ and neglecting $O(\varepsilon^2)$ and higher terms, we get

$$\begin{aligned} &y^{(4)} + \mu_1^2\ddot{y} + \mu_2^2(\ddot{y} + \mu_1^2 y) \\ &= 5\varepsilon a \mu_1^2 \mu_1' \sin\varphi + 2\varepsilon \mu_1^3 (B_1 \sin\varphi + aC_1 \sin\varphi) \\ &\quad + \mu_2^2(-\varepsilon a \mu_1' \sin\varphi - 2\varepsilon \mu_1 (B_1 \sin\varphi + aC_1 \cos\varphi)). \end{aligned} \tag{4.10}$$

where μ_1' represents the derivative with respect to slowly varying time τ and μ_2 is reduced angular frequency of the nonlinear differential systems.

equation (4.10) can be re-written as

$$\begin{aligned} &y^{(4)} + (\mu_1^2 + \mu_2^2)\ddot{y} + \mu_1^2\mu_2^2 y \\ &= \varepsilon\mu_1'(5\mu_1^2 - \mu_2^2)a\sin\varphi + 2\varepsilon\mu_1(\mu_1^2 - \mu_2^2)(B_1\sin\varphi + aC_1\cos\varphi). \end{aligned} \tag{4.11}$$

Now if

$$y^{(4)} + (\mu_1^2 + \mu_2^2)\ddot{y} + \mu_1^2\mu_2^2 y = 0, \tag{4.12}$$

then equation (4.11) becomes

$$\varepsilon\mu_1'(5\mu_1^2 - \mu_2^2)a\sin\varphi + 2\varepsilon\mu_1(\mu_1^2 - \mu_2^2)(B_1\sin\varphi + aC_1\cos\varphi) = 0. \tag{4.13}$$

Equating the coefficients of $\sin\varphi$ and $\cos\varphi$ from both side of equation (4.13), we get the following functions

$$B_1 = -\frac{\mu_1'(5\mu_1^2 - \mu_2^2)a}{2\mu_1(\mu_1^2 - \mu_2^2)}, \quad C_1 = 0. \tag{4.14}$$

Again comparing equation (4.11) with equation (4.6), we obtain

$$\mu_1^2 + \mu_2^2 = \omega_1^2 + \omega_2^2 - 6k^2, \tag{4.15}$$

$$\mu_1^2\mu_2^2 = 5k^4 - k^2(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2 + \lambda. \tag{4.16}$$

Also we get the following relation by using equations (4.15)- (4.16)

$$\mu_1^2 - \mu_2^2 = \sqrt{(\omega_1^2 - \omega_2^2)^2 + 16k^4 - 8k^2(\omega_1^2 + \omega_2^2) - 4\lambda}. \tag{4.17}$$

By solving equation (4.15) and equation (4.17), we write

$$\mu_1 = \sqrt{\frac{1}{2}\left(\omega_1^2 + \omega_2^2 - 6k^2 + \sqrt{(\omega_1^2 - \omega_2^2)^2 + 16k^4 - 8k^2(\omega_1^2 + \omega_2^2) - 4\lambda}\right)}, \tag{4.18}$$

$$\mu_2 = \sqrt{\frac{1}{2}\left(\omega_1^2 + \omega_2^2 - 6k^2 - \sqrt{(\omega_1^2 - \omega_2^2)^2 + 16k^4 - 8k^2(\omega_1^2 + \omega_2^2) - 4\lambda}\right)}. \tag{4.19}$$

Putting the value of y from equation (4.8) into equation (4.3) and the values of B_1 and C_1 from equation (4.14) into equation (4.9), we obtain the following final solution

$$x(t,\varepsilon) = a e^{-kt}\cos\varphi, \tag{4.20}$$

$$\dot{a} = -\frac{\varepsilon\,\mu_1'(5\mu_1^2 - \mu_2^2)\,a}{2\mu_1(\mu_1^2 - \mu_2^2)}, \qquad \dot{\varphi} = \mu_1(\tau). \tag{4.21}$$

Thus, the first order approximate analytical solution of equation (4.1) is obtained by equation (4.20) by the presented coupling technique. Usually, the integrations of equation (4.21) are carried out according to the well-known techniques of calculus [69-70], but sometimes they are carried out by a numerical procedure [17-18, 24, 28, 34, 44, 47-68, 77-82] with the help of equations (4.18) - (4.19). Hence the determination of the first order analytical approximate solution of equation (4.1) is completed by the proposed method.

4.3 Example

To apply the above procedure, we are going to assume the fourth order strongly generalized nonlinear damped oscillatory differential system in the following form

$$x^{(4)} + ({\omega_1}^2 + {\omega_2}^2)\ddot{x} + {\omega_1}^2{\omega_2}^2 x = -(p\dddot{x} + q\dot{x}) - \varepsilon(\alpha_3 x^3 + \alpha_5 x^5 + \alpha_7 x^7), \tag{4.22}$$

where $f_3(x)=x^3$, $f_5(x)=x^5$, $f_7(x)=x^7$ are the given nonlinear functions. By using the substitution equation (4.3) [78-82] and then simplifying them and in accordance to the homotopy perturbation [74-82], equation (4.22) can be written as

$$y^{(4)}+(\mu_1^2+\mu_2^2)\ddot{y}+\mu_1^2\mu_2^2 y=\lambda y-\varepsilon(\alpha_3 y^3 e^{-2kt}+\alpha_5 y^5 e^{-4kt}+\alpha_7 y^7 e^{-6kt}). \tag{4.23}$$

In accordance to the extended form of the KBM [2-3] method, the solution of equation (4.23) is considered as follows

$$y=a\cos\varphi, \tag{4.24}$$

where a and φ are obtained by the following relations

$$\dot{a}=-\frac{\varepsilon\mu_1'(5\mu_1^2-\mu_2^2)a}{2\mu_1(\mu_1^2-\mu_2^2)}, \qquad \dot{\varphi}=\mu_1(\tau). \tag{4.25}$$

According to the trigonometric identity, we know

$$\cos^n\varphi=\frac{1}{2^{n-1}}\begin{bmatrix}\cos n\varphi+n\cos(n-2)\varphi+\dfrac{n(n-1)}{2!}\cos(n-4)\varphi \\ +\dfrac{n(n-1)(n-2)}{3!}\cos(n-6)\varphi+\cdots\end{bmatrix}, \tag{4.26}$$

for all odd n. Now using the value of y from equation (4.24) into the right hand side of equation (4.23) and using the trigonometric identity equation (4.26) and rearranging, we obtain

$$\begin{aligned}&y^{(4)}+(\mu_1^2+\mu_2^2)\ddot{y}+\mu_1^2\mu_2^2 y\\&=\left(\lambda a-\frac{3\varepsilon\alpha_3 a^3 e^{-2kt}}{4}-\frac{5\varepsilon\alpha_5 a^5 e^{-4kt}}{8}-\frac{35\varepsilon\alpha_7 a^7 e^{-6kt}}{64}\right)\cos\varphi\\&-\varepsilon\left(\frac{\alpha_3 a^3 e^{-2kt}}{4}+\frac{5\alpha_5 a^5 e^{-4kt}}{16}+\frac{21\alpha_7 a^7 e^{-6kt}}{64}\right)\cos 3\varphi+\cdots.\end{aligned} \tag{4.27}$$

Since in presence of secular terms, the solutions are non-uniform. So, to obtain the uniform solution of the system, the requirement of no secular terms in particular solution of equation (4.27) implies that the coefficients of the $\cos\varphi$ term are zero. Setting these terms to zero, we obtain

$$\lambda a-\frac{3\varepsilon\alpha_1 a^3 e^{-2kt}}{4}-\frac{5\varepsilon\alpha_2 a^5 e^{-4kt}}{8}-\frac{35\varepsilon\alpha_3 a^7 e^{-6kt}}{64}=0. \tag{4.28}$$

We are looking the nontrivial solution of the system. So, for the nontrivial solution (*i.e.*, $a\neq 0$), equation (4.28) leads to

$$\lambda = \frac{3\varepsilon\alpha_1 a^2 e^{-2kt}}{4} + \frac{5\varepsilon\alpha_2 a^4 e^{-4kt}}{8} + \frac{35\varepsilon\alpha_3 a^6 e^{-6kt}}{64}. \tag{4.29}$$

Now substituting the value of λ from equation (4.29) into equation (4.18) – (4.19) we get

$$\mu_1 = \sqrt{\frac{1}{2}\left(\omega_1^2 + \omega_2^2 - 6k^2 + \sqrt{\begin{matrix}(\omega_1^2 - \omega_2^2)^2 + 16k^4 - 8k^2(\omega_1^2 + \omega_2^2) - 3\varepsilon\alpha_3 a^2 e^{-2kt} \\ -5\varepsilon\alpha_5 a^4 e^{-4kt}/2 - 35\varepsilon\alpha_7 a^6 e^{-6kt}/16\end{matrix}}\right)}, \tag{4.30}$$

$$\mu_2 = \sqrt{\frac{1}{2}\left(\omega_1^2 + \omega_2^2 - 6k^2 - \sqrt{\begin{matrix}(\omega_1^2 - \omega_2^2)^2 + 16k^4 - 8k^2(\omega_1^2 + \omega_2^2) - 3\varepsilon\alpha_3 a^2 e^{-2kt} \\ -5\varepsilon\alpha_5 a^4 e^{-4kt}/2 - 35\varepsilon\alpha_7 a^6 e^{-6kt}/16\end{matrix}}\right)}. \tag{4.31}$$

Squaring equation (4.30) and expanding according to the binomial theorem and then simplifying, we can write

$$\mu_1^2 = \omega_1^2 - 3k^2 + \frac{4k^4}{\omega_1^2 - \omega_2^2} - \frac{2k^2(\omega_1^2 + \omega_2^2)}{\omega_1^2 - \omega_2^2} - \frac{3\varepsilon\alpha_3 a^2 e^{-2kt}}{4(\omega_1^2 - \omega_2^2)} - \frac{5\varepsilon\alpha_5 a^4 e^{-4kt}}{8(\omega_1^2 - \omega_2^2)} - \frac{35\varepsilon\alpha_7 a^6 e^{-6kt}}{64(\omega_1^2 - \omega_2^2)}. \tag{4.32}$$

Differentiating equation (4.32) with respect to t and treating a as constant, we get

$$2\mu_1 \mu_1' = \frac{3k\alpha_3 a_0^2 e^{-2kt}}{2(\omega_1^2 - \omega_2^2)} + \frac{5k\alpha_5 a_0^4 e^{-4kt}}{2(\omega_1^2 - \omega_2^2)} + \frac{105k\alpha_7 a_0^6 e^{-6kt}}{32(\omega_1^2 - \omega_2^2)}. \tag{4.33}$$

Dividing equation (4.33) by equation (4.32) we obtain

$$\frac{\mu_1'}{\mu_1} = \left(\frac{3k\alpha_3 a_0^2 e^{-2kt}}{4(\omega_1^2 - \omega_2^2)} + \frac{5k\alpha_5 a_0^4 e^{-4kt}}{4(\omega_1^2 - \omega_2^2)} + \frac{105k\alpha_7 a_0^6 e^{-6kt}}{64(\omega_1^2 - \omega_2^2)}\right) \times \left(\begin{matrix}\frac{(\omega_1^2 - \omega_2^2)(\omega_1^2 - 5k^2) + 4k^4 - 4k^2\omega_2^2}{(\omega_1^2 - \omega_2^2)} - \frac{3\varepsilon\alpha_3 a_0^2 e^{-2kt}}{4(\omega_1^2 - \omega_2^2)} \\ -\frac{5\varepsilon k\alpha_5 a_0^4 e^{-4kt}}{8(\omega_1^2 - \omega_2^2)} - \frac{35\varepsilon k\alpha_7 a_0^6 e^{-6kt}}{64(\omega_1^2 - \omega_2^2)}\end{matrix}\right)^{-1}, \tag{4.34}$$

which can be rewritten as the following form

$$\frac{\mu_1'}{2\mu_1} = \frac{1}{\{(\omega_1^2 - \omega_2^2)(\omega_1^2 - 5k^2) + 4k^4 - 4k^2\omega_2^2\}} \left(\frac{3k\alpha_3 a_0^2 e^{-2kt}}{8} + \frac{5k\alpha_5 a_0^4 e^{-4kt}}{8} + \frac{105k\alpha_7 a_0^6 e^{-6kt}}{128}\right) \times \left(\begin{matrix}1 + \frac{1}{\{(\omega_1^2 - \omega_2^2)(\omega_1^2 - 5k^2) + 4k^4 - 4k^2\omega_2^2\}} \times \\ \left(\frac{3\varepsilon\alpha_3 a^2 e^{-2kt}}{4} + \frac{5\varepsilon\alpha_5 a^4 e^{-4kt}}{8} + \frac{35\varepsilon\alpha_7 a^6 e^{-6kt}}{64}\right)\end{matrix}\right). \tag{4.35}$$

Now putting equation (4.35) into equation (4.14) and collecting the terms up to $O(\varepsilon)$, we get

$$B_1 = R_1 a e^{-2kt} + R_2 a e^{-4kt} + R_3 a e^{-6kt}, \tag{4.36}$$

where

$$\begin{aligned} R_1 &= -\frac{3k\alpha_3 a_0^2 (5\mu_1^2 - \mu_2^2)}{8(\mu_1^2 - \mu_2^2)\{(\omega_1^2 - \omega_2^2)(\omega_1^2 - 5k^2) + 4k^4 - 4k^2\omega_2^2\}}, \\ R_2 &= -\frac{5k\alpha_5 a_0^4 (5\mu_1^2 - \mu_2^2)}{8(\mu_1^2 - \mu_2^2)\{(\omega_1^2 - \omega_2^2)(\omega_1^2 - 5k^2) + 4k^4 - 4k^2\omega_2^2\}}, \\ R_3 &= -\frac{105k\alpha_7 a_0^6 (5\mu_1^2 - \mu_2^2)}{128(\mu_1^2 - \mu_2^2)\{(\omega_1^2 - \omega_2^2)(\omega_1^2 - 5k^2) + 4k^4 - 4k^2\omega_2^2\}}. \end{aligned} \tag{4.37}$$

Thus, equation (4.21) reduces to the following simple form

$$\dot{a} = \varepsilon(R_1 a e^{-2kt} + R_2 a e^{-4kt} + R_3 a e^{-6kt}), \qquad \dot{\varphi} = \mu_1(\tau). \tag{4.38}$$

After carrying out the integrations of equation (4.38), we obtain the following equations for the amplitude (a) and phase (φ) variables

$$\begin{aligned} a &= a_0 \exp\left(\frac{\varepsilon R_1}{2k}(1 - e^{-2kt}) + \frac{\varepsilon R_2}{4k}(1 - e^{-4kt}) + \frac{\varepsilon R_3}{6k}(1 - e^{-6kt}) \right), \\ \varphi &= \varphi_0 + \int_0^t \mu_1(\tau)dt, \end{aligned} \tag{4.39}$$

where $\tau = \varepsilon t$, a_0 and φ_0 are the initial amplitude and phase variables for the dynamical systems respectively. Thus, the first order approximate analytical solution of equation (4.22) is obtained by

$$x = a e^{-kt} \cos\varphi, \tag{4.40}$$

where a and φ are carrying out from equation (4.39) with the help of equations (4.30), (4.31) and (4.37).

4.4 Results and Discussion

In this chapter, a coupled approximate analytical technique has been extended to obtain the first order approximate analytical solutions for certain type of fourth order strongly generalized nonlinear oscillatory differential system with damping and the method has been successfully implemented to illustrate the effectiveness and convenience of the proposed method. The first order approximate analytical solutions of equation (4.22) are computed by equation (4.40) with the help of equation (4.39) and the corresponding numerical solutions are obtained by the well known fourth order **Runge-Kutta** method.

Moreover, the presented method is simple and the advantage of this method is that the first order approximate solutions show good agreement with the corresponding numerical solutions for several damping effects (**Figs. 4.1-4.6**). The initial approximation can be freely chosen, which is identified via various methods [2-4, 17-83]. The approximate solutions obtained by the presented method are valid not only for fourth order strongly generalized nonlinear damped oscillatory differential systems, but also for weakly one. **Figs. 4.1-4.3** are provided to compare the solutions obtained by the presented method to the corresponding numerical solutions with small damping for fourth order strongly generalized nonlinear oscillatory differential systems. Also **Figs. 4.4-4.5** are cited to compare the solutions obtained by the presented method to the corresponding numerical solutions for fourth order weakly generalized nonlinear oscillatory differential systems with damping effects. However, the proposed method is able to give the desired results for fourth order strongly nonlinear damped oscillatory differential systems (**Fig. 4.6.**) with cubic nonlinearity. From the **Figs. 4.1-4.6,** it is noticed that the obtained approximate analytical solutions for both strongly and weakly generalized nonlinear damped oscillatory differential systems show good agreement with those solutions obtained by the fourth order **Runge-Kutta** method. The average percentage errors have been calculated between numerical and approximate solutions in table 4.1. From the table 4.1, it is clear that the average percentage errors except one case are negligible.

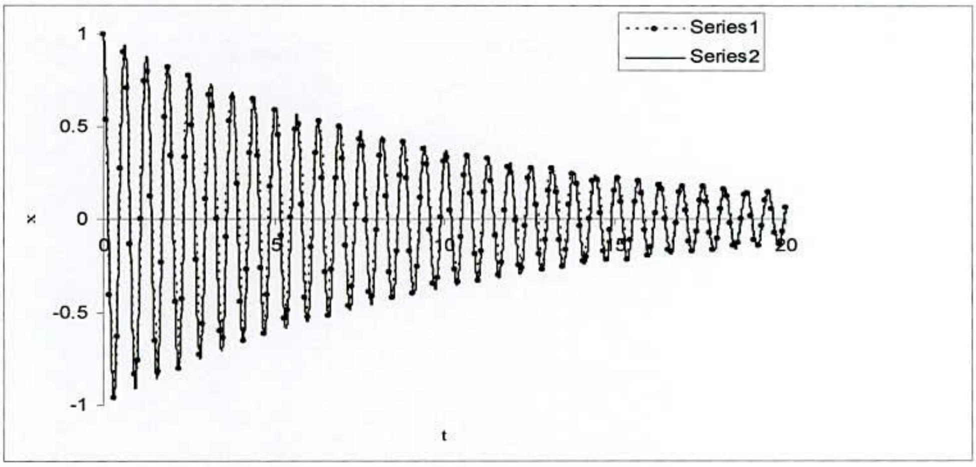

Fig. 4.1. First approximate solution of equation (4.22) is denoted by $-\bullet-$ (dashed lines) by the presented coupling technique with the initial conditions $a_0 = 1.0$, $\varphi_0 = 0$ or $[x(0) = 1.00000$, $\dot{x}(0) = -0.09985$, $\ddot{x}(0) = -99.95239$, $\dddot{x}(0) = 29.95643]$ when $\omega_1 = 10.0$, $\omega_2 = 5.0$, $k = 0.1$, $\varepsilon = 1.0, l_3 = l_5 = l_7 = 1.0$ and $f_3 = x^3$, $f_5 = x^5$, $f_7 = x^7$. Corresponding numerical solution is denoted by - (solid line).

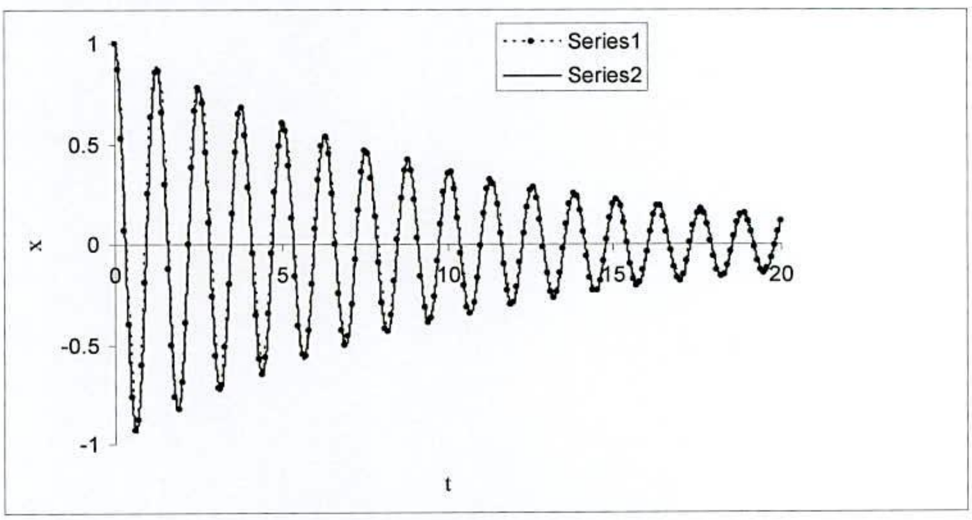

Fig. 4.2. First approximate solution of equation (4.22) is denoted by $-\bullet-$ (dashed lines) by the presented coupling technique with the initial conditions $a_0 = 1.0$, $\varphi_0 = 0$ or $[x(0) = 1.00000$, $\dot{x}(0) = -0.09843$, $\ddot{x}(0) = -25.01930$, $\dddot{x}(0) = 7.43632]$ when $\omega_1 = 5.0$, $\omega_2 = 1.0$, $k = 0.1$, $\varepsilon = 1.0, l_3 = l_5 = l_7 = 1.0$ and $f_3 = x^3$, $f_5 = x^5$, $f_7 = x^7$. Corresponding numerical solution is denoted by - (solid line).

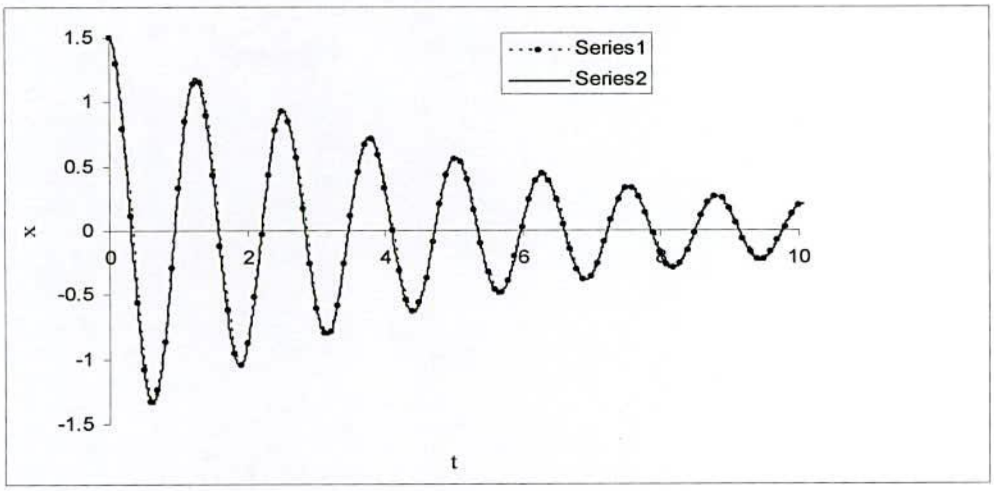

Fig. 4.3. First approximate solution of equation (4.22) is denoted by $-\bullet-$ (dashed lines) by the presented coupling technique with the initial conditions $a_0 = 1.5$, $\varphi_0 = 0$ or $[x(0) = 1.50000,\ \dot{x}(0) = -0.26579,\ \ddot{x}(0) = -37.86418,\ \dddot{x}(0) = 21.22602]$ when $\omega_1 = 5.0$, $\omega_2 = 1.0$, $k = 0.2$, $\varepsilon = 1.0, l_3 = l_5 = l_7 = 1.0$ and $f_3 = x^3$, $f_5 = x^5$, $f_7 = x^7$. Corresponding numerical solution is denoted by - (solid line)

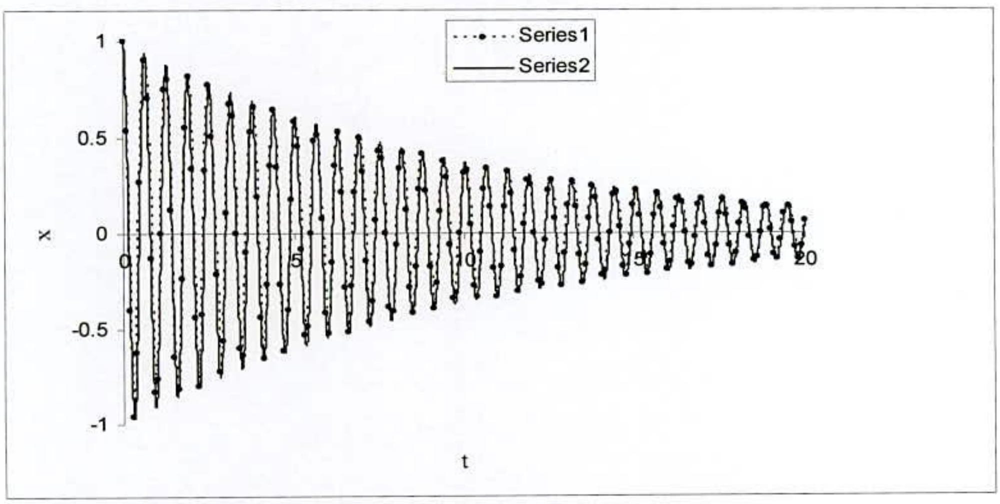

Fig. 4.4. First approximate solution of equation (4.22) is denoted by $-\bullet-$ (dashed lines) by the presented coupling technique with the initial conditions $a_0 = 1.0$, $\varphi_0 = 0$ or $[x(0) = 1.00000,\ \dot{x}(0) = -0.09998,\ \ddot{x}(0) = -99.92923,\ \dddot{x}(0) = 29.97764]$ when $\omega_1 = 10.0$, $\omega_2 = 5.0$, $k = 0.1$, $\varepsilon = 0.1, l_3 = l_5 = l_7 = 1.0$ and $f_3 = x^3$, $f_5 = x^5$, $f_7 = x^7$. Corresponding numerical solution is denoted by - (solid line).

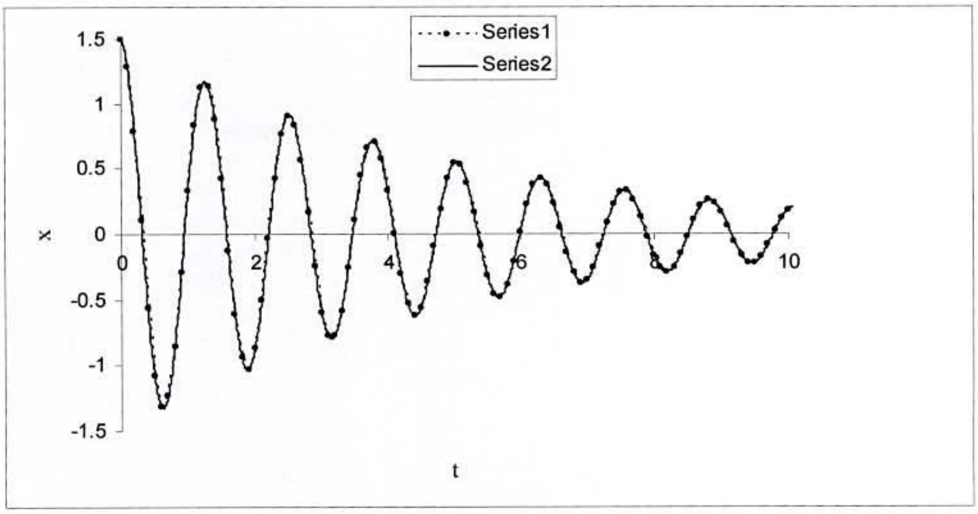

Fig. 4.5. First approximate solution of equation (4.22) is denoted by $-\bullet-$ (dashed lines) by the presented coupling technique with the initial conditions $a_0 = 1.5$, $\varphi_0 = 0$ or $[x(0) = 1.50000, \quad \dot{x}(0) = -0.29653, \ddot{x}(0) = -37.20460, \dddot{x}(0) = 22.19241]$ when $\omega_1 = 5.0$, $\omega_2 = 1.0$, $k = 0.2$, $\varepsilon = 0.1$, $l_3 = l_5 = l_7 = 1.0$ and $f_3 = x^3$, $f_5 = x^5$, $f_7 = x^7$. Corresponding numerical solution is denoted by - (solid line)

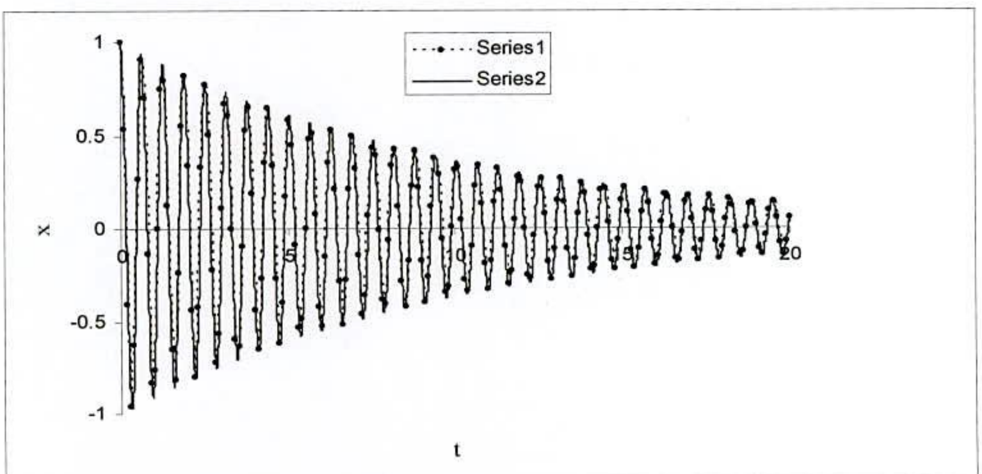

Fig. 4.6. First approximate solution of equation (4.22) is denoted by $-\bullet-$ (dashed lines) by the presented coupling technique with the initial conditions $a_0 = 1.0$, $\varphi_0 = 0$ or $[x(0) = 1.00000, \quad \dot{x}(0) = -0.09997, \ddot{x}(0) = -99.93667, \dddot{x}(0) = 29.97655]$ when $\omega_1 = 10.0$, $\omega_2 = 5.0$, $k = 0.1$, $\varepsilon = 1.0$, $l_3 = 1.0$, $l_5 = l_7 = 0.0$ and $f_3 = x^3$, $f_5 = x^5$, $f_7 = x^7$. Corresponding numerical solution is denoted by - (solid line).

Table 4.1: Average percentage error between **numerical** and **approximate** solutions:

Figures No.	**Initial conditions for different values of the parameters**	**Average Percentage Error**
Fig. 4.1	$a_0 = 1.0,\ \varphi_0 = 0$ or $[x(0) = 1.00000,\ \dot{x}(0) = -0.09985,\ \ddot{x}(0) = -99.95239,\ \dddot{x}(0) = 29.95643]$ when $\omega_1 = 10.0$, $\omega_2 = 5.0$, $k = 0.1$, $\varepsilon = 1.0$, $l_3 = l_5 = l_7 = 1.0$ and $f_3 = x^3$, $f_5 = x^5$, $f_7 = x^7$.	-0.26013%
Fig. 4.2	$a_0 = 1.0,\ \varphi_0 = 0$ or $[x(0) = 1.00000, \dot{x}(0) = -0.09843, \ddot{x}(0) = -25.01930,\ \dddot{x}(0) = 7.43632]$ when $\omega_1 = 5.0$, $\omega_2 = 1.0$, $k = 0.1$, $\varepsilon = 1.0$, $l_3 = l_5 = l_7 = 1.0$ and $f_3 = x^3$, $f_5 = x^5$, $f_7 = x^7$.	-0.27229%
Fig. 4.3	$a_0 = 1.5,\ \varphi_0 = 0$ or $[x(0) = 1.50000, \dot{x}(0) = -0.26579, \ddot{x}(0) = -37.86418,\ \dddot{x}(0) = 21.22602]$ when $\omega_1 = 5.0$, $\omega_2 = 1.0$, $k = 0.2$, $\varepsilon = 1.0$, $l_3 = l_5 = l_7 = 1.0$ and $f_3 = x^3$, $f_5 = x^5$, $f_7 = x^7$.	4.471953%
Fig. 4.4	$a_0 = 1.0,\ \varphi_0 = 0$ or $[x(0) = 1.00000, \dot{x}(0) = -0.09998, \ddot{x}(0) = -99.92923,\ \dddot{x}(0) = 29.97764]$ when $\omega_1 = 10.0$, $\omega_2 = 5.0$, $k = 0.1$, $\varepsilon = 0.1$, $l_3 = l_5 = l_7 = 1.0$ and $f_3 = x^3$, $f_5 = x^5$, $f_7 = x^7$	0.46932%
Fig. 4.5	$a_0 = 1.5,\ \varphi_0 = 0$ or $[x(0) = 1.50000,\ \dot{x}(0) = -0.29653, \ddot{x}(0) = -37.20460,\ \dddot{x}(0) = 22.19241]$ when $\omega_1 = 5.0$, $\omega_2 = 1.0$, $k = 0.2$, $\varepsilon = 0.1$, $l_3 = l_5 = l_7 = 1.0$ and $f_3 = x^3$, $f_5 = x^5$, $f_7 = x^7$	-0.44104%
Fig. 4.6	$a_0 = 1.0,\ \varphi_0 = 0$ or $[x(0) = 1.00000, \dot{x}(0) = -0.09997, \ddot{x}(0) = -99.93667,\ \dddot{x}(0) = 29.97655]$ when $\omega_1 = 10.0$, $\omega_2 = 5.0$, $k = 0.1$, $\varepsilon = 1.0$, $l_3 = 1.0$, $l_5 = l_7 = 0.0$ and $f_3 = x^3$, $f_5 = x^5$, $f_7 = x^7$.	-0.74605%

CHAPTER V

Conclusions

The determination of amplitude and phase variables is important in strongly and weakly generalized nonlinear damped oscillatory differential systems and they play very important role for any physical problem. The amplitude and phase variables characterize the oscillatory processes. In presence of damping, amplitude $a \to 0$ as $t \to \infty$ (i.e., for large time t).

The presented technique in **chapter III** is able to give the desired results for second order strongly generalized nonlinear differential systems with damping and it is also noticed that the first order analytical approximate solutions show good agreement (**Figs. 3.1-3.3**) with those solutions obtained by the numerical procedure for second order strongly and weakly generalized nonlinear differential systems with damping.

Also in **chapter IV**, the presented technique is able to give the desired results for fourth order strongly generalized nonlinear differential systems with small damping effects. The graphical representations show good agreement (**Figs. 4.1- 4.6**) between the first order approximate analytical solutions and the corresponding numerical solutions for fourth order strongly and weakly generalized nonlinear differential systems.

It is also mentioned that, the classical KBM method is failed to tackle for both second and fourth order strongly and weakly generalized nonlinear differential systems with damping and He's homotopy perturbation method is failed to handle nonlinear systems with damping. Some limitations of He's homotopy perturbation (without damping) technique and the KBM method (weak nonlinearity) have been overcome by the methods presented in **chapter III and chapter IV.**

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