Analytical Solutions of Third and Fourth Order Weakly Nonlinear Ordinary Differential Systems with Damping and Slowly Varying Coefficients by the Unified KBM Method

By

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Dedicated to My Parents

Late. Md. Julfikar Ali Biswas & Rumicha Begum

and

Elder Brother B. M. Rahmat.

Those who have chosen underprivileged life to continue my smile.

Declaration

This is to certify that the thesis work entitled "**Analytical Solutions of Third and Fourth Order Weakly Nonlinear Ordinary Differential Systems with Damping and Slowly Varying Coefficients by the Unified KBM Method**" has been carried out by Md. Eabad Ali, in the Department of Mathematics, Khulna University of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of the thesis work has not been submitted anywhere for the award of any degree or diploma.

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Abstract

A perturbation method known as "the asymptotic averaging method" in the theory of nonlinear oscillations was first presented by Krylov and Bogoliubov (KB) in 1947. Primarily, the method was developed only to obtain the periodic solutions of the second order weakly nonlinear differential systems. Later, the method of KB has been improved and justified by Bogoliubov and Mitropolaskii in 1967. In literature, this method is known as the Krylov-Bogoliubov-Mitropolaskii (KBM) method. Now a days this method is used for obtaining the solutions of second, third and fourth order nonlinear differential systems for oscillatory, damped oscillatory, over damped, critically damped and more critically damped cases by imposing some proper restrictions. In this thesis, an analytical approximate technique is extended to find out the second approximate solutions of third order weakly nonlinear differential systems in the presence of strong linear damping and slowly varying coefficients based on the KBM method. Also, the KBM method is presented to find out the solutions of a fourth order weakly nonlinear differential systems in the presence of strong linear damping and slowly varying coefficients including some limitations. To justify the presented method, the approximate solutions have been compared to those solutions obtained by the fourth order **Runge-Kutta** method graphically.

Publication

The following paper has been extracted from this thesis:

1. M. Alhaz Uddin, M. Eabad Ali, M. Wali Ullah and Rehana Sultana Bipasha, Analytical approximate solution of fourth order weakly nonlinear differential systems based on unified KBM method with strong damping and slowly varying coefficients including some limitations. Indian J. Theoretical Physics, Vol.62 No.1-2 (2014). **(Accepted).**

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CHAPTER I

Introduction

Differential equation is a mathematical tool, which has its application in many branches of knowledge of mankind. Numerous physical, mathematical, economical, chemical, biological, biochemical, and many other relations appear mathematically in the form of differential equations that are linear or nonlinear, autonomous or non-autonomous. Generally, in many physical phenomena, such as spring-mass systems, resistor-capacitor-inductor circuits, bending of beams, chemical reactions, the motion of pendulums, the motion of the rotating mass around another body, etc, the differential equations are occurred. Also, in ecology and economics the differential equations are vastly used. Basically, many differential equations involving physical phenomena are nonlinear. Differential equations, which are linear, are comparatively easy to solve and nonlinear are laborious and in some cases it is impossible to solve them analytically. In such situations mathematicians, physicists and engineers convert the nonlinear equations into linear equations by imposing some conditions. In case of small oscillation, linearization is a well known technique to solve the problems. But, such a linearization is not always possible and when it is not possible, then the original nonlinear equation itself must be used. The study of nonlinear equations is generally confined to a variety of rather special cases, and one must resort to various methods of approximation.

At first van der Pol [1] paid attention to the new (self-excitations) oscillations and indicated that their existence is inherent in the nonlinearity of the differential systems characterizing the procedure. This nonlinearity appears, thus, as the very essence of these phenomena and by linearizing the differential equation in the sense of the method of small oscillation, one simply eliminates the possibility of investigating such problems. Thus, it is necessary to deal with the nonlinear problems directly instead of evading them by dropping the nonlinear terms. To solve nonlinear differential equations, there exist some methods such perturbation, technique, harmonic balance, multiple time scale, homotopy perturbation etc. Among the methods, the method of perturbations, i.e., asymptotic expansions in terms of a small parameter are foremost. Perturbation methods have received much attention as these methods for accuracy and quickly computing numerical solutions of dynamic, stochastic, economic equilibrium models for both single-agent or rational expectations models and multi-agent or game theory models. A perturbation method is based on the following aspects: the equations to be solved are sufficiently "smooth" or sufficiently differentiable a number of times in the required regions of variables and parameters.

The Krylov-Bogoliubov-Mitropolaskii (KBM) [3, 4] method was developed for obtaining the periodic solutions of second order nonlinear differential equations. Now a days, the KBM method is used to obtain oscillatory as well as damped, critically damped, over damped, near critically damped, more critically damped oscillatory and non-oscillatory solutions of second, third, fourth etc, order nonlinear differential systems by imposing some restrictions to obtain the uniform solution. The method of KB [3] is an asymptotic method in the sense that $\varepsilon \to 0$. An asymptotic series itself may not be convergent, but for a fixed number of terms, the approximate solution tends to the exact solution as $\varepsilon \to \infty$. It may be noted that the term asymptotic is frequently used in the theory of oscillations in the sense that $\varepsilon \to 0$. But, in this case, the mathematical method is quite different. It is an important approach to the study of such nonlinear oscillations in the small parameter expansion. Two widely spread methods in this theory are mainly used in literature; one is averaging asymptotic KBM method and other is multiple time scale method [13]. The KBM method is particularly convenient and is the extensively used technique to obtain the approximate solutions among the methods used to study the nonlinear differential systems with small nonlinearity. The KBM method starts with the solution of linear equation (sometimes called the generating solution of the linear equation), assuming that in the nonlinear case, the amplitude and the phase variables in the solution of the linear differential equations are time dependent functions instead of constants. So, this method introduces an additional condition on the first derivative of the assumed solution for determining the solution of a second order nonlinear differential equation. The KBM [3, 4] method requires that the asymptotic solutions are free from secular terms. These assumptions are mainly valid for second and third order nonlinear differential equations. But for the fourth order differential equation the correction terms sometimes contain secular terms, although the solution is generated by the classical KBM asymptotic method. For this reason, the traditional solutions fail to explain the proper situation of the systems. To remove the presence of secular terms and to obtain the desired results, we need to impose some conditions. The main objective of this thesis is to find out these limitations and to fill these gaps and to determine the desired solutions under some special conditions. In this thesis, KBM method has been extended for obtaining the second order analytical approximate solution of the third order weakly nonlinear ordinary differential systems in the presence of strong linear damping and slowly varying coefficients including the effects of small damping. Also the KBM method has been presented to solve the fourth order weakly nonlinear ordinary differential systems in t he presence of strong linear damping and slowly varying coefficients including the effects of small damping. The results may be used in mechanics, physics, chemistry, plasma physics, circuit and control theory, population dynamics, economics, etc.

The chapter outline of this thesis is as follows**: In chapter II**, the review of literature is presented. **In chapter III**, the KBM method has been developed for obtaining the second approximate solution of third order weakly nonlinear differential system in the presence of strong linear damping and slowly varying coefficients. First approximate solution of a fourth order weakly nonlinear differential systems has been presented in the presence of strong linear damping and slowly varying coefficients based on the KBM method **in chapter IV**. Finally, some concluding remarks are included **in chapter V**.

CHAPTER II

Literature Review

The characteristics of nonlinear differential equations are peculiar. But mathematical formulations of physical and engineering problems often results in differential equations that are nonlinear. However, in many cases, it is possible to replace a nonlinear differential equation with a related linear differential equation that approximates the actual equations closely enough to give useful results. In many cases, such a linearization is not possible or feasible, when it is not, the original nonlinear differential equations must be tackled directly. During the last several decades a number of famous Russian scientists, Mandelstam and Papalexi [2], Krylov and Bogoliubov [3], Bogoliubov and Mitropolaskii [4] worked jointly and investigated nonlinear mechanics. To solve nonlinear differential equations there exist several methods. Among the methods, the method of perturbations, i.e. an asymptotic expansion in terms of small parameter is foremost. Firstly, Krylov and Bogoliubov [3] considered the following nonlinear differential equation of the form

$$\ddot{x}+\omega^2 x=\varepsilon f(x,\dot{x},t,\varepsilon), \tag{2.1}$$

where over dotes denote ordinary derivative with respect to t, ε is a small positive parameter and f is a power series in ε, whose coefficients are polynomials in $x, \dot{x}, \sin t$ and $\cos t$. In general, f does not contain either ε or t. To describe the behavior of nonlinear oscillations by the solutions obtained by perturbation method, Lindstedt [6], Glyden [7], Liapounoff [8], Poincare [9] discussed only periodic solutions, transient were not considered. Most probably, Poisson initiated to find the approximate solutions of nonlinear differential equations around 1830 and the technique was established by Liouville. The KBM [3, 4] method starts with the solution of the linear equation assuming that in nonlinear systems the amplitude and phase variables in the solution of linear equation are time dependent functions rather than constants. This procedure introduces an additional condition on the first derivative of the assumed solution for determining the desired results. Some meritful works are done and the elaborative uses have been made by Stoker [10], McLachlan [11], Minorsky [12], Nayfeh [13], and Bellman [14]. Duffing [15] has investigated many significant results about the periodic solutions of the following nonlinear damped differential equation named after him.

$$\ddot{x} + 2k\dot{x} + \omega^2 x = -\varepsilon x^3. \tag{2.2}$$

Sometimes different types of nonlinear phenomena occur, when the amplitude of the dependent variable of the dynamic system is less than or greater than unity. The damping is negative when the amplitude is less than unity and the damping is positive when the amplitude is greater than unity. The governing nonlinear differential equation having these phenomena is

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0. \tag{2.3}$$

The equation (2.3) is known as van der Pol equation. Kruskal [16] has extended the KB [3] method to solve the fully nonlinear differential equation of the following form

$$\ddot{x} = F(x, \dot{x}, \varepsilon). \tag{2.4a}$$

Cap [17] has studied nonlinear systems of the form

$$\ddot{x} + \omega^2 f(x) = \varepsilon F(x, \dot{x}). \tag{2.4b}$$

Generally, since f does not contain either ε or t thus the equation (2.1) becomes

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}). \tag{2.5}$$

In the treatment of nonlinear oscillations by perturbation method, only periodic solutions were discussed, transients were not considered by different investigators, where as Krylov and Bogoliubov [3] have discussed transient response firstly. When $\varepsilon = 0$, the equation (2.5) reduces to linear equation and its solution is

$$x = a\cos(\omega t + \varphi). \tag{2.6}$$

where a and φ are arbitrary constants to be determined by using the given initial conditions. When $\varepsilon \neq 0$, it is sufficiently small, then Krylov and Bogoliubov [3] assume that the solution of equation (2.5) is still given by equation (2.6) together with the derivative of the form

$$\dot{x} = -a\omega\sin(\omega t + \varphi). \tag{2.7}$$

where a and φ are the function of t, rather than being constants. In this case, the solution of equation (2.5) is

$$x = a(t)\cos(\omega t + \varphi(t)). \tag{2.8}$$

and the derivative of the solution is

$$\dot{x} = -a(t)\omega\sin(\omega t + \varphi(t)). \tag{2.9}$$

Differentiating the assumed solution equation (2.8) with respect to time t, we obtain

$$\dot{x} = -\dot{a}\cos\psi - a\ \omega\sin\psi - a\dot{\varphi}.\sin\psi, where \psi = \omega t + \varphi(t) \tag{2.10}$$

Using the equations (2.7) and (2.10), we get

$$\dot{a}\cos\psi = a\dot{\varphi}\sin\psi. \tag{2.11}$$

Again, differentiating equation (2.9) with respect to time t, we have

$$\ddot{x} = -\dot{a}\omega\sin\psi - a\omega^2\cos\psi - a\omega\dot{\varphi}\cos\psi. \tag{2.12}$$

Putting the value of time $\ddot{x}$ from equation (2.12) into the equation (2.5) and using equations (2.8) and (2.9), we obtain

$$\dot{a}\,\omega\sin\psi + a\omega\dot{\varphi}\cos\psi = -\varepsilon\, f(a\cos\psi, -a\,\omega\sin\psi). \tag{2.13}$$

Solving equations (2.11) and (2.13), we have

$$\dot{a} = -\frac{\varepsilon}{\omega}\sin\psi\, f(a\cos\psi, -a\omega\sin\psi), \tag{2.14}$$

$$\dot{\varphi} = -\frac{\varepsilon}{a\omega}\cos\psi\, f(a\cos\psi, -a\omega\sin\psi). \tag{2.15}$$

It is observed that, a basic differential equation (2.5) of the second order in the unknown x, reduces to two first order differential equations (2.14) and (2.15) in the unknown a and φ. Moreover, a and φ are proportional to ε; a and φ are slowly varying functions of the time t with period $T = \frac{2\pi}{\omega}$. It is noted that these first order equations are now writing in terms of the amplitude a and phase φ as dependent variables. Therefore, the right sides of equations (2.14) and (2.15) show that both a and φ are periodic functions of period T. In this case, the right-hand terms of these equations contain a small parameter ε and also contain both a and

φ, which are slowly varying functions of the time t, with period $T=\dfrac{2\pi}{\omega}$. We can transfer the equations (2.14) and (2.15) into more convenient form.

Now, expanding $\sin\psi f(a\cos\psi,-a\omega\sin\psi)$ and $\cos\psi\, f(a\cos\psi,-a\omega\sin\psi)$ in Fourier series in the total phase ψ, the first approximate solution of equation (2.5), by averaging equations (2.14) and (2.15) with period $T=\dfrac{2\pi}{\omega}$ is

$$\begin{aligned}\langle\dot{a}\rangle &= -\frac{\varepsilon}{2\pi\,\omega}\int_0^{2\pi}\sin\psi\, f(a\cos\psi,-a\omega\sin\psi)\,d\psi,\\ \langle\dot{\phi}\rangle &= -\frac{\varepsilon}{2\pi\,\omega\,a}\int_0^{2\pi}\cos\psi\, f(a\cos\psi,-a\omega\sin\psi)\,d\psi\end{aligned} \tag{2.16}$$

where a and φ are independent of time t under the integrals. Later, the KB[3] technique has been extended and justified by Bogolibov and Mitropolskii [4], and has been extended to non-stationary vibrations by Mitropolskii [5]. They have assumed the solution of the nonlinear differential equation (2.5) of the following form

$$x=a\cos\psi+\varepsilon u_1(a,\psi)+\varepsilon^2u_2(a,\psi)+\cdots+\varepsilon^n u_n(a,\psi)+o(\varepsilon^{n+1}), \tag{2.17}$$

where $u_{k,}(k=1,2,\cdots,n)$ are periodic functions of ψ with a period 2π, and the quantities a and ψ are functions of time t and defined by the following first order ordinary differential equations

$$\begin{aligned}\dot{a}&=\varepsilon A_1(a)+\varepsilon^2A_2(a)+\cdots+\varepsilon^nA_n(a)+o(\varepsilon^{n+1}),\\ \dot{\psi}&=\omega+\varepsilon B_1(a)+\varepsilon^2B_2(a)+\cdots+\varepsilon^nB_n(a)+o(\varepsilon^{n+1}).\end{aligned} \tag{2.18}$$

The functions u_k, A_k and $B_k,(k=1,2,\cdots,n)$ φ are to be chosen in such a way that the equation (2.17), after replacing a and ψ by the functions defined in equation (2.18), is a solution of equation (2.5). Since there are no restrictions in choosing functions A_k and B_k, it generates the arbitrariness in the definition of the functions u_k (Bogoluibov and Mitropolskii [4]). To remove this arbitrariness, the following additional conditions are imposed

$$\begin{aligned} &\int_0^{2\pi} u_k(b,\psi)\cos\psi \, d\psi = 0, \\ &\int_0^{2\pi} u_k(a,\psi)\sin\psi \, d\psi = 0. \end{aligned} \tag{2.19}$$

Absences of secular terms in the all successive approximations are guaranteed by these conditions. Differentiating equation (2.17) two times with respect to t, substituting the values of $x, \dot{x}$ and $\ddot{x}$ into equation (2.5), using these relations in equation (2.18) and equating the coefficients of $\varepsilon^k, (k = 1, 2, \cdots, n)$ one will obtain

$$\omega^2\big((u_k)_{\psi\psi} + u_k\big) = f^{(k-1)}(a,\psi) + 2\omega(aB_k \cos\psi + A_k \sin\psi), \tag{2.20}$$

where $(u_k)_\psi$ denotes partial derivatives with respect to ψ,

$$\begin{aligned} &f^{(0)}(a,\psi) = f(a\cos\psi, -a\omega\sin\psi), \\ &f^{(1)}(a,\psi) = u_1 f_x(a\cos\psi, -a\omega\sin\psi) + (A_1\cos\psi - aB_1\sin\psi + \omega\frac{\partial u_1}{\partial\psi}) \\ &\qquad \times f_{\dot{x}}(a\cos\psi, -a\omega\sin\psi) + (aB_1{}^2 - A_1\frac{dA_1}{d\psi})\cos\psi \\ &\qquad + (2A_1B_1 - aA_1\frac{dB_1}{d\psi})\sin\psi - 2\omega(A_1\frac{\partial^2 u_1}{\partial a \partial\psi} + \frac{\partial^2 u_1}{\partial\psi^2}). \end{aligned} \tag{2.21}$$

Here $f^{(k-1)}$ is a periodic function of ψ with period 2π which depends also on the amplitude a. Therefore, $f^{(k-1)}$ and u_k can be expanded in the Fourier series as

$$\begin{aligned} &f^{(k-1)}(a,\psi) = g_0^{(k-1)}(a) + \sum_{n=1}^{\infty}(g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi), \\ &u_k(a,\psi) = v_0^{(k-1)}(a) + \sum_{n=1}^{\infty}(v_n^{(k-1)}(a)\cos n\psi + \omega_n^{(k-1)}(a)\sin n\psi), \end{aligned} \tag{2.22}$$

where,

$$g_0^{(k-1)} = \frac{1}{2\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\,d\psi,$$
$$g_n^{(k-1)} = \frac{1}{\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\cos n\psi\,d\psi, \tag{2.23}$$
$$h_n^{(k-1)} = \frac{1}{\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\sin n\psi\,d\psi, \qquad n \geq 1.$$

Here $v_1^{(k-1)} = \omega_1^{(k-1)} = 0$ for all values of k, because both integrals of equation (2.19) are vanished. Substituting these values into the equation (2.20), we obtain

$$\begin{aligned}
&\omega^2 v_0^{(k-1)}(a) + \sum_{n=1}^{\infty}\omega^2(1-n^2)[v_n^{(k-1)}(a)\cos n\psi + w_n^{(k-1)}(a)\sin n\psi] \\
&= g_0^{(k-1)}(a) + (g_1^{(k-1)}(a) + 2\omega a B_i)\cos n\psi + (h_1^{(k-1)}(a) + 2\omega A_i)\sin n\psi) \\
&+ \sum_{n=2}^{\infty}(g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi).
\end{aligned} \tag{2.24}$$

Now, equating the coefficients of the harmonics of the same order, yield

$$\begin{aligned}
&(g_1^{(k-1)}(a) + 2\omega a B_k) = 0, \qquad (h_1^{(k-1)}(a) + 2\omega A_k) = 0, \\
&v_0^{(k-1)}(a) = \frac{g_0^{(k-1)}(a)}{\omega^2}, \qquad v_n^{(k-1)}(a) = \frac{g_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \\
&\omega_n^{(k-1)}(a) = \frac{h_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \qquad n > 1.
\end{aligned} \tag{2.25}$$

These are the sufficient conditions to obtain the desired order of approximation. For the first approximation, we have

$$\begin{aligned}
A_1 &= -\frac{h_1^{(0)}(a)}{2\omega} = -\frac{1}{2\pi\omega}\int_0^{2\pi} f(a\cos t\psi, -a\omega\sin\psi)\sin\psi\,d\psi, \\
B_1 &= -\frac{g_1^{(0)}(a)}{2a\omega} = -\frac{1}{2\pi a\omega}\int_0^{2\pi} f(a\cos t\psi, -a\omega\sin\psi)\cos\psi\,d\psi.
\end{aligned} \tag{2.26}$$

Thus, the variational equations in (2.18) become

$$\begin{aligned}
\dot{a} &= -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\,d\psi, \\
\dot{\psi} &= \omega - \frac{\varepsilon}{2\pi a\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi\,d\psi.
\end{aligned} \tag{2.27}$$

It is seen that, the equations (2.27) are similar to the equation (2.16). Thus, the first approximate solution obtained by Bogoluibov and Mitropolskii [4] is identical to the original solution obtained by Krylov and Bogoliubov [3]. In literature, this method is known as Krylov-Bogoliubov-Mitropolskii (KBM) [3, 4] method. Also, higher approximate solutions can be found easily. The correction term u_1 is obtained by equation (2.22) by using equation (2.25) in the following form

$$u_1 = \frac{g_0^{(0)}(a)}{\omega^2} + \sum_{n=2}^{\infty} \frac{g_n^{(0)} \cos n\psi + h_n^{(0)}(a) \sin n\psi}{\omega^2(1-n^2)} \tag{2.28}$$

The solution equation (2.17) together with u_1 is known as the first order improved solution in which a and ψ are obtained from the equation (2.27). If the values of the functions A_1 and B_1 are substituted from equation (2.26) into the second relation of the equation (2.21), the function $f^{(1)}$ and in the similar way, the functions A_2, B_2 and u_2 can be found. Therefore, the determination of the higher order approximation is completed. The KB [3] method is very similar to that of the van der Pol [1] and related to it. van der Pol [1] has applied the method of variation of constants to the basic solution $x = a\cos\omega t + b\sin\omega t$ of $\ddot{x} + \omega^2 x = 0$, on the other hand Krylov-Bogoliubov [3] have applied the same method to the basic solution $x = a\cos(\omega t + \phi)$ of the same equation. Thus, in the KB [3] method the varied constants are a and φ, while in the van der Pol's [1] method the constants are a and b. The method of KB [3] seems more interesting from the point of view of applications, since it deals directly with the amplitude and phase of the quasi-harmonic oscillation.

The solution of the equation (2.4a) is based on recurrent relations and is given as the power series of the small parameter. Cap [17] has solved the equation (2.4b) by using elliptic functions in the sense of Krylov-Bogoliubov [3]. The KBM [3, 4] method has been extended by Popov [18] to damped nonlinear differential systems represented by the following equation

$$\ddot{x} + 2k\dot{x} + \omega^2 x = \varepsilon f(\dot{x}, x), \tag{2.29}$$

where $-2k\dot{x}$ is the linear damping force and $0 < k < \omega$. It is noteworthy that, because of the importance of the Popov's [18] method in the physical systems, involving damping force, Mendelson [19] and Bojadziev [20] have retrieved Popov's [18] results. Bojadziev [20] has used the KBM [3, 4] method to investigate the solutions of nonlinear differential systems raised from biological and biochemical systems. In case of damped nonlinear differential systems, the first of equation (2.18) has been replaced by

$$\dot{a} = -k\,a + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \cdots + \varepsilon^n A_n(a) + o(\varepsilon^{n+1}). \tag{2.18a}$$

Murty and Deekshatulu [21] have developed a simple analytical method to obtain the time response of second order nonlinear over damped systems with small nonlinearity represented by the equation (2.29), based on the KBM [3, 4] method of variation of parameters. Alam [22] has extended the KBM method to find the solutions of over damped nonlinear differential systems, when one root of the auxiliary equation becomes much smaller than the other root. According to the KBM method, Murty et al. [23] have found a hyperbolic type asymptotic solution of an over damped system represented by the nonlinear differential equation (2.29), i.e., in the case $k > \omega$. They have used hyperbolic functions, $\cosh\varphi$ and $\sinh\varphi$ instead of their circular counterpart, which are used by Krylov-Bogoliubov [3], Mitropolskii [5], Popov [18] and Mendelson [19]. In case of oscillatory or damped oscillatory process these may be used arbitrarily for all kinds of initial conditions. But, the in case of non-oscillatory systems, $\cosh\varphi$ or $\sinh\varphi$ should be used depending on the given set of initial conditions. Bojadziev and Edwards [24] have investigated solutions of oscillatory and non-oscillatory systems represented by equation (2.29) when k and ω are slowly varying functions of time t. Murty [25] has presented a unified KBM method for solving the second order nonlinear differential systems represented by the equation (2.29), which covers the undammed, damped and over damped cases. Arya and Bojadziev [26, 27] have examined damped oscillatory systems and time dependent oscillating systems with slowly varying parameters and delay. Sattar [28] has developed an asymptotic method to solve a second order critically damped nonlinear differential system represented by equation (2.29). He has found the asymptotic solution of the equation (2.29) in the following form

$$x = a(1+\psi) + \varepsilon u_1(a,\psi) + \cdots + \varepsilon^n u_n(a,\psi) + o(\varepsilon^{n+1}), \tag{2.30}$$

where a is defined by the first equation of (2.18) and ψ is defined by

$$\dot{\psi} = 1 + \varepsilon C_1(a) + \varepsilon^2 C_2(a) + \cdots + \varepsilon^n C_n(a) + o(\varepsilon^{n+1}). \tag{2.18b}$$

Osiniskii [29] has extended the KBM [3, 4] method to the following third order nonlinear differential equation

$$\dddot{x} + c_1 \ddot{x} + c_2 \dot{x} + c_3 x = \varepsilon f(\ddot{x}, \dot{x}, x) \tag{2.31}$$

where ε is a small positive parameter and f is a given nonlinear function. He has assumed the asymptotic solution of equation (2.31) in the form

$$x = a + b\cos\psi + \varepsilon u_1(a,b,\psi) + \cdots + \varepsilon^n u_n(a,b,\psi) + o(\varepsilon^{n+1}), \tag{2.32}$$

where each $u_k, (k = 1,2,\cdots,n)$ is a periodic function of ψ with period 2π and a, b and ψ are functions of time t, and they are given by

$$\begin{aligned}
\dot{a} &= -\lambda a + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \cdots + \varepsilon^n A_n(a) + o(\varepsilon^{n+1}), \\
\dot{b} &= -\mu a + \varepsilon B_1(b) + \varepsilon^2 B_2(b) + \cdots + \varepsilon^n B_n(b) + o(\varepsilon^{n+1}), \\
\dot{\psi} &= \omega + \varepsilon C_1(b) + \varepsilon^2 C_2(b) + \cdots + \varepsilon^n C_n(b) + o(\varepsilon^{n+1}),
\end{aligned} \tag{2.33}$$

where $-\lambda$, $-\lambda \pm i\omega$ are the eigen values of the equation (2.31) when $\varepsilon = 0$.

By using the KBM [3, 4] method, Bojadziev [30] has investigated asymptotic solutions of nonlinear differential equation with time lag delay function. Bojadziev and Lardner [31] have also found solutions in mechanical systems governed by hyperbolic differential equation with small nonlinearities. Bojadziev [32], Bojadziev and Chan [33] have applied the KBM [3, 4] method to solve the problem of population dynamics. Lin and Khan [34] have also used the KBM method to some biological problems. Proskurjakov [35] and Bojadziev et al. [36] have investigated periodic solutions of nonlinear systems by the KBM [3, 4] and Poincare [9] methods and they have compared the two solutions. Bojadziev and Lardner [37, 38] have investigated mono-frequent oscillations in mechanical systems including the case of internal resonance, governed by hyperbolic differential equations with small nonlinearities. Bojadziev and Lardner [38] have also investigated solution for a certain hyperbolic partial differential equation with small nonlinearity and large time delay including unperturbed and perturbed parts of the equation. Rauch [39] has studied oscillations of a third order nonlinear autonomous system. Bojadziev [40] and Bojadziev and Hung [41] have developed a technique by using the method of KBM [3, 4] to investigate a weakly nonlinear differential system with strong damping. Osiniskii [42] has also extended the KBM [3, 4] method to third order nonlinear partial differential equation with internal friction and relaxation. Mulholland [43] has studied nonlinear oscillations governed by third order differential equation. Lardner and Bojadziev [44] have investigated nonlinear damped oscillations governed by a third order partial differential equation. They have introduced the concept of "couple amplitude" where the unknown functions A_k, B_k and C_k depend on both the amplitudes a and b. Alam [45] has used the KBM method for solving *nth* order nonlinear differential system with slowly varying coefficients. Alam et al. [46] have presented a general form of the KBM [3, 4] method for solving nonlinear partial differential equations. Raymond and Cabak [47] have examined the effects of internal resonance on impulsive forced nonlinear systems with two-degree-of-freedom. Alam [48] has also presented a compact form of the KBM [3, 4] unified method for solving an *nth*, $n \geq 2,3$ order nonlinear differential systems. The formula presented in [48] is compact, systematic, practical, and easier. Bojadziev [49] has presented a damped forced nonlinear vibration of systems with delay. Alam [50] has presented a method to obtain the solution of *nth*, $n \geq 2,3$ order over damped nonlinear systems under some special conditions. Later, Alam [50, 53] has extended the KBM method to *nth*, $n \geq 2,3$ order nonlinear differential systems. Alam [51] has presented a perturbation method based on the KBM [3, 4] method to find the approximate solutions of second order nonlinear differential systems with large damping. Alam et al. [52] have investigated perturbation solution of a second order time dependent nonlinear system based on the KBM method. Alam and Alam [53] have developed an asymptotic method for certain third-order non-oscillatory non-linear systems. Lim and Wu [54] have also presented a new analytical approach to the Duffing harmonic oscillator. Uddin and Sattar [55] have presented an approximate solution of a fourth order weakly nonlinear differential system with strong damping slowly varying coefficients by unified KBM method but they have not followed their impose restriction strictly. Alam and Sattar [56] have developed a simple method to obtain the time response of third order over damped nonlinear systems together with slowly varying coefficients under some special conditions. Alam and Sattar [57] have presented a unified KBM [3, 4] method for solving third order nonlinear systems. Alam [58] has also presented a unified KBM [3, 4] method, which is not the formal form of the original KBM method for solving *nth*, $n \geq 2,3$

order nonlinear systems. The solution contains some unusual variables, yet this solution is very important. Alam [59] has redeveloped the KBM method presented in [56] to find the approximate solutions of critically damped nonlinear systems in the presence of different damping forces by considering different sets of variational equations. Alam [59] has also extended the KBM method for a third order over damped nonlinear system when two of the eigen values are almost equal (i.e., the system is near to the critically damped) and the rest is small. Alam [60] has presented oscillating processes of third-order non-linear differential systems. Uddin and Sattar [61] have developed an approximate technique for solving Duffing type equation with small damping and slowly varying coefficients. Akbar et al. [62] have presented the KBM method for solving fourth order more critically damped nonlinear systems. Uddin and Sattar [63] have developed an approximate technique for solving strongly nonlinear biological systems with small damping effects. Alam [64] has presented a perturbation theory of *nth* order nonlinear differential systems with large damping. Uddin et al. [65] have developed an approximate technique for solving strongly nonlinear differential systems with small damping effects. Alam [66] has presented a unified KBM method for solving *nth*, $n \geq 2, 3$ order nonlinear differential systems. Recently, Alom and Uddin [67] have presented an approximate technique for solving fourth order near critically damped nonlinear systems with special conditions based on the KBM method.

CHAPTER III

Second approximate solution of a third order weakly nonlinear differential system in the presence of strong linear damping and slowly varying coefficients based on the KBM method

3.1 Introduction

The study of nonlinear problems is of crucial importance not only in different areas of physics but also in engineering and applied mathematics, since most phenomena in our world are nonlinear and are described by nonlinear differential equations. It is very difficult to solve nonlinear problems and in general, it is often more difficult to get an analytical approximate solution than a numerical one for a given nonlinear problem. The several methods are used to find approximate solutions of nonlinear problems, such as the perturbation techniques [24, 45] and harmonic balance method [68], etc. Bojadziev and Edward [24] have presented an asymptotic method for non-oscillatory and oscillatory processes. Arya and Bojadziev [26] have studied a system of second order nonlinear hyperbolic differential equation with slowly varying coefficients. Arya and Bojadziev [27] have also studied a time-dependent nonlinear oscillatory system with damping, slowly varying coefficients and delay. Alam [45] has investigated a unified KBM method for obtaining the first approximate solution of nth order nonlinear systems with slowly varying coefficients. Uddin and Sattar [55] have obtained an approximate solution of a fourth order weakly non-linear differential system with strong damping and slowly varying coefficients by the unified KBM method. Roy and Alam [69] have studied the effect of higher approximation of Krylov-Bogoliubov-Mitropolskii's solution and matched asymptotic solution for second order nonlinear differential system with slowly varying coefficients and damping near to a turning point. Alam and Sattar [70] have also presented an asymptotic method for obtaining the first approximate solution of a third order nonlinear differential system with varying coefficients. Akbar et al. [71] has established a technique for solving nth order nonlinear differential equation under some special conditions including the case of internal resonance. Feshchenko et al. [72] have presented a brief way to determine the KBM solution (first order) of nth, $n = 2, 3, \ldots$ order linear differential systems. Sometimes the first approximate solutions give desired results when the linear damping effect is very small. Otherwise, the solutions give incorrect results after a long time $t >> 1$, where the reduced frequency becomes small. From our study, it is seen that most of the researchers have been carried out to obtain the first approximate solutions for both constants and varying coefficients [24, 45, 46, 72]. The complicated but not less important case of second order approximate solution of a third order nonlinear differential systems in the presence of strong linear damping and slowly varying coefficients has remained almost untouched. The main goal of this chapter is to fill this gap based on the KBM method.

3.2 The method

Let us consider a third-order weakly nonlinear ordinary differential equation with slowly varying coefficients in the following form

$$\dddot{x} + k_1(\tau)\ddot{x} + k_2(\tau)\dot{x} + k_3(\tau)x = \varepsilon f(x,\dot{x},\ddot{x},\tau) \tag{3.1}$$

where the over dots represent the time derivatives, ε is a small positive parameter which measures the strength of the nonlinearity, $\tau = \varepsilon t$ slowly varying time, $k_j(\tau) \geq 0, j = 1,2,3$ and f is a given nonlinear function which satisfies $f(-x,-\dot{x},-\ddot{x},\tau) = -f(x,\dot{x},\ddot{x},\tau)$. The coefficients are varying slowly in the sense that their time derivatives are proportional to ε [57].

By putting $\varepsilon = 0, \tau = \tau_0 =$ constant in equation (3.1), we obtain the solution of the unperturbed equation (i.e., linear equation) with constant coefficients. The unperturbed equation of equation (3.1) has three eigen values $\lambda_j(\tau_0), j = 1,2,3,$ where $\lambda_j(\tau_0)$ are constants, but if $\varepsilon \neq 0$ then it is assumed that $\lambda_j(\tau)$ are varying slowly with time t. The solution of the linearrized equation of equation (3.1) is obtained in the following form

$$x(t,0) = \sum_{j=1}^{3} a_{j,0} e^{\lambda_j(\tau_0)t}, \tag{3.2}$$

where $a_{j,0}$, $j = 1,2,3$ are arbitrary constants.

Now we are going to choose a solution of equation (3.1) that reduces to equation (3.2) as a limit $\varepsilon \to 0$ in accordance with the KBM method in the following form:

$$x(t,\varepsilon) = \sum_{j=1}^{3} a_j(t) + \varepsilon u_1(a_1,a_2,a_3,\tau) + \varepsilon^2 u_2(a_1,a_2,a_3,\tau) + \cdots, \tag{3.3}$$

where each a_j satisfies the following first order differential equation

$$\dot{a}_j = \lambda_j a_j + \varepsilon A_j(a_1, a_2, a_3) + \varepsilon^2 B_j(a_1, a_2, a_3) + \cdots . \tag{3.4}$$

Confining only to the first few terms, (1, 2, 3) in the series expansions of equation(3.3) and equation (3.4), we evaluate the functions $u_1,\ u_2, \cdots$ and $A_j,\ B_j, \cdots, j = 1, 2, 3$ such that each $a_j(t)$ appearing in equation (3.3) and equation (3.4) satisfy the given differential equation (3.1) with an accuracy of ε^{m+1} [66]. Theoretically, the solution can be obtained up to any order of approximations but owing to the rapidly growing algebraic complexity for the derivation of the formula, the solution is in general confined to a low order, usually the first order. In order to determine these functions it is assumed that the functions $u_1,\ u_2$ do not contain the fundamental terms which included in the series expansions equation (3.3) at order ε^0. Now differentiating equation (3.3) three times with respect to time t and using the relations equation (3.4) and substituting the values of $\dddot{x}, \ddot{x}, \dot{x}$ together with x into the original equation (3.1) with the slowly varying coefficients $k_1(\tau) = -(\lambda_1(\tau) + \lambda_2(\tau) + \lambda_3(\tau)),\ k_2(\tau) = \lambda_1(\tau)\lambda_2(\tau) + \lambda_2(\tau)\lambda_3(\tau) + \lambda_1(\tau)\lambda_3(\tau)$, $k_3(\tau) = -\lambda_1(\tau)\lambda_2(\tau)\lambda_3(\tau)$ and expanding the right hand side of equation (3.1) by Taylor's series and equating the coefficients of ε and ε^2 on both sides we obtain the following equations

$$\begin{aligned}&\prod_{j=1}^{3}(\Omega - \lambda_j)u_1 + \sum_{j=1}^{3}\Big(\prod_{k=1,k\neq j}^{3}(\Omega - \lambda_k)A_j\Big)\\ &+ \sum_{j=1}^{n}\frac{1}{2}\Big(\sum_{k=0}^{n-2}(n-k)(n-k-1)\,c_k\,\lambda_j^{(n-k-2)}\Big)\lambda_j'\ a_j = f^{(0)}(a_1,a_2,a_3,\tau)\end{aligned} \tag{3.5}$$

$$\prod_{j=1}^{3}(\Omega - \lambda_j)u_2 + \sum_{j=1}^{3}\Big(\prod_{k=1,k\neq j}^{3}(\Omega - \lambda_k)B_j\Big) + \sum_{j=1}^{3}\lambda_j'\ A_j + \sum_{j=1}^{3}\lambda_j''\ a_j = f^{(1)}(a_1,a_2,a_3,\tau) \tag{3.6}$$

where $f(x_0,\dot{x}_0,\ddot{x}_0,\tau) = f^{(0)}(a_1,a_2,a_3,\tau) + \varepsilon f^{(1)}(a_1,a_2,a_3,\tau)$, $x_0 = \sum_{j=1}^{3} a_j$, $\Omega = \sum_{j=1}^{3}\lambda_j a_j \frac{\partial}{\partial a_j}$,

$\lambda_j' = \dfrac{d\lambda_j}{d\tau}$ and $n = 3$ is the order of the differential equation.

We have already assumed that u_1 and u_2 do not contain the fundamental terms and for this reason the solution will be free from secular terms, namely $t\cos t,\ t\sin t$ and $t\,e^{-t}$.

Since the solution will be non-uniform in the presence of secular terms. Under these restrictions, we are able to solve equation (3.5) and equation (3.6), by separating this into $n+1$ individual equations for the unknown functions u_1, u_2, A_j and B_j. In general, the functions $f^{(0)}$, $f^{(1)}$, u_1 and u_2 are expanded in Taylor series in the following forms

$$f^{(0)} = \sum_{m_1=0,m_2=0\dots m_n=0}^{\infty,\infty,\dots,\infty} F_{m_1,m_2,m_3}(\tau) a_1^{m_1} a_2^{m_2} a_3^{m_3} \tag{3.7}$$

$$u_1 = \sum_{m_1=0,m_2=0,\dots\dots,m_n=0}^{\infty,\infty,\dots,\infty} U_{m_1,m_2 m_3}(\tau) a_1^{m_1} a_2^{m_2} a_3^{m_3} \tag{3.8}$$

$$f^{(1)} = \sum_{m_1=0,m_2=0,\dots\dots,m_n=0}^{\infty,\infty,\dots,\infty} G_{m_1,m_2 m_3}(\tau) a_1^{m_1} a_2^{m_2} a_3^{m_3} \tag{3.9}$$

and

$$u_2 = \sum_{m_1=0,m_2=0,m_3=0}^{\infty,\infty,\infty} V_{m_1,m_2,m_3}(\tau) a_1^{m_1} a_2^{m_2} a_3^{m_3}. \tag{3.10}$$

The eigen values of the unperturbed equation can be written as $\lambda(\tau_0)$ and $\mu_l(\tau_0) \pm i\omega_l(\tau_0)$ where $l=1$. For the above restrictions, it guarantees that u_1 and u_2 must be excluded all terms with $a_{2l}^{m_{2l}} a_{2l+1}^{m_{2l+1}}$ of $f^{(0)}$ and $f^{(1)}$ where $m_{2l} - m_{2l+1} = \pm 1$. Since as a linear approximation (*i.e.* $\varepsilon \to 0$) $a_{2l}^{m_{2l}} a_{2l+1}^{m_{2l+1}}$ becomes $e^{\omega_l t}$ when $m_{2l} - m_{2l+1} = 1$ or $e^{-\omega_l t}$ when $m_{2l} - m_{2l+1} = -1$. It is noticed that $e^{\pm\omega_l t}$ are known as the fundamental terms [3, 4]. Usually these are included in equations A_j and B_j. Moreover, it is restricted (by Krylov and Bogoliubov [3]) that the functions A_j and B_j are independent of the fundamental terms. Now to determine the equations for A_1 and B_1, we follow the assumption of Bojadziev [40] that u_1 and u_2 do not contain a term te^{-t} (as limit $\mu_l \to 0$) and we obtain the following equations

$$\begin{aligned}(\prod_{k=2}^{n}(\Omega - \lambda_k)) A_1 + \frac{1}{2}(\sum_{k=0}^{n-2}(n-k)(n-k-1)c_k \lambda_{2l-1}^{(2l-k-2)})\lambda_1' a_1 \\ = \sum_{m_1=0,m_{2l}=0,m_{2l+1}=0}^{\infty,\infty,\infty} F_{m_1,m_{2l},m_{2l+1}} a_1^{m_1} a_{2l}^{m_{2l}} a_{2l+1}^{m_{2l+1}},\ m_{2l} = m_{2l+1}\end{aligned} \tag{3.11}$$

$$(\prod_{k=2}^{n}(\Omega-\lambda_k))B_1+\frac{1}{2}(\sum_{k=0}^{n-2}(n-k)(n-k-1)c_k\,\lambda_{2l-1}{}^{(2l-k-2)})\lambda_1'\,a_1$$
$$=\sum_{m_1=0,m_{2l}=0,m_{2l+1}=0}^{\infty,\infty,\infty}G_{m_1,m_{2l},m_{2l+1}}a_1^{m_1}a_{2l}^{m_{2l}}a_{2l+1}^{m_{2l+1}},\ \ m_{2l}=m_{2l+1}. \tag{3.12}$$

Then the equations for $u_1,\ u_2,\ A_j$ and $B_j,\ j=1,2,\ldots n$ are obtained as

$$\prod_{j=1}^{n}(\Omega-\lambda_j)u_1=\sum_{m_1=0,m_{2l}=0,m_{2l+1}}^{\infty,\infty,\infty}F_{m_1,m_{2l},m_{2l+1}}(\tau)a_1^{m_1}a_{2l}^{m_{2l}}a_{2l+1}^{m_{2l+1}}\qquad m_{2l}-m_{2l+1}\neq 0,\pm 1 \tag{3.13}$$

$$(\prod_{k=1,k\neq 2l}^{n}(\Omega-\lambda_k))A_{2l}+\frac{1}{2}(\sum_{k=0}^{n-2}(n-k)(n-k-1)c_k\lambda_{2l}^{2l-k-2})\lambda_{2l}'\,a_{2l}$$
$$=\sum_{m_{2l}=0,m_{2l+1}=0}^{\infty,\infty}F_{m_{2l}m_{2l+1}}a_{2l}^{m_{2l}}a_{2l+1}^{m_{2l+1}},\ \ m_{2l}-m_{2l+1}=1 \tag{3.14}$$

$$(\prod_{k=1,k\neq 2l+1}^{n}(\Omega-\lambda_k))A_{2l+1}+\frac{1}{2}(\sum_{k=0}^{n-2}(n-k)(n-k-1)c_k\lambda_{2l}^{2l-k-2})\lambda_{2l+1}'\,a_{2l+1}$$
$$=\sum_{m_{2l}=0,m_{2l+1}=0}^{\infty,\infty}F_{m_{2l}m_{2l+1}.0}a_{2l}^{m_{2l}}a_{2l+1}^{m_{2l+1}},\ \ m_{2l}-m_{2l+1}=-1 \tag{3.15}$$

and

$$\prod_{j=1}^{n}(\Omega-\lambda_j)u_2=\sum_{m_1=0,m_{2l}=0,\ldots m_{2l+1}}^{\infty,\infty,\infty}G_{m_1,m_{2l},m_{2l+1}}(\tau)a_1^{m_1}a_{2l}^{m_{2l}}a_{2l+1}^{m_{2l+1}},\ \ m_{2l}-m_{2l+1}\neq 0,\pm 1 \tag{3.16}$$

$$(\prod_{k=1,k\neq 2l-1}^{n}(\Omega-\lambda_k))B_{2l}+\frac{1}{2}(\sum_{k=0}^{n=2}(n-k)(n-k-1)c_k\lambda_{2l}^{2l-k-2})\lambda_{2l}'\,a_{2l}$$
$$=\sum_{m_{2l}=0,\ m_{2l+1}=0}^{\infty,\infty}G_{m_{2l},m_{2l-1}}a_{2l}^{m_{2l}}a_{2l+1}^{m_{2l+1}},\ \ m_{2l}-m_{2l+1}=1 \tag{3.17}$$

and

$$(\prod_{k=1,k\neq 2l+1}^{n}(\Omega-\lambda_k))B_{2l+1}+\frac{1}{2}(\sum_{k=0}^{n-2}(n-k)(n-k-1)c_k\lambda_{2l}^{2l-k-2})\lambda_{2l+1}'\,a_{2l+1}$$
$$=\sum_{m_{2l}=0,\ m_{2l+1}=0}^{\infty,\infty}G_{m_{2l}m_{2l+1}}a_{2l}^{m_{2l}}a_{2l+1}^{m_{2l+1}},\ \ m_{2l}-m_{2l+1}=-1. \tag{3.18}$$

To obtain the particular solutions of equations. (3.11)- (3.18), we replace the operator Ω by $\sum_{j=1}^{3}m_j\lambda_j$, since we know that $\Omega(a_1^{m_1}a_{2l}^{m_{2l}}a_{2l+1}^{m_{2l+1}})=\sum_{j=1}^{n}m_j\lambda_j(a_1^{m_1}a_{2l}^{m_{2l}}a_{2l+1}^{m_{2l+1}})$. Hence the determination of second order approximate solution of equation (3.1) is completely determined.

But it is noticed that the solution equation (3.3) is not a standard form of the KBM method. To reduce the solution of equation (3.3) to the standard form of the KBM method, we need to use the following substitutions

$$a_1 = a$$
$$a_{2l} = \frac{1}{2} b e^{i\varphi t}, \tag{3.19}$$
$$a_{2l+1} = \pm \frac{1}{2} b e^{-i\varphi t}, l = (n-1)/2,$$

where a, b represent the amplitudes and φ represents the phase of the nonlinear physical differential systems.

3.3 Example

For the practical importance of the above method, we consider the following third order weakly nonlinear differential equation in the presence of strong linear damping and slowly varying coefficients

$$\dddot{x} + k_1(\tau)\ddot{x} + k_2(\tau)\dot{x} + k_3(\tau)x = \varepsilon x^3 \tag{3.20}$$

Comparing this with equation (3.1) we have $n = 3,\ j = 1,\ 2,\ 3;\ f(x, \dot{x}, \ddot{x}, \tau) = x^3$ and $x_0 = a_1 + a_2 + a_3$. Now we obtain

$$\begin{aligned} f(x,\dot{x}) &= f(a_1 + a_2 + a_3 + \varepsilon u_1, \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \varepsilon(A_1 + A_2 + A_3 + \Omega u_1)) \\ &= f(a_1 + a_2 + a_3, \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) + \varepsilon u_1 f_x(a_1 + a_2 + a_3, \\ &\quad \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) + \varepsilon(A_1 + A_2 + A_3 + \Omega u_1) \\ &\quad \times f_{\dot{x}}(a_1 + a_2 + a_3, \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) + \cdots \\ &= f^{(0)} + \varepsilon f^{(1)} \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} f^{(0)} &= f(a_1 + a_2 + a_3, \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) = a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + a_2^3 + 3a_1^2 a_3 \\ &\quad + 6a_1 a_2 a_3 + 3a_2^2 a_3 + 3a_1 a_3^2 + 3a_2 a_3^2 + a_3^3, \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} f^{(1)} &= 3u_1 f_x(a_1 + a_2 + a_3, \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) \\ &= 3u_1(a_1 + a_2 + a_3)^2 \\ &= 3[r_1 a_1^3 a_2^2 + (r_1 + 2r_3)a_1 a_2^4 + (r_1 + r_2)a_1 a_2^2 a_3^2 + (2r_1 + r_2)a_1^2 a_2^3 + 2(r_1 + r_3)a_1 a_2^3 a_3 \\ &\quad + 2r_1 a_1^2 a_2^2 a_3 + r_2 a_1^3 a_3^2 + (r_2 + 2r_4)a_1 a_3^4 + 2r_2 a_1^2 a_2 a_3^2 + 2(r_2 + r_4)a_1 a_2 a_3^3 \\ &\quad + (2r_2 + r_4)a_1^2 a_3^3 + r_3 a_2^5 + r_3 a_2^3 a_3^2 + 2r_3 a_2^4 a_3 + r_4 a_2^2 a_3^3 + r_4 a_3^5 + 2r_4 a_2 a_3^4], \end{aligned} \tag{3.23}$$

where

$$r_1 = \frac{3}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)}, \qquad r_2 = \frac{3}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1-\lambda_2+2\lambda_3)}$$
$$r_3 = \frac{1}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}, \qquad r_4 = \frac{1}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)} \tag{3.24}$$

Substituting the values of n, j and $f^{(0)}$ in equation (3.5) and according to our restrictions [equations (3.13)-(3.18)], we obtain four equations for A_1, A_2, A_3 and u_1 whose solutions are respectively given by the following equations

$$A_1 = -\frac{(2\lambda_1-\lambda_2-\lambda_3)\lambda_1' a_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{a_1^3}{(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)} + \frac{6a_1a_2a_3}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)},$$
$$A_2 = -\frac{(2\lambda_2-\lambda_1-\lambda_3)\lambda_2' a_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{3a_1^2a_2}{(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)} + \frac{3a_2^2a_3}{2\lambda_2)(2\lambda_2+\lambda_3-\lambda_1)}, \tag{3.25}$$
$$A_3 = -\frac{(2\lambda_3-\lambda_1-\lambda_2)\lambda_3' a_3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} + \frac{3a_1^2a_3}{(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3-\lambda_2)} + \frac{3a_2a_3^2}{2\lambda_3(2\lambda_3+\lambda_2-\lambda_1)},$$

and

$$u_1 = \frac{3a_1a_2^2}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)} + \frac{3a_1a_3^2}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1-\lambda_2+2\lambda_3)}$$
$$+\frac{a_2^3}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)} + \frac{a_3^2}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}. \tag{3.26}$$

Also substituting equation (3.25) into equations (3.17)- (3.18) and according to our restrictions [equations (3.13)-(3.18)], we obtain three equations for B_1, B_2 and B_3 whose solutions are respectively given by the following equations

$$B_1 = \frac{9}{(\lambda_1+\lambda_2+2\lambda_3)(\lambda_1+2\lambda_2+\lambda_3)}[\frac{1}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)}$$
$$+\frac{1}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1-\lambda_2+2\lambda_3)}]a_1a_2^2a_3^2 + \frac{(2\lambda_1-\lambda_2-\lambda_3)\lambda_1'^2 a_1}{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2}$$
$$-\frac{\lambda_1' a_1^3}{(3\lambda_1-\lambda_2)^2(3\lambda_1-\lambda_3)^2} - \frac{6\lambda_1' a_1a_2a_3}{(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)^2} - \frac{\lambda_1'' a_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)},$$

$$B_2 = \frac{3a_2^3a_3^2}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(3\lambda_2+\lambda_3)(3\lambda_3+2\lambda_2-\lambda_1)}$$
$$+\frac{18a_1^2a_2^2a_3}{4\lambda_2(\lambda_1+\lambda_2)^2(\lambda_1+2\lambda_2+\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)}+\frac{(2\lambda_2-\lambda_1-\lambda_3)\lambda_2'^2a_2}{(\lambda_2-\lambda_1)^2(\lambda_2-\lambda_3)^2}$$
$$-\frac{3\lambda_2'a_1^2a_2}{(\lambda_1+\lambda_2)^2(2\lambda_1+\lambda_2-\lambda_3)^2}-\frac{3\lambda_2'a_2^2a_3}{4\lambda_2^2(2\lambda_2+\lambda_3-\lambda_1)^2}-\frac{\lambda_2''a_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)},$$
$$B_3 = \frac{a_2^2a_3^3}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(\lambda_2+3\lambda_3)(2\lambda_2+3\lambda_3-\lambda_1)} \tag{3.27}$$
$$+\frac{3a_1^2a_2a_3^2}{2\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_1+\lambda_2+2\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)}+\frac{(2\lambda_3-\lambda_1-\lambda_2)\lambda_3'^2a_3}{(\lambda_3-\lambda_1)^2(\lambda_3-\lambda_2)^2}$$
$$-\frac{3\lambda_3'a_1^2a_3}{(\lambda_1+\lambda_3)^2(2\lambda_1+\lambda_3-\lambda_2)^2}-\frac{3\lambda_3'a_2a_3^2}{4\lambda_3^2(\lambda_2+2\lambda_3-\lambda_1)^2}-\frac{\lambda_3''a_3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}.$$

We are not interested to determine the correction term u_2. So, we are ignoring it. Now substituting the values of A_1, A_2, B_1 and B_2 from equation (3.25) and equation (3.27) into equation (3.4), we obtain the following equations

$$\dot{a}_1 = \lambda_1a_1+\varepsilon(-\frac{(2\lambda_1-\lambda_2-\lambda_3)\lambda_1'a_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{a_1^3}{(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)}+\frac{6a_1a_2a_3}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)})$$
$$+\varepsilon^2[\frac{9}{(\lambda_1+\lambda_2+2\lambda_3)(\lambda_1+2\lambda_2+\lambda_3)}[\frac{1}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)}$$
$$+\frac{1}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1-\lambda_2+2\lambda_3)}]a_1a_2^2a_3^2+\frac{(2\lambda_1-\lambda_2-\lambda_3)\lambda_1'^2a_1}{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2} \tag{3.28}$$
$$-\frac{\lambda_1'a_1^3}{(3\lambda_1-\lambda_2)^2(3\lambda_1-\lambda_3)^2}-\frac{6\lambda_1'a_1a_2a_3}{(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)^2}-\frac{\lambda_1''a_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}]$$

$$\dot{a}_2 = \lambda_2a_2+\varepsilon(-\frac{(2\lambda_2-\lambda_1-\lambda_3)\lambda_2'a_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}+\frac{3a_1^2a_2}{(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)}+\frac{3a_2^2a_3}{2\lambda_2)(2\lambda_2+\lambda_3-\lambda_1)})$$
$$+\varepsilon^2[\frac{3a_2^3a_3^2}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(3\lambda_2+\lambda_3)(3\lambda_3+2\lambda_2-\lambda_1)}$$
$$+\frac{18a_1^2a_2^2a_3}{4\lambda_2(\lambda_1+\lambda_2)^2(\lambda_1+2\lambda_2+\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)}+\frac{(2\lambda_2-\lambda_1-\lambda_3)\lambda_2'^2a_2}{(\lambda_2-\lambda_1)^2(\lambda_2-\lambda_3)^2} \tag{3.29}$$
$$-\frac{3\lambda_2'a_1^2a_2}{(\lambda_1+\lambda_2)^2(2\lambda_1+\lambda_2-\lambda_3)^2}-\frac{3\lambda_2'a_2^2a_3}{4\lambda_2^2(2\lambda_2+\lambda_3-\lambda_1)^2}-\frac{\lambda_2''a_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}]$$

For a damped nonlinear system, substituting $\lambda_1=-\lambda(\tau),\ \lambda_{1,2}=-\mu(\tau)\pm i\omega(\tau)$ and $a_1=a,\ a_2=\frac{1}{2}be^{i\varphi},\ a_3=\frac{1}{2}be^{-i\varphi}$ into equation (3.26) and equations (3.28) - (3.29) and then simplifying them, we obtain the following equations for the amplitudes, phase variable and the correction terms as the forms

$$
\begin{aligned}
\dot{a} &= -\lambda(\tau)a + \varepsilon(l_0 a + l_1 a^3 + l_2 ab^2) + \varepsilon^2(l_3 a + l_4 a^3 + l_5 ab^2 + l_6 ab^4),\\
\dot{b} &= -\mu(\tau)b + \varepsilon(m_0 b + m_1 a^2 b + m_2 b^3) + \varepsilon^2(m_3 b + m_4 a^2 b + m_5 b^3 + m_6 a^2 b^3 + m_7 b^5),\\
\dot{\varphi} &= \omega(\tau) + \varepsilon(n_0 + n_1 a^2 + n_2 b^2) + \varepsilon^2(n_3 + n_4 a^2 + n_5 b^2 + n_6 a^2 b^2 + n_7 b^4),
\end{aligned}
\tag{3.30}
$$

and

$$
u_1 = ab^2(c_2\cos 2\varphi + d_2\sin 2\varphi) + b^3(c_3\cos 3\varphi + d_3\sin 3\varphi), \tag{3.31}
$$

where

$$
l_0 = -\frac{2(\lambda-\mu)\lambda'}{((\lambda-\mu)^2+\omega^2)^2},\quad l_1 = \frac{1}{(3\lambda-\mu)^2+\omega^2},\quad l_2 = \frac{3}{2((\lambda+\mu)^2+\omega^2)},
$$

$$
l_3 = -\frac{2(\lambda-\mu)\lambda'^2 - ((\lambda-\mu)^2+\omega^2)\lambda''}{((\lambda-\mu)^2+\omega^2)^2},\ l_4 = \frac{\lambda'}{((3\lambda-\mu)^2+\omega^2)^2},\ l_5 = \frac{3\lambda'}{2((\lambda+\mu)^2+\omega^2)^2},
$$

$$
l_6 = \frac{-9(\mu((\lambda+\mu)^2-3\omega^2)-4\omega^2(\lambda+\mu))}{16(\mu^2+\omega^2)((\lambda+\mu)^2+\omega^2)((\lambda+\mu)^2+9\omega^2)((\lambda+3\mu)^2+\omega^2)},
$$

$$
m_0 = \frac{(3\mu'\omega-\omega'(\lambda-\mu))(\lambda-\mu)-\omega(\mu'(\lambda-\mu)+3\omega\omega')}{2\omega((\lambda-\mu)^2+\omega^2)},\quad m_1 = \frac{3(\lambda(\lambda+\mu)-\omega^2)}{2(\lambda^2+\omega^2)((\lambda+\mu)^2+\omega^2)}
$$

$$
m_2 = \frac{-3(\mu(\lambda-3\mu)+\omega^2)}{8(\mu^2+\omega^2)((\lambda-3\mu)^2+\omega^2)},
$$

$$
m_3 = \frac{\begin{aligned}&-((\lambda-\mu)((\lambda-\mu)^2+5\omega^2)(\mu'^2-\omega'^2)+2\mu'\omega\omega'((\lambda-\mu)^2-3\omega^2)\\ &+2\omega(\mu''\omega+\omega''(\lambda-\mu))((\lambda+\mu)^2+\omega^2))\end{aligned}}{4\omega^2((\lambda-\mu)^2+\omega^2)^2},
$$

$$
m_4 = \frac{\begin{aligned}&3(\mu'((\lambda^2-\omega^2)((\lambda+\mu)^2-\omega^2)-4\lambda\omega^2(\lambda+\mu))+2\omega\omega'((\lambda+\mu)(\lambda^2-\omega^2)\\ &+\lambda((\lambda+\mu)^2-\omega^2)))\end{aligned}}{4(\lambda^2+\omega^2)^2((\lambda+\mu)^2+\omega^2)^2},
$$

$$
m_5 = \frac{\begin{aligned}&3(\mu'((\mu^2-\omega^2)((\lambda-3\mu)^2-\omega^2)+4\mu\omega^2(\lambda-3\mu))+2\omega\omega'(\mu((\lambda-3\mu)^2-\omega^2)\\ &-(\mu^2-\omega^2)(\lambda-3\mu)))\end{aligned}}{16(\mu^2+\omega^2)^2((\lambda-3\mu)^2+\omega^2)^2},
$$

$$
m_6 = \frac{\begin{aligned}&-9((\mu((\lambda+\mu)^2-\omega^2)-2\omega^2(\lambda+\mu))((\lambda+\mu)(\lambda+3\mu)-3\omega^2)\\ &-\omega^2(4\lambda+10\mu)((\lambda+\mu)(\lambda+3\mu)-\omega^2))\end{aligned}}{8(\mu^2+\omega^2)((\lambda+\mu)^2+\omega^2)((\lambda+3\mu)^2+\omega^2)((\lambda+\mu)^2+9\omega^2)},
$$

$$m_7 = \frac{\begin{array}{l}-3((2(\mu(\lambda-3\mu)+3\omega^2)(\mu^2-\omega^2)-5\mu\omega^2(\lambda-6\mu))(\lambda-5\mu) \\ +\omega^2(2(\lambda-6\mu)(\mu^2-\omega^2)+5\mu(\mu(\lambda-3\mu)+3\omega^2)))\end{array}}{128(\mu^2+\omega^2)(\mu^2+4\omega^2)(4\mu^2+\omega^2)((\lambda-3\mu)^2+9\omega^2)((\lambda-5\mu)^2+\omega^2)},$$

$$n_0 = -\frac{(\mu'(\lambda-\mu)+3\omega\omega')(\lambda-\mu)+\omega(3\mu'\omega-\omega'(\lambda-\mu)))}{2\omega((\lambda-\mu)^2+\omega^2)}, n_1 = \frac{3\omega(2\lambda+\mu))}{2(\lambda^2+\omega^2)((\lambda+\mu)^2+\omega^2)}$$

$$n_2 = \frac{-3\omega(\lambda-4\mu)}{8(\mu^2+\omega^2)((\lambda-3\mu)^2+\omega^2)},$$

$$n_3 = \frac{\begin{array}{l}-(\omega((\lambda-\mu)^2-3\omega^2)(\mu'^2-\omega'^2)-2\mu'\omega'(\lambda-\mu)((\lambda-\mu)^2+5\omega^2) \\ +2\omega(\mu''(\lambda-\mu)-\omega\omega'')((\lambda-\mu)^2+\omega^2))\end{array}}{4\omega^2((\lambda-\mu)^2+\omega^2)^2},$$

$$n_4 = \frac{\begin{array}{l}3(2\mu'\omega((\lambda+\mu)(\lambda^2-\omega^2)+\lambda((\lambda+\mu)^2-\omega^2)) \\ -\omega'(((\lambda+\mu)^2-\omega^2)(\lambda^2-\omega^2)-4\lambda\omega^2(\lambda+\mu)))\end{array}}{4(\lambda^2+\omega^2)^2((\lambda+\mu)^2+\omega^2)^2},$$

$$n_5 = \frac{\begin{array}{l}3(2\mu'\omega(\mu((\lambda-3\mu)^2-\omega^2)-(\mu^2-\omega^2)(\lambda-3\mu))-\omega'(\mu^2-\omega^2)((\lambda-3\mu)^2-\omega^2) \\ +4\mu\omega^2(\lambda-3\mu)))\end{array}}{16(\mu^2+\omega^2)^2((\lambda-3\mu)^2+\omega^2)^2},$$

$$n_6 = \frac{-9\omega((((\lambda+\mu)(\lambda+3\mu)-\omega^2)^2+(4\lambda+10\mu)(\mu((\lambda+\mu)^2-\omega^2)-2\omega^2(\lambda+\mu)))}{8(\mu^2+\omega^2)((\lambda+\mu)^2+\omega^2)((\lambda+3\mu)^2+\omega^2)((\lambda+\mu)^2+9\omega^2)},$$

$$n_7 = \frac{\begin{array}{l}-3\omega((2(\lambda-6\mu)(\mu^2-\omega^2)+5\mu(\mu(\lambda-3\mu)+3\omega^2))(\lambda-5\mu) \\ -2(\mu^2-\omega^2)(\mu(\lambda-3\mu)+3\omega^2)+5\mu\omega^2(\lambda-6\mu))\end{array}}{128(\mu^2+\omega^2)(\mu^2+4\omega^2)(4\mu^2+\omega^2)((\lambda-3\mu)^2+9\omega^2)((\lambda-5\mu)^2+\omega^2)}, \tag{3.32}$$

and

$$\begin{aligned}
c_2 &= \frac{3(-\mu(\lambda+\mu)^2+\omega^2(4\lambda+7\mu))}{4(\mu^2+\omega^2)((\lambda+\mu)^2+\omega^2)((\lambda+\mu)^2+9\omega^2)}, \\
d_2 &= \frac{3\omega((\lambda+\mu)(\lambda+5\mu)-3\omega^2)}{4(\mu^2+\omega^2)((\lambda+\mu)^2+\omega^2)((\lambda+\mu)^2+9\omega^2)}, \\
c_3 &= \frac{\mu^2(\lambda-3\mu)+\omega^2(-2\lambda+15\mu))}{16(\mu^2+\omega^2)(\mu^2+4\omega^2)((\lambda-3\mu)^2+9\omega^2)}, \\
d_3 &= \frac{-3\omega(\mu(\lambda-4\mu)+2\omega^2)}{16(\mu^2+\omega^2)(\mu^2+4\omega^2)((\lambda-3\mu)^2+9\omega^2)}.
\end{aligned} \tag{3.33}$$

Thus, the second approximate solution of equation (3.20) is obtained by

$$x(t,\varepsilon) = a + b\cos\varphi + \varepsilon u_1 \tag{3.34}$$

where a, b and φ are the solutions of equation (3.30) and u_1 is given by equation (3.31).

3.4 Results and Discussion

We have solved two simultaneous differential equations for the amplitude and phase variables, and a partial differential equation for u_1 involving three independent variables, amplitude and phase. We are also able to solve all the equations of A_j, B_j, $j = 1,2,3$ and u_1. In particular case, we are forced to assume that $\lambda(\tau)$, $\mu(\tau)$ are constants and $\omega(\tau) = \omega_0 e^{-h\tau}$ is varying slowly with time t, where ω_0 is constant. The amplitude and phase variables change slowly with time t. The behavior of amplitudes and phase variables characterizes the oscillating processes and they keep an important role to the nonlinear dynamical systems. The amplitudes tend to zero as $t \to \infty$ (i.e. when time is very large) in the presence of damping. Figures (3.1, 3.3, 3.5) are drawn to compare between the first approximate solutions obtained by the KBM method and those solutions obtained by the numerical procedure for several damping. We observe that the first approximate solutions show good agreement with those solutions obtained by the numerical procedure in the presence of strong linear damping with slowly varying coefficients. Also figures (3.2, 3.4, 3.8) are drawn to compare between the second approximate solutions obtained by the KBM method and those solutions obtained by the corresponding numerical solution (fourth order Runge-Kutta method) for several damping. We observe that the second approximate solutions also show good agreement with those obtained by the numerical procedure in the presence of strong linear damping with slowly varying coefficients but the analytical approximate solutions (1^{st} & 2^{nd} approximate solutions) deviate from the numerical solutions when the linear damping effect is small (Figs. 3.5, 3.6). Moreover, this method is able to give the required result when the coefficients of the given nonlinear systems become constants ($h = 0$, Figs. 3.7, 3.8). The limitation of the presented method is that it is valid only for weakly nonlinear system in the presence strong linear damping and converges rapidly to the numerical solution otherwise it deviates from the numerical solution. Most of the researchers [24, 45, 57, 72] have not discussed this limitation of the KBM [3, 4] method. According to the theory of nonlinear oscillations, higher order approximate solutions give the better results.

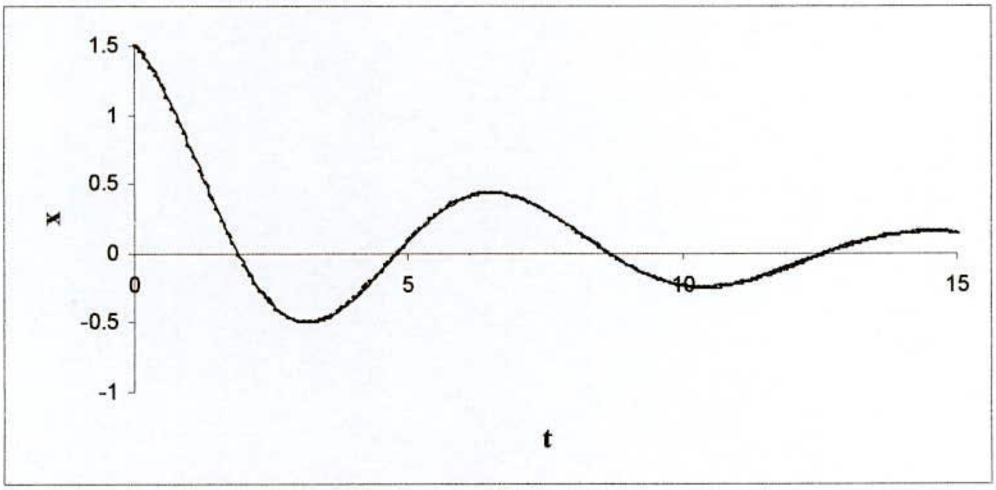

Fig.3.1 First approximate solution ($-\bullet-$ dotted lines) of equation (3.20) is compared with the corresponding numerical solution (- solid line) obtained by fourth-order Runge-Kutta method for $\lambda = 0.5$, $\mu = 0.15$, $\omega_0 = 1.0$, $h = 0.25, \varepsilon = 0.1$ and $f = x^3$ with the initial conditions $[x(0) = 1.50838, \dot{x}(0) = -0.38079, \ddot{x}(0) = -0.97857]$ or $a_0 = 0.5, b_0 = 1.0$ and $\varphi_0 = 0$.

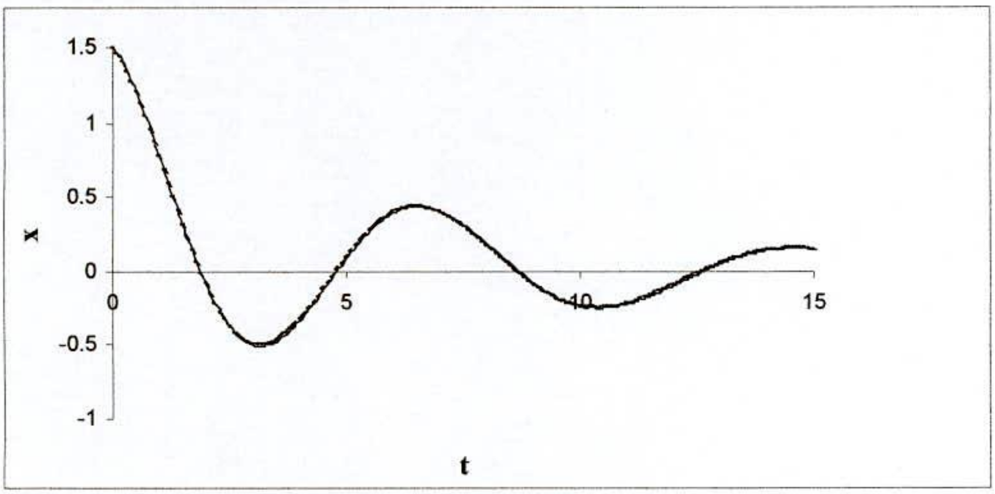

Fig.3.2 Second approximate solution ($-\bullet-$ dotted lines) of equation (3.20) is compared with the corresponding numerical solution (- solid line) obtained by fourth-order Runge-Kutta method for $\lambda = 0.5$, $\mu = 0.15$, $\omega_0 = 1.0$, $h = 0.25, \varepsilon = 0.1$ and $f = x^3$ with the initial conditions $[x(0) = 1.50838, \dot{x}(0) = -0.37974, \ddot{x}(0) = -0.98498]$ or $a_0 = 0.5, b_0 = 1.0$ and $\varphi_0 = 0$.

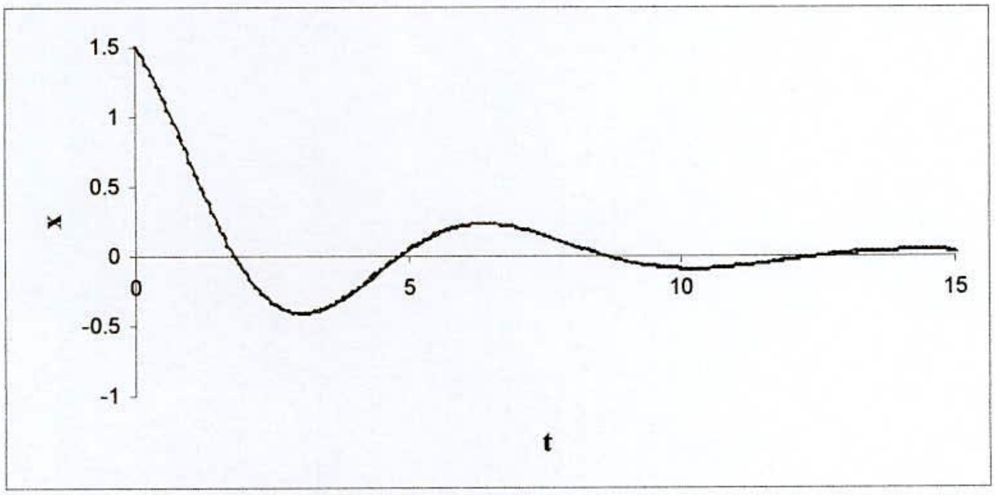

Fig.3.3 First approximate solution ($-\bullet-$ dotted lines) of equation (3.20) is compared with the corresponding numerical solution (- solid line) obtained by fourth-order Runge-Kutta method for $\lambda = 0.7$, $\mu = 0.25$, $\omega_0 = 1.0$, $h = 0.25, \varepsilon = 0.1$ and $f = x^3$ with the initial conditions $[x(0) = 1.50848, \dot{x}(0) = -0.58075, \ddot{x}(0) = -0.84169]$ or $a_0 = 0.5$, $b_0 = 1.0$ and $\varphi_0 = 0$.

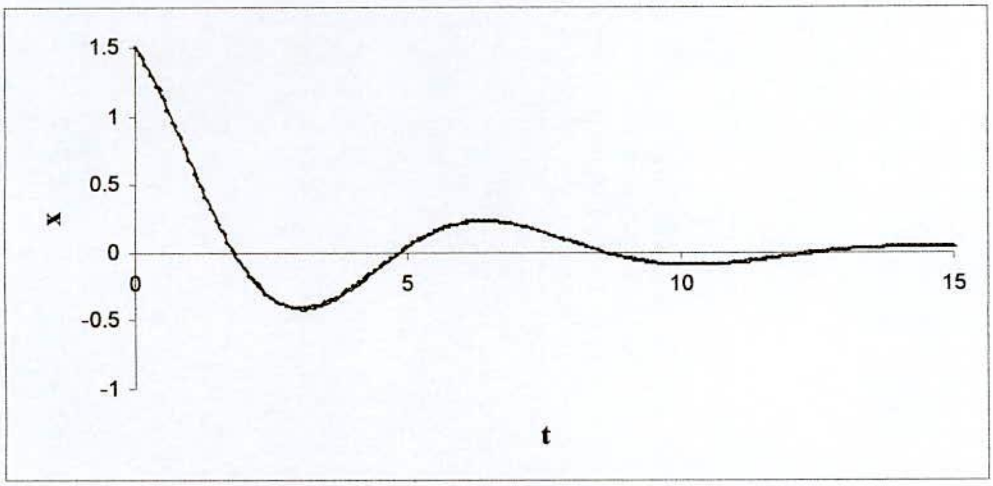

Fig.3.4 Second approximate solution ($-\bullet-$ dotted lines) of equation (3.20) is compared with the corresponding numerical solution (- solid line) obtained by fourth-order Runge-Kutta method for $\lambda = 0.7$, $\mu = 0.25$, $\omega_0 = 1.0$, $h = 0.25, \varepsilon = 0.1$ and $f = x^3$ with the initial conditions $[x(0) = 1.50838, \dot{x}(0) = -0.57933, \ddot{x}(0) = -0.84885]$ or $a_0 = 0.5$, $b_0 = 1.0$ and $\varphi_0 = 0$.

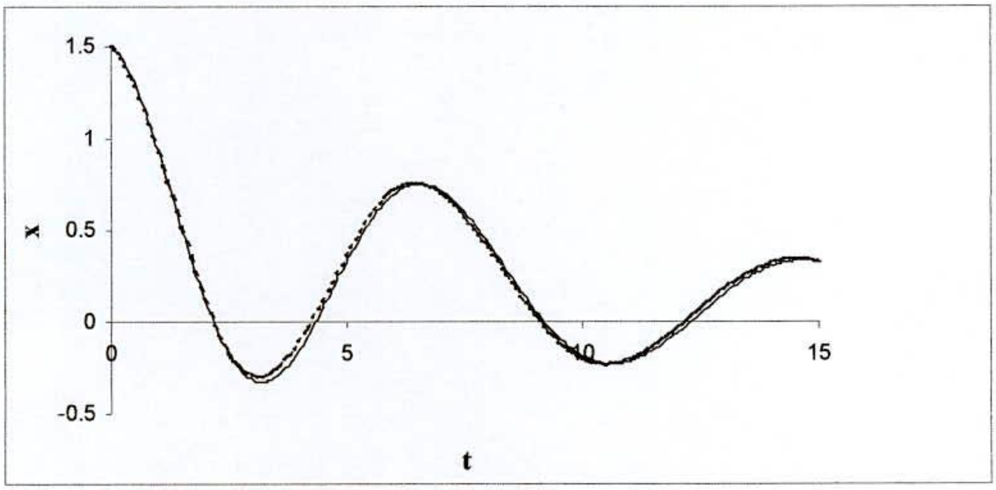

Fig.3.5 First approximate solution ($-\bullet-$ dotted lines) of equation (3.20) is compared with the corresponding numerical solution (- solid line) obtained by fourth-order Runge-Kutta method for $\lambda = 0.2$, $\mu = 0.1$, $\omega_0 = 1.0$, $h = 0.25, \varepsilon = 0.1$ and $f = x^3$ with the initial conditions $[x(0) = 1.50578,\ \dot{x}(0) = -0.17768,\ \ddot{x}(0) = -1.05083]$ or $a_0 = 0.5, b_0 = 1.0$ and $\varphi_0 = 0$.

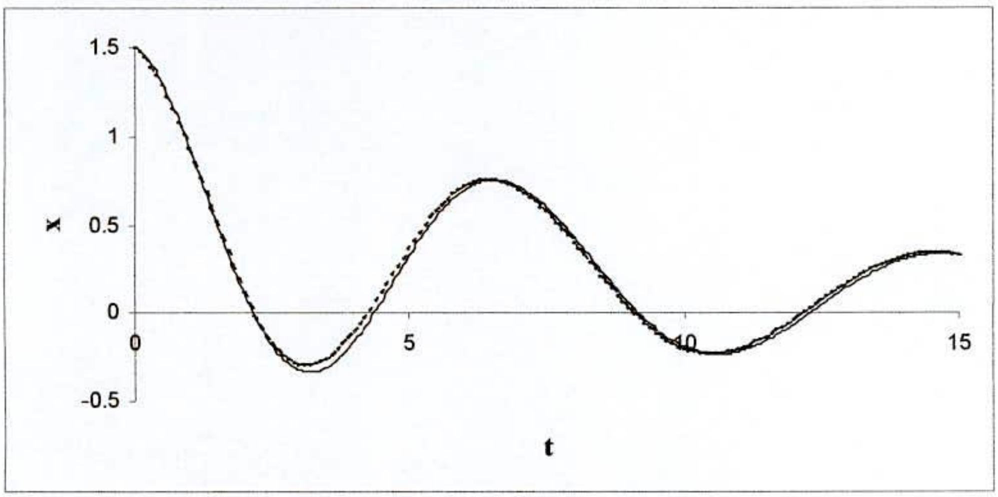

Fig.3.6 Second approximate solution ($-\bullet-$ dotted lines) of equation (3.20) is compared with the corresponding numerical solution (- solid line) obtained by fourth-order Runge-Kutta method for $\lambda = 0.2$, $\mu = 0.1$, $\omega_0 = 1.0$, $h = 0.25, \varepsilon = 0.1$ and $f = x^3$ with the initial conditions $[x(0) = 1.50578,\ \dot{x}(0) = -0.17715,\ \ddot{x}(0) = -1.05717]$ or $a_0 = 0.5, b_0 = 1.0$ and $\varphi_0 = 0$.

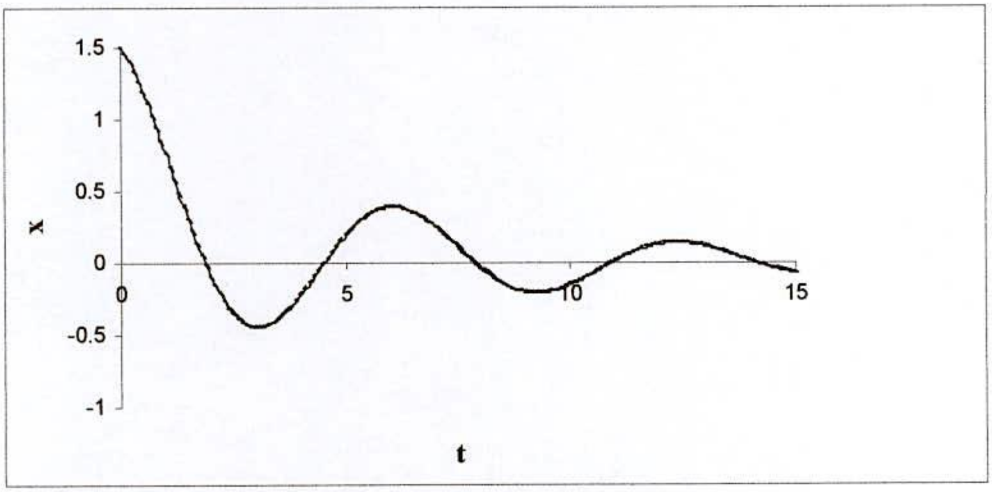

Fig.3.7 First approximate solution ($-\bullet-$ dotted lines) of equation (3.20) is compared with the corresponding numerical solution (- solid line) obtained by fourth-order Runge-Kutta method for $\lambda = 0.5$, $\mu = 0.15$, $\omega_0 = 1.0$, $h = 0., \varepsilon = 0.1$ and $f = x^3$ with the initial conditions $[x(0) = 1.50838, \dot{x}(0) = -0.41556, \ddot{x}(0) = -0.95255]$ or $a_0 = 0.5, b_0 = 1.0$ and $\varphi_0 = 0$.

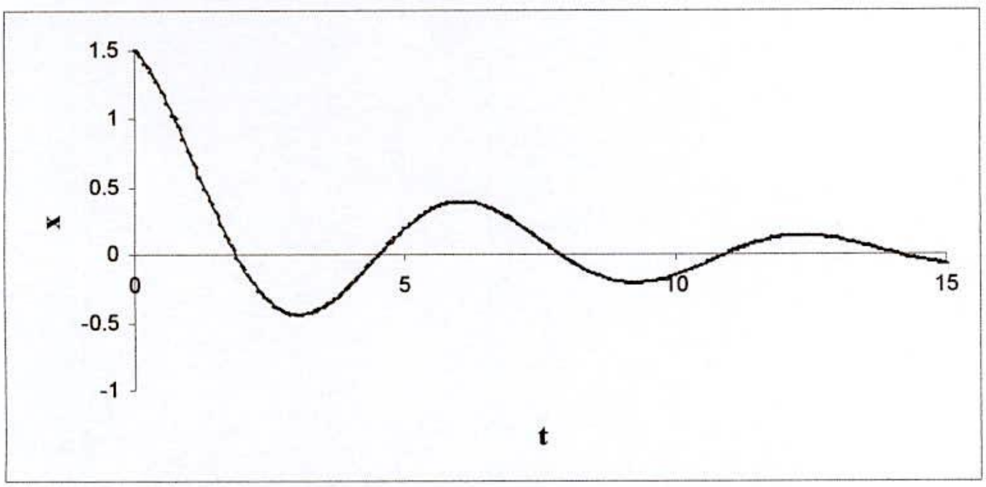

Fig.3.8 Second approximate solution ($-\bullet-$ dotted lines) of equation (3.20) is compared with the corresponding numerical solution (- solid line) obtained by fourth-order Runge-Kutta method for $\lambda = 0.5$, $\mu = 0.15$, $\omega_0 = 1.0$, $h = 0., \varepsilon = 0.1$ and $f = x^3$ with the initial conditions $[x(0) = 1.50838, \dot{x}(0) = -0.41611, \ddot{x}(0) = -0.94904]$ or $a_0 = 0.5, b_0 = 1.0$ and $\varphi_0 = 0$.

CHAPTER IV

First approximate solution of a fourth order weakly nonlinear differential in the presence of strong linear damping and slowly varying coefficients based on the KBM method

4.1 Introduction

KBM [3, 4] method is convenient and one of the widely used and popular technique to obtain the analytical approximate solutions of weakly nonlinear differential systems. It is perhaps noteworthy that because of importance of physical problems involving damping, Popov [18] have extended this method to weakly damped nonlinear oscillatory differential systems. Murty [25] has used Popov's method to obtain over-damped solutions of weakly nonlinear differential equations based on unified theory of Murty et al. [23]. Later, this method has been extended to damped oscillatory and purely non oscillatory differential systems with slowly varying coefficients by Bojadziev and Edward [24]. Arya and Bojadziev. [26] have studied a system of second order weakly nonlinear hyperbolic partial differential equation with slowly varying coefficients. Arya and Bojadziev [27] have also studied a time-dependent nonlinear oscillatory system with damping, slowly varying coefficients and delay. Alam [45] has investigated a unified KBM method for solving weakly nonlinear system of order $n \geq 3$ with slowly varying coefficients. Uddin and Sattar [55] have obtained an approximate solution of a fourth order weakly non-linear differential system with strong damping and slowly varying coefficients by unified KBM method but they have not followed strictly their imposed restrictions. Akbar et al. [62] have studied a fourth order weakly nonlinear differential equation with constant coefficients. Alam [66] has investigated a unified KBM method for solving *nth* order weakly nonlinear differential system. Alam and Sattar. [70] have also presented an asymptotic method for third order weakly nonlinear system with varying coefficients. Feshchenko et al. [72] have presented a brief way to determine KBM solution for weakly nonlinear differential systems. Most of the researchers have studied the second and third order weakly nonlinear differential systems for both constant and varying coefficients to obtain the first approximate solutions.

4.2 The method

Let us assume a fourth order weakly nonlinear ordinary differential equation with slowly varying coefficients in the following form

$$x^{(4)}+k_1(\tau)\dddot{x}+k_2(\tau)\ddot{x}+k_3(\tau)\dot{x}+k_4(\tau)x=\varepsilon f(x,\dot{x},\ddot{x},\dddot{x},\tau), \tag{4.1}$$

where the over dots represent the time derivatives, $\varepsilon << 1$ is a small positive parameter which measures the strength of the nonlinearity, $\tau=\varepsilon t$ is the slowly varying time, $k_j(\tau)\geq 0, \quad j=1, 2, 3,4$ are slowly varying coefficients and f is a given nonlinear function. The coefficients are slowly varying in the sense that their time derivatives are proportional to ε [57].

The unperturbed solution of equation (4.1) is obtained by setting $\varepsilon=0, \tau=\tau_0=$ constant. We assume that the unperturbed equation of equation (4.1) has four eigen values $\lambda_j(\tau_0), j=1, 2, 3,4$; where $\lambda_j(\tau_0)$ are constants, but if $\varepsilon \neq 0$ then $\lambda_j(\tau)$ are varying slowly with time t. The solution of the linearized equation of equation (4.1) has the following form:

$$x(t,0)=\sum_{j=1}^{4} a_{j,0}e^{\lambda_j(\tau_0)t}, \tag{4.2}$$

where $a_{j,0}$, $j=1, 2, 3,4$ are arbitrary constants.

Now we are going to choose a solution of equation (4.1) that reduces to equation (4.2) as a limit $\varepsilon \to 0$ in accordance with the KBM method in the following form:

$$x(t,\varepsilon)=\sum_{j=1}^{4} a_j(t)+\varepsilon u_1(a_1,a_2,a_3,a_4,\tau)+\varepsilon^2 u_2(a_1,a_2,a_3,a_4,\tau)+\cdots, \tag{4.3}$$

where $u_1, u_2,\cdots$ are small correction terms. Each a_j satisfies the following first order ordinary differential equation:

$$\dot{a}_j=\lambda_j a_j+\varepsilon A_j(a_1,a_2,a_3,a_4,\tau)+\varepsilon^2 B_j(a_1,a_2,a_3,a_4,\tau)+\cdots \tag{4.4}$$

Confining only to the first few terms (1, 2, 3…) in the series expansions of equation (4.3) and equation (4.4), we evaluate the functions u_1, $u_2,\cdots$ and A_j, $B_j,\cdots, j=1, 2, 3,4$ such that each $a_j(t)$ appearing in equation (4.3) and equation (4.4) satisfies the given

differential equation (4.1) with an accuracy of ε^{m+1} [66]. In order to determine these functions, it is assumed that the functions $u_1,\ u_2,\cdots$ do not contain the fundamental terms which are included in the series expansions equation (4.3) at order ε^0. Differentiating equation (4.3) four-times with respect to time t and using the relations equation (4.4) and substituting the values of $x^{(4)},\ \dddot{x},\ \ddot{x},\ \dot{x}$ and x into the original equation (4.1) with the slowly varying coefficients $k_1(\tau)=-(\lambda_1(\tau)+\lambda_2(\tau)+\lambda_3(\tau)+\lambda_4(\tau))$,
$k_2(\tau)=\lambda_1(\tau)\lambda_2(\tau)+\lambda_1(\tau)\lambda_3(\tau)+\lambda_1(\tau)\lambda_4(\tau)+\lambda_2(\tau)\lambda_3(\tau)+\lambda_2(\tau)\lambda_4(\tau)+\lambda_3(\tau)\lambda_4(\tau)$,
$k_3(\tau)=-(\lambda_1(\tau)\lambda_2(\tau)\lambda_3(\tau)+\lambda_1(\tau)\lambda_2(\tau)\lambda_4(\tau)+\lambda_1(\tau)\lambda_3(\tau)\lambda_4(\tau)+\lambda_2(\tau)\lambda_3(\tau)\lambda_4(\tau))$ and $k_4(\tau)=\lambda_1(\tau)\lambda_2(\tau)\lambda_3(\tau)\lambda_4(\tau)$ and expanding the right hand side of equation (4.1) by Taylor series and equating the coefficients of ε on both sides, we obtain the following equation:

$$\prod_{j=1}^{4}(\Delta-\lambda_j)u_1+\sum_{j=1}^{4}\Big(\prod_{k=1,k\neq j}^{4}(\Delta-\lambda_k)A_j\Big)+\sum_{j=1}^{4}\frac{1}{2}\Big(\sum_{k=0}^{2}(4-k)(3-k)c_k\,\lambda_j^{(2-k)}\Big)\lambda_j'\,a_j \\ =f^{(0)}(a_1,a_2,a_3,a_4,\tau), \tag{4.5}$$

where $\Delta=\sum_{j=1}^{4}\lambda_j a_j\dfrac{\partial}{\partial a_j},\ \lambda_j'=\dfrac{d\lambda_j}{d\tau},\ j=1,2,3,4,\ f^{(0)}(a_1,a_2,a_3,a_4,\tau)=f(x_0,\dot{x}_0,\ddot{x}_0,\dddot{x}_0,\tau)$,

and $x_0=\sum_{j=1}^{4}a_j$.

We have already assumed that u_1 does not contain the fundamental terms and for this reason the solution will be free from secular terms, namely $t\cos t,\ t\sin t$ and te^{-t}. In the presence of secular terms, the solutions will be non uniform, so we have to ignore these terms. According to these restrictions, we are able to solve equation (4.5) by separating this into five individual equations for the unknown functions u_1 and A_j. In general, the functions $f^{(0)}$ and u_1 are expanded in Taylor's series in the following forms:

$$f^{(0)}=\sum_{m_1=0,m_2=0\ldots m_n=0}^{\infty,\infty,\ldots,\infty}F_{m_1,m_2,m_3}(\tau)a_1^{m_1}a_2^{m_2}a_3^{m_3}a_4^{m_4} \tag{4.6}$$

$$u_1=\sum_{m_1=0,m_2=0,\ldots\ldots,m_n=0}^{\infty,\infty,\ldots,\infty}U_{m_1,m_2m_3}(\tau)a_1^{m_1}a_2^{m_2}a_3^{m_3}a_4^{m_4} \tag{4.7}$$

The eigen values of the unperturbed equation can be written as $-\mu_l(\tau_0)\pm i\omega_l(\tau_0)$, where $l=1,2$. For the above restrictions, it guaranties that u_1 must exclude all terms with ${a_{2l-1}}^{m_{2l-1}}{a_{2l}}^{m_{2l}}$ of $f^{(0)}$, where $m_{2l-1}-m_{2l}=\pm1$. Since according to the linear approximation $(i.e.\ \varepsilon\to 0)$, ${a_{2l-1}}^{m_{2l-1}}{a_{2l}}^{m_{2l}}$ becomes $e^{\omega_l t}$ when $m_{2l-1}-m_{2l}=1$ or $e^{-\omega_l t}$ when $m_{2l-1}-m_{2l}=-1$. It is noticed that $e^{\pm\omega_l t}$ are known as the fundamental terms [3, 4]. Usually these are included in equations A_j. Also, it is restricted (by Krylov and Bogoliubov [3]) that the functions A_j are independent of the fundamental terms.

Then the equations for u_1 and A_j, $j=1,2,3,4$ are written as

$$\prod_{j=1}^{4}(\Delta-\lambda_j)u_1=\sum_{m_{2l-1}=0,m_{2l}=0}^{\infty,\infty}F_{,m_{2l-1},\,m_{2l}}(\tau){a_{2l-1}}^{m_{2l-1}}{a_{2l}}^{m_{2l}},\quad m_{2l-1}-m_{2l}\neq 0,\ \pm1 \tag{4.8}$$

and

$$\begin{aligned}(\prod_{k=1,k\neq 2l-1}^{4}(\Delta-\lambda_k))A_{2l-1}+\frac{1}{2}(\sum_{k=0}^{2}(4-k)(3-k)c_k{\lambda_{2l-1}}^{(2l-k-2)})\lambda'_{2l-1}a_{2l-1}\\ =\sum_{m_{2l-1}=0,m_{2l}=0}^{\infty,\infty}F_{m_{2l-1}\,m_{2l}}{a_{2l-1}}^{m_{2l-1}}{a_{2l}}^{m_{2l}},\ m_{2l-1}-m_{2l}=1,\end{aligned} \tag{4.9}$$

$$\begin{aligned}(\prod_{k=1,k\neq 2l}^{4}(\Delta-\lambda_k))A_{2l}+\frac{1}{2}(\sum_{k=0}^{2}(4-k)(3-k)c_k{\lambda_{2l}}^{(2l-k-2)})\lambda'_{2l}a_{2l}\\ =\sum_{m_{2l-1}=0,m_{2l}=0}^{\infty,\infty}F_{m_{2l-1}m_{2l}}{a_{2l-1}}^{m_{2l-1}}{a_{2l}}^{m_{2l}},\ m_{2l-1}-m_{2l}=-1.\end{aligned} \tag{4.10}$$

The particular solutions of equations (4.8) - (4.10), are determined by replacing the operator Δ by $\sum_{j=1}^{4}m_j\lambda_j$, (since $\Delta({a_{2l-1}}^{m_{2l-1}}{a_{2l}}^{m_{2l+1}})=\sum_{j=1}^{n}m_j\lambda_j({a_{2l-1}}^{m_{2l-1}}{a_{2l}}^{m_{2l}})$. equation (4.3) is not a standard form of KBM [3, 4] method and is presented in terms of some unusual variables. Therefore, the solution obtained by formula of equation (4.1) is transformed to the formal form by replacing the unusual variables by amplitudes and phases in the form: $a_{2l-1}=\frac{1}{2}b_l e^{i\varphi_1}$, $\qquad a_{2l}=\pm\frac{1}{2}b_l e^{-i\varphi_2}$, $l=1,2$ (4.11)

Thus, the first order analytical approximate solution of equation (4.1) is completed. The method can be carried out to higher order approximations in a similar manner.

4.3 Example

Let us assume the following fourth order weakly nonlinear differential equation with slowly varying coefficients

$$x^{(4)} + k_1(\tau)\dddot{x} + k_2(\tau)\ddot{x} + k_3(\tau)\dot{x} + k_4(\tau)x = \varepsilon x^3, \tag{4.12}$$

where $f(x,\dot{x},\ddot{x},\dddot{x},\tau) = x^3$ and $x_0 = a_1 + a_2 + a_3 + a_4$.

Now

$$\begin{aligned} f^{(0)} &= a_1^3 + a_2^3 + a_3^3 + a_4^3 + 3(a_1^2 a_2 + a_1 a_2^2 + a_1^2 a_3 + 2a_1 a_2 a_3 + a_2^2 a_3 \\ &\quad + a_1^2 a_4 + 2a_1 a_2 a_4 + a_2^2 a_4 + a_1 a_3^2 + 2a_1 a_3 a_4 + a_1 a_4^2 \\ &\quad + a_2 a_3^2 + 2a_2 a_3 a_4 + a_2 a_4^2 + a_3^2 a_4 + a_3 a_4^2). \end{aligned} \tag{4.13}$$

Putting the values of $f^{(0)}$ in equation (4.5) and imposing restrictions [equations (4.8)-(4.10)], we obtain five equations for A_1, A_2, A_3, A_4 and u_1 whose solutions are obtained as follows:

$$\begin{aligned} A_1 &= -\frac{(3\lambda_1^2 - 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3 - 2\lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4)\lambda_1' a_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)} \\ &\quad + \frac{3a_1^2 a_2}{2\lambda_1(2\lambda_1 + \lambda_2 - \lambda_3)(2\lambda_1 + \lambda_2 - \lambda_4)} + \frac{6a_1 a_3 a_4}{(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_4)(\lambda_1 + \lambda_3 + \lambda_4 - \lambda_2)}, \\ A_2 &= -\frac{(3\lambda_2^2 - 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 - 2\lambda_2\lambda_4 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_3\lambda_4)\lambda_2' a_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)} \\ &\quad + \frac{3a_1 a_2^2}{2\lambda_2(\lambda_1 + 2\lambda_2 - \lambda_3)(\lambda_1 + 2\lambda_2 - \lambda_4)} + \frac{6a_2 a_3 a_4}{(\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4)(\lambda_2 + \lambda_3 + \lambda_4 - \lambda_1)}, \\ A_3 &= -\frac{(3\lambda_3^2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3 - 2\lambda_3\lambda_4 + \lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4)\lambda_3' a_3}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_4)} \\ &\quad + \frac{3a_3^2 a_4}{2\lambda_3(2\lambda_3 + \lambda_4 - \lambda_1)(2\lambda_3 + \lambda_4 - \lambda_2)} + \frac{6a_1 a_3 a_4}{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4)}, \\ A_4 &= -\frac{(3\lambda_4^2 - 2\lambda_1\lambda_4 - 2\lambda_2\lambda_4 - 2\lambda_3\lambda_4 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda_4' a_4}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)} \\ &\quad + \frac{3a_3 a_4^2}{2\lambda_4(\lambda_3 + 2\lambda_4 - \lambda_1)(\lambda_3 + 2\lambda_4 - \lambda_2)} + \frac{6a_1 a_2 a_4}{(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_4)(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)}, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} u_1 &= C_1 a_1^3 + C_1^* a_2^3 + C_2 a_3^3 + C_2^* a_4^3 + C_3 a_1 a_3^2 + C_3^* a_2 a_4^2 + C_4 a_1 a_4^2 + C_4^* a_2 a_3^2 \\ &\quad + C_5 a_1^2 a_3 + C_5^* a_2^2 a_3 + C_6 a_1^2 a_4 + C_5^* a_2^2 a_4, \end{aligned} \tag{4.15}$$

where

$$C_1 = \frac{1}{2\lambda_1(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)},$$

$$C_1^* = \frac{1}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(3\lambda_2-\lambda_3)},$$

$$C_2 = \frac{1}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(3\lambda_3-\lambda_4)},$$

$$C_2^* = \frac{1}{2\lambda_4(3\lambda_4-\lambda_1)(3\lambda_4-\lambda_2)(3\lambda_4-\lambda_3)}$$

$$C_3 = \frac{3}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(\lambda_1+2\lambda_3-\lambda_4)},$$

$$C_3^* = \frac{3}{2\lambda_4(\lambda_2+\lambda_4)(\lambda_2+2\lambda_4-\lambda_1)(\lambda_2+2\lambda_4-\lambda_3)},$$

$$C_4 = \frac{3}{2\lambda_4(\lambda_1+\lambda_4)(\lambda_1+2\lambda_4-\lambda_2)(\lambda_1+2\lambda_4-\lambda_3)},$$

$$C_4^* = \frac{3}{2\lambda_3(\lambda_2+\lambda_3)(\lambda_2+2\lambda_3-\lambda_1)(\lambda_2+2\lambda_3-\lambda_4)},$$

$$C_5 = \frac{3}{2\lambda_1(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3-\lambda_2)(2\lambda_1+\lambda_3-\lambda_4)},$$

$$C_5^* = \frac{3}{2\lambda_2(\lambda_2+\lambda_3)(2\lambda_2+\lambda_3-\lambda_1)(2\lambda_2+\lambda_3-\lambda_4)}, \qquad (4.16)$$

$$C_6 = \frac{3}{2\lambda_1(\lambda_1+\lambda_4)(2\lambda_1+\lambda_4-\lambda_2)(2\lambda_1+\lambda_4-\lambda_3)}$$

$$C_5^* = \frac{3}{2\lambda_2(\lambda_2+\lambda_4)(2\lambda_2+\lambda_4-\lambda_1)(2\lambda_2+\lambda_4-\lambda_3)}.$$

Inserting the values of A_1, A_2, A_3 and A_4 from equation (4.15) into equation (4.4), we obtain

$$\dot{a}_1 = \lambda_1 a_1 + \varepsilon\Big(-\frac{(3\lambda_1^2 - 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3 - 2\lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4)\lambda_1' a_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}$$
$$+\frac{3a_1^2 a_2}{2\lambda_1(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1+\lambda_2-\lambda_4)} + \frac{6a_1 a_3 a_4}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_4)(\lambda_1+\lambda_3+\lambda_4-\lambda_2)}\Big),$$

$$\dot{a}_2 = \lambda_2 a_2 + \varepsilon\Big(-\frac{(3\lambda_2^2 - 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 - 2\lambda_2\lambda_4 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_3\lambda_4)\lambda_2' a_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}$$
$$+\frac{3a_1 a_2^2}{2\lambda_2(\lambda_1+2\lambda_2-\lambda_3)(\lambda_1+2\lambda_2-\lambda_4)} + \frac{6a_2 a_3 a_4}{(\lambda_2+\lambda_3)(\lambda_2+\lambda_4)(\lambda_2+\lambda_3+\lambda_4-\lambda_1)}\Big),$$

$$\dot{a}_3 = \lambda_3 a_3 + \varepsilon\Big(-\frac{(3\lambda_3^2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3 - 2\lambda_3\lambda_4 + \lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4)\lambda_3' a_3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}$$
$$+\frac{3a_3^2 a_4}{2\lambda_3(2\lambda_3+\lambda_4-\lambda_1)(2\lambda_3+\lambda_4-\lambda_2)} + \frac{6a_1 a_3 a_4}{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}\Big),$$
$$\dot{a}_4 = \lambda_4 a_4 + \varepsilon\Big(-\frac{(3\lambda_4^2 - 2\lambda_1\lambda_4 - 2\lambda_2\lambda_4 - 2\lambda_3\lambda_4 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda_4' a_4}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}$$
$$+\frac{3a_3 a_4^2}{2\lambda_4(\lambda_3+2\lambda_4-\lambda_1)(\lambda_3+2\lambda_4-\lambda_2)} + \frac{6a_1 a_2 a_4}{(\lambda_1+\lambda_4)(\lambda_2+\lambda_4)(\lambda_1+\lambda_2+\lambda_4-\lambda_3)}\Big). \tag{4.17}$$

For a damped solution of equation (4.12), substituting $\lambda_{1,2} = -\mu_1(\tau) \pm i\,\omega_1(\tau)$, $\lambda_{3,4} = -\mu_2(\tau) \pm i\,\omega_2(\tau)$ and using the transformation equations $a_1 = \frac{1}{2}a\,e^{i\varphi_1}, a_2 = \frac{1}{2}a\,e^{-i\varphi_1}$, $a_3 = \frac{1}{2}b\,e^{i\varphi_2}$, $a_4 = \frac{1}{2}b\,e^{-i\varphi_2}$ into equation (4.17) and then simplifying and separating the real and imaginary parts, we obtain the following variational equations for the amplitudes and phase variables and these forms are very important for any physical system as the systems are characterized by amplitude and phase variables:

$$\begin{aligned}
\dot{a} &= -\mu_1 a + \varepsilon(l_1 a + l_2 a^3 + l_3 a b^2),\\
\dot{\varphi}_1 &= \omega_1(\tau) + \varepsilon(m_1 + m_2 a^2 + m_3 b^2),\\
\dot{b} &= -\mu_2 b + \varepsilon(p_1 b + p_2 a^2 b + p_3 b^3),\\
\dot{\varphi}_2 &= \omega_2(\tau) + \varepsilon(q_1 + q_2 a^2 + q_3 b^2),
\end{aligned} \tag{4.18}$$

and the first correction term u_1 is obtained as

$$u_1 = a^3(c_1\cos 3\varphi_1 + d_1\sin 3\varphi_1) + b^3(c_2\cos 3\varphi_2 + d_2\sin 3\varphi_2) + \cdots, \tag{4.19}$$

where

$$l_1 = -\frac{\begin{array}{l}(\omega_1'(((\mu_1-\mu_2)^2 - 5\omega_1^2 + \omega_2^2)((\mu_1-\mu_2)^2 - \omega_1^2 + \omega_2^2) + 12\omega_1^2(\mu_1-\mu_2)^2)\\ + 4\mu_1'\omega_1(\mu_1-\mu_2)((\mu_1-\mu_2)^2 + \omega_1^2 + \omega_2^2))\end{array}}{2\omega_1((\mu_1-\mu_2)^2 + (\omega_1-\omega_2)^2)((\mu_1-\mu_2)^2 + (\omega_1+\omega_2)^2)},$$

$$m_1 = \frac{\begin{array}{l}4\omega_1\omega_1'(\mu_1-\mu_2)((\mu_1-\mu_2)^2 + \omega_1^2 + \omega_2^2) - \mu_1'(((\mu_1-\mu_2)^2 - 5\omega_1^2 + \omega_2^2)((\mu_1-\mu_2)^2\\ - \omega_1^2 + \omega_2^2) + 12\omega_1^2(\mu_1-\mu_2)^2)\end{array}}{2\omega_1((\mu_1-\mu_2)^2 + (\omega_1-\omega_2)^2)((\mu_1-\mu_2)^2 + (\omega_1+\omega_2)^2)}$$

$$l_2 = -\frac{3(\mu_1((3\mu_1-\mu_2)^2 - \omega_1^2 + \omega_2^2) - 2\omega_1^2(3\mu_1-\mu_2))}{8(\mu_1^2 + \omega_1^2)((3\mu_1-\mu_2)^2 + (\omega_1-\omega_2)^2)((3\mu_1-\mu_2)^2 + (\omega_1+\omega_2)^2)},$$

$$m_2 = -\frac{3\omega_1((3\mu_1-\mu_2)(5\mu_1-\mu_2)-\omega_1^2+\omega_2^2)}{8(\mu_1^2+\omega_1^2)((3\mu_1-\mu_2)^2+(\omega_1-\omega_2)^2)((3\mu_1-\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$l_3 = -\frac{3(\mu_2((\mu_1+\mu_2)^2-\omega_1^2+\omega_2^2)-2\omega_1^2(\mu_1+\mu_2))}{4(\mu_2^2+\omega_1^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$m_3 = -\frac{3\omega_1((\mu_1+\mu_2)(\mu_1+3\mu_2)-\omega_1^2+\omega_2^2)}{4(\mu_2^2+\omega_1^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$p_1 = -\frac{\begin{array}{l}(\omega_2'(((\mu_1-\mu_2)^2+\omega_1^2-\omega_2^2)((\mu_1-\mu_2)^2+\omega_1^2-5\omega_2^2)+12\omega_2^2(\mu_1-\mu_2)^2)\\ -4\mu_2'\omega_2(\mu_1-\mu_2)((\mu_1-\mu_2)^2+\omega_1^2+\omega_2^2))\end{array}}{2\omega_2((\mu_1-\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1-\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$q_1 = -\frac{\begin{array}{l}(4\omega_2\omega_2'(\mu_1-\mu_2)((\mu_1-\mu_2)^2+\omega_1^2+\omega_2^2)+\mu_2'(((\mu_1-\mu_2)^2+\omega_1^2-\omega_2^2)((\mu_1-\mu_2)^2\\ +\omega_1^2-5\omega_2^2)+12\omega_2^2(\mu_1-\mu_2)^2)\end{array}}{2\omega_2((\mu_1-\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1-\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$p_2 = -\frac{3(\mu_1((\mu_1+\mu_2)^2+\omega_1^2-\omega_2^2)-2\omega_2^2(\mu_1+\mu_2))}{4(\mu_1^2+\omega_2^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)},$$

$$q_2 = -\frac{3\omega_2((\mu_1+\mu_2)(3\mu_1+\mu_2)+\omega_1^2-\omega_2^2)}{4(\mu_1^2+\omega_2^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)},$$

$$p_3 = -\frac{3(\mu_2((\mu_1-3\mu_2)^2+\omega_1^2-\omega_2^2)+2\omega_2^2(\mu_1-3\mu_2))}{8(\mu_2^2+\omega_2^2)((\mu_1-3\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1-3\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$q_3 = -\frac{3\omega_2((\mu_1-3\mu_2)(\mu_1-5\mu_2)+\omega_1^2-\omega_2^2)}{8(\mu_2^2+\omega_2^2)((\mu_1-3\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1-3\mu_2)^2+(\omega_1+\omega_2)^2)}, \quad (4.20)$$

and

$$c_1 = \frac{(\mu_1^2-2\omega_1^2)((3\mu_1-\mu_2)^2-9\omega_1^2+\omega_2^2)-18\mu_1\omega_1^2(3\mu_1-\mu_2)}{16(\mu_1^2+\omega_1^2)(\mu_1^2+4\omega_1^2)((3\mu_1-\mu_2)^2+(3\omega_1-\omega_2)^2)((3\mu_1-\mu_2)^2+(3\omega_1+\omega_2)^2)},$$

$$d_1 = -\frac{3\omega_1(\mu_1((3\mu_1-\mu_2)^2-9\omega_1^2+\omega_2^2)+2(3\mu_1-\mu_2)(\mu_1^2-2\omega_1^2))}{16(\mu_1^2+\omega_1^2)(\mu_1^2+4\omega_1^2)((3\mu_1-\mu_2)^2+(3\omega_1-\omega_2)^2)((3\mu_1-\mu_2)^2+(3\omega_1+\omega_2)^2)}, \quad (4.21)$$

$$c_2 = \frac{(\mu_2^2-2\omega_2^2)((\mu_1-3\mu_2)^2+\omega_1^2-9\omega_2^2)+18\mu_2\omega_2^2(\mu_1-3\mu_2)}{16(\mu_2^2+\omega_2^2)(\mu_2^2+4\omega_2^2)((\mu_1-3\mu_2)^2+(\omega_1-3\omega_2)^2)((\mu_1-3\mu_2)^2+(\omega_1+3\omega_2)^2)},$$

$$d_2 = -\frac{3\omega_2(\mu_2((\mu_1-3\mu_2)^2+\omega_1^2-9\omega_2^2)-2(\mu_1-3\mu_2)(\mu_2^2-2\omega_2^2))}{16(\mu_2^2+\omega_2^2)(\mu_2^2+4\omega_2^2)((\mu_1-3\mu_2)^2+(\omega_1-3\omega_2)^2)((\mu_1-3\mu_2)^2+(\omega_1+3\omega_2)^2)}$$

....

Thus, the first order analytical approximate solution of equation (4.12) is obtained by

$$x(t,\varepsilon) = a\cos\varphi_1 + b\cos\varphi_2 + \varepsilon\, u_1, \quad (4.22)$$

where the amplitudes a, b and phases φ_1, φ_2 are the solutions of equation (4.18) and u_1 is given by equation (4.19).

4.4 Results and Discussion

Based on the KBM method, first order analytical approximate solution is obtained for fourth order weakly nonlinear differential equations in the presence of strong linear damping and slowly varying coefficients. We have solved four simultaneous differential equations for amplitude and phase variables and a partial differential equation for u_1 involving four independent variables of amplitudes and phases. Also we are able to solve all the equations of A_j, $j = 1, 2, 3, 4$ and u_1 by a unified formula. In a particular case, we are forced to assume that $\mu_l(\tau), l = 1, 2$ are constants and $\omega_1(\tau) = \omega_0 e^{-h\tau}$ and $\omega_2(\tau) = 2\omega_1(\tau)$ are varying slowly with time t, where ω_0 and h are constants. Figs.4.1-4.4 are plotted to compare between the first approximate solutions obtained by the KBM method and those obtained by the numerical procedure for several damping effects since the graphical representation is very important to visualize the physical systems. Moreover, this method is also able to give the desired results when the coefficients of the given nonlinear differential equation become constants $(h = 0)$. It is also noticed that the presented method is valid only in the presence of strong linear damping and deviate from the numerical solutions in the presence of small linear damping. From the Figs. (4.1)-(4.2), it is seen that the first approximate solutions show a good agreement with the corresponding numerical solutions obtained by the fourth order Runge-Kutta method in the presence of strong linear damping effects and it is deviated from the numerical solutions in the presence of small linear damping effects (Figs. 4.3-4.4). Also it is mentioned that u_1 is small correction term, so we can ignore this term as it has no appreciable effect on the solution. We have observed that the researchers [45, 57, 66, 71] have not discussed the limitation of their presented methods which is found in our study.

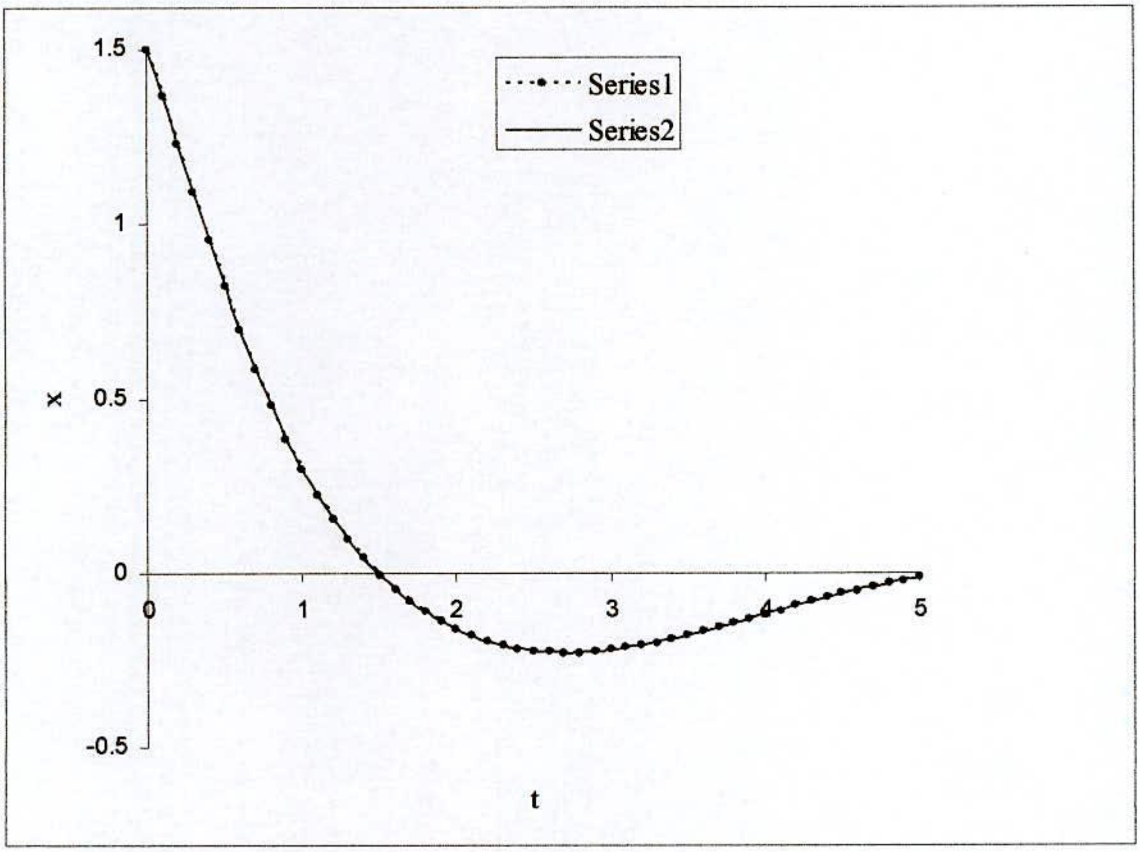

Fig.4.1 First approximate solution (denoted by $-\bullet-$) of equation (4.12), with the initial conditions $[x(0)=1.49992,\ \dot{x}(0)=-1.43063,\ \ddot{x}(0)=-1.44426,\ \dddot{x}(0)=8.77149]$ or $a_0=0.5,\ \varphi_1=0,\ b=1.0,\ \varphi_2=0, \mu_1=1.5,\ \mu_2=0.75,\ \omega_0=1.0,\ h=0.5,\ \varepsilon=0.1, \omega_1=\omega_0 e^{-h\tau}$, $\omega_2=2\omega_1, \tau=\varepsilon t$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

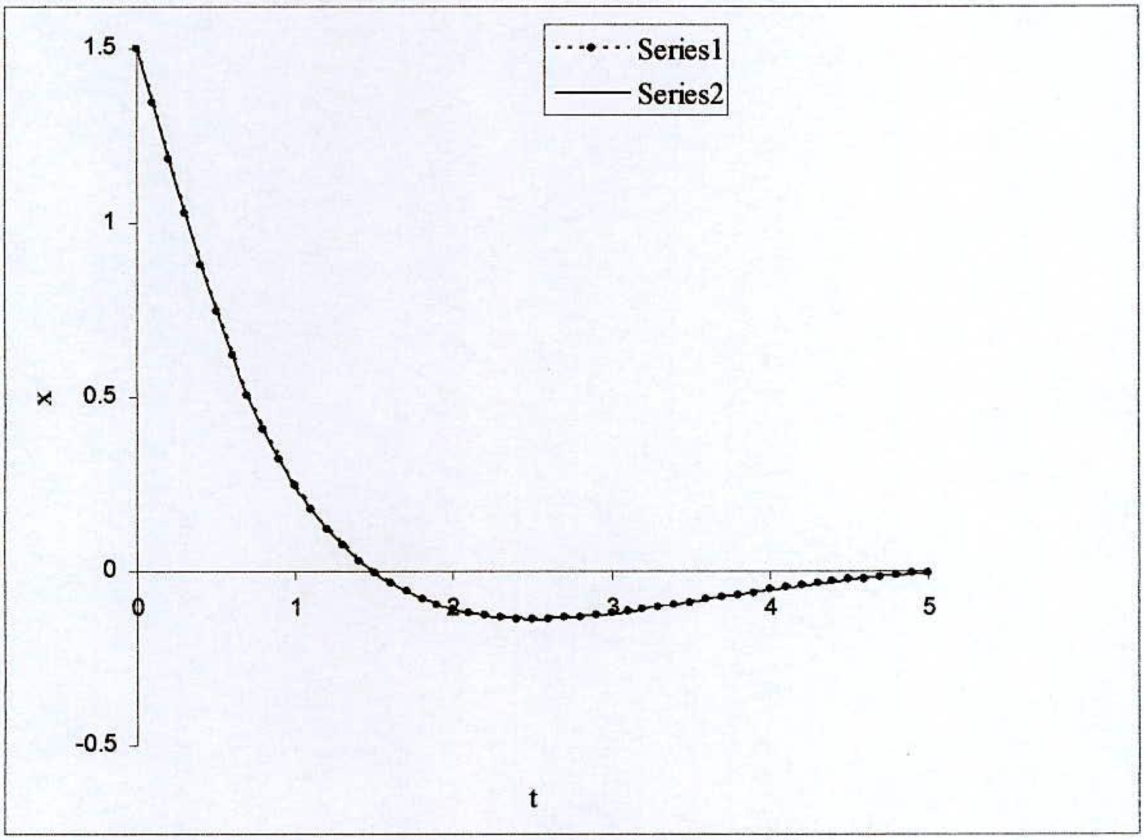

Fig.4.2 First approximate solution (denoted by $-\bullet-$) of equation (4.12), with the initial conditions $[x(0)=1.50000,\ \dot{x}(0)=-1.55838,\ \ddot{x}(0)=-1.02503,\ \dddot{x}(0)=9.27385]$ or $a_0=0.5,\ \varphi_1=0,\ b=1.0,\ \varphi_2=0, \mu_1=1.75,\ \mu_2=0.75,\ \omega_0=1.0,\ h=0.5,\ \varepsilon=0.1, \omega_1=\omega_0 e^{-h\tau}$, $,\omega_2=2\omega_1, \tau=\varepsilon t$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

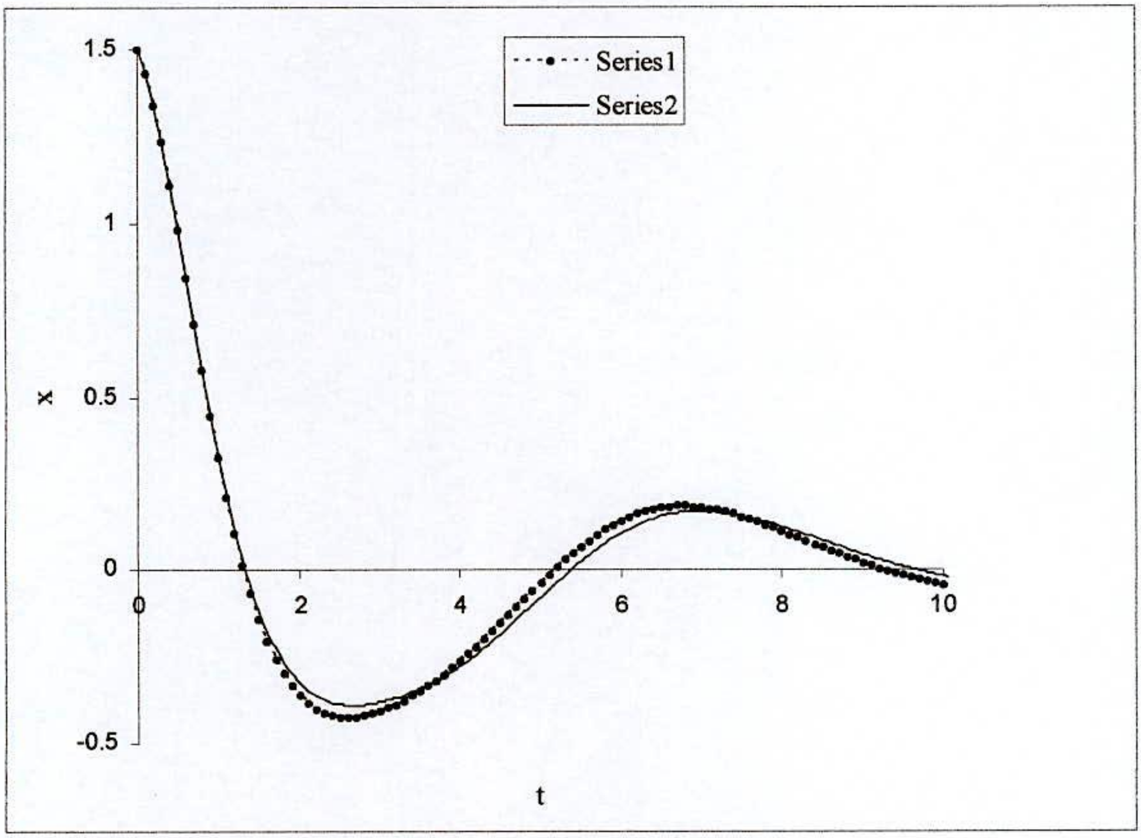

Fig.4.3 First approximate solution (denoted by $-\bullet-$) of equation (4.12), with the initial conditions $[x(0)=1.50056, \dot{x}(0)=-0.53420, \ddot{x}(0)=-2.71155, \dddot{x}(0)=4.55716]$ or $a_0=0.5,\ \varphi_1=0, b=1.0, \varphi_2=0,\ \mu_1=0.75,\ \mu_2=0.5,\ \omega_0=1.0,\ h=0.5,\ \varepsilon=0.1, \omega_1=\omega_0 e^{-h\tau}$, $\omega_2=2\omega_1,\ \tau=\varepsilon t$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

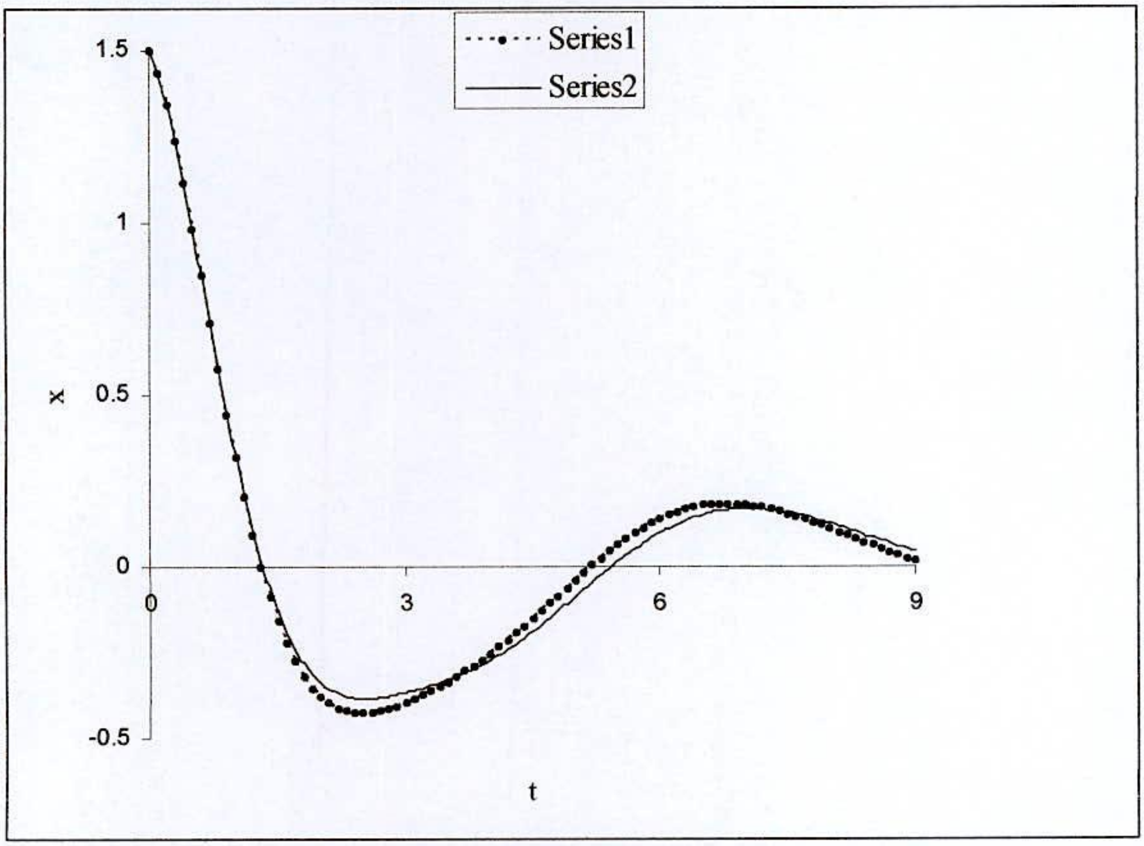

Fig.4.4 First approximate solution (denoted by $-\bullet-$) of equation (4.12), with the initial conditions $[x(0)=1.50055,\ \dot{x}(0)=-0.52839,\ \ddot{x}(0)=-2.75067,\ \dddot{x}(0)=4.30002]$ or $a_0=0.5,\ \varphi_1=0,\ b=1.0,\ \varphi_2=0,\ \mu_1=0.7,\ \mu_2=0.25,\ \omega_0=1.0,\ h=0.5,\ \varepsilon=0.1,\ \omega_1=\omega_0 e^{-h\tau}$, $\omega_2=2\omega_1, \tau=\varepsilon t$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

CHAPTER V

CONCLUSIONS

Usually, it is so much difficult to formulate the KBM method for obtaining the higher order approximate solutions of a third order nonlinear differential systems. We have presented a general formula for the second approximate solutions by the KBM method for obtaining the transient's response of a third order nonlinear differential systems with slowly varying coefficients in the presence of strong linear damping. We have also presented the KBM method for solving fourth order weakly nonlinear differential systems in the presence of strong linear damping and slowly varying coefficients. Also we have followed the impose restriction strictly. From the figures, it is seen that the analytical approximate solutions in the presence of strong linear damping obtained by the KBM method are shown good agreement with those numerical solutions obtained by the fourth order Range-Kutta method. It is also noticed that the analytical approximate solutions deviate from the numerical solutions in the presence of small linear damping obtained the presented method.

The determination of amplitudes and phases variables is very important in all physical problems. The amplitudes and phases characterize the oscillating processes. Moreover, the variational equations of amplitudes and phases are important to investigate the stability of differential systems. In general, the variational equations for the amplitudes and phases are solved numerically. In this case, the KBM method facilitates the numerical method and it also requires the numerical calculation of a few numbers of points. On the contrary, a direct attempts dealing with some harmonic terms requires the numerical calculation of a great number of points.

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