Dufour Effect on Combined Heat and Mass Transfer by Laminar Mixed Convection Flow from a Vertical Surface under the Influence of an Induced Magnetic Field

by

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Declaration

This is to certify that the thesis work entitled "Dufour Effect on Combined Heat and Mass Transfer by Laminar Mixed Convection Flow from a Vertical Surface under the Influence of an Induced Magnetic Field" has been carried out by Mst. Moslema Khatun in the Department of Mathematics, Khulna university of Engineering & Technology, Khulna, Bangladesh. The above thesis work or any part of this work has not been submitted anywhere for the award of any degree or diploma.

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Dedication

To my respectable late Father and Mother whose constant guidance and inspirations helped me to choose the correct path of life.

&

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Abstract

In this study the laminar mixed free and force convection flow and heat transfer of viscous incompressible electrically conducting fluid above a vertical porous continuously moving surface is considered under the action of a transverse applied magnetic field. The Dufour or diffusion-thermo effect in the presence of induced magnetic filed is taken into account. The governing differential equations relevant to the problem are solved by using the perturbation technique. On introducing the non-dimensional concept and initiating the idea of usual Boussinesq's approximation, the solutions for velocity field, temperature distribution, induced magnetic field and current density are obtained under certain assumptions. The influences of various establish dimensionless parameters on the velocity and temperature profiles, induced magnetic fields as well as on the shear stress are studied graphically. The numerical results have also shown that the diffusion-thermo (Dufour) effect has a great influence in the study of flow and heat transfer process of some types of fluids considered.

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Contents

LIST OF FIGURES

CHAPTER 1

Available Information Regarding MHD Heat and Mass Transfer Flows

In this chapter we have discussed some fundamental topics related to the problems of solving the equations of the fluid mechanics, Magnetohydrodynamics (MHD), heat and mass transfer process viz. fundamental equations of fluid dynamics, MHD approximations, MHD equations, dimensionless parameters, free-forced convections, heat and mass transfer flows, suction etc., which are of interest of this investigation.

1.1 Magnetohydrodynamics (MHD)

Magnetohydrodynamics is that branch of continuum mechanics which deals with the flow of electrically conducting fluids in presence of electric and magnetic fields. Many natural phenomena and engineering problems are susceptible to MHD analysis.

Faraday [23] carried out experiments with the flow of mercury in glass tubes placed between poles of a magnet, and discovered that a voltage was induced across the tube due to the motion of the mercury across the magnetic field, perpendicular to the direction of flow and to the magnetic field. He observed that the current generated by this induced voltage interacted with the magnetic field to slow down the motion of the fluid, and this current produced its own magnetic field that obeyed Ampere's right hand rule and thus, in turn distorted the magnetic field.

The first astronomical application of the MHD theory occurred in 1899 when Bigalow suggested that the sun was a gigantic magnetic system. Alfaven [12] discovered MHD waves in the sun. These waves are produced by disturbance that propagates simultaneously in the conducting fluid and the magnetic field. The largest interests on MHD in the fluid of aerodynamics have been presented by Rossow [57] for incompressible fluid of constant property flat plate boundary layer flow. His results indicated that the skin frictions and the heat transfer were reduced substantially whiten a transverse magnetic field way applied to the fluid.

The current trend for the application of magneto fluid dynamics is toward a strong magnetic field (so that the influence of the electromagnetic force is noticeable) and toward a low density of the gas (such as in space flight and in nuclear fusion research). Under these conditions the Hall current and ion slip current become important.

1.2 Electromagnetic Equations

Magnetohydrodynamic equations are the ordinary electromagnetic and hydrodynamic equations which have been modified to take account of the interaction between the motion of the fluid and electromagnetic field. The basic laws of electromagnetic theory are all contained in special theory of relativity. But it is always assumed that all velocities are small in comparison to the speed of light.

Before writing down the MHD equations we will first of all notice the ordinary electromagnetic equations and hydromagnetic equations (Cramer and Pai, [18]). The mathematical formulation of the electromagnetic theory is known as Maxwell's equations which explore the relation of basic field quantities and their production. The Maxwell's electromagnetic equations are given by

Charge continuity $$\nabla . \boldsymbol{D} = \rho_c \qquad (1.1)$$

Current continuity $$\nabla . \boldsymbol{J} = -\frac{\partial \rho_c}{\partial t} \qquad (1.2)$$

Magnetic field continuity $$\nabla . \boldsymbol{B} = 0 \qquad (1.3)$$

Ampere's law $$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial \mathbf{t}} \qquad (1.4)$$

Faraday's law $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial \mathbf{t}} \qquad (1.5)$$

Constitutive equations for **D** and **B** $$\mathbf{D} = \varepsilon' \mathbf{E} \qquad (1.6)$$

$$\mathbf{B} = \mu_{\mathrm{c}} \mathbf{H} \qquad (1.7)$$

Lorentz force on a change $$\mathbf{F}_{\mathrm{p}} = \mathbf{q}'(\mathbf{E} + \mathbf{q}_{\mathrm{p}} \times \mathbf{B}) \qquad (1.8)$$

Total current density flow $$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{q} \times \mathbf{B}) + \rho_{\mathrm{c}} \mathbf{q} \qquad (1.9)$$

In equations (1.1) – (1.9), **D** is the displacement current, ρ_c is the charge density, **J** is the current density, **B** is the magnetic induction, **H** is the magnetic field, **E** is the electric field, ε' is the electrical permeability of the medium, μ_e is the magnetic permeability of medium, $\boldsymbol{q}_p$ velocity of the charge, σ is the electrical conductivity, **q** is the velocity of the fluid and $\rho_c \mathbf{q}$ is the convection current due to charges moving with the fluid.

1.3 Fundamental Equations of Fluid Dynamics of Viscous Fluids

In the study of fluid flow one determines the velocity distribution as well as the states of the fluid over the whole space for all time. There are six unknowns namely, the three components (u, v, w) of velocity $\mathbf{q}$, the temperature T, the pressure p and the density ρ of the fluid, which are function of spatial co-ordinates and time. In order to determine these unknown we have the following equations:

(a) Equation of state, which connects the temperature, the pressure and the density of the fluid.

$$p = \rho RT \tag{1.10}$$

For an incompressible fluid the equation of state simply

$$\rho = \text{constant} \tag{1.11}$$

(b) Equation of continuity, which gives relation of conservation of mass of the fluid. The equation of continuity for a viscous incompressible fluid is

$$\nabla.\mathbf{q} = 0 \tag{1.12}$$

(c) Equation of motion, also known as the Navier-Stokes equations, which give the relations of the conservation of momentum of the fluid. For a viscous incompressible fluid the equation of motion is

$$\rho \frac{D\mathrm{q}}{Dt} = \mathrm{F} - \nabla p + \mu \nabla^2 \mathrm{q} \tag{1.13}$$

where **F** is the body force per unit volume and the last term on the right hand side represents the force per unit volume due to viscous stresses and p is the pressure. The operator,

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

is known as the material derivative or total derivative with respect to time which gives the variation of a certain quantity of the fluid particle with respect to time. Also ∇^2 represents the Laplacian operator.

(d) The equation of energy, which gives the relation of conservation of energy of the fluid. For an incompressible fluid with constant viscosity and heat conductivity the energy equation is

$$\rho C_p \frac{DT}{Dt} = \frac{\partial Q}{\partial t} + k\nabla^2 T + \phi \tag{1.14}$$

C_p is the specific heat at constant pressure, $\frac{\partial Q}{\partial t}$ is the rate of heat produced per unit volume by external agencies, k is the thermal conductivity of the fluid, ϕ is the viscous dissipation function for an incompressible fluid

$$\phi = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{1}{2}\left(Y_{xy}^2 + Y_{yz}^2 + Y_{zx}^2\right)\right]$$

where

$$Y_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$Y_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$Y_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

(e) The concentration equation for viscous incompressible fluid is

$$\frac{DC}{Dt} = D_M \nabla^2 C \tag{1.15}$$

C is the concentration and D_M is the chemical molecular diffusivity.

1.4 MHD Approximations

The electromagnetic equation as given in (1.1) – (1.9) are not usually applied in their present form and requires interpretation and several assumptions to provide the set to be used in MHD. In MHD we consider a fluid that is grossly neutral. The charge density ρ_e in Maxwell's equations must then be interpreted as an excess charge density which is in general not large. If we disregard the excess charge density, we must disregard the displacement current. In most problems the displacement current, the excess charge density and the current due to convection of the excess charge are small (Cramer and Pai, [18]).

The electromagnetic equations to be used are then as follows:

$$\nabla.\mathbf{D} = 0 \tag{1.16}$$

$$\nabla.\mathbf{J} = 0 \tag{1.17}$$

$$\nabla.\mathbf{B} = 0 \tag{1.18}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{1.19}$$

$$\nabla \times \mathbf{E} = 0 \tag{1.20}$$

$$\mathbf{D} = \varepsilon' \mathbf{E} \tag{1.21}$$

$$\mathbf{B} = \mu_e \mathbf{H} \tag{1.22}$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{q} \times \mathbf{B}) \tag{1.23}$$

1.5 MHD Equations

We will now modify the equations of fluid dynamics suitably to take account of the electromagnetic phenomena.

(1) The MHD equation of continuity for viscous incompressible electrically conducting fluid remains the same

$$\nabla.\mathbf{q} = 0 \tag{1.24}$$

(2) The MHD momentum equation for a viscous incompressible and electrically conducting fluid is

$$\rho \frac{D\mathbf{q}}{Dt} = \mathbf{F} - \nabla p + \mu \nabla^2 \mathbf{q} + \mathbf{J} \times \mathbf{B} \tag{1.25}$$

where **F** is the body force term per unit volume corresponding to the usual viscous fluid dynamic equations and the new term $\mathbf{J} \times \mathbf{B}$ is the force on the fluid per unit volume produced by the interaction of the current and magnetic field (called a Lorentz force).

(3) The MHD energy equation for a viscous incompressible electrically conducting fluid is

$$\rho C_p \frac{DT}{Dt} = \frac{\partial Q}{\partial t} + k\nabla^2 T + \phi + \frac{J^2}{\sigma} \tag{1.26}$$

The new term $\frac{J^2}{\sigma}$ is the Joule heating term and is due to the resistance of the fluid to the flow of current.

(4) The MHD equation of concentration for viscous incompressible electrically conducting fluid remains the same as

$$\frac{DC}{Dt} = D_M \nabla^2 C \tag{1.27}$$

1.6 Some Important Dimensionless Parameters of Fluid Dynamics and Magneto hydrodynamics.

(1) Reynolds Number, Re:

It is the most important parameter of the fluid dynamics of a viscous fluid. It represents the ratio of the inertia force to viscous force and is defined as

$$\mathrm{Re} = \frac{\text{inertial force}}{\text{viscous force}} = \frac{\rho U^2 L^2}{\mu UL} = \frac{UL}{\upsilon}$$

where U, L, ρ and μ are the characteristic values of velocity, length, density and coefficient of viscosity of the fluid respectively. When the Reynolds number of the system is small the viscous force is predominant and the effect of viscosity is important in the whole velocity field. When the Reynolds number is large the inertia force is predominant, and the effects of viscosity is important only in a narrow region near the solid wall or other restricted region which is known as boundary layer. If the Reynolds number is enormously large, the flow becomes turbulent.

(2) Magnetic Reynolds Number, R_σ:

It is the ratio of the fluid flux to the magnetic diffusivity and is given by

$$R_\sigma = \frac{UL}{(\mu_e \sigma)^{-1}}$$

It is one of the most important parameter of MHD. The magnetic Reynolds number determines the diffusion of the magnetic field along the stream lines. R_σ is a measure of the effect of the flow on the magnetic field. If it is very small compared to unity, the magnetic field is not distorted by the flow when it is very large. The magnetic field moves with the flow and is called frozen in.

(3) Prandtl Number, P_r :

The Prandtl number is the ratio of kinematics viscosity to thermal diffusivity and may be written as follows

$$P_r = \frac{\text{kinematic viscosity}}{\text{thermal diffusivity}} = \frac{\upsilon}{\left(\frac{k}{\rho C_p}\right)}$$

The value of υ shows the effect of viscosity of fluid. The smaller the value of υ the narrower is the region which is affected by viscosity and which is known as the boundary layer region when υ is very small. The value of $\frac{k}{\rho C_p}$ shows the thermal diffusivity due to heat conduction. The smaller the value of $\frac{k}{\rho C_p}$ is the narrower is the region which is affected by the heat conduction and which is known as thermal boundary layer when $\frac{k}{\rho C_p}$ is small. Thus the Prandtl number shows the relative importance of heat conduction and viscosity of a fluid. For a gas the Prandtl number is of order of unity.

(4) Magnetic Prandtl Number, P_m :

The magnetic Prandtl number is a measure of the relative magnitude of the fluid boundary layer thickness. It is the ratio of the viscous diffusivity to the magnetic diffusivity and is given by

$$P_m = \frac{\upsilon}{\upsilon_H} = \frac{R_\sigma}{R_e}$$

$\mathbf{P_m}$ is generally small and is a measure of the relative magnitude of the fluid boundary layer thickness to the magnetic boundary layer thickness. However, when the magnetic Reynolds number is large, the boundary layer thickness is small and is of nearly the same size as viscous boundary layer thickness. In this case $\mathbf{P_m}$ is not small.

(5) Schmidt Number, S_c :

This is the ratio of the kinematic viscosity to the chemical molecular diffusivity and is defined as

$$S_c = \frac{\upsilon}{D_m} = \frac{(\text{kinematic viscosity})}{(\text{chemical molecular diffusivity})}$$

(6) Grashof Number, G_r :

This is defined as

$$G_r = \frac{g_0 \beta' (\Delta T) L^3}{\upsilon^2}$$

and is a measure of the relative importance of the buoyancy of the force and viscous force. The larger it is the stronger is the convective current. In the above g_0 is the acceleration due to gravity, β' is the co-efficient of volume expansion and T is the temperature of the flow field.

(7) Modified Grashof Number, G_m :

This is defined as

$$G_m = \frac{g_0 \beta^* (\Delta T) L^3}{\upsilon^2}$$

where β^* is the co-efficient of expansion with concentration and C is the species concentration.

(8) Eckert Number, E_c :

Eckert number can be interpreted as the addition of heat due to viscous dissipation. Thus it is the ratio of the kinetic energy and thermal energy and is defined as

$$E_c = \frac{U^2}{C_\rho (T_w - T_\infty)}$$

where U is some reference velocity and $T_w - T_\infty$ is the difference between two reference temperatures. It is very small for incompressible fluid and for low motion.

(9) Soret Number, S_0:

This is defined as

$$S_0 = \frac{D_T \left(T_w - T_\infty\right)}{\upsilon\left(C_w - C_\infty\right)}$$

where D_T is the thermal diffusivity, T_w is the temperature at the plate, T_∞ is the temperature of the fluid at infinity, C_w is the concentration at the plate and C_∞ is the species concentration at infinity.

(10) Duffer Number, D_f:

This is defined as $D_f = \dfrac{mk_t \rho}{C_s Q}$

where k_t is the thermal diffusion ratio, m is the constant mass flux per unit area, ρ is the density of the fluid, Q is the constant heat flux per unit area, C_s is the concentration susceptibility.

(11) Magnetic Parameter, M:

This is obtained from the ratio of the magnetic force to the inertia force and is defined as $M = \dfrac{\mu_c H_0 \left(\sigma L\right)^{\frac{1}{2}}}{\left(\rho U\right)^{\frac{1}{2}}}$

where H_0 is applied magnetic field.

1.7 Suction and Injection

For boundary layer flows with adverse pressure gradients, the boundary layer will eventually separate from the surface. Separation of the flow causes many undesirable features over the whole field; for instance if separation occurs on the surface of an airfoil,

the lift of the airfoil will decrease and the drag will enormously increase. In some problems we wish to maintain laminar flow without separation. Various means have been proposed to prevent the separation of boundary layer; suction and injection are two of them.
The stabilizing effect of the boundary layer development has been well known for several years and till to date suction is still the most of efficient, simple and common method of boundary layer control. Hence, the effect of suction on hydro-magnetic boundary layer is of great interest in astrophysics. It is often necessary to prevent separation of the boundary layer to reduce the drag and attain high lift values.

Many authors have made mathematical studies on these problems, especially in the case of steady flow. Among them the name of Cobble [20] may be cited who obtained the conditions under which similarity solutions exist for hydro-magnetic boundary layer flow past a semi-infinite flat plate with or without suction. Following this, Soundalgekar and Ramanamurthy [65] analyzed the thermal boundary layer. Then Singh [61] studied this problem for large values of suction velocity employing asymptotic analysis in the spirit of Nanbu [47]. Singh and Dikshit [62] have again adopted the asymptotic method to study the hydro-magnetic effect on the boundary layer development over a continuously moving plate. In a similar way Bestman [14] studied the boundary layer flow past a semi-infinite heated porous plate for two-component plasma.

On the other hand, one of the important problems faced by the engineers engaged in high-speed flow in the cooling of the surface to avoid the structural failures as a result of frictional heating and other factors. In this respect the possibility of using injection at the surface is a measure to cool the body in the high temperature fluid. Injection of secondary fluid through porous walls is of practical importance in film cooling of turbine blades combustion chambers. In such application injection usually occurs normal to the surface and the injected fluid may be similar to or different from the primary fluid. In some recent applications, however, it has been recognized that the cooling efficiency can be enhanced by vectored injection at an angle other than 90^0 to the surface. A few workers including Inger and Swearn [34] have theoretically proved this feature for a boundary layer. In addition, most previous calculations have been limited to injection rates ranging from small to moderate. Raptis and Kafoussis [54] studied the free convection effects on the flow field of an incompressible, viscous dissipative fluid, past an infinite vertical porous plate, which is accelerated in its own plane. He considered that the fluid is subjected to a normal velocity of suction/injection proportional to $t^{-\frac{1}{2}}$ and the plate is perfectly insulated, i.e., there is no heat transfer between the fluid and the plate. Hasimoto [30] studied the boundary layer growth on an infinite flat plate started at time t = 0, with uniform suction or injection. Exact solutions of the Navier-Stokes equation of motion were derived for the case of uniform suction and injection, which was taken to be steady or proportional to $t^{-\frac{1}{2}}$. Numerical calculations are also made for the case of impulsive motion of the plate. In the case of injection, velocity profiles have injection points. The qualitative nature of the flow on both the suction the cases are obtained form the result of the corresponding studies on steady boundary layer, so far obtained.

1.8 Large Suction

When the rate of suction is very high then it is called large suction. Singh [61] studied the problem of Soundalgeker and Ramanamurthy [65] for large value of suction parameter by making use of the perturbation technique, as has been done by Nanbu [47]. Later Singh and Dikshit [62] studied the hydro-magnetic flow past a continuously moving semi-infinite porous plate employing the same perturbation technique. They also derived similarity solutions for large suction. The large suction in fact enabled them to obtain analytical solutions those are of immense value that compliment various numerical solutions. For the present problem studying on MHD free convection and mass transfer flow with thermal diffusion, Duffer effect and large suction we have to use the shooting method for getting the numerical solutions.

1.9 Free and Force Convection

Free or natural convection is a mechanism, or type of heat transport, in which the fluid motion is not generated by any external source (like a pump, fan, suction device, etc.) but only by density differences in the fluid occurring due to temperature gradients. In natural convection, fluid surrounding a heat source receives heat, becomes less dense and rises. The surrounding, cooler fluid then moves to replace it. This cooler fluid is then heated and the process continues, forming convection current; this process transfers heat energy from the bottom of the convection cell to top. The driving force for natural convection is buoyancy, a result of differences in fluid density. Because of this, the presence of a proper acceleration such as arises from resistance to gravity, or an equivalent force (arising from acceleration, centrifugal force or Coriolis force), is essential for natural convection. For example, natural convection essentially does not operate in free-fall (inertial) environments, such as that of the orbiting International Space Station, where other heat transfer mechanisms are required to prevent electronic components from overheating.

Natural convection has attracted a great deal of attention from researchers because of its presence both in nature and engineering applications. In nature, convection cells formed from air raising above sunlight warmed land or water, are a major feature all weather systems. Convection is also seen in the rising plume of hot air from fire, oceanic currents, and sea-wind formation (where upward convection is also modified by Coriolis forces). In engineering applications, convection is commonly visualized in the formation of microstructures during the cooling of molten metals, and fluid flows around shrouded heat-dissipation fins, and solar ponds. A very common industrial application of natural convection is free air cooling without the aid of fans: this can happen on small scales (computer chips) to large scale process equipment.

Forced convection is a mechanism, or type of heat transport in which fluid motion is generated by an external source (like a pump, fan, suction device, etc.). It should be considered as one of the main methods of useful heat transfer as significant amounts of heat energy can be transported very efficiently and this mechanism is found very commonly in everyday life, including central heating, air conditioning, steam turbines and in many other machines. Forced convection is often encountered by engineers designing or analyzing heat exchangers, pipe flow, and flow over a plate at a different temperature than the stream (the case of a shuttle wing during re-entry, for example). However, in any forced convection situation, some amount of natural convection is always present whenever there are g-forces present (i.e., unless the system is in free fall). When the natural convection is not negligible, such flows are typically referred to as mixed convection.

When analyzing potentially mixed convection, a parameter called the Archimedes number (Ar) parametrizes the relative strength of free and forced convection. The Archimedes number is the ratio of Grashof number and the square of Reynolds number $\left(Ar = \frac{G_r}{\mathrm{Re}^2}\right)$, which represents the ratio of buoyancy force and inertia force, and which stands in for the contribution of natural convection. When Ar >> 1, natural convection dominates and when Ar << 1, forced convection dominates.

When natural convection isn't a significant factor, mathematical analysis with forced convection theories typically yields accurate results. The parameter of importance in forced convection is the Peclet number $\left(Pe = \frac{UL}{\alpha}\right)$, which is the ratio of advection (movement by currents) and diffusion (movement from high to low concentrations) of heat.

1.10 Porous Medium:

A porous medium is a material containing pores (voids). The pores are typically filled with a fluid (liquid or gas). The skeletal material is usually a solid, but structures like foams are often also usefully analyzed using concept of porous media.

A porous medium is most often characterized by its porosity. Other properties of the medium (e.g., permeability, tensile strength, electrical conductivity) can sometimes be derived from the respective properties of its constituents and the media porosity and pores structure, but such a derivation is usually complex. Even the concept of porosity is only straight-forward for a poroelastic medium.

Many natural substances such as rocks, soils, biological tissues (e.g. bones, wood), and man made materials such as cements and ceramics can be considered as porous media. Many of their important properties can only be rationalized by considering them to be porous media.

The concept of porous media is used in many areas of applied science and engineering: filtration, mechanics (acoustics, geomechanics, soil mechanics, rock mechanics), engineering (petroleum engineering, bio-remediation, construction engineering), geosciences (hydrogeology, petroleum geology, geophysics), biology and biophysics, material science, etc. Fluid flow through porous media is a subject of most common interest and has emerged a separate field of study. The study of more general behaviour of porous media involving deformation of the solid frame is called poromechanics.

1.11 MHD Boundary Layer and Related Transfer Phenomena

Boundary layer phenomena occur when the influence of a physical quantity is restricted to small regions near confining boundaries. This phenomenon occurs when the non-dimensional diffusion parameters such as the Reynolds number and the Peclet number of the magnetic Reynolds number are large. The boundary layers are then the velocity and thermal or magnetic boundary layers, and each thickness is inversely proportional to the square root of the associated diffusion number. Prandtl fathered classical fluid dynamic boundary layer theory by observing, from experimental flows that for large Reynolds number, the viscosity and thermal conductivity appreciably influenced the flow only near a wall. When distant measurements in the flow direction are compared with a characteristic dimension in that direction, transverse measurements compared with the boundary layer thickness, and velocities compared with the free stream velocity, the Navier Stokes and energy equations can be considerably simplified by neglecting small quantities. The number of component equations is reduced to those in the flow direction and pressure is then only a function of the flow direction and can be determined from the inviscid flow solution. Also the number of viscous term is reduced to the dominant term and the heat conduction in the flow direction is negligible.

MHD boundary layer flows are separated in two types by considering the limiting cases of a very large or a negligible small magnetic Reynolds number. When the magnetic field is oriented in an arbitrary direction relative to a confining surface and the magnetic Reynolds number. When the magnetic field is oriented in an arbitrary direction relative to a confining surface and the magnetic Reynolds number is very small; the flow direction component of the magnetic interaction and the corresponding Joule heating is only a function of the transverse magnetic field component and local velocity in the flow direction. Changes in the transverse magnetic boundary layer are negligible. The thickness of magnetic boundary layer is very large and the induced magnetic field is negligible. However, when the magnetic Reynolds numbers is large, the magnetic boundary layer thickness is small and is of nearly the same size as the viscous and thermal boundary layers and then the MHD boundary layer equations must be solved simultaneously. In this case, the magnetic field moves with the flow and is called frozen mass.

1.12 MHD and Heat Transfer

With the advent of hypersonic flight, the field of MHD, as defined above, which has been associated largely with liquid-metal pumping, has attracted the interest of aero dynamists. It is possible to alter the flow and the heat transfer around high-velocity vehicles provided that the air is sufficiently ionized. Further more, the invention of high temperature facilities such as the shock tube and plasma jet has provided laboratory sources of flowing ionized gas, which provide and incentive for the study of plasma accelerators and generators.

As a result of this, many of the classical problems of fluid mechanics have been reinvestigated. Some of these analyses arouse out of the natural tendency of scientists to investigate a new subject. In this case it was the academic problem of solving the equations of fluid mechanics with a new body force and another source of dissipation in the energy equation. Sometimes their were no practical application for these results. For example, natural convection MHD flows have been of interest to the engineering community only since the investigations are directly applicable to the problems in geophysics and astrophysics. But it was in the field of aerodynamic heating that the largest interest was aroused. Rossow [57] presented the first paper on this subject. His result, for incompressible constant-property flat plate boundary layer flow, indicated that the skin friction and heat transfer were reduced substantially when a transverse magnetic field was applied to the fluid. This encouraged a multitude analysis for every conceivable type of aerodynamic flow, and most of the research centered on the stagnation point where, in hypersonic flight, the highest degree of ionization could be expected. The results of these studies were sometimes contradictory concerning the amount by which the heat transfer would be reduced (Some of this was due to misinterpretations and invalid comparisons). Eventually, however, it was concluded that the field strengths, necessary to provide sufficient shielding against heat fluxes during atmospheric flight, were not competitive (in terms of weight) with other methods of cooling (Sutton and Gloersen, [66]). However, the invention of new lightweight super conduction magnets has recently revived interests in the problem of providing heat protection during the very high velocity reentry from orbital and supper orbital flight (Levy and Petschek, [41]).

Processes, heat transfer considerations arise owing to chemical reaction and are often due to the nature of the process. In processes such as drying, evaporation at the surface of water body, energy transfer in a wet cooling tower and the flow in a desert cooler, heat and mass transfer occur simultaneously. In many of these processes, interest lies in the determination of the total energy transfer although in processes such as drying, the interest lies mainly in the overall mass transfer for moisture removal. Natural convection processes involving the combined mechanisms are also encountered in many natural processes, such as evaporation, condensation and agricultural drying, in many industrial applications involving solution and mixtures in the absence of an externally induced flow and in many chemical processing systems. In many processes such as the curing of plastics, cleaning and chemical processing of materials relevant to the manufacture of printed circuits, manufacture of pulp-insulated cables etc, the combined buoyancy mechanisms arise and the total energy and material transfer resulting from the combined mechanisms have to be determined.

The basic problem is governed by the combined buoyancy effects arising from the simultaneous diffusion of thermal energy and of chemical species. Therefore the equations of continuity, momentum, energy and mass diffusions are coupled through the buoyancy terms alone, if there are other effects, such as the Soret and Duffer effects, they are neglected. This would again be valid for low species concentration levels. These additional effects have also been considered in several investigations, for example, the work of the Caldwell [17]. Groots and Mozur [28], Hurle and Jakeman [33] and Legros, *et al.* [46]).
Somers [63] considered combined boundary mechanisms for flow adjacent to a wet isothermal vertical surface in an unsaturated environment. Uniform temperature and uniform species concentration at the surface were assumed and an integral analysis was carried out to obtain the result which is expected to be valid for P_r and S_c values around 1.0 with one buoyancy effect being small compared with the other. Mathers *et al.* [45] treated the problem as a boundary layer flow for low species concentration, neglecting inertia effects. Results were obtained numerically for P_r=1.0 and S_c varying from 0.5 to 10. Lowell and Adams [43] and Gill *et al.* [27] also considered this problem, including additional effects such as appreciable normal velocity at the surface and comparable species concentrations in the mixture. Similar solutions were investigated by Lowell and Adams [43], Lightfoot [42] and Saville and Churchill [60] considered come asymptotic

solutions. Adams and Mc Fadden [1] presented experimental measurements of heat and mass transfer parameters, with opposed buoyancy effects. Gebhart and Pera [24] studied laminar vertical natural convection flows resulting from the combined buoyancy mechanisms in terms of similarity solutions. Similar analyses have been carried out by Pera and Gebhart [52] for flow over horizontal surfaces and by Mollendrof and Gebhart [46] for axisymmetric flows, particularly for the axisymmetric Plume.

Mollendrof and Gebhart [46] carried out an analysis for axis symmetric flows. The governing equations were solved for the combined effects of thermal and mass diffusion in an axisymmetric plume flow. Boura and Gebhart [15], Hubbel and Gebhart [32] and Tenner and Gebhart (1771) have studied buoyant free boundary flows in a concentration stratified medium. Agrawal el al. [2] have studied the combined buoyancy effects on the thermal and mass diffusion on MHD natural flows, and it is observed that, for the fixed G_r and P_r the value of X_t (dimensionless length parameter) decreases as the strength of the magnetic parameter increases. Georgantopoulos *et al.* [26] discussed the effects of free convective and mass transfer flow in a conducting liquid, when the fluid is subject to a transverse magnetic field. Haldavnekar and Soundalgekar [29] studied the effects of mass transfer on free convective flow of an electrically conducting viscous fluid past an infinite porous plate with constant suction and transversely applied magnetic field. An exact analysis was made by Soundalgekar and Gupta [64] of the effects of mass transfer and the free convection currents on the MHD Stokes (Rayleigh) problem for the flow of an electrically conducting incompressible viscous fluid past an impulsively started vertical plate under the action of a transversely applied magnetic field. The heat due to viscous and Joule dissipation and induced magnetic field are neglected.

During the course of discussion, the effects of heating $G_r < 0$ of the plate by free convection currents, and G_m (modified Grashof number), S_c and M on the velocity and the skin friction are studied. Nunousis and Goudas [49] have studied the effects of mass transfer on free convective problem in the Stokes problem for an infinite vertical limiting surface. Georgantopolous and Nanousis [25] have considered the effects of the mass transfer on free convection flow of an electrically conducting viscous fluid (e.g. of a stellar atmosphere, of star) in the presence of transverse magnetic field. Solution for the velocity and skin friction in closed from are obtained with the help of the Laplace transform technique, and the results obtained for the various values of the parameters, S_c , P_r and M are given in graphical form. Raptis and Kafoussias [54] presented the analysis of free convection and mass transfer steady hydro magnetic flow of an electrically conducting viscous incompressible fluid, through a porous medium, occupying a semi infinite region of the space bounded by an infinite vertical and porous plate under the action of transverse magnetic field. Approximate solution has been obtained for the velocity, temperature, concentration field and the rate of heat transfer. The effects of different parameters on the velocity field and the rate of heat transfer are discussed for the case of air (Prandtle number P_r = 0.71) and the water vapor (Schmidt number S_c = 0.60), Raptis and Tzivanidis [55] considered the effects of variable suction/ injection on the unsteady two dimensional free convective flow with mass transfer of an electrically conducting fluid past vertical accelerated plate in the presence of transverse magnetic field. Solutions of the governing equations of the flow are obtained with the power series. An analysis of two dimensional steady free convective flow of a conducting fluid, in the presence of a magnetic field and a foreign mass, past an infinite vertical porous and unmoving surface is carried out by Raptis (1983), when the heat flux is constant at the limiting surface and the magnetic Reynolds number of the flow is not small. Assuming constant suction at the surface, approximate solutions of the coupled non-linear equations are derived for the velocity field, the temperature field, the magnetic field and for their related quantities. Agrawal *et al.* [5] considered the steady laminar free convection flow with mass transfer of an electrically conducting liquid along a plane wall with periodic suction. The considered sinusoidal suction velocity distribution is of the form $V' = V_0\left\{1+\varepsilon\cos\frac{\pi z'}{l}\right\}$, where $v_0 > 0$, is the wavelength of the periodic suction velocity distribution, and ε is the amplitude of the suction velocity variation which is assumed to be small quantity. It is observed that near the plate the velocity is a maximum and decreases as y increases. Also, an increase in the magnetic parameter the velocity decreases. Agrawal *et al.* [4] have investigated the effect of Hall current on the combined effect of thermal and mass diffusion of an electrically conducting liquid past an infinite vertical porous plate, when the free stream oscillates about constant nonzero mean. The velocity and temperature distributions are shown on graphs for different values of parameters. The value of P_r is chosen as 0.71 for air. In selecting the values of S_c, the Schmidt number, the diffusing chemical species of most common interest in air are considered. From the figures it is seen that, with the increase air. Hall parameter, the mean primary velocity decreases, where as the mean secondary velocity increases for a fixed magnetic parameter M and S_c. However, for a fixed m, and increase air magnetic parameter M or S_c, leads to a decrease in both the primary and the secondary velocities. The mean shear stresses at the plate due to primary and secondary velocity and the mean rate of heat transfer from the plate are also given. To study the behavior of the oscillatory and transient part of the velocity and temperature distribution, curves are drawn for various values of parameters that describe the flow at $Wt = \frac{\pi}{2}$, The non-dimensional shear stress and the rate of heat transfer are obtained. The above problem has been extended by the same authors (Agrawal *et al.* [3]) when the plate temperature oscillates in time a constant nonzero mean, while the free stream is isothermal. The velocity, temperature and concentration distribution, together with the heat and mass transfer results, have been computed for different values of P_r, G_r, M and m.

1.13 Soret and Duffer Effect

In the above mentioned studies, heat and mass transfer occur simultaneously in a moving fluid where the relations between the fluxes and the driving potentials are of more complicated nature. In general the thermal-diffusion effects is of a smaller order of magnitude than the effects described by Fourier or Flick's laws and is often neglected in heat and mass transfer process. Mass fluxes can also be created by temperature gradients and this is Soret or Thermal diffusion effect. There are, however, exceptions. The thermal-diffusion effect, (commonly known as Soret effect) for instance, has been utilized for isotope separation and in mixtures between gases with very light molecular weight (H_2, He) and of medium molecular weight (N_2, air). The diffusion thermo effect was found to be of such a magnitude that it could not be neglected (Eckert and Drake, [22]). In view of the importance of the diffusion thermo effect, Jha and Singh [35] presented an analytical study for free convection and mass transfer flow for an infinite vertical plate moving impulsively in its own plane, taking into account the Soret effect. Kaffoussias [36] studied the MHD free convection and mass transfer flow, past an infinite vertical plate moving on its own plane, taken into account the thermal diffusion when (i) the boundary surface is impulsively started moving in its own plane (ISP) and (ii) it is uniformly accelerated (UAP). The problem is solved with the help of Laplace transfer method and analytical expressions are given for the velocity field as well as for the skin friction for the above mentioned two different cases. The effects of the velocity and skin friction of the various dimensionless parameters entering into the problem are discussed with the help of graphs. For the I.S.P. and U.A.P. cases, it is seen from the figures that the effect of magnetic parameters M is to decrease the fluid (water) velocity inside the boundary layer. This influence of the magnetic field on the velocity field is more evident in the presence of thermal diffusion. From the same figures it is also concluded that the fluid velocity rises due to greater thermal diffusion. Hence, the velocity field is considerably affected by the magnetic field and the thermal diffusion. Nanousis [48] extended the work of Kafoussias [36] to the case of rotating fluid taking into account the Soret effect. The plate is assumed to be moving on its own plane with arbitrary velocity $U_0 f(t')$ where U_0 is a constant velocity and $f(t')$ a non-dimensional function of the time t'. The solution of the problem is obtained with the help of Laplace transform technique. Analytical expression is given for the velocity field and for skin friction for two different cases, e.g., when the plate is impulsively stared, moving on its own plane (case I) and when it is uniformly accelerated (case II). The effects on the velocity field and skin friction, of various dimensionless parameters entering into the problem, especially of the Soret number S_0, are discussed with the help of graphs. In case of an impulsively started plate and uniformly accelerated plate (case I and case II), it is seen that the primary velocity increase with the increase of S_0 and the magnetic parameter M. It has been observed that energy can be generated not only by temperature gradients but also by composition gradients. The energy flux caused by composition gradients is called the Duffer or diffusion thermo effect. On the other hand, mass fluxes can also be created by temperature gradients and this is the Soret or thermal diffusion effect.

CHAPTER 2

Introduction and Literature Review

The convective heat and mass transfer process takes place due to the buoyancy effects owing to the differences of temperature and concentration, respectively. In dealing with the transport phenomena, the thermal and mass diffusions occurring by the simultaneous action of buoyancy forces are of considerable interest in nature and in many engineering practices, such as geophysics, oceanography etc. Many theoretical and experimental investigations have been established in the literature involving the studies of heat and mass transfer over plates by natural, forced or combined convection, and most of these studies are based upon the laminar boundary layer approach. The mixed free-forced convective and mass transfer flow has essential applications in separation processes in chemical engineering or drying processes. The natural convection boundary layer flows induced by the combined buoyancy effects of thermal and mass diffusion has been investigated primarily by Gebhart and Pera [24] and Pera and Gebhart [52]. Furthermore, convective flow through porous media is very important in many physical and natural applications, namely, heat transfer associated with heat recovery from geothermal systems and particularly in the field of large storage systems of agricultural products, storage of nuclear waste, heat removal from nuclear fuel debris, petroleum extraction, control of pollutant spread in groundwater, etc. The free convection flow past an infinite vertical porous plate with suction velocity perpendicular to the plate surface was studied by Brezovsky *et al.* [16], Kawase and Ulbrecht [38], Martynenko *et al.* [44] and further extended by Weiss *et al.* [67]. A finite difference numerical scheme was considered by Pantokratoras [50] in order to study the laminar free convection boundary layer flow past over an isothermal vertical plate with uniform suction or injection. Hassanien *et al.* [31] has investigated the natural convection boundary layer flow of micropolar fluid over a semi-infinite plate embedded in a saturated porous medium where the plate maintained at a constant heat flux with uniform suction/injection velocity.

In recent times, the problems of natural convective heat and mass transfer flows through a porous medium under the influence of a magnetic field have been paid attention of a number of researchers because of their possible applications in many branches of science, engineering and geophysical process. Considering these numerous applications, MHD free convective heat and mass transfer flow in a porous medium have been studied by among others Raptis and Kafoussias [54], Sattar [58], sattar and Hossain [59] etc. Besides, Kim [39] has been studied the effect of MHD of a micropolar fluid on coupled heat and mass transfer, flowing on a vertical porous plate moving in a porous medium. However, Pantokratoras [51] showed that a moving electrically conducting fluid induced a new magnetic field, which interacts with the applied external magnetic fields and the relative importance of this induced magnetic field depends on the relative value of the magnetic Reynolds number ($\boldsymbol{R_m} >> 1$).

Nevertheless, more complicate phenomenon arises between the fluxes and the driving potentials when heat and mass transfer occur simultaneously in case of a moving fluid. It has been observed that an energy flux can be generated not only due to the temperature gradients but also by composition gradients. The energy flux caused by a composition gradient is called the Dufour or diffusion-thermo effect, whereas, mass flux caused by temperature gradients is known as the Soret or thermal-diffusion effect. In general, the thermal-diffusion and diffusion-thermo effects are of a smaller order of magnitude than the effects described by Fourier's or Fick's law. This is why, most of the studies of heat and mass transfer processes, however, considered constant plate temperature and concentration and have neglected the diffusion-thermo and thermal-diffusion terms from the energy and concentration equations, respectively. Ignoring the Soret and Dufour effects, Choudhary and Sharma [19] studied the mixed convection flow over a continuously moving porous vertical plate with combined buoyancy effects of thermal and mass diffusion under the action of a transverse magnetic field, when the plate is subjected to constant heat and mass flux. But in some exceptional cases, for instance, in mixture between gases with very light molecular weight (H_2, He) and of medium molecular weight (N_2, air) the diffusion-thermo (Dufour) effect and in isotope separation the thermal-diffusion (Soret) effect was found to be of a considerable magnitude such that they cannot be ignored. In view of the relative importance of these above mentioned effects many researchers have studied and reported results for these flows of whom the names are, Eckert and Drake [22], Dursunkaya and Worek [21], etc. Whereas, Kafoussias and Williams [37] studied the same effects on mixed free-forced convective and mass transfer boundary layer flow with temperature dependent viscosity. Later, Anghel *et al.* [13] has investigated the composite Dufour and Soret effects on free convection boundary layer heat and mass transfer over a vertical surface in a Darcian porous regime. A theoretical steady with numerical solution of two-dimensional free convection and mass transfer flow past a continuously moving semi-infinite vertical porous plate in a porous medium is presented by Alam *et al.* [11], taking into account the Dufour and Soret effects. Later, a numerical study based on Nachtsheim-Swigert shooting iteration technique together with sixth order Runge-Kutta integration scheme have been carried out by Alam and Rahman [8] in order to investigate the Dufour and Soret effects on mixed convection flow past a vertical porous flat plate with variable fluid suction. Recently, Alam *et al.* [9] investigated the diffusion-thermo and thermal-diffusion effects on unsteady free convection and mass transfer flow past an accelerated vertical porous flat plate embedded in a porous medium with time dependent temperature and concentration. Very recently, a mathematical model and numerical study based on the finite element method has been implemented by Rawat and Bhargava [56] for the viscous, incompressible heat and mass transfer of a micropolar fluid through a Darcian porous medium with the presence of buoyancy, Soret/Dufour diffusion, viscous heating and wall transpiration.

The Dufour and Soret effects on steady MHD free convective heat and mass transfer flow past a semi-infinite vertical porous plate embedded in a porous medium have been studied by Alam and Rahman [7]. Next, Alam *et al.* [10] studied the Dufour and Soret effects on unsteady MHD free convection and mass transfer flow past an infinite vertical flat plate. Alam *et al.* [11] further extensively investigated the Dufour and Soret effects on steady MHD free-forced convective and mass transfer flow past a semi-infinite vertical plate. In recent times, a numerical study of the natural convection heat and mass transfer about a vertical surface embedded in a saturated porous medium under the influence of a magnetic field has been done by Postelnicu [53], taking into account the diffusion-thermo and thermal-diffusion effects. Following the study to those of Choudhary and Sharma [19], Pantokratoras [51] and Postelnicu [53], our main aim is to investigate the Dufour effect on combined heat and mass transfer of a steady laminar mixed free-forced convective flow of viscous incompressible electrically conducting fluid above a semi-infinite vertical porous surface under the influence of an induced magnetic field.

CHAPTER 3

Governing Equations and Solutions

Consider a model of steady mixed convection and mass transfer flow of a viscous, incompressible, electrically conducting fluid past a continuously moving infinite vertical electrically nonconducting porous plate under the influence of a transversely applied magnetic field. Introducing the cartesian coordinate system in which the axes x and y are chosen to be along and normal to the plate, respectively. The flow is assumed to be in the x-direction, which is taken along the vertical plate in the upward direction. Based on the assumptions that the magnetic Reynolds number of the flow is not taken to be of considerable magnitude so that the induced magnetic field is taken into account. The magnetic field is of the form $\mathbf{H} = \left(H_x, H_y, 0\right)$. The equation of conservation of electric charge $\nabla.\mathbf{J} = 0$, where $\mathbf{J} = \left(J_x, J_y, J_z\right)$. Since the direction of propagation of electric charge is along the y-axis and the plate is electrically nonconducting, $J_y = 0$ every where within the flow. It is also assumed that the Joule heating effect is small enough and divergence equation for the magnetic field $\nabla.\mathbf{H} = \mathbf{0}$ is of the form $H_y = H_0$. The schematic view of the flow configuration and coordinate system of the problem are shown in Figure-3.1.

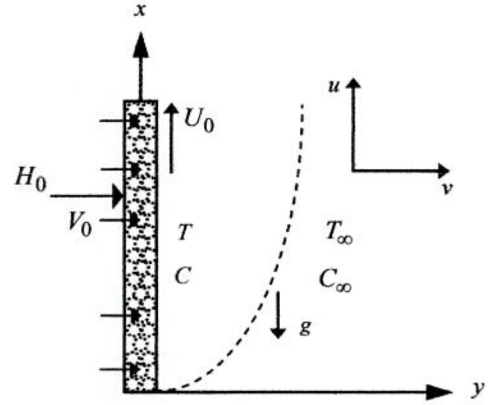

Figure 3.1 Flow configuration and coordinate system

Further, as the plate is infinite extent, all physical variables depend on y only and therefore the equation of continuity is given by

$$\frac{dv}{dy}=0 \tag{3.1}$$

whose solution gives $v=-V_0$, where V_0 is the constant velocity of suction normal to the plate and the negative sign indicates that the suction velocity is directed towards the plate surface. In accordance with the above assumptions and initiating the concept of usual Boussinesq's approximation, the basic equations related to the problem incorporating with the Maxwell's equations and generalized Oham's law can be put in the following form:

$$-V_0\frac{du}{dy}=g\beta(T-T_\infty)+g\beta^*(C-C_\infty)+\upsilon\frac{d^2u}{dy^2}+\frac{\mu_e H_o}{\rho}\frac{dH_x}{dy} \tag{3.2}$$

$$-V_0\frac{dH_x}{dy}=H_0\frac{du}{dy}+\frac{1}{\sigma\mu_e}\frac{d^2H_x}{dy^2} \tag{3.3}$$

$$-V_0\frac{dT}{dy}=\frac{k}{\rho C_p}\frac{d^2T}{dy^2}+\frac{\upsilon}{C_p}\left(\frac{du}{dy}\right)^2+\frac{1}{\sigma\rho C_p}\left(\frac{dH_x}{dy}\right)^2+\frac{Dk_T}{C_s C_p}\frac{d^2C}{dy^2} \tag{3.4}$$

$$-V_0\frac{dC}{dy}=D\frac{d^2C}{dy^2} \tag{3.5}$$

The relevant boundary conditions on the vertical surface and in the uniform stream are defined as follows:

$$\left.\begin{aligned} &u=U_0,\ \frac{dT}{dy}=-\frac{Q}{k},\ \frac{dC}{dy}=-\frac{m}{D},\ H_x=H_w \text{ at } y=0\\ &u=0,\ T=T_\infty,\ C=C_\infty,\ H_x=0 \text{ when } y\to\infty \end{aligned}\right\} \tag{3.6}$$

where g is the acceleration due to gravity, β is the coefficient of thermal expansion, T denotes fluid temperature, C is concentration of species, T_∞ and C_∞ are the temperature and species concentration of the uniform flow, β^* is the concentration expansion coefficient, υ is the Newtonian kinematic viscosity of the fluid, μ_e is the magnetic permeability, H_0 is the applied constant magnetic field, H_x is induced magnetic field, ρ is the density of the fluid, σ is the electrical conductivity, k is the thermal conductivity, D is the chemical molecular diffusivity, C_p is the specific heat capacity of the fluid at constant pressure, C_s is the concentration susceptibility, k_T is the thermal diffusion ratio, U_0 is the uniform velocity, Q is the constant heat flux per unit area, m is the constant mass flux per unit area and H_w is the induced magnetic field at the wall, respectively. In order to simplify a numerical solution, we introduce the following transformations, viz:

$$u^* = \frac{u}{V_0};\ y^* = \frac{V_0 y}{\upsilon};\ U^* = \frac{U_0}{V_0};\ k^* = \frac{V_0^2 k}{\upsilon^2};\ C^* = \frac{DV_0\left(C - C_\infty\right)}{\upsilon m};$$

$$\theta^* = \frac{kV_0(T - T_\infty)}{\upsilon Q};\ H^* = \sqrt{\frac{\mu_e}{\rho}}\frac{H_x}{V_0} \tag{3.7}$$

and defining the following dimensionless parameters

$P_r = \frac{\rho \upsilon C_p}{k}$ (Prandtl number)

$G_r = \frac{\upsilon^2 g \beta Q}{kV_0^4}$ (Grashoff number)

$G_m = \frac{\upsilon g \beta^* (C_w - C_\infty)}{V_0^3}$ (modified Grashoff number)

$E_c = \frac{kV_0^3}{\upsilon Q C_p}$ (Eckert number)

$D_f = \frac{\rho m k_T}{Q C_s}$ (Dufour number)

$S_c = \frac{\upsilon}{D}$ (Schmidt number)

$P_m = \sigma \upsilon \mu_e$ (Magnetic diffusivity)

$M = \frac{H_0}{V_0}\sqrt{\frac{\mu_e}{\rho}}$ (Magnetic parameter), where C_w is the concentration on the plate wall.

On introducing the above non-dimensional quantities, in the equations (3.2) – (3.5) with boundary conditions (3.6) we have

$$u^* = \frac{u}{V_0}$$

or, $u = V_0 u^*$

$$\frac{du}{dy} = V_0 \frac{du^*}{dy}$$

$$y^* = \frac{V_0 y}{\upsilon}$$

$$\therefore \frac{dy^*}{dy} = \frac{V_0}{\upsilon}$$

Then $\frac{du}{dy} = V_0 \frac{du^*}{dy^*} \cdot \frac{dy^*}{dy}$

$$= \frac{V_0^2}{\upsilon} \frac{du^*}{dy^*} \tag{3.8}$$

$$\frac{d^2u}{dy^2} = \frac{d}{dy^*}\left(\frac{V_0^2}{\upsilon}\frac{du^*}{dy^*}\right) \cdot \frac{dy^*}{dy}$$

$$= \frac{V_0}{\upsilon^2} \frac{d^2u^*}{dy^{*2}} \tag{3.9}$$

$$\frac{H_x}{V_0}\sqrt{\frac{\mu_e}{\rho}} = H^*$$

or, $H_x = \sqrt{\frac{\rho}{\mu_e}} V_0 H^*$

Now $\frac{dH_x}{dy} = \frac{dH_x}{dy^*} \cdot \frac{dy^*}{dy}$

$$= \sqrt{\frac{\rho}{\mu_e}} V_0 \frac{dH^*}{dy^*} \cdot \frac{V_0}{\upsilon}$$

$$= \frac{V_0^2}{\upsilon} \sqrt{\frac{\rho}{\mu_e}} \frac{dH^*}{dy^*} \tag{3.10}$$

$$\frac{d^2H_x}{dy^2} = \frac{d}{dy^*}\left(\frac{V_0^2}{\upsilon}\sqrt{\frac{\rho}{\mu_e}}\frac{dH^*}{dy^*}\right)\frac{dy^*}{dy}$$

$$= \frac{V_0^3}{\upsilon^2} \sqrt{\frac{\rho}{\mu_e}} \frac{d^2H^*}{dy^{*2}} \tag{3.11}$$

Again,

$$C^* = \frac{V_0 D (C - C_\infty)}{m\upsilon}$$

or, $C - C_\infty = \frac{m\upsilon C^*}{V_0 D}$ (3.12)

and $\theta^* = \frac{kV_0 (T - T_\infty)}{\upsilon Q}$

$$\text{or, } T-T_\infty=\frac{Q\upsilon\theta^*}{kV_0} \tag{3.13}$$

Substituting (3.8) – (3.10) and (3.12) – (3.13) in equation (3.2) we get,

$$-V_0.\frac{V_0^2}{\upsilon}\frac{du^*}{dy^*}=g\beta\left(\frac{\theta^* Q\upsilon}{kV_0}\right)+g\left(\frac{\beta^* m\upsilon C^*}{V_0 D}\right)+\upsilon\left(\frac{V_0^3}{\upsilon^2}.\frac{d^2u^*}{dy^{*2}}\right)+\frac{\mu_e H_0}{\rho}.\frac{V_0^2}{\upsilon}\sqrt{\frac{\rho}{\mu_e}}\frac{dH^*}{dy^*}$$

$$\text{or } \frac{-V_0^3}{\nu}\frac{du^*}{dy^*}=\frac{g\beta\nu Q\theta^*}{kV_0}+\frac{g\beta^* m\upsilon C^*}{V_0 D}+\frac{V_0^3}{\upsilon}\frac{d^2u^*}{dy^{*2}}+\frac{V_0^2}{\upsilon}\frac{\mu_e H_0}{\rho}\sqrt{\frac{\rho}{\mu_e}}\frac{dH^*}{dy^*}$$

$$\text{or, } \frac{d^2u^*}{dy^{*2}}+\frac{H_0}{V_0}\sqrt{\frac{\mu_e}{\rho}}\frac{dH^*}{dy^*}+\frac{du^*}{dy^*}=-\frac{g\beta\upsilon^2 Q\theta^*}{kV_0^4}-\frac{g\beta^*\upsilon^2 mC^*}{DV_0^4}$$

$$\text{or, } \frac{d^2u^*}{dy^{*2}}+M\frac{dH^*}{dy^*}+\frac{du^*}{dy^*}=-\frac{g\beta\upsilon^2 Q\theta^*}{kV_0^4}-\frac{g\beta^*\upsilon^2 C^* V_0 D(C-C_\infty)}{\upsilon C^* DV_0^4}$$

$$\text{or, } \frac{d^2u^*}{dy^{*2}}+M\frac{dH^*}{dy^*}+\frac{du^*}{dy^*}=-G_r\theta^*-G_m C^* \tag{3.14}$$

where

$$M=\frac{H_0}{V_0}\sqrt{\frac{\mu_e}{\rho}}$$

$$G_r=\frac{g\beta\upsilon^2 Q}{kV_0^4}$$

$$G_m=g\beta^*\upsilon^2(C-C_\infty)\frac{D}{V_0^3}$$

Putting (3.8), (3.10) – (3.11) in equation (3.3) we have

$$-V_0\frac{V_0^2}{\upsilon}\sqrt{\frac{\rho}{\mu_e}}\frac{dH_x}{dy^*}=H_0\frac{V_0^2}{\upsilon}\frac{du^*}{dy^*}+\frac{1}{\sigma\mu_e}\frac{V_0^3}{\upsilon^2}\sqrt{\frac{\rho}{\mu_e}}\frac{d^2H_x}{dy^{*2}}$$

$$\text{or, } -\frac{dH_x}{dy^*}=\frac{H_0}{V_0}\sqrt{\frac{\mu_e}{\rho}}\frac{du^*}{dy^*}+\frac{1}{\sigma\mu_e\upsilon}\frac{d^2H_x}{dy^{*2}} \tag{3.15}$$

where $M=\frac{H_0}{V_0}\sqrt{\frac{\mu_e}{\rho}}$

and $P_m=\sigma\upsilon\mu_e$.

Again equation (3.4) gives

$$-V_0\frac{dT}{dy}=\frac{k}{\rho C_p}\frac{d^2T}{dy^2}+\frac{\upsilon}{C_p}\left(\frac{du}{dy}\right)^2+\frac{1}{\sigma\rho}\left(\frac{dH_x}{dy}\right)^2+\frac{Dk_t}{C_s C_p}\frac{d^2C}{dy^2}$$

From equation (3.13) we have

$$T = \frac{Q\upsilon\theta^*}{kV_0} + T_\infty$$

$$\frac{dT}{dy} = \frac{d}{dy^*}\left(\frac{Q\upsilon\theta^*}{kV_0} + T_\infty\right).\frac{dy^*}{dy}$$

$$= \frac{Q\upsilon}{kV_0}\frac{d\theta^*}{dy^*}.\frac{V_0}{\upsilon}$$

$$= \frac{Q}{k}\frac{d\theta^*}{dy^*}$$

$$\therefore \frac{d^2T}{dy^2} = \frac{d}{dy^*}\left(\frac{Q}{k}\frac{d\theta^*}{dy^*}\right)\frac{dy^*}{dy}$$

$$= \frac{Q}{k}\frac{d^2\theta^*}{dy^{*2}}.\frac{V_0}{\upsilon}$$

$$= \frac{QV_0}{k\upsilon}\frac{d^2\theta^*}{dy^{*2}}$$

Further from equation (3.12)

$$C = \frac{m\upsilon C^*}{V_0 D} + C_\infty$$

or, $$\frac{dC}{dy} = \frac{d}{dy^*}\left(\frac{m\upsilon}{V_0 D}C^* + C_\infty\right).\frac{dy^*}{dy}$$

$$= \frac{m\upsilon}{V_0 D}\frac{dC^*}{dy^*}\frac{dy^*}{dy}$$

$$= \frac{m\upsilon}{V_0 D}\frac{dC^*}{dy^*}.\frac{V_0}{\upsilon}$$

$$= \frac{m}{D}\frac{dC^*}{dy^*}$$

Hence

$$\frac{d^2C}{dy^2} = \frac{d}{dy^*}\left(\frac{m}{D}\frac{dC^*}{dy^*}\right)\frac{dy^*}{dy}$$

$$= \frac{m}{D}\frac{d^2C^*}{dy^{*2}}.\frac{V_0}{\upsilon}$$

$$= \frac{mV_0}{D\upsilon}\frac{d^2C^*}{dy^{*2}}$$

Therefore equation (3.4) becomes

$$-V_0\frac{Q}{k}\frac{d\theta^*}{dy^*}=\frac{k}{\rho C_p}\frac{QV_0}{k\upsilon}\frac{d^2\theta^*}{dy^{*2}}+\frac{\upsilon}{C_p}\left(\frac{V_0^2}{\upsilon}\frac{du^*}{dy^*}\right)^2+\frac{1}{\sigma\rho C_p}\left(\frac{V_0^2}{\upsilon}\sqrt{\frac{\rho}{\mu_e}}\frac{dH_x}{dy^*}\right)^2+\frac{Dk_t}{C_sC_p}\frac{mV_0}{D\upsilon}\frac{d^2C^*}{dy^{*2}}$$

or, $$\frac{QV_0}{\rho\upsilon C_p}\frac{d^2\theta^*}{dy^{*2}}+\frac{QV_0}{k}\frac{d\theta^*}{dy^*}=-\frac{V_0^4}{\upsilon C_p}\left(\frac{du^*}{dy^*}\right)^2-\frac{1}{\sigma\rho\upsilon^2\mu_eC_p}\left(\frac{dH_x}{dy^*}\right)^2+\frac{Dk_t}{C_sC_p}\frac{mV_0}{D\upsilon}\frac{d^2C^*}{dy^{*2}}$$

or,

$$\frac{d^2\theta^*}{dy^{*2}}+\frac{\rho\upsilon C_p}{k}\frac{d\theta^*}{dy^*}=\frac{-kV_0^3}{Q\upsilon C_p}.\frac{\rho\upsilon C_p}{k}\left(\frac{du^*}{dy^*}\right)^2-\frac{kV_0^3}{Q\upsilon C_p}.\frac{\rho\upsilon C_p}{k}.\frac{1}{\sigma\upsilon C_p}\left(\frac{dH_x}{dy^*}\right)^2$$
$$+\frac{Dk_t}{C_sC_p}\frac{\rho\upsilon C_p}{QV_0}\frac{mV_0}{D\upsilon}\frac{d^2C^*}{dy^{*2}}$$

or,

$$\frac{d^2\theta^*}{dy^{*2}}+\frac{\rho\upsilon C_p}{k}\frac{d\theta^*}{dy^*}=-\frac{kV_0^3}{Q\upsilon C_p}.\frac{\rho\upsilon C_p}{k}\left(\frac{du^*}{dy^*}\right)^2-\frac{kV_0^3}{Q\upsilon C_p}.\frac{\rho\upsilon C_p}{k}.\frac{1}{\sigma\upsilon\mu_e}\left(\frac{dH_x}{dy^*}\right)^2+\frac{mk_t\rho}{C_sQ}\frac{d^2C^*}{dy^{*2}}$$

$$\therefore\frac{d^2\theta^*}{dy^{*2}}+P_r\frac{d\theta^*}{dy^*}=-P_rE_c\left[\left(\frac{du^*}{dy^*}\right)^2+\frac{1}{P_m}\left(\frac{dH_x}{dy^*}\right)^2\right]+D_f\frac{d^2C^*}{dy^{*2}} \quad (3.16)$$

where $\frac{\rho\upsilon C_p}{k}=\frac{\mu C_p}{k}=P_r$, $E_c=\frac{kV_0^3}{Q\upsilon C_p}$, $P_m=\sigma\upsilon\mu_e$

and $D_f=\frac{mk_t\rho}{C_sQ}$ is the Dufour number.

Then equation (3.5) becomes

$$-\frac{V_0m}{D}\frac{dC^*}{dy^*}=D\frac{mV_0}{D\upsilon}\frac{d^2C^*}{dy^{*2}}$$

or, $$\frac{dC^*}{dy^*}+\frac{D}{\upsilon}\frac{d^2C^*}{dy^{*2}}=0 \quad (3.17)$$

where $\frac{\upsilon}{D}=S_c$

The corresponding boundary conditions are now transformed as follows:

$y=0\Rightarrow y^*=0$

$u=U_0$ at $y=0\Rightarrow V_0u^*=U_0\Rightarrow u^*=\frac{U_0}{V_0}$, i.e., $u^*=U^*$ at $y^*=0$

$$\frac{dT}{dy} = -\frac{Q}{k} \text{ at } y = 0 \Rightarrow \frac{Q}{k}\frac{d\theta^*}{dy^*} = -\frac{Q}{k}, \text{ i.e., } \frac{d\theta^*}{dy^*} = -1 \text{ at } y^* = 0$$

$$\frac{dC}{dy} = -\frac{m}{D} \text{ at } y = 0 \Rightarrow \frac{m}{D}\frac{dC^*}{dy^*} = -\frac{m}{D}, \text{ i.e., } \frac{dC^*}{dy^*} = -1 \text{ at } y^* = 0$$

$$H_x = H_w \text{ at } y = 0 \Rightarrow \frac{H^*}{\sqrt{\frac{\mu_e}{\rho}\frac{1}{V_0}}} = H_w \Rightarrow \frac{H_0 H^*}{\sqrt{\frac{\mu_e}{\rho}\frac{H_0}{V_0}}} = H_w \Rightarrow \frac{H_0 H^*}{M} = H_w, \text{ i.e., } H^* = h \text{ at } y^* = 0$$

where $h = \frac{MH_w}{H_0}$.

Alos $y \to \infty \Rightarrow y^* \to \infty$

$u = 0$ when $y \to \infty \Rightarrow V_0 u^* = 0$, i.e., $u^* = 0$ when $y^* \to \infty$

$T = T_\infty$ when $y \to \infty \Rightarrow \frac{Q\upsilon\theta^*}{kV_0} = 0$, i.e., $\theta^* = 0$ when $y^* \to \infty$

$C = C_\infty$ when $y \to \infty \Rightarrow \frac{m\upsilon C^*}{DV_0} = 0$, i.e., $C^* = 0$ when $y^* \to \infty$

and $H_x = 0$ when $y \to \infty \Rightarrow \frac{H^*}{\sqrt{\frac{\mu_e}{\rho}\frac{1}{V_0}}} = 0$, i.e., $H^* = 0$ when $y^* \to \infty$

Therefore, ignoring the asterisk (*), we obtain (3.14) – (3.17) as follows:

$$\frac{d^2u}{dy^2} + M\frac{dH}{dy} + \frac{du}{dy} = -G_r\theta - G_m C \tag{3.18}$$

$$\frac{1}{P_m}\frac{d^2H}{dy^2} + M\frac{du}{dy} + \frac{dH}{dy} = 0 \tag{3.19}$$

$$\frac{d^2\theta}{dy^2} + P_r\frac{d\theta}{dy} = -P_r E_c\left[\left(\frac{du}{dy}\right)^2 + \frac{1}{P_m}\left(\frac{dH}{dy}\right)^2\right] + D_f\frac{d^2C}{dy^2} \tag{3.20}$$

$$\frac{dC}{dy} + \frac{1}{S_c}\frac{d^2C}{dy^2} = 0 \tag{3.21}$$

with corresponding boundary conditions

$$\left.\begin{aligned} y = 0 &: u = U, \ \frac{d\theta}{dy} = -1, \ \frac{dC}{dy} = -1, \ H = h \ \text{(say)} \\ y \to \infty &: u = 0, \ \theta = 0, \ C = 0, \ H = 0 \end{aligned}\right\} \tag{3.22}$$

CHAPTER 4

Numerical Solution and Discussion

4.1 Numerical Solution

The simplest solution of equation (3.21) can be obtained as follows:

$$\frac{dC}{dy}+\frac{1}{S_c}\frac{d^2C}{dy^2}=0 \tag{4.1}$$

Let the trial solution of (4.1) be $C=e^{mx}$ (4.2)

So the auxiliary equation of equation (4.1) is

$$\frac{1}{S_c}m^2+m=0$$

$$\text{i.e. } m\left(1+\frac{m}{S_c}\right)=0$$

$$\therefore\ m=0 \text{ and } m=-S_c$$

Therefore, we have the trial solution $C=A+Be^{-s_c y}$

Applying boundary conditions on C from (3.22) we have

$C=0$ as $y\to\infty \Rightarrow 0=A$ i.e. $A=0$

$\frac{dC}{dy}=-1$ at $y=0$: then $-1=-S_cB$ i.e. $B=\frac{1}{S_c}$

Hence

$$C=\frac{1}{S_c}e^{-S_c y} \tag{4.3}$$

To obtain a complete solution of the coupled nonlinear system of equations (3.18) – (3.20) under boundary conditions (3.22), we introduce the perturbation approximation. Since the dependent variables u, H and θ mostly dependent on y only and the fluid is purely incompressible one, we expand the dependent variables in powers of Eckert number E_c which is small enough such that the terms in E_c^2 and its higher order can be neglected. Thus we assume

$$u(y)=u_1(y)+E_c u_2(y)+O\left(E_c^{\,2}\right)+\ldots\ldots\ldots\ldots$$
$$H(y)=H_1(y)+E_c H_2(y)+O\left(E_c^{\,2}\right)+\ldots\ldots\qquad(4.4)$$
$$\theta(y)=\theta_1(y)+E_c\theta_2(y)+O\left(E_c^{\,2}\right)+\ldots\ldots\ldots\ldots$$

Now, $\dfrac{du}{dy}=u_1'(y)+E_c u_2'(y)+\ldots\ldots\ldots$

$$\frac{d^2u}{dy^2}=u_1''(y)+E_c u_2''(y)+\ldots\ldots\ldots$$

$$\frac{dH}{dy}=H_1'(y)+E_c H_2'(y)+\ldots\ldots\ldots\ldots\ldots$$

$$\frac{d^2H}{dy^2}=H_1''(y)+E_c H_2''(y)+\ldots\ldots\ldots$$

$$\frac{d\theta}{dy}=\theta_1'(y)+E_c\theta_2'(y)+\ldots\ldots\ldots$$

$$\frac{d^2\theta}{dy^2}=\theta_1''(y)+E_c\theta_2''(y)+\ldots\ldots\ldots$$

Substituting equation (4.4) into (3.18), (3.19) and (3.20), we get-

From (3.18):

$$\frac{d^2u}{dy^2}+M\frac{dH}{dy}+\frac{du}{dy}=-G_r\theta-G_mC$$

or, $\left\{u_1''(y)+E_c u_2''(y)+\ldots\right\}+M\left\{H_1'(y)+E_c H_2'(y)+\ldots\right\}+\left\{u_1'(y)+E_c u_2'(y)+\ldots\right\}$

$$=-G_r\left\{\theta_1(y)+E_c\theta_2(y)+\ldots\right\}-G_mC$$

Equating the co-efficient of the like powers of E_c and neglecting the terms in E_c^2 and higher order, we have

$$u_1''+MH_1'+u_1'=-G_r\theta_1-G_mC\qquad(4.5)$$

$$u_2''+MH_2'+u_2'=-G_r\theta_2\qquad(4.6)$$

From (3.19):

$$\frac{1}{P_m}\frac{d^2H}{dy^2}+\frac{dH}{dy}+M\frac{du}{dy}=0$$

or, $\dfrac{1}{P_m}\left\{H_1''(y)+E_c H_2''(y)+\ldots\right\}+\left\{H_1'(y)+E_c H_2'(y)+\ldots\right\}+M\left\{u_1'(y)+E_c u_2'(y)+\ldots\right\}=0$

Separating like terms we get,

$$\frac{1}{P_m}H_1'' + H_1' + Mu_1' = 0$$

$$\therefore\ H_1'' + P_m H_1' + MP_m u_1' = 0 \tag{4.7}$$

$$\text{and } H_2'' + P_m H_2' + MP_m u_2' = 0 \tag{4.8}$$

From (3.20):

$$\theta_1'' + P_r\theta_1' = D_f C''$$

$$\text{and} \quad \theta_2'' + P_r\theta_2' = -P_r u_1'^2 - \frac{P_r}{P_m}H_1'^2$$

So substituting (4.4) into (3.18), (3.19) and (3.20) and equating the co-efficient of the like powers of E_c and neglecting the terms of E_c^2 and higher order, we obtain the equations of zero and first order approximations as follows:

Coefficient of $(E_c)^0$:

$$u_1'' + MH_1' + u_1' = -G_r\theta_1 - G_m C \tag{4.9}$$

$$H_1'' + P_m H_1' + MP_m u_1' = 0 \tag{4.10}$$

$$\theta_1'' + P_r\theta_1' = D_f C'' \tag{4.11}$$

and coefficient of $(E_c)^1$:

$$u_2'' + MH_2' + u_2' = -G_r\theta_2 \tag{4.12}$$

$$H_2'' + P_m H_2' + MP_m u_2' = 0 \tag{4.13}$$

$$\theta_2'' + P_r\theta_2' = -P_r u_1'^2 - \frac{P_r}{P_m}H_1'^2 \tag{4.14}$$

with the corresponding boundary conditions

$$\left.\begin{array}{l} u_1 = U,\ u_2 = 0,\ H_1 = h,\ H_2 = 0,\ \theta_1' = -1,\ \ \theta_2' = 0 \ \text{ at } y = 0 \\ u_1 = 0,\ u_2 = 0,\ H_1 \to 0,\ H_2 \to 0,\ \theta_1 \to 0,\ \theta_2 \to 0 \ \text{ as } y \to \infty \end{array}\right\} \tag{4.15}$$

Finally, equations (4.9) – (4.14) together with the boundary conditions (4.15) can be written separately as follows:

$$\frac{d^2u_1}{dy^2} + M\frac{dH_1}{dy} + \frac{du_1}{dy} = -G_r\theta_1 - \frac{G_m}{S_c}e^{-S_c y} \tag{4.16}$$

$$\frac{d^2H_1}{dy^2}+P_m\frac{dH_1}{dy}+MP_m\frac{du_1}{dy}=0 \quad (4.17)$$

$$\frac{d^2\theta_1}{dy^2}+P_r\frac{d\theta_1}{dy}=S_cD_fe^{-S_cy} \quad (4.18)$$

with boundary conditions

$$\left.\begin{array}{l} y=0\ : u_1=U,\ \dfrac{d\theta_1}{dy}=-1,\ H_1=h \\ y\to\infty : u_1=0,\ \theta_1=0,\ H_1=0 \end{array}\right\} \quad (4.19)$$

$$\frac{d^2u_2}{dy^2}+M\frac{dH_2}{dy}+\frac{du_2}{dy}=-G_r\theta_2 \quad (4.20)$$

$$\frac{d^2H_2}{dy^2}+P_m\frac{dH_2}{dy}+MP_m\frac{du_2}{dy}=0 \quad (4.21)$$

$$\frac{d^2\theta_2}{dy^2}+P_r\frac{d\theta_2}{dy}=-P_r\left[\left(\frac{du_1}{dy}\right)^2+\frac{1}{P_m}\left(\frac{dH_1}{dy}\right)^2\right] \quad (4.22)$$

with boundary conditions

$$\left.\begin{array}{l} y=0\ : u_2=0,\ \dfrac{d\theta_2}{dy}=0,\ H_2=0 \\ y\to\infty : u_2=0,\ \theta_2=0,\ H_2=0 \end{array}\right\} \quad (4.23)$$

Now we are interested to solve equation (4.16) – (4.18) with boundary conditions (4.19) and equation (4.20) – (4.22) with boundary conditions (4.23).

From equation (4.18) we have

$$\frac{d^2\theta_1}{dy^2}+P_r\frac{d\theta_1}{dy}=S_cD_fe^{-S_cy}$$

Here the auxiliary equation is obtained by

$$m^2+P_rm=0$$

$$\text{or, } m(m+P_r)=0$$

$$\therefore m=0 \text{ and } m=-P_r$$

The complementary function is obtained by $\theta_{1c}=A+Be^{-P_ry}$

Now the particular integral $\theta_{1p}=\dfrac{D_fS_ce^{-S_cy}}{(-S_c)^2-P_rS_c}$

$$= -\frac{D_f e^{-S_c y}}{(P_r - S_c)}$$

General solution is $\theta_1 = \theta_{1c} + \theta_{1p} = A + Be^{-P_r y} - \frac{D_f e^{-S_c y}}{P_r - S_c}$

Applying boundary conditions:

$\theta_1 = 0$ as $y \to \infty \Rightarrow A = 0$

and

$\theta_1' = -1$ at $y = 0$ gives

$$-1 = -BP_r + \frac{D_f S_c}{P_r - S_c}$$

$$\therefore B = \frac{1}{P_r}\left(1 + \frac{D_f S_c}{P_r - S_c}\right)$$

Hence $\theta_1 = \frac{1}{P_r}\left(1 + \frac{D_f S_c}{P_r - S_c}\right)e^{-P_r y} - \frac{D_f e^{-S_c y}}{(P_r - S_c)}$

or, $\theta_1 = \frac{1}{P_r}\left(1 + \frac{D_f S_c}{P_r - S_c}\right)e^{-P_r y} - \frac{D_f S_c e^{-S_c y}}{S_c(P_r - S_c)}$

Let us consider $\frac{D_f S_c}{P_r - S_c} = n_7$

Then $\theta_1 = \frac{1}{P_r}(1 + n_7)e^{-P_r y} - \frac{1}{S_c} n_7 e^{-S_c y}$ (4.24)

Now we have from (4.16) and (4.17),

$$u_1'' + MH_1' + u_1' = -G_r \theta_1 - G_m C \tag{4.25}$$

$$H_1'' + P_m H_1' + MP_m u_1' = 0 \tag{4.26}$$

Now from (4.26) we obtain

$$u_1' = -\frac{1}{MP_m}\left(H_1'' + P_m H_1'\right)$$

$$\therefore u_1'' = -\frac{1}{MP_m}\left(H_1''' + P_m H_1''\right)$$

Substituting the above two relations together with (4.3) and (4.24) into equation (4.25) we have

$$-\frac{1}{MP_m}\left(H_1''' + P_m H_1''\right) + MH_1' - \frac{1}{MP_m}\left(H_1'' + P_m H_1'\right) = -G_r\left\{\frac{(1+n_7)}{P_r}e^{-P_r y} - \frac{n_7}{S_c}e^{-S_c y}\right\} - \frac{G_m}{S_c}e^{-S_c y}$$

or,

$$H_1''' + P_m H_1'' - M^2 P_m H_1' + H_1'' + P_m H_1' = G_r MP_m\left\{\frac{(1+n_7)}{P_r}e^{-P_r y} - \frac{n_7}{S_c}e^{-S_c y}\right\} + \frac{G_m MP_m}{S_c}e^{-S_c y}$$

or, $$H_1''' + (1+P_m)H_1'' + P_m\left(1-M^2\right)H_1' = G_r MP_m\left\{\frac{(1+n_7)}{P_r}e^{-P_r y} - \frac{n_7}{S_c}e^{-S_c y}\right\} + \frac{G_m MP_m}{S_c}e^{-S_c y}$$

Here auxiliary equation is $m^3 + (1+P_m)m^2 + P_m\left(1-M^2\right)m = 0$

For simplicity, let us consider, $1+P_m = n_1$ and $P_m(1-M^2) = n_2$

Then $m^3 + n_1 m^2 + n_2 m = 0$

or, $m\left(m^2 + n_1 m + n_2\right) = 0$

i.e. $m = 0$ and $m = \dfrac{-n_1 \pm \sqrt{n_1^2 - 4n_2}}{2}$

The complementary function is $H_{1c} = A + Be^{\frac{-n_1-\sqrt{n_1^2-4n_2}}{2}y} + Ce^{\frac{-n_1+\sqrt{n_1^2-4n_2}}{2}y}$

Particular integral is $$H_{1p} = \frac{G_r MP_m\left\{\frac{(1+n_7)}{P_r}e^{-P_r y} - \frac{n_7}{S_c}e^{-S_c y}\right\} + \frac{G_m MP_m}{S_c}e^{-S_c y}}{D^3 + n_1 D^2 + n_2 D}$$

or, $$H_{1p} = \frac{\frac{G_r MP_m}{P_r}(1+n_7)e^{-P_r y}}{(-P_r)^3 + n_1(-P_r)^2 - n_2 P_r} - \frac{\frac{G_r MP_m}{S_c}n_7 e^{-S_c y}}{(-S_c)^3 + n_1(-S_c)^2 - n_2 S_c} + \frac{\frac{G_m MP_m}{S_c}e^{-S_c y}}{(-S_c)^3 + n_1(-S_c)^2 - n_2 S_c}$$

$$= \frac{MP_m G_r (1+n_7)e^{-P_r y}}{-P_r^2\left(P_r^2 - n_1 P_r + n_2\right)} - \frac{MP_m G_r\, n_7 e^{-S_c y}}{-S_c^2\left(S_c^2 - n_1 S_c + n_2\right)} + \frac{MP_m G_m\, e^{-Scy}}{-S_c^2\left(S_c^2 - n_1 S_c + n_2\right)}$$

Choosing $\dfrac{G_r MP_m}{P_r^2} = n_3$; $\dfrac{G_m MP_m}{S_c^2} = n_4$; $\dfrac{n_3}{P_r^2 - n_1 P_r + n_2} = n_5$ and $\dfrac{n_4}{S_c^2 - n_1 S_c + n_2} = n_6$

we have $H_{1p} = -n_5(1+n_7)e^{-P_r y} + \dfrac{G_r}{G_m}n_6 n_7 e^{-S_c y} - n_6 e^{-S_c y}$

$$= -n_5(1+n_7)e^{-P_r y} - n_6\left(1 - \frac{G_r}{G_m}n_7\right)e^{-S_c y}$$

So the general solution is

$$H_1 = H_{1c} + H_{1p} = A + Be^{\frac{-n_1-\sqrt{n_1^2-4n_2}}{2}y} + Ce^{\frac{-n_1+\sqrt{n_1^2-4n_2}}{2}y} - n_5(1+n_7)e^{-P_r y} - n_6\left(1-\frac{G_r}{G_m}n_7\right)e^{-S_c y}$$

Applying boundary conditions:

$H_1 = h$ at $y = 0$, we have

$$h = A + B + C - n_5(1+n_7) - n_6\left(1-\frac{G_r}{G_m}n_7\right)$$

and at $y \to \infty$, $H_1 = 0 \Rightarrow C = 0$

which gives $A = 0$.

Therefore we have $B = h + n_5(1+n_7) + n_6\left(1-\frac{G_m}{G_r}n_7\right)$

If we consider $\frac{n_1+\sqrt{n_1^2-4n_2}}{2} = A_1$, then

$$H_1(y) = \left\{h + n_5(1+n_7) + n_6\left(1-\frac{G_r}{G_m}n_7\right)\right\}e^{-A_1 y} - n_5(1+n_7)e^{-P_r y} - n_6\left(1-\frac{G_r}{G_m}n_7\right)e^{-S_c y}$$

Further from (4.25) we have $u_1'' + u_1' = -G_r\theta_1 - G_m C - MH_1'$

i.e. $u_1'' + u_1' = -\frac{G_r}{P_r}(1+n_7)e^{-P_r y} + \frac{G_r}{S_c}n_7 e^{-S_c y} - \frac{G_m}{S_c}e^{-S_c y} - M\Big[-A_1\Big\{h + n_5(1+n_7)$

$$+n_6\left(1-\frac{G_r}{G_m}n_7\right)\Big\}e^{-A_1 y} + P_r n_5(1+n_7)e^{-P_r y} + S_c n_6\left(1-\frac{G_r}{G_m}n_7\right)e^{-S_c y}\Big]$$

$$\left[\because \theta_1 = \frac{1}{P_r}(1+n_7)e^{-P_r y} - \frac{1}{S_c}n_7 e^{-S_c y}\right]$$

Here, auxiliary equation is $m^2 + m = 0$

or, $m(m+1) = 0$

i.e. $m = 0$ and $m = -1$

Complementary function is $u_{1c} = A + Be^{-y}$

Particular integral is

$$u_{1p} = \frac{-M\left[-A_1\left\{h + n_5(1+n_7) + n_6\left(1-\frac{G_r}{G_m}n_7\right)\right\}e^{-A_1 y} + P_r n_5(1+n_7)e^{-P_r y} + S_c n_6\left(1-\frac{G_r}{G_m}n_7\right)e^{-S_c y}\right]}{D^2 + D}$$

$$= \frac{-\left\{\frac{G_r}{P_r}(1+n_7) + MP_r n_5 (1+n_7)\right\} e^{-P_r y}}{(-P_r)^2 - P_r} - \frac{\left\{\frac{G_m}{S_c} - \frac{G_r}{S_c} n_7 + M S_c n_6 \left(1 - \frac{G_r}{G_m} n_6\right) e^{-S_c y}\right\}}{(-S_c)^2 - S_c}$$

$$- \frac{MA_1 \left\{h + n_5 (1+n_7) + n_6 \left(1 - \frac{G_r}{G_m} n_7\right)\right\} e^{-A_1} y}{(-A_1)^2 - A_1}$$

$$= \frac{-\left\{G_r (1+n_7) + MP_r^2 n_5 (1+n_7)\right\} e^{-P_r y}}{-P_r^{\,2}(1-P_r)} - \frac{\left\{G_m - G_r n_7 + MS_c^2 n_6 \left(1 - \frac{G_r}{G_m} n_7\right)\right\} e^{-S_c y}}{-S_c^2 (1-S_c)}$$

$$+ \frac{MA_1 \left\{h + n_5 (1+n_7) + n_6 \left(1 - \frac{G_r}{G_m} n_7\right)\right\} e^{-A_1 y}}{-A_1 (1-A_1)}$$

Choosing

$$\frac{M\left\{h + n_5 (1+n_7) + n_6 \left(1 - \frac{G_r}{G_m} n_7\right)\right\}}{(1-A_1)} = n_8 \, ;$$

$$\frac{G_r (1+n_7) + MP_r^2 n_5 (1+n_7)}{P_r^2 (1-P_r)} = n_9 \, ;$$

$$\frac{G_m - G_r n_7 + MS_c^2 n_6 \left(1 - \frac{G_r}{G_m} n_7\right)}{S_c^2 (1-S_c)} = n_{10} \, ;$$

The particular integral is $u_{1p} = -n_8 e^{-A_1 y} + n_9 e^{-P_r y} + n_{10} e^{-S_c y}$

The general solution is $u_1 = u_{1c} + u_{1p} = A + Be^{-y} - n_8 e^{-A_1 y} + n_9 e^{-P_r y} + n_{10} e^{-S_c y}$

Using boundary conditions:

At $y = 0$, $u_1 = U \Rightarrow U = A + B - n_8 + n_9 + n_{10}$

i.e. $U = A + B - n_8 + n_9 + n_{10}$

and at $y \to \infty$, $u_1 = 0 \Rightarrow A = 0$

Hence $B = U + n_8 - n_9 - n_{10}$

Therefore,

$u_1 = (U + n_8 - n_9 - n_{10}) e^{-y} - n_8 e^{-A_1 y} + n_9 e^{-P_r y} + n_{10} e^{-S_c y}$

Thus solving the above equations (4.16) – (4.18) under boundary conditions (4.19), we get

$$u_1(y) = (U + n_8 - n_9 - n_{10})e^{-y} - n_8 e^{-A_1 y} + n_9 e^{-P_r y} + n_{10} e^{-S_c y} \tag{4.27}$$

$$H_1(y) = \left\{ h + n_5(1+n_7) + n_6\left(1 - \frac{G_r}{G_m} n_7\right)\right\} e^{-A_1 y} - n_5(1+n_7)e^{-P_r y} - n_6\left(1 - \frac{G_r}{G_m} n_7\right) e^{-S_c y} \tag{4.28}$$

$$\theta_1(y) = \frac{1}{P_r}(1+n_7)e^{-P_r y} - \frac{1}{S_c} n_7 e^{-S_c y} \tag{4.29}$$

Similarly solving equations (4.20) – (4.22) under boundary conditions (4.23), we get

$$u_2'' + u_2' + MH_2' = -G_r \theta_2 \tag{4.30}$$

$$H_2'' + P_m H_2' + MP_m u_2' = 0 \tag{4.31}$$

$$\theta_2'' + P_r \theta_2' = -P_r\left(u_1'\right)^2 - \frac{P_r}{P_m}\left(H_1'\right)^2 \tag{4.32}$$

From equation (4.32) we have auxiliary equation

$$m^2 + P_r m = 0$$

or, $m(m + P_r) = 0$

i.e. $m = 0$ and $m = -P_r$

Hence the complementary function is $\theta_{2c} = A + Be^{-P_r y}$

Now the particular integral is $\theta_{2p} = \dfrac{-P_r\left(u_1'\right)^2 - \dfrac{P_r}{P_m}\left(H_1'\right)^2}{D^2 + P_r D}$

$$\theta_{2p} = \frac{-P_r\left\{-(U + n_8 - n_9 - n_{10})e^{-y} + A_1 n_8 e^{-A_1 y} - P_r n_9 e^{-P_r y} - S_c n_{10} e^{-S_c y}\right\}^2}{D^2 + P_r D}$$

$$-\frac{\dfrac{P_r}{P_m}\left[-A_1\left\{h + n_5(1+n_7) + n_6\left(1 - \dfrac{G_r}{G_m} n_7\right)\right\} e^{-A_1 y} + P_r n_5(1+n_7)e^{-P_r y} + S_c n_6\left(1 - \dfrac{G_r}{G_m} n_7\right) e^{-S_c y}\right]^2}{D^2 + P_r D}$$

$$= -P_r\Big[(U + n_8 - n_9 - n_{10})^2 e^{-2y} + A_1^2 n_8^2 e^{-2A_1 y} + P_r^2 n_9^2 e^{-2P_r y} + S_c^2 n_{10}^2 e^{-2S_c y}$$

$$-2A_1n_8\left(U+n_8-n_9-n_{10}\right)e^{-\left(1+A_1\right)y}-2A_1P_rn_8n_9e^{-\left(A_1+P_r\right)y}$$

$$+2P_rn_9\left(U+n_8-n_9-n_{10}\right)e^{-(1+P_r)y}+2S_cn_{10}\left(U+n_8-n_9-n_{10}\right)e^{-(1+S_c)y}$$

$$\left.-2A_1S_cn_8n_{10}e^{-\left(A_1+S_c\right)y}+2P_rS_cn_9n_{10}e^{-(P_r+S_c)y}\right]$$

$$-\frac{P_r}{P_m}\left[A_1^{\,2}\left\{h+n_5\left(1+n_7\right)+n_6\left(1-\frac{G_r}{G_m}n_7\right)\right\}^2e^{-2A_1y}+P_r^2n_5^2\left(1+n_7\right)^2e^{-2P_ry}\right.$$

$$+S_c^{\,2}n_6^{\,2}\left(1-n_7\frac{G_r}{G_m}\right)^2e^{-2S_cy}-2A_1P_rn_5\left(1+n_7\right)\left\{h+n_5\left(1+n_7\right)+n_6\left(1-n_7\frac{G_r}{G_m}\right)\right\}e^{-\left(A_1+P_r\right)y}$$

$$\frac{\left.+2P_rS_cn_5n_6\left(1+n_7\right)\left(1-\frac{G_r}{G_m}n_7\right)e^{-(P_r+S_c)y}-2A_1S_cn_6\left(1-\frac{G_r}{G_m}n_7\right)\left\{h+n_5\left(1+n_7\right)+n_6\left(1-\frac{G_r}{G_m}n_7\right)\right\}e^{-\left(A_1+S_c\right)y}\right]}{D^2+P_rD}$$

If we choose $P_m=1$, the particular integral is

$$\theta_{2p}=-P_r\left[\frac{\left(U+n_8-n_9-n_{10}\right)^2e^{-2y}}{(-2)^2-2P_r}+\frac{A_1^2\left[\left\{h+n_5\left(1+n_7\right)+n_6\left(1-\frac{G_r}{G_m}n_7\right)\right\}^2+n_8^2\right]e^{-2A_1y}}{\left(-2A_1\right)^2-2P_rA_1}\right.$$

$$+\frac{P_r^2\left\{n_5^2\left(1+n_7\right)^2+n_9^2\right\}e^{-2P_ry}}{\left(-2P_r\right)^2-2P_r^{\,2}}+\frac{S_c^2\left\{n_6^2\left(1-\frac{G_r}{G_m}n_7\right)^2+n_{10}^2\right\}e^{-2S_cy}}{\left(-2S_c\right)^2-2S_cP_r}$$

$$-\frac{2A_1n_8\left(U+n_8-n_9-n_{10}\right)e^{-\left(1+A_1\right)y}}{\left\{-\left(1+A_1\right)\right\}^2-\left(1+A_1\right)P_r}+\frac{2P_rn_9\left(U+n_8-n_9-n_{10}\right)e^{-(1+P_r)y}}{\left\{-\left(1+P_r\right)\right\}^2-P_r\left(1+P_r\right)}$$

$$+\frac{2S_cn_{10}\left(U+n_8-n_9-n_{10}\right)e^{-(1+S_c)y}}{\left\{-\left(1+S_c\right)\right\}^2-P_r\left(1+S_c\right)}$$

$$+\frac{2A_1P_r\left[n_5\left(1+n_7\right)\left\{h+n_5\left(1+n_7\right)+n_6\left(1-\frac{G_r}{G_m}n_7\right)\right\}+n_8n_9\right]e^{-\left(A_1+P_r\right)y}}{\left\{-\left(A_1+P_r\right)\right\}^2-P_r\left(A_1+P_r\right)}$$

$$+\frac{2P_rS_c\left\{n_5n_6\left(1+n_7\right)\left(1-\frac{G_r}{G_m}n_7\right)+n_9n_{10}\right\}\ e^{-(P_r+S_c)y}}{\left\{-\left(P_r+S_c\right)\right\}^2-P_r\left(P_r+S_c\right)}$$

$$\left.-\frac{2A_1S_c\left[n_6\left(1-\frac{G_r}{G_m}n_7\right)\left\{h+n_5(1+n_7)+n_6\left(1-\frac{G_r}{G_m}n_7\right)\right\}+n_8n_{10}\right]e^{-(A_1+S_c)y}}{\left\{-\left(A_1+S_c\right)\right\}^2-P_r\left(A_1+S_c\right)}\right]$$

$$=-P_r\left[\frac{\left(U+n_8-n_9-n_{10}\right)^2e^{-2y}}{-2\left(P_r-2\right)}+\frac{A_1^2\left[\left\{h+n_5(1+n_7)+n_6\left(1-\frac{G_r}{G_m}n_7\right)\right\}+n_8^2\right]e^{-2A_1y}}{-2A_1\left(P_r-2A_1\right)}\right.$$

$$+\frac{P_r^2\left\{n_5^2\left(1+n_7\right)^2+n_9^2\right\}e^{-2P_ry}}{2P_r^2}+\frac{S_c^2\left\{n_6^2\left(1-\frac{G_r}{G_m}n_7\right)^2+n_{10}^2\right\}e^{-2S_cy}}{-2S_c\left(P_r-2S_c\right)}$$

$$-\frac{2A_1n_8\left(U+n_8-n_9-n_{10}\right)e^{-(1+A_1)y}}{-\left(1+A_1\right)\left(P_r-1-A_1\right)}+\frac{2P_rn_9\left(U+n_8-n_9-n_{10}\right)e^{-(1+P_r)y}}{1+P_r}$$

$$+\frac{2S_cn_{10}\left(U+n_8-n_9-n_{10}\right)e^{-(1+S_c)y}}{-\left(1+S_c\right)\left(P_r-1-S_c\right)}$$

$$-\frac{2A_1P_r\left[n_5\left(1+n_7\right)\left\{h+n_5\left(1+n_7\right)+n_6\left(1-\frac{G_r}{G_m}n_7\right)\right\}+n_8n_9\right]e^{-(A_1+P_r)y}}{A_1\left(A_1+P_r\right)}$$

$$+\frac{2P_rS_c\left\{n_5n_6\left(1+n_7\right)\left(1-\frac{G_r}{G_m}n_7\right)+n_9n_{10}\right\}e^{-(P_r+S_c)y}}{S_c\left(P_r+S_c\right)}$$

$$\left.-\frac{2A_1S_c\left[n_6\left(1-\frac{G_r}{G_m}n_7\right)\left\{h+n_5\left(1+n_7\right)+n_6\left(1-\frac{G_r}{G_m}n_7\right)\right\}+n_8n_{10}\right]e^{-(A_1+S_c)y}}{-\left(A_1+S_c\right)\left(P_r-A_1-S_c\right)}\right]$$

Further choosing

$$\frac{(U+n_8-n_9-n_{10})^2}{-2(P_r-2)}=n_{11}; \quad \frac{A_1\left\{h+n_5(1+n_7)+n_6\left(1-\frac{G_r}{G_m}n_7\right)+n_8^2\right\}}{2(P_r-2A_1)}=n_{12};$$

$$\frac{n_5^2(1+n_7)^2+n_9^2}{2}=n_{13}; \quad \frac{S_c\left\{n_6^2\left(1-\frac{G_r}{G_m}n_7\right)^2+n_{10}^2\right\}}{2(P_r-2S_c)}=n_{14};$$

$$\frac{2A_1n_8(U+n_8-n_9-n_{10})}{-(1+A_1)(P_r-1-A_1)}=n_{15}; \quad \frac{2P_rn_9(U+n_8-n_9-n_{10})}{1+P_r}=n_{16};$$

$$\frac{2S_cn_{10}(U+n_8-n_9-n_{10})}{-(1+S_c)(P_r-1-S_c)}=n_{17};$$

$$\frac{2P_r\left[n_5(1+n_7)\left\{h+n_5(1+n_7)+n_6\left(1-\frac{G_r}{G_m}n_7\right)\right\}+n_8n_9\right]}{A_1+P_r}=n_{18};$$

$$\frac{2P_r\left\{n_5n_6(1+n_7)\left(1-\frac{G_r}{G_m}n_7\right)+n_9n_{10}\right\}}{P_r+S_c}=n_{19};$$

$$\frac{2A_1S_c\left[n_6\left(1-\frac{G_r}{G_m}n_7\right)\left\{h+n_5(1+n_7)+n_6\left(1-\frac{G_r}{G_m}n_7\right)\right\}+n_8n_{10}\right]}{-(A_1+S_c)(P_r-A_1-S_c)}=n_{20}$$

the particular integral is now

$$\theta_{2p}=-P_r\left[n_{11}e^{-2y}-n_{12}^{-A_1y}+n_{13}e^{-2P_ry}-n_{14}e^{-2S_cy}-n_{15}e^{-(1+A_1)y}+n_{16}e^{-(1+P_r)y}\right.$$
$$\left.+n_{17}e^{-(1+S_c)y}-n_{18}e^{(A_1+P_r)y}+n_{19}e^{(P_r+S_c)y}-n_{20}e^{-(A_1+S_c)y}\right]$$

The general solution of θ_2 is

$$\theta_2=\theta_{2c}+\theta_{2p}=A+Be^{-P_ry}-P_r\left[n_{11}e^{-2y}-n_{12}^{-2A_1y}+n_{13}e^{-2P_ry}-n_{14}e^{-2S_cy}-n_{15}e^{-(1+A_1)y}\right.$$
$$\left.+n_{16}e^{-(1+P_r)y}+n_{17}e^{-(1+S_c)y}-n_{18}e^{-(A_1+P_r)y}+n_{19}e^{-(P_r+S_c)y}-n_{20}e^{-(A_1+S_c)y}\right]$$

Applying the boundary conditions:

$\theta_2'=0$ at $y=0$ implies

$$0=-BP_r-P_r\Big[-2n_{11}+2A_1n_{12}-2P_rn_{13}+2S_cn_{14}+\left(1+A_1\right)n_{15}-\left(1+P_r\right)n_{16}-\left(1+S_c\right)n_{17}$$
$$+\left(A_1+P_r\right)n_{18}-\left(P_r+S_c\right)n_{19}+\left(A_1+S_c\right)n_{20}\Big]$$

$\theta_2=0$ as $y\to\infty$ implies $A=0$

Then $B=2n_{11}-2A_1n_{12}+2P_rn_{13}-2S_cn_{14}-\left(1+A_1\right)n_{15}+\left(1+P_r\right)n_{16}+\left(1+S_c\right)n_{17}$
$$-\left(A_1+P_r\right)n_{18}+\left(P_r+S_c\right)n_{19}-\left(A_1+S_c\right)n_{20}\Big]=A_2\quad\text{(Say)}$$

Hence $\theta_2(y)=A_2e^{-P_ry}-P_r\Big[n_{11}e^{-2y}-n_{12}{}^{-2A_1y}+n_{13}e^{-2P_ry}-n_{14}e^{-2S_cy}-n_{15}e^{-(1+A_1)y}+n_{16}e^{-(1+P_r)y}$
$$+n_{17}e^{-(1+S_c)y}-n_{18}e^{-(A_1+P_r)y}+n_{19}e^{-(P_r+S_c)y}-n_{20}e^{-(A_1+S_c)y}\Big]\quad(4.33)$$

From (4.31) we have

$$u_2'=-\frac{1}{MP_m}\left(H_2''+P_mH_2'\right)$$

Substituting u_2' and u_2'' in equation (4.20) we get,

$$u_2''=-\frac{1}{MP_m}\left(H_2'''+P_mH_2''\right)-\frac{1}{MP_m}\left(H_2'''+P_mH_2''\right)-\frac{1}{MP_m}\left(H_2''+P_mH_2'\right)+MH_2'=-G_r\theta_2$$

or, $H_2'''+P_mH_2''+H_2''+P_mH_2'-M^2P_mH_2'=MP_mG_r\theta_2$

or,

$$H_2'''+\left(1+P_m\right)H_2''+P_m\left(1-M^2\right)H_2'=MP_mG_r\Big[A_2e^{-P_ry}-P_r\Big\{n_{11}e^{-2y}-n_{12}{}^{-2A_1y}+n_{13}e^{-2P_ry}$$
$$-n_{14}e^{-2S_cy}-n_{15}e^{-(1+A_1)y}+n_{16}e^{-(1+P_r)y}+n_{17}e^{-(1+S_c)y}-n_{18}e^{-(A_1+P_r)y}+n_{19}e^{-(P_r+S_c)y}-n_{20}e^{-(A_1+S_c)y}\Big\}\Big]$$

Now auxiliary equation is

$$m^3+n_1m^2+n_2m=0$$

or, $m\left(m^2+n_1m+n_2\right)=0$

i.e. $m=0$ and $m=\dfrac{-n_1\pm\sqrt{n_1^2-4n_2}}{2}$

Therefore the complementary function is $H_{2c}=A+Be^{\frac{-n_1-\sqrt{n_1^2-4n_2}}{2}y}+Ce^{\frac{-n_1+\sqrt{n_1^2-4n_2}}{2}y}$

and the particular integral is

$$H_{2p}=\frac{MP_mG_rA_2e^{-P_ry}}{(-P_r)^3+(-P_r)^2n_1-n_2P_r}-MP_mP_rG_r\left\{\frac{n_{11}e^{-2y}}{(-2)^3+(-2)^2n_1-2n_2}\right.$$

$$-\frac{n_{12}e^{-2A_1y}}{(-2A_1)^3+(-2A_1)^2n_1-2A_1n_2}+\frac{n_{13}e^{-2P_ry}}{(-2P_r)^3+(-2P_r)^2n_1-2P_rn_2}$$

$$-\frac{n_{14}e^{-2S_cy}}{(-2S_c)^3+(-2S_c)^2n_1-2S_cn_2}-\frac{n_{15}e^{-(1+A_1)y}}{\{-(1+A_1)\}^3+(1+A_1)^2n_1-(1+A_1)n_2}$$

$$+\frac{n_{16}e^{-(1+P_r)y}}{\{-(1+P_r)\}^3+(1+P_r)^2n_1-(1+P_r)n_2}+\frac{n_{17}e^{-(1+S_c)y}}{\{-(1+S_c)\}^3+(1+S_c)^2n_1-(1+S_c)n_2}$$

$$-\frac{n_{18}e^{-(A_1+P_r)y}}{\{-(A_1+P_r)\}^3+(A_1+P_r)^2n_1-(A_1+P_r)n_2}+\frac{n_{19}e^{-(P_r+S_c)y}}{\{-(P_r+S_c)\}^3+(P_r+S_c)^2n_1-(P_r+S_c)n_2}$$

$$\left.-\frac{n_{20}e^{-(A_1+S_c)y}}{\{-(A_1+S_c)\}^3+(A_1+S_c)^2n_1-(A_1+S_c)n_2}\right\}$$

$$=\frac{MP_mP_rG_rA_2e^{-P_ry}}{-P_r^2\left(P_r^2-n_1P_r+n_2\right)}-MP_mP_rG_r\left[\frac{n_{11}e^{-2y}}{-2(4-2n_1+n_2)}-\frac{n_{12}e^{-2A_1y}}{-2A_1\left(4A_1^2-2A_1n_1+n_2\right)}\right.$$

$$+\frac{n_{13}e^{-P_ry}}{-2P_r\left(4P_r^2-2P_rn_1+n_2\right)}+\frac{n_{14}e^{-2S_cy}}{-2S_c\left(4S_c^2-2S_cn_1+n_2\right)}$$

$$-\frac{n_{15}e^{-(1+A_1)y}}{-(1+A_1)\left\{(1+A_1)^2-(1+A_1)n_1+n_2\right\}}+\frac{n_{16}e^{-(1+P_r)y}}{-(1+P_r)\left\{(1+P_r)^2-(1+P_r)n_1+n_2\right\}}$$

$$+\frac{n_{17}e^{-(1+S_c)}}{-(1+S_c)\left\{(1+S_c)^2-(1+S_c)n_1+n_2\right\}}-\frac{n_{18}e^{-(A_1+P_r)y}}{-(A_1+P_r)\left\{(A_1+P_r)^2-(A_1+P_r)n_1+n_2\right\}}$$

$$\left.+\frac{n_{19}e^{-(P_r+S_c)y}}{-(P_r+S_c)\left\{(P_r+S_c)^2-(P_r+\mathrm{S_c})n_1+n_2\right\}}-\frac{n_{20}e^{-(A_1+S_c)y}}{-(A_1+S_c)\left\{(A_1+S_c)^2-(A_1+S_c)n_1+n_2\right\}}\right]$$

If we choose

$$\frac{A_2}{P_r\left(P_r^2-P_rn_1+n_2\right)}=n_{21};\ \frac{n_{11}}{2(4-2n_1+n_2)}=n_{22};\ \frac{n_{12}}{2A_1\left(4A_1^2-2A_1n_1+n_2\right)}=n_{23};$$

$$\frac{n_{13}}{2P_r\left(4P_r^2-2P_rn_1+n_2\right)}=n_{24};\ \frac{n_{14}}{2S_c\left(4S_c^{\,2}-2S_cn_1+n_2\right)}=n_{25};$$

$$\frac{n_{15}}{\left(1+A_1\right)\left\{\left(1+A_1\right)^2-\left(1+A_1\right)P_r+n_2\right\}}=n_{26};\ \frac{n_{16}}{\left(1+P_r\right)\left\{\left(1+P_r\right)^2-\left(1+P_r\right)n_1+n_2\right\}}=n_{27};$$

$$\frac{n_{17}}{\left(1+S_c\right)\left\{\left(1+S_c\right)^2-\left(1+S_c\right)n_1+n_2\right\}}=n_{28};\ \frac{n_{18}}{\left(A_1+P_r\right)\left\{\left(A_1+P_r\right)^2-\left(A_1+P_r\right)n_1+n_2\right\}}=n_{29};$$

$$\frac{n_{19}}{\left(P_r+S_c\right)\left\{\left(P_r+S_c\right)^2-\left(P_r+S_c\right)n_1+n_2\right\}}=n_{30};\ \frac{n_{20}}{\left(A_1+S_c\right)\left\{\left(A_1+S_c\right)^2-\left(A_1+S_c\right)n_1+n_2\right\}}=n_{31}$$

and $P_m=1$, then the particular integral is

$$H_{2p}=MP_rG_r\Big[-n_{21}e^{-P_ry}+n_{22}e^{-2y}-n_{23}e^{-2A_1y}+n_{24}e^{-2P_ry}-n_{25}e^{-2S_cy}-n_{26}e^{-\left(1+A_1\right)y}+n_{27}e^{-\left(1+P_r\right)y}$$

$$+n_{28}e^{-\left(1+S_c\right)y}-n_{29}e^{-\left(A_1+P_r\right)y}+n_{30}e^{-\left(P_r+S_c\right)y}-n_{31}e^{-\left(A_1+S_c\right)y}\Big]$$

General solution of H_2 is

$$H_2=H_{2c}+H_{2p}=A+Be^{\frac{-n_1-\sqrt{n_1^{\,2}-4n_2}}{2}y}+Ce^{\frac{-n_1+\sqrt{n_1^{\,2}-4n_2}}{2}y}+Mp_rG_r\Big[-n_{21}e^{-p_ry}$$

$$+n_{22}e^{-2y}-n_{23}e^{-2A_1y}+n_{24}e^{-2P_ry}-n_{25}e^{-2S_cy}-n_{26}e^{-\left(1+A_1\right)y}+n_{27}e^{-\left(1+P_r\right)y}$$

$$+n_{28}e^{-\left(1+S_c\right)y}-n_{29}e^{-\left(A_1+P_r\right)y}+n_{30}e^{-\left(P_r+S_c\right)y}-n_{31}e^{-\left(A_1+S_c\right)y}\Big]$$

with respect to boundary conditions:

$H_2=0$ at $y=0$ implies

$$A+B+C+Mp_rG_r\left[-n_{21}+n_{22}-n_{23}+n_{24}-n_{25}-n_{26}+n_{27}+n_{28}-n_{29}+n_{30}-n_{31}\right]=0$$

and $H_2=0$ as $y\to\infty \ \Rightarrow C=0$.

Hence $A=0$.

Then

$$B=-MP_rG_r\left[-n_{21}+n_{22}-n_{23}+n_{24}-n_{25}-n_{26}+n_{27}+n_{28}-n_{29}+n_{30}-n_{31}\right]$$

or, $B=MP_rG_rA_3$

where $A_3=n_{21}-n_{22}+n_{23}-n_{24}+n_{25}+n_{26}-n_{27}-n_{28}+n_{29}-n_{30}+n_{31}$

So $H_2(y)=MP_rG_r\Big[A_3e^{-A_1y}-n_{21}e^{-P_ry}+n_{22}e^{-2y}-n_{23}e^{-2A_1y}+n_{24}e^{-2P_ry}-n_{25}e^{2S_cy}$

$$-n_{26}e^{-(1+A_1)y}+n_{27}e^{-(1+P_r)y}+n_{28}e^{-(1+S_c)y}-n_{29}{}^{-(A_1+P_r)y}+n_{30}e^{-(P_r+S_c)y}-n_{31}e^{-(A_1+S_c)y}\Big] \qquad (4.34)$$

From (4.20) we have $u_2''+u_2'=-G_r\theta_2-MH_2'$

$$u_2''+u_2'=-G_r\Big[A_2e^{-P_ry}-P_r\Big\{n_{11}e^{-2y}-n_{12}e^{2A_1y}+n_{13}e^{-2P_ry}-n_{14}e^{-2S_cy}-n_{15}e^{-(1+A_1)y}$$

$$+n_{16}e^{-(1+P_r)y}+n_{17}e^{-(1+S_c)y}-n_{18}e^{-(A_1+P_r)y}+n_{19}e^{-(P_r+S_c)y}-n_{20}e^{-(A_1+S_c)y}\Big\}\Big]$$

$$-M^2P_rG_r\Big[-A_1A_3e^{-A_1y}+P_rn_{21}e^{-P_ry}-2n_{22}e^{-2y}+2n_{23}e^{-2A_1y}-2P_rn_{24}e^{-2P_ry}$$

$$+2S_cn_{25}e^{-2S_cy}+(1+A_1)n_{26}e^{-(1+A_1)y}-(1+P_r)n_{27}e^{-(1+P_r)y}-(1+S_c)n_{28}e^{-(1+S_c)y}$$

$$+(A_1+P_r)n_{29}e^{-(A_1+P_r)y}-(P_r+S_c)n_{30}e^{-(P_r+S_c)y}+(A_1+S_c)n_{31}e^{-(A_1+S_c)y}\Big]$$

$$=-G_rA_2e^{-P_ry}+G_rP_r\Big\{n_{11}e^{-24}-n_{12}e^{-2A_1y}+n_{13}e^{-2P_ry}-n_{14}e^{-2S_cy}-n_{15}e^{-(1+A_1)y}$$

$$+n_{16}e^{-(1+P_r)y}+n_{17}e^{(1+S_c)y}-n_{18}e^{-(A_1+P_r)y}+n_{19}e^{-(P_r+S_c)y}-n_{20}e^{-(A_1+S_c)y}\Big\}$$

$$-M^2P_rG_r\Big\{A_1A_3e^{-A_1y}+P_rn_{21}e^{-P_ry}-2n_{22}e^{-2y}+2A_1n_{23}e^{-2A_1y}-2P_rn_{24}e^{-2P_ry}$$

$$+2S_cn_{25}e^{-2S_cy}+(1+A_1)n_{26}e^{-(1+A_1)y}-(1+P_r)n_{27}e^{-(1+P_r)y}-(1+S_c)n_{28}e^{-(1+S_c)y}$$

$$+(A_1+P_r)n_{29}e^{-(A_1+P_r)y}-(P_r+S_c)n_{30}e^{-(P_r+S_c)y}+(A_1+S_c)n_{31}e^{-(A_1+S_c)y}\Big\}$$

$$=-\left(G_rA_2+M^2P_rG_rn_{21}\right)e^{-P_ry}+P_rG_r\Big[\left\{n_{11}+2M^2n_{22}\right\}e^{-2y}-\left(n_{12}+2M^2A_1n_{23}\right)e^{-2A_1y}$$

$$+\left(n_{13}+2M^2P_rn_{24}\right)e^{-2P_ry}-\left(n_{14}+2M^2S_cn_{25}\right)e^{-2S_cy}-\left\{n_{15}+M^2(1+A_1)n_{26}\right\}e^{-(1+A_1)y}$$

$$+\left\{n_{16}+M^2(1+P_r)n_{27}\right\}e^{-(1+P_r)y}+\left\{n_{17}+(1+s_c)M^2n_{28}\right\}e^{-(1+S_c)y}$$

$$-\left\{n_{18}+(A_1+P_r)M^2n_{29}\right\}e^{-(A_1+P_r)y}+\left\{n_{19}+(P_r+S_c)M^2n_{30}\right\}e^{-(P_r+S_c)y}$$

$$-\left\{n_{20}+(A_1+S_c)M^2n_{31}\right\}e^{-(A_1+S_c)y}+M^2P_rG_rA_1A_3e^{-A_1y}\Big]$$

The auxiliary equation is $m^2+m=0\Rightarrow m=0,-1$

Hence the complementary function is $u_{2c} = A + Be^{-y}$

and the particular integral is

$$u_{2p} = -\frac{\left(G_r A_2 + M^2 P_r^2 G_r n_{21}\right)e^{-P_r y}}{\left(-P_r\right)^2 - P_r} + P_r G_r \left[\frac{\left(n_{11} + 2M^2 n_{22}\right)e^{-2y}}{(-2)^2 - 2} - \frac{\left(n_{12} + 2M^2 A_1 n_{23}\right)e^{-2A_1 y}}{\left(-2A_1\right)^2 - 2A_1}\right.$$

$$+\frac{\left(n_{13} + 2M^2 P_r n_{24}\right)e^{-2P_r y}}{\left(-2P_r\right)^2 - 2P_r} - \frac{\left(n_{14} + 2M^2 S_c n_{25}\right)e^{-2S_c y}}{\left(-2S_c\right)^2 - 2S_c} - \frac{\left\{n_{15} + \left(1 + A_1\right)M^2 n_{26}\right\}e^{-\left(1+A_1\right)y}}{\left\{-\left(1 + A_1\right)\right\}^2 - \left(1 + A_1\right)}$$

$$+\frac{\left\{n_{16} + M^2\left(1 + P_r\right)^2 n_{27}\right\}e^{-\left(1+P_r\right)y}}{\left\{-\left(1 + P_r\right)\right\}^2 - \left(1 + P_r\right)} + \frac{\left\{n_{17} + M^2\left(1 + S_c\right)n_{28}\right\}e^{-\left(1+S_c\right)y}}{\left\{-\left(1 + S_c\right)\right\}^2 - \left(1 + S_c\right)} - \frac{\left\{n_{18} + \left(A_1 + P_r\right)M^2 n_{29}\right\}e^{-\left(A_1+P_r\right)}}{\left\{-\left(A_1 + P_r\right)\right\}^2 - \left(A_1 + P_r\right)}$$

$$\left.+\frac{\left\{n_{19} + M^2\left(P_r + S_c\right)n_{30}\right\}e^{-\left(P_r+S_c\right)y}}{\left\{-\left(P_r + S_c\right)\right\}^2 - \left(P_r + S_c\right)} - \frac{\left\{n_{20} + M^2\left(A_1 + S_c\right)n_{31}\right\}e^{-\left(A_1+S_c\right)}}{\left\{-\left(A_1 + S_c\right)\right\}^2 - \left(A_1 + S_c\right)} + \frac{M^2 P_r G_r A_1 A_3 e^{-A_1 y}}{\left(-A_1\right)^2 - A_1}\right]$$

$$= -\frac{\left(G_r A_2 + M^2 P_r^2 G_r n_{21}\right)e^{-P_r y}}{-P_r\left(1 + P_r\right)} + P_r G_r \left[\frac{\left(n_{11} + 2M^2 n_{22}\right)e^{-2y}}{2} - \frac{\left(n_{12} + 2M^2 A_1 n_{23}\right)e^{-2A_1 y}}{-2A_1\left(1 - 2A_1\right)}\right.$$

$$+\frac{\left(n_{13} + 2M^2 P_r n_{24}\right)e^{-2P_r y}}{-2P_r\left(1 - 2P_r\right)} - \frac{\left(n_{14} + 2M^2 S_c n_{25}\right)e^{-2S_c y}}{-2S_c\left(1 - 2S_c\right)} - \frac{\left\{n_{15} + \left(1 + A_1\right)M^2 n_{26}\right\}e^{-\left(1+A_1\right)y}}{A_1\left(1 + A_1\right)}$$

$$+\frac{\left\{n_{16} + M^2\left(1 + P_r\right)^2 n_{27}\right\}e^{-\left(1+P_r\right)y}}{P_r\left(1 + P_r\right)} + \frac{\left\{n_{17} + M^2\left(1 + S_c\right)n_{28}\right\}e^{-\left(1+S_c\right)y}}{S_c\left(1 + 2S_c\right)} - \frac{\left\{n_{18} + \left(A_1 + P_r\right)M^2 n_{29}\right\}e^{-\left(A_1+P_r\right)}}{-\left(A_{1+}P_r\right)\left(1 - A_1 - P_r\right)}$$

$$\left.+\frac{\left\{n_{19} + M^2\left(P_r + S_c\right)n_{30}\right\}e^{-\left(P_r+S_c\right)y}}{-\left(P_r + S_c\right)\left(1 - P_r - S_c\right)} - \frac{\left\{n_{20} + M^2\left(A_1 + S_c\right)n_{31}\right\}e^{-\left(A_1+S_c\right)}}{-\left(A_1 + S_c\right)\left(1 - A_1 - S_c\right)} + \frac{M^2 P_r G_r A_1 A_3 e^{-A_1 y}}{-A_1\left(1 - A_1\right)}\right]$$

If we substitute

$$\frac{G_r A_2 + M^2 P_r^2 G_m n_{21}}{P_r\left(1 - P_r\right)} = n_{32}; \qquad \frac{P_r G_r\left(n_{11} + 2M^2 n_{22}\right)}{2} = n_{33};$$

$$\frac{P_r G_r\left(n_{12} + 2M^2 A_1 n_{23}\right)}{2A_1\left(1 - 2A_1\right)} = n_{34}; \qquad \frac{P_r G_r\left(n_{13} + M^2 P_r n_{24}\right)}{2P_r\left(1 - 2P_r\right)} = n_{35};$$

$$\frac{P_r G_r\left(n_{14} + 2M^2 S_c n_{25}\right)}{2S_c\left(1 - 2S_c\right)} = n_{36}; \qquad \frac{P_r G_r\left\{n_{15} + M^2\left(1 + A_1\right)n_{26}\right\}}{A_1\left(1 + A_1\right)} = n_{37};$$

$$\frac{P_r G_r\left\{n_{16} + \left(1 + P_r\right)M^2 n_{27}\right\}}{P_r\left(1 + P_r\right)} = n_{38}; \qquad \frac{P_r G_r\left\{n_{17} + \left(1 + S_c\right)M^2 n_{28}\right\}}{S_c\left(1 + S_c\right)} = n_{39};$$

$$\frac{P_rG_r\left\{n_{18}+\left(A_1+P_r\right)M^2n_{29}\right\}}{\left(A_1+P_r\right)\left(1-A_1-P_r\right)}=n_{40}; \qquad \frac{P_rG_r\left\{n_{19}+\left(P_r+S_c\right)M^2n_{30}\right\}}{\left(P_r+S_c\right)\left(1-P_r-S_c\right)}=n_{41};$$

$$\frac{P_rG_r\left\{n_{20}+\left(A_1+S_c\right)M^2n_{31}\right\}}{\left(A_1+S_c\right)\left(1-A_1-S_c\right)}=n_{42}; \qquad \frac{M^2P_rG_rA_3}{1-A_1}=n_{43};$$

Then the particular integral is

$$u_{2p}=n_{32}e^{-P_ry}+n_{33}e^{-2y}+n_{34}e^{-2A_1y}-n_{35}e^{-2P_ry}+n_{36}e^{2S_cy}-n_{37}e^{-\left(1+A_1\right)y}+n_{38}e^{-\left(1+P_r\right)y}$$

$$+n_{39}e^{-\left(1+S_c\right)y}+n_{40}e^{-\left(A_1+P_r\right)y}-n_{41}e^{-\left(P_r+S_c\right)y}+n_{42}e^{-\left(A_1+S_c\right)y}-n_{43}e^{A_1y}$$

So the general solution is

$$u_2=u_{2c}+u_{2p}=A+Be^{-y}+n_{32}e^{-P_ry}+n_{33}e^{-2y}+n_{34}e^{-2A_1y}-n_{35}e^{-2P_ry}$$

$$+n_{36}e^{-2S_cy}-n_{37}e^{-\left(1+A_1\right)y}-n_{37}e^{-\left(1+A_1\right)y}+n_{38}e^{-\left(1+P_r\right)y}+n_{39}e^{-\left(1+S_c\right)y}$$

$$+n_{40}e^{-\left(A_1+P_r\right)y}-n_{41}e^{-\left(P_r+S_c\right)y}+n_{42}e^{-\left(A_1+S_c\right)y}-n_{43}e^{-A_1y}$$

Applying the boundary conditions

$u_2=0$ at $y=0$, we have

$$A+B+n_{32}+n_{33}+n_{34}-n_{35}+n_{36}-n_{37}+n_{38}+n_{39}+n_{40}-n_{41}+n_{42}-n_{43}=0$$

as $y\to\infty$ then $u_2=0$

Hence $A=0$.

Therefore, $B=-\left(n_{32}+n_{33}+n_{34}-n_{35}+n_{36}-n_{37}+n_{38}+n_{39}+n_{40}-n_{41}+n_{42}-n_{43}\right)$

If $A_4=-\left(n_{32}+n_{33}+n_{34}-n_{35}+n_{36}-n_{37}+n_{38}+n_{39}+n_{40}-n_{41}+n_{42}-n_{43}\right)$, then

$$u_2\left(y\right)=A_4e^{-y}+n_{32}e^{-P_ry}+n_{33}e^{-2y}+n_{34}e^{-2A_1y}-n_{35}e^{-2P_ry}+n_{36}e^{-2S_cy}-n_{37}e^{-\left(1+A_1\right)y}$$

$$+n_{38}e^{-\left(1+P_r\right)y}+n_{39}e^{-\left(1+S_c\right)y}+n_{40}e^{-\left(A_1+P_r\right)y}-n_{41}e^{-\left(P_r+S_c\right)y}+n_{42}e^{-\left(A_1+S_c\right)y}-n_{43}e^{-A_1y} \quad (4.35)$$

Therefore, the solution of above equations (4.16) – (4.18) with boundary conditions (4.19) and equation (4.20) – (4.22) with boundary conditions (4.23) correspond to the following:

$$\theta_1\left(y\right)=\frac{1}{P_r}\left(1+n_7\right)e^{-P_ry}-\frac{1}{S_c}n_7e^{-S_cy} \quad (4.36)$$

$$H_1(y)=\left[h+n_5(1+n_7)+n_6\left(1-\frac{G_r}{G_m}n_7\right)\right]e^{-A_1y}-n_5(1+n_7)e^{-P_ry}-n_6\left(1-\frac{G_r}{G_m}n_7\right)e^{-S_cy} \quad (4.37)$$

$$u_1(y)=(U+n_8-n_9-n_{10})e^{-y}-n_8e^{-A_1y}+n_9e^{-P_ry}+n_{10}e^{-S_cy} \quad (4.38)$$

$$\theta_2(y)=A_2e^{-P_ry}-P_r\Big[n_{11}e^{-2y}-n_{12}e^{-2A_1y}+n_{13}e^{-2P_ry}-n_{14}e^{-2S_cy}-n_{15}e^{-(1+A_1)y}+n_{16}e^{-(1+P_r)y}$$
$$+n_{17}e^{-(1+S_c)}-n_{18}e^{-(A_1+P_r)y}+n_{19}e^{-(P_r+S_c)y}-n_{20}e^{-(A_1+S_c)y}\Big] \quad (4.39)$$

$$H_2(y)=MP_rG_r\Big[A_3e^{-A_1y}-n_{21}e^{-2P_ry}+n_{22}e^{-2y}-n_{23}e^{-2A_1y}+n_{24}e^{-2P_ry}-n_{25}e^{-2S_cy}-n_{26}e^{-(1+A_1)y}$$
$$+n_{27}e^{-(1+P_r)y}+n_{28}e^{-(1+S_c)y}-n_{29}e^{-(A_1+P_r)y}+n_{30}e^{-(P_r+S_c)y}-n_{31}e^{-(A_1+S_c)y}\Big] \quad (4.40)$$

$$u_2(y)=A_4e^{-y}+n_{32}e^{-P_ry}+n_{33}e^{-2y}+n_{34}e^{-2A_1y}-n_{35}e^{-2P_ry}+n_{36}e^{-2S_cy}-n_{37}e^{-(1+A_1)y}$$
$$+n_{38}e^{-(1+P_r)y}+n_{39}e^{-(1+S_c)y}+n_{40}e^{-(A_1+P_r)y}-n_{41}e^{-(P_r+S_c)y}+n_{42}e^{-(A_1+S_c)y}-n_{43}e^{-A_1y} \quad (4.41)$$

where

$n_1=(1+P_m)$; $\quad n_2=P_m(1-M^2)$; $\quad n_3=\dfrac{MP_mG_r}{P_r^2}$;

$n_4=\dfrac{MP_mG_m}{S_c^2}$; $\quad n_5=\dfrac{n_3}{P_r^2-P_rn_1+n_2}$; $\quad n_6=\dfrac{n_4}{S_c^2-S_cn_1+n_2}$;

$n_7=\dfrac{D_fS_c}{P_r-S_c}$; $\quad n_8=\dfrac{M\left\{h+n_5(1+n_7)+n_6\left(1-\frac{G_r}{G_m}n_7\right)\right\}}{1-A_1}$;

$n_9=\dfrac{G_r(1+n_7)+MP_r^2n_5(1+n_7)}{P_r^2(1-P_r)}$; $\quad n_{10}=\dfrac{G_m-G_rn_7+S_c^2n_6\left(1-\frac{G_r}{G_m}n_7\right)}{S_c(1-S_c)}$;

$n_{11}=-\dfrac{(U+n_8-n_9-n_{10})^2}{2(P_r-2)}$; $\quad n_{12}=\dfrac{A_1\left[\left\{h+n_5(1+n_7)+n_6\left(1-\frac{G_r}{G_m}n_7\right)\right\}^2+n_8^2\right]}{2(P_r-2A_1)}$;

$n_{13}=\dfrac{n_5^2(1+n_7)^2+n_9^2}{2}$; $\quad n_{14}=\dfrac{S_c\left\{n_6^2\left(1-\frac{G_r}{G_m}n_7\right)^2+n_{10}^2\right\}}{2(P_r-2S_c)}$;

$$n_{15} = \frac{2A_1 n_8 \left(U + n_8 - n_9 - n_{10}\right)}{\left(1 + A_1\right)\left(1 + A_1 - P_r\right)} \quad ; \qquad n_{16} = \frac{2P_r n_9 \left(U + n_8 - n_9 - n_{10}\right)}{1 + P_r};$$

$$n_{17} = \frac{2S_c n_{10} \left(U + n_8 - n_9 - n_{10}\right)}{-\left(1 + S_c\right)\left(1 + S_c - P_r\right)};$$

$$n_{18} = \frac{2P_r \left[n_5 \left(1 + n_7\right) \left\{ h + n_5 \left(1 + n_7\right) + n_6 \left(1 - \frac{G_r}{G_m} n_7\right) \right\} + n_8 n_9 \right]}{A_1 + P_r};$$

$$n_{19} = \frac{2P_r \left\{ n_5 n_6 \left(1 + n_7\right) \left(1 - \frac{G_r}{G_m} n_7\right) + n_9 n_{10} \right\}}{P_r + S_c};$$

$$n_{20} = \frac{2A_1 S_c \left[n_6 \left(1 - \frac{G_r}{G_m} n_7\right) \left\{ h + n_5 \left(1 + n_7\right) + n_6 \left(1 - \frac{G_r}{G_m} n_7\right) \right\} + n_8 n_{10} \right]}{\left(A_1 + S_c\right)\left(A_1 + S_c - P_r\right)};$$

$$n_{21} = \frac{A_2}{P_r \left(P_r^2 - n_1 P_r + n_2\right)}; \qquad n_{22} = \frac{n_{11}}{2\left(4 - 2n_1 + n_2\right)};$$

$$n_{23} = \frac{n_{12}}{2A_1 \left(4A_1^2 - 2A_1 n_1 + n_2\right)}; \qquad n_{24} = \frac{n_{13}}{2P_r \left(4P_r^2 - 2P_r n_1 + n_2\right)};$$

$$n_{25} = \frac{n_{14}}{2s_c \left(4s_c^2 - 2S_c n_1 + n_2\right)};$$

$$n_{26} = \frac{n_{15}}{\left(1 + A_1\right)\left\{\left(1 + A_1\right)^2 - \left(1 + A_1\right) P_r + n_2\right\}};$$

$$n_{27} = \frac{n_{16}}{\left(1 + P_r\right)\left\{\left(1 + P_r\right)^2 - \left(1 + P_r\right) n_1 + n_2\right\}}; \quad n_{28} = \frac{n_{17}}{\left(1 + S_c\right)\left\{\left(1 + S_c\right)^2 - \left(1 + S_c\right) n_1 + n_2\right\}};$$

$$n_{29} = \frac{n_{18}}{\left(A_1 + P_r\right)\left\{\left(A_1 + P_r\right)^2 - \left(A_1 + P_r\right) n_1 + n_2\right\}};$$

$$n_{30} = \frac{n_{19}}{\left(P_r + S_c\right)\left\{\left(P_r + S_c\right)^2 - \left(P_r + S_c\right) n_1 + n_2\right\}};$$

$$n_{31} = \frac{n_{20}}{\left(A_1 + S_c\right)\left\{\left(A_1 + S_c\right)^2 - \left(A_1 + S_c\right) n_1 + n_2\right\}}; \quad n_{32} = \frac{G_r A_2 + M^2 P_r^2 G_m n_{21}}{P_r \left(1 - P_r\right)};$$

$$n_{33} = \frac{P_r G_r \left\{ n_{11} + 2M^2 n_{22} \right\}}{2}; \qquad n_{34} = \frac{P_r G_r \left(n_{12} + 2M^2 A_1 n_{23} \right)}{2A_1 \left(1 - 2A_1 \right)};$$

$$n_{35} = \frac{P_r G_r \left(n_{13} + 2M^2 P_r n_{24} \right)}{2P_r \left(1 - 2P_r \right)}; \qquad n_{36} = \frac{P_r G_r \left(n_{14} + 2M^2 S_c n_{25} \right)}{2S_c \left(1 - 2S_c \right)};$$

$$n_{37} = \frac{P_r G_r \left\{ n_{15} + \left(1 + A_1 \right) M^2 n_{26} \right\}}{A_1 \left(1 + A_1 \right)}; \qquad n_{38} = \frac{P_r G_r \left\{ n_{16} + \left(1 + P_r \right) M^2 n_{27} \right\}}{P_r \left(1 + P_r \right)};$$

$$n_{39} = \frac{P_r G_r \left\{ n_{17} + \left(1 + S_c \right) M^2 n_{28} \right\}}{S_c \left(1 + S_c \right)}; \qquad n_{40} = \frac{P_r G_r \left\{ n_{18} + \left(A_1 + P_r \right) M^2 n_{29} \right\}}{\left(A_1 + P_r \right)\left(1 - A_1 - P_r \right)};$$

$$n_{41} = \frac{P_r G_r \left\{ n_{19} + \left(P_r + S_c \right) M^2 n_{30} \right\}}{\left(P_r + S_c \right)\left(1 - P_r - S_c \right)}; \qquad n_{42} = \frac{P_r G_r \left\{ n_{20} + \left(A_1 + S_c \right) M^2 n_{31} \right\}}{\left(A_1 + S_c \right)\left(1 - A_1 - S_c \right)};$$

$$n_{43} = \frac{M^2 P_r G_r A_3}{1 - A_1};$$

$$A_1 = \frac{n_1 + \sqrt{n_1^2 - 4n_2}}{2}$$

$$A_2 = 2n_{11} - 2A_1 n_{12} + 2P_r n_{13} - 2S_c n_{14} - \left(1 + A_1 \right) n_{15} + \left(1 + P_r \right) n_{16} + \left(1 + S_c \right) n_{17} - \left(A_1 + P_r \right) n_{18} + \left(P_r + S_c \right) n_{19} - \left(A_1 + S_c \right) n_{20}$$

$$A_3 = n_{21} - n_{22} + n_{23} - n_{24} + n_{25} + n_{26} - n_{27} - n_{28} + n_{29} - n_{30} + n_{31}$$

$$A_4 = -\left(n_{32} + n_{33} + n_{34} - n_{35} + n_{36} - n_{37} + n_{38} + n_{39} + n_{40} - n_{41} + n_{42} - n_{43} \right)$$

The shearing stress at the plate is given by

$$\tau_s = \left(\frac{\partial u}{\partial y} \right)_{y=0} = \left(\frac{\partial u_1}{\partial y} \right)_{y=0} + E_c \left(\frac{\partial u_2}{\partial y} \right)_{y=0}$$

and the rate of heat transfer per unit area of the plate is given by $Q^* = -k \left(\frac{\partial T^*}{\partial y^*} \right)_{y^*=0}$.

Therefore, the dimensionless heat transfer coefficient, which is usually known as the Nusselt number (Nu), is obtained by $Nu = \frac{Q^* \upsilon}{k V_0 \left(T^* - T_\infty^* \right)} = \left(\frac{1}{\theta} \right)_{y=0}$.

The rate of mass transfer is given by $m^* = -\rho D\left(\frac{\partial C^*}{\partial y^*}\right)_{y^*=0}$ and hence the dimensionless mass transfer coefficient, which is generally known as the Sherwood number (S_h), is obtained by $S_h = \frac{m^* \upsilon}{\rho V_0 D\left(C^* - C^*_\infty\right)} = \left(\frac{1}{C}\right)_{y=0} = S_c$. Consequently, it is observed from this equation that there exists a linear relationship between the Sherwood number and the Schmidt number.

4.2 Numerical Results and Discussion

The system of coupled, nonlinear, ordinary differential equations (4.16) – (4.18) and (4.20) – (4.23) governed by the boundary conditions (4.19) and (4.23), respectively are obtained by using perturbation technique. In order to get insight into the physical phenomena of the problem, the approximate numerical results of the first order solutions (4.39) – (4.41) concerning the velocity, temperature and induced magnetic field have been carried out for small values of Eckert number E_c (which is the measure of the heat produced by friction) with different selected values of the established dimensionless parameters like Dufour number (D_f), Grashof number (G_r), Schmidt number (S_c), magnetic parameter (*M*), etc. Since the two most important fluids are atmospheric air and water, the values of the Prandtl number (P_r) are limited to 0.71 for air (at 20^0 C) and 7.0 for water (at 20^0 C) for numerical investigation. Following the work of Choudhary and Sharma (2006), other parameters like magnetic diffusivity (P_m), modified Grashof number (G_m) for mass transfer, and *h* are chosen to be fixed values 1.0, 3.0, and 1.0, respectively. It is also mentioned here that for the sake of brevity the values of the viscosity/temperature parameter are taken to be positive ($G_r > 0$, for a cooling Newtonian fluid), which correspond to a cooling problem that is generally encountered in engineering in connection with the cooling of reactors. The numerical results obtained for dimensionless velocity, temperature and induced magnetic field versus *y* for different selected values of the parameters $\left(D_f, G_r, P_r, S_c \text{ and } M\right)$ are presented in Figures 4.1 – 4.19.

The effect of Dufour number on the velocity and temperature profiles and induced magnetic fields are illustrated in Figures 4.1 – 4.3 for Prandtl number $P_r = 0.71$ (air), $G_r = 3.0$, $S_c = 0.30$ and $M = 5.0$. It is observed from Figure 4.1 that the velocity decreases quantitatively with the increase of D_f from 0.00 to 0.80. Figure 4.2 shows the effect of Dufour number on the temperature fields. Like velocity, we observe that the temperature profiles decrease with the increasing value of D_f. It is also observed from Figure 4.3 that an increase in the Dufour number leads to a decrease in the induced magnetic field.

Figures 4.4 – 4.8 exhibit the behaviors of velocity, temperature and induced magnetic field profiles respectively for air $(P_r = 0.71)$ and water $(P_r = 7.0)$ taking the different values of S_c ($S_c = 0.30$ for helium at 25^0 C temperature and 1 atmospheric pressure and $S_c = 0.78$ for ammonia) with $G_r = 3.0$, $D_f = 0.05$ and $M = 3.0$.

Figures 4.4 shows the velocity profiles for the variation of Schmidt number (S_c) from 0.30 to 0.78 with $\Pr = 0.71$, $G_r = 3.0$, $D_f = 0.05$ and $M = 3.0$. We observe that the velocity is high for helium $(S_c = 0.30)$ than ammonia $(S_c = 0.78)$ for air at 25^0 C temperature and 1 atmospheric pressure. But it is observed from Figure 4.5 that for a fixed value of Schmidt number $(S_c = 0.30)$ the velocities are found to be dominated highly in the case of air $(P_r = 0.71)$ than of water $(P_r = 7.0)$. A comparison of the variation of velocity profiles for different Prandtl number (P_r) and Schmidt number (S_c) with $G_r = 3.0$, $D_f = 0.05$ and $M = 5.0$ is shown in Figure 4.6. As might be expected, the velocity profiles increase gradually near the plate, become maximum in the vicinity of the plate, and then decrease slowly away from the plate.

The effects of S_c on the temperature profiles are shown for both air $(P_r = 0.71)$ and water $(P_r = 7.0)$ in Figures 4.7 and 4.8 respectively. With the increase of S_c, the temperature profiles decreases for both air and water. Figure 4.9 shows a comparison of temperature profiles for the variation of both P_r and Sc. It is observed that the influence of Prandtl number on the temperature profiles is very significant. A rise in P_r through $P_r = 0.71$ (air) to $P_r = 7.0$ (water) corresponds to a dramatic decrease of temperature throughout the domain and therefore leads to a decrease in thermal conductivity of the fluid.

Further, Figure 4.10 exhibits that the induced magnetic fields extensively decreased with increasing S_c but the rate of decrease is superior for water, where $P_r = 7.0$, than air, where $P_r = 0.71$.

Figures 4.11 and 4.12 investigate the variation of velocity profiles for varying G_r corresponding to the values of $P_r = 0.71$ (air) and $P_r = 7.0$ (water), respectively, with $S_c = 0.30$, $D_f = 0.05$ and $M = 3.0$. A comparative study of the curves is shown in Figure 4.13, which reveal that the values of the velocity increased with an increase in G_r for $P_r = 0.71$, where as, a reverse effect is observed for $P_r = 7.0$. Therefore, it implies the physical fact that higher G_r values boost up flow velocities through air $(P_r = 0.71)$ but slow down through the water $(P_r = 7.0)$.

The variation of temperature profiles for different Grashof numbers in case of air and water are plotted separately in Figure 4.14 and 4.15 with $S_c = 0.30$, $D_f = 0.05$ and $M = 6.0$, where as a comparison of the variation of temperature profiles for different Grashof number (G_r) and Prandtl number (P_r) is shown in Figure 4.16 . We observed from Figure 4.16 that temperature decrease with the decrease of the Grashof number G_r for air $(P_r = 0.71)$ but the variation is negligible in case of water.

Figure 4.17 exhibits the variation of velocity profiles for different Magnetic parameter (M) and Grashof number (G_r) with $P_r = 0.71$, $S_c = 0.30$ and $D_f = 0.05$. From the figure it is noticed that an increase in M gives rise to a decrease of the velocity.

Figure 4.18 observes the effect of the magnetic parameter M on the temperature profiles for $P_r = 0.71$, $S_c = 0.30$, $G_r = 3.0$ and $D_f = 0.05$. Here we see that the temperature decreases with the increase in M.

Figure 4.19 shows the variation of magnetic fields for different values of Magnetic parameter M with $P_r = 0.71$ $S_c = 0.30$, $G_r = 3.0$ and $D_f = 0.05$. It is observed from the figure that as M rises, the induced magnetic fields are decreased substantially.

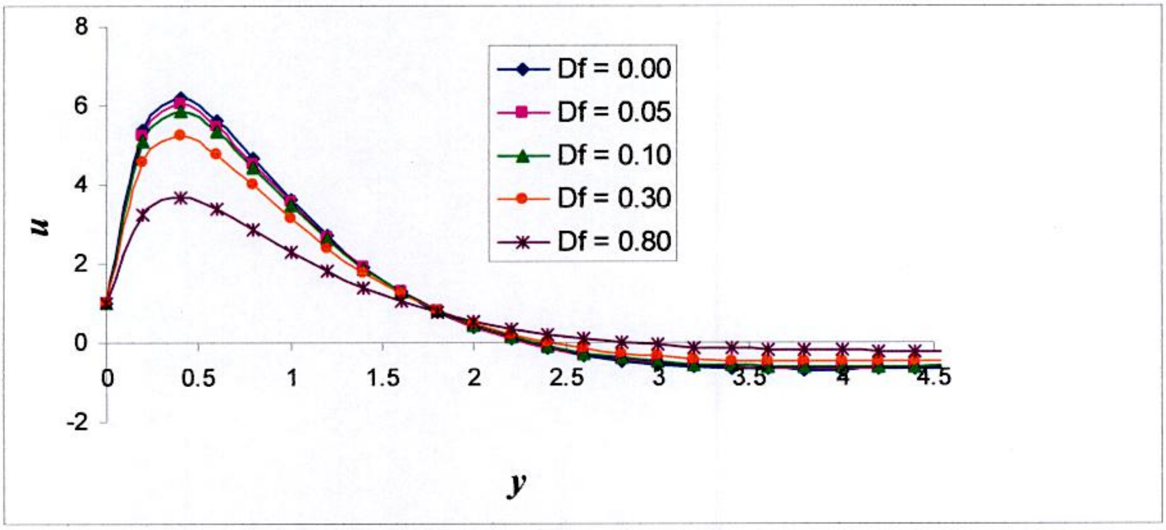

Figure 4.1 Variation of velocity profiles for different values of Dufour number $\left(D_f\right)$ with $P_r = 0.71$, $G_r = 3.0$, $S_c = 0.30$ and $M = 5.0$.

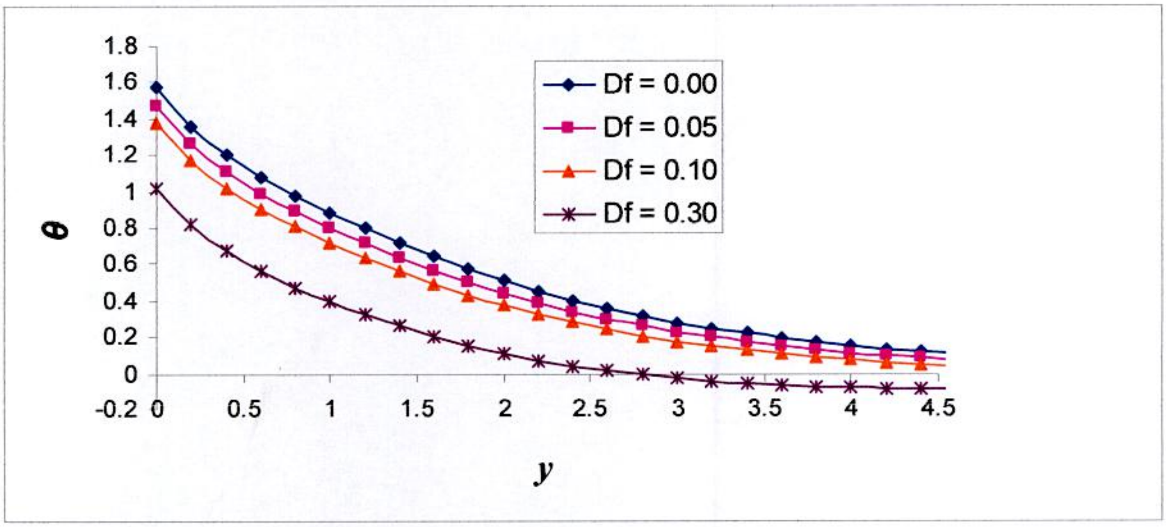

Figure 4.2 Variation of temperature profiles for different values of Dufour number $\left(D_f\right)$ with $P_r = 0.71$, $G_r = 3.0$, $S_c = 0.30$ and $M = 5.0$.

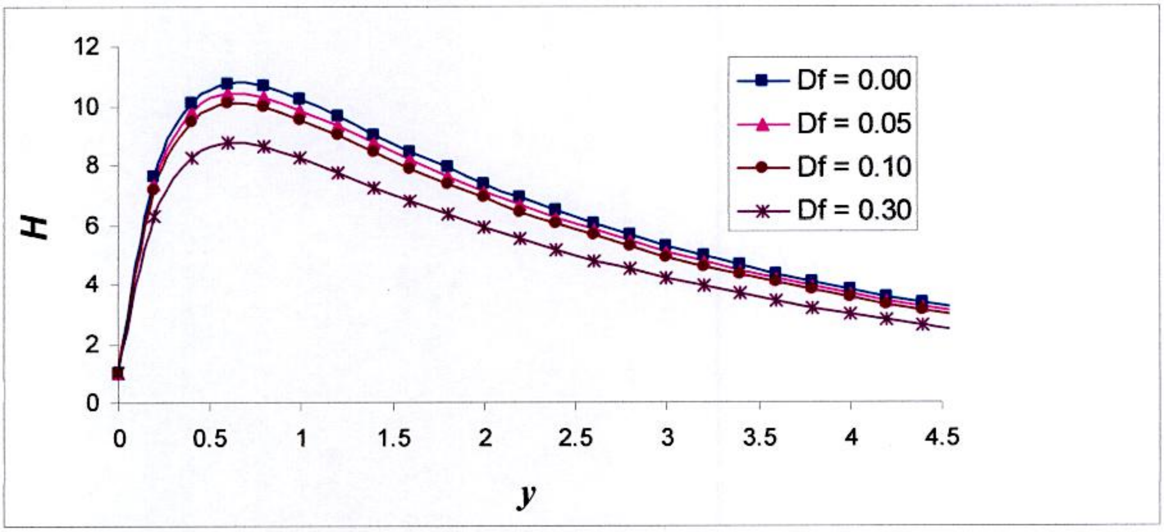

Figure 4.3 Variation of induced magnetic fields for different values of Dufour number (D_f) with $P_r = 0.71$, $G_r = 3.0$, $S_c = 0.30$ and $M = 5.0$.

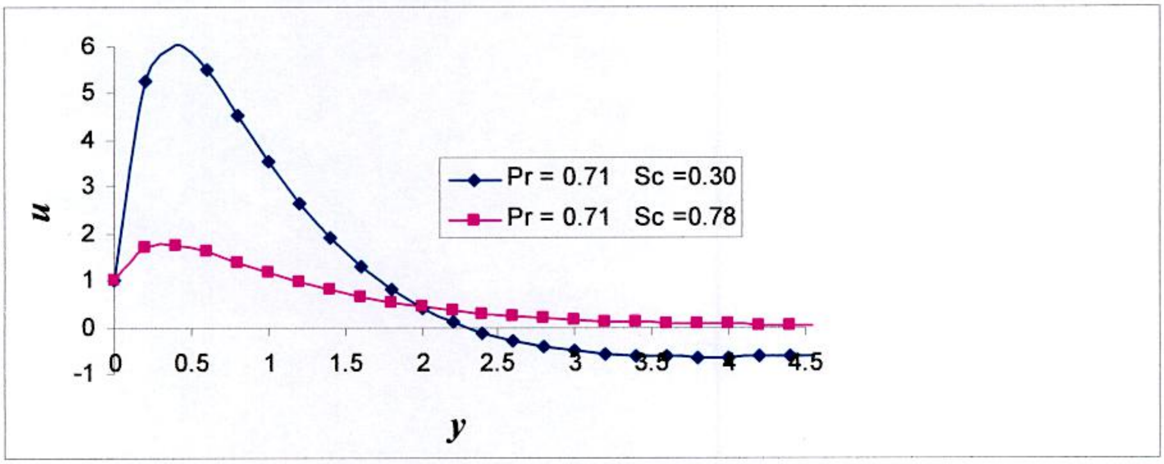

Figure 4.4 Variation of velocity profiles for different values of Schmidt number (S_c) with $\Pr = 0.71$, $G_r = 3.0$, $D_f = 0.05$ and $M = 3.0$.

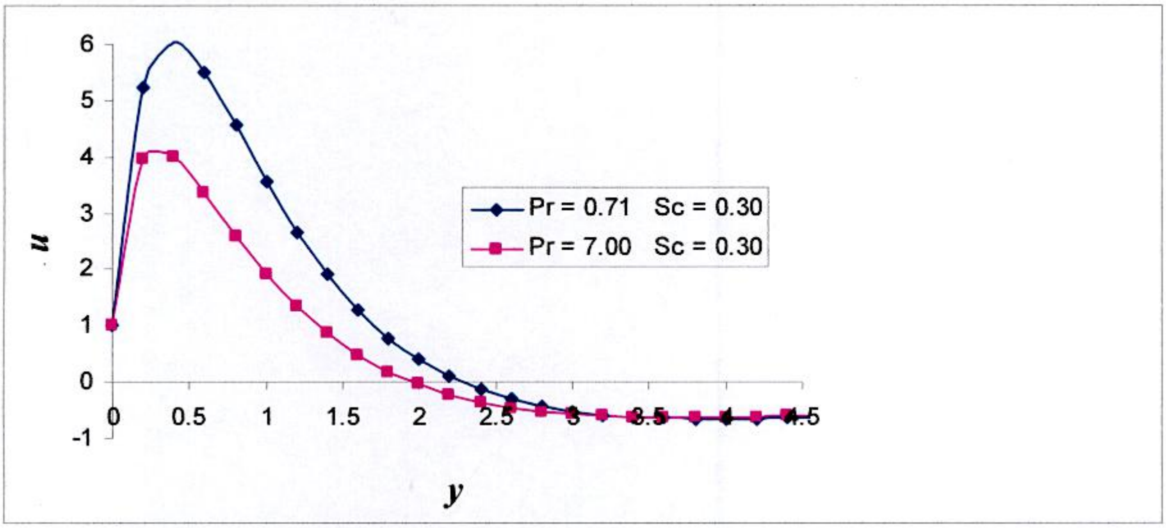

Figure 4.5 Variation of velocity profiles for different Prandtl number (P_r) with $S_c = 0.30$, $G_r = 3.0$, $D_f = 0.05$ and $M = 3.0$.

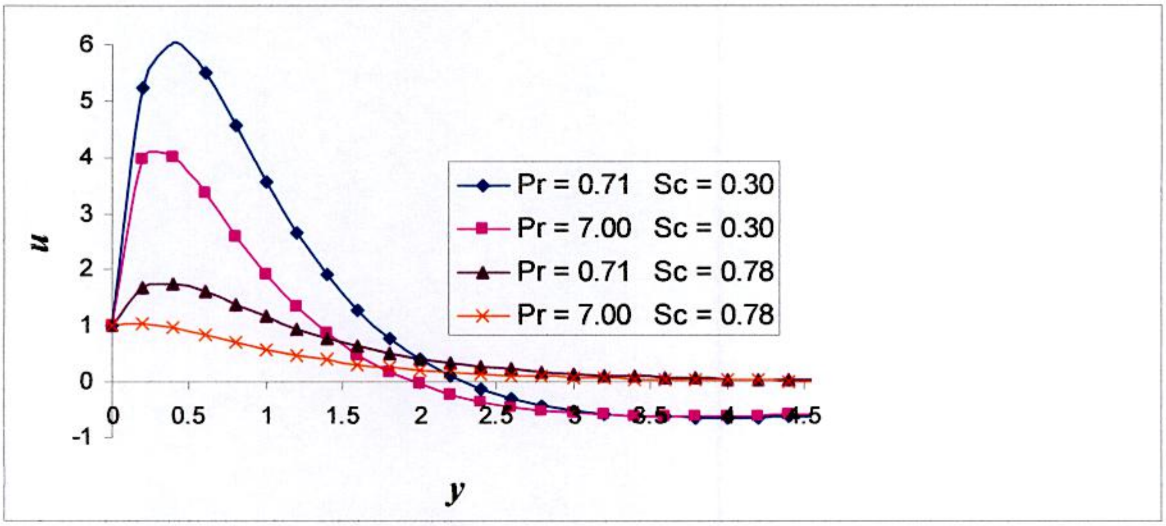

Figure 4.6 Comparison of the variation of velocity profiles for different Prandtl number (P_r) and Schmidt number (S_c) with $G_r = 3.0$, $D_f = 0.05$ and $M = 5.0$.

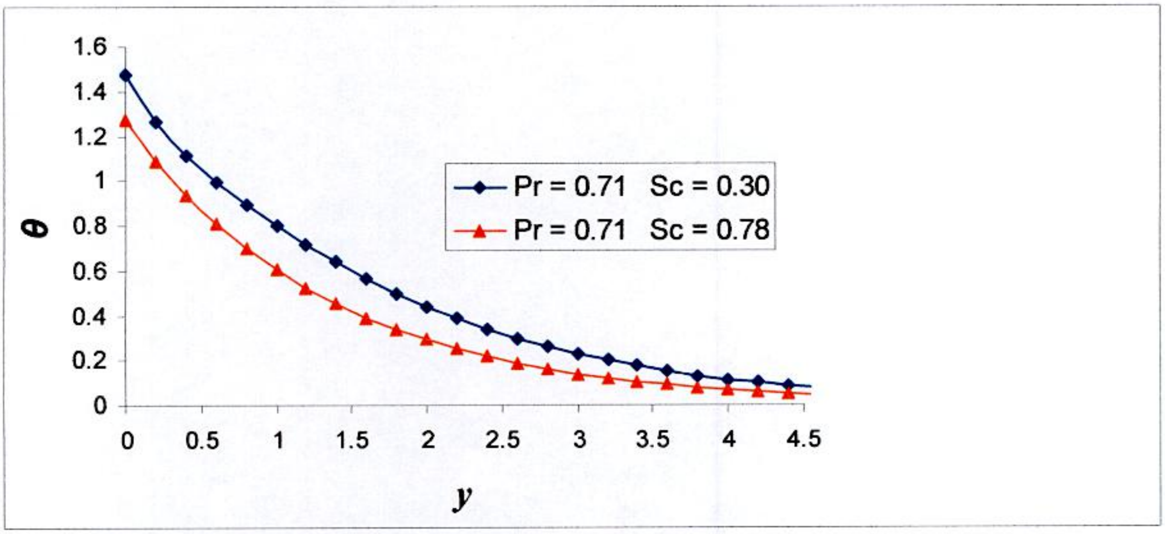

Figure 4.7 Variation of temperature profiles for different values of Schmidt number (S_c) with $\Pr = 0.71$, $G_r = 3.0$, $D_f = 0.05$ and $M = 3.0$.

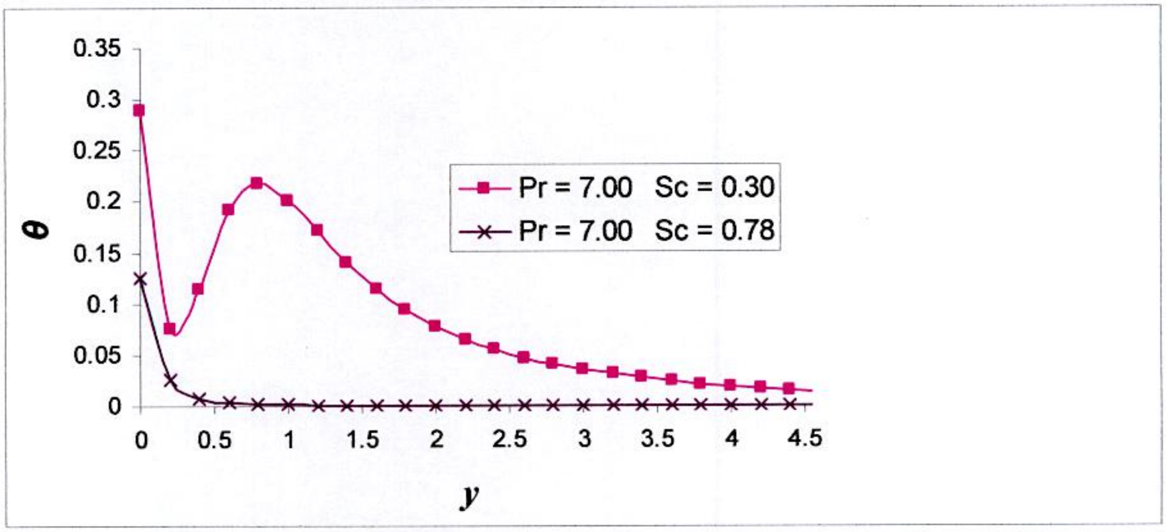

Figure 4.8 Variation of temperature profiles for different values of Schmidt number (S_c) with $\Pr = 7.0$, $G_r = 3.0$, $D_f = 0.05$ and $M = 3.0$.

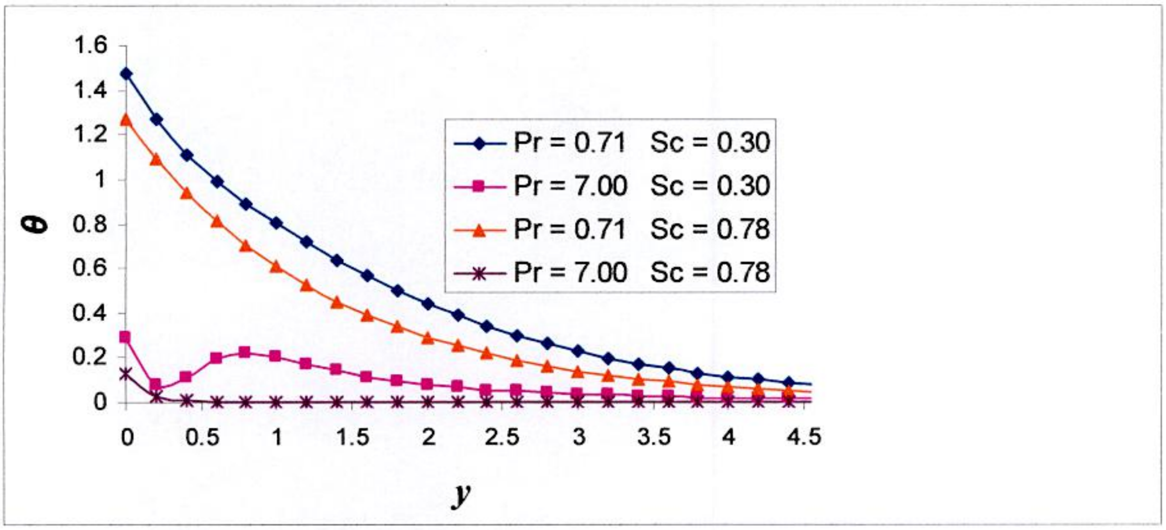

Figure 4.9 Comparison of the variation of temperature profiles for different Prandtl number (P_r) and Schmidt number (S_c) with $G_r = 3.0$, $D_f = 0.05$ and $M = 3.0$.

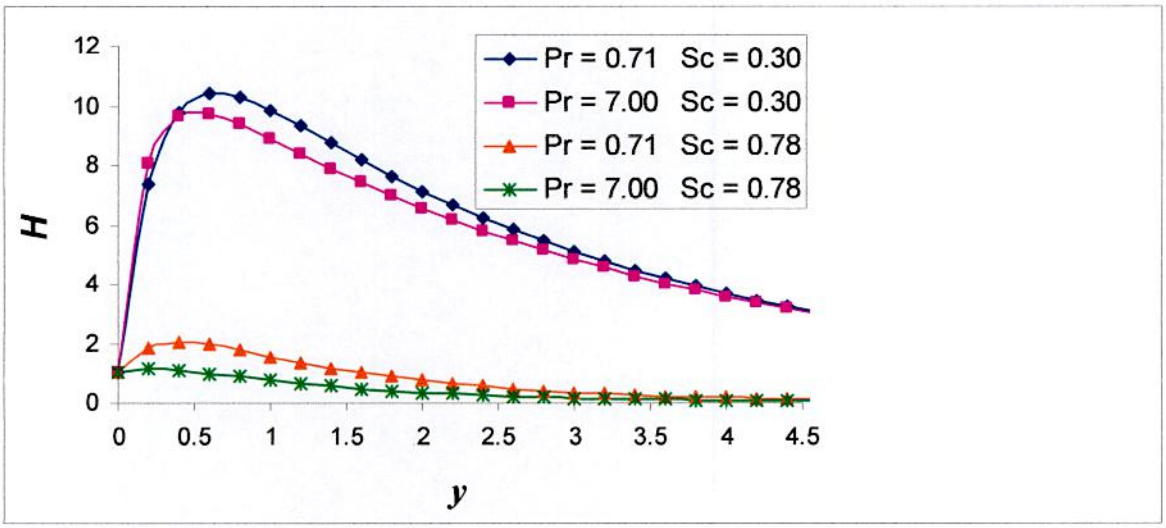

Figure 4.10 Variation of induced magnetic fields for different Prandtl number (P_r) and Schmidt number (S_c) with $G_r = 3.0$ and $M = 3.0$.

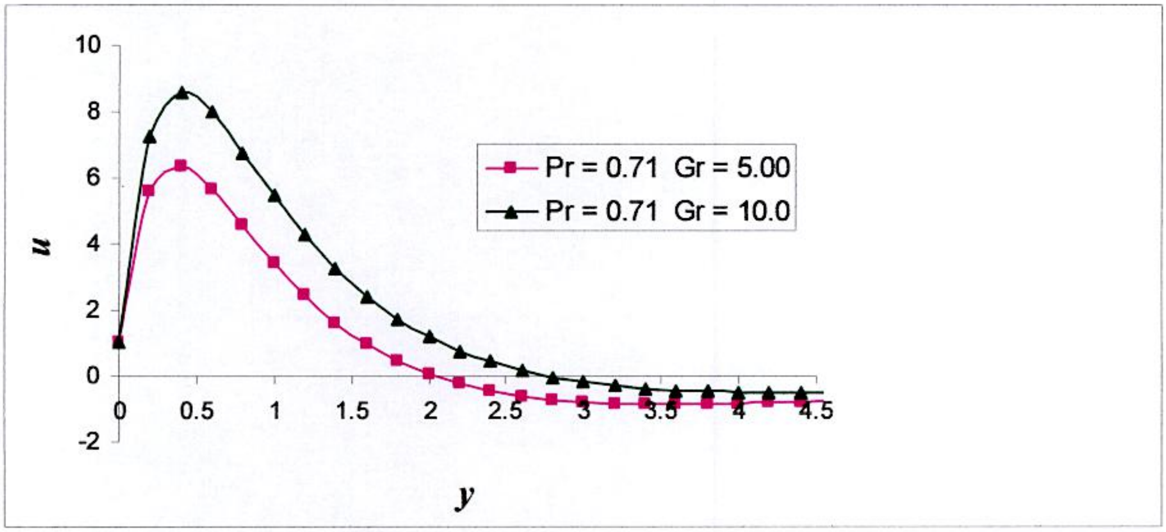

Figure 4.11 Variation of velocity profiles for different Grashof number (G_r) with $P_r = 0.71$, $S_c = 0.30$, $D_f = 0.05$ and $M = 3.0$.

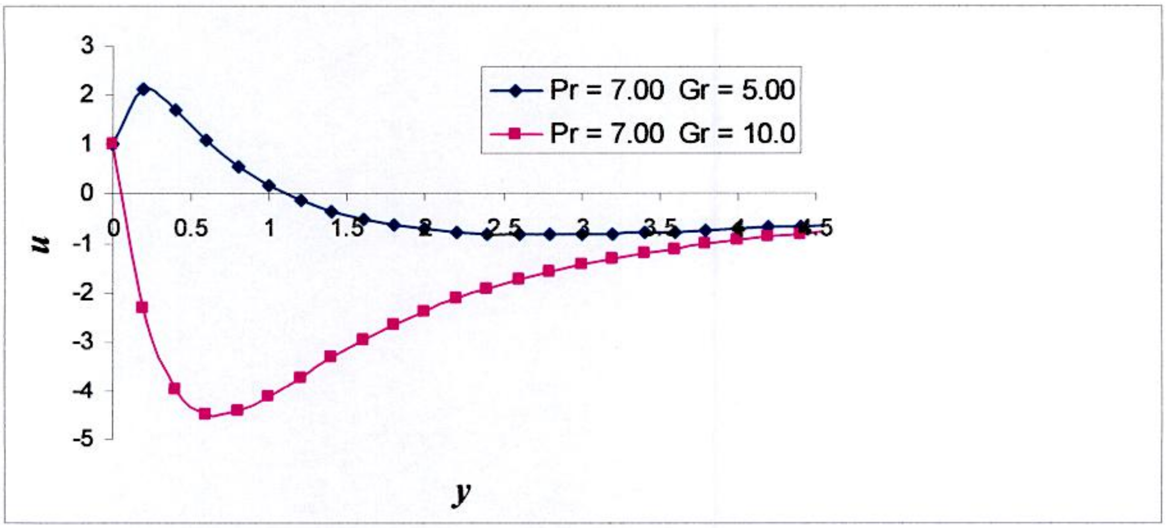

Figure 4.12 Variation of velocity profiles for different Grashof number (G_r) with $P_r = 7.0$, $S_c = 0.30$, $D_f = 0.05$ and $M = 3.0$.

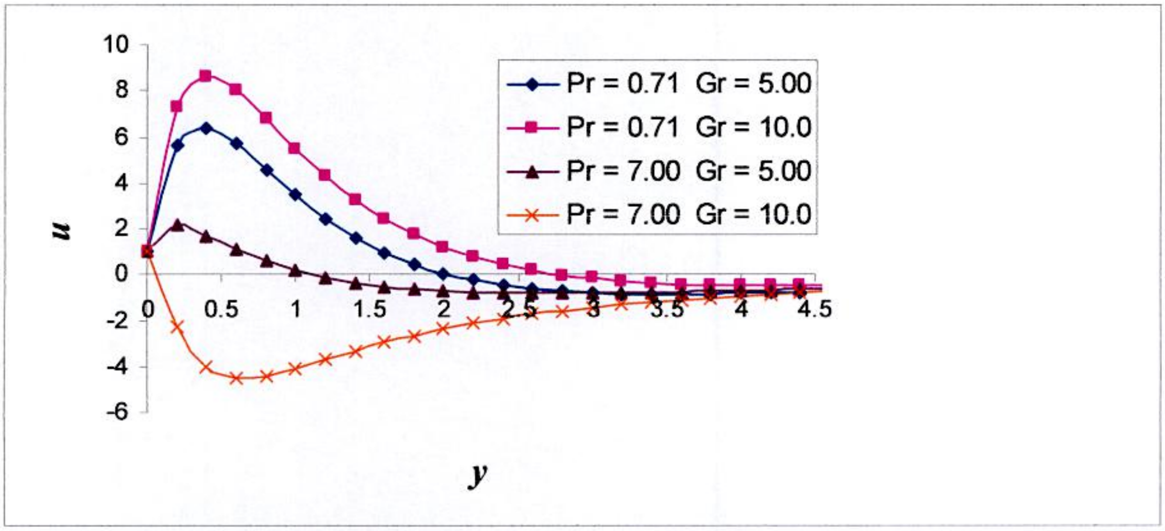

Figure 4.13 Comparison of the variation of velocity profiles for different Prandtl number (P_r) and Grashof number (G_r) with $S_c = 0.30$, $D_f = 0.05$ and $M = 6.0$.

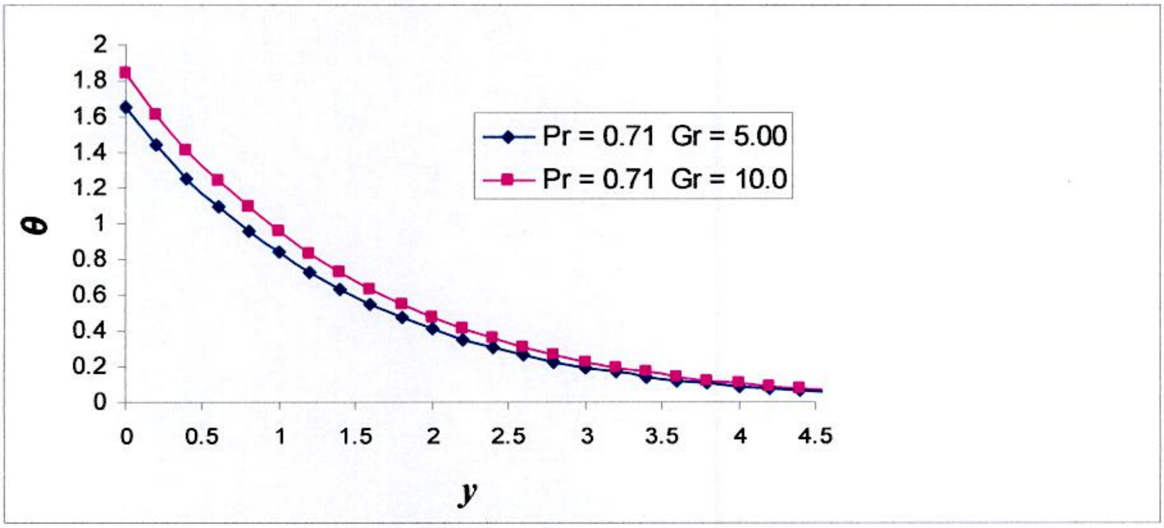

Figure 4.14 Variation of temperature profiles for different Grashof number (G_r) with $P_r = 0.71$, $S_c = 0.30$, $D_f = 0.05$ and $M = 6.0$.

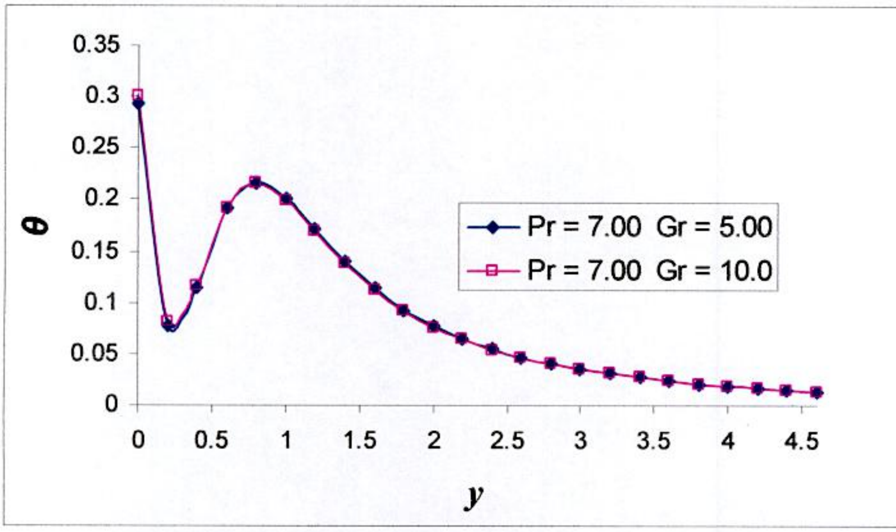

Figure 4.15 Variation of temperature profiles for different Grashof number (G_r) with $P_r = 7.0$, $S_c = 0.30$, $D_f = 0.05$ and $M = 6.0$.

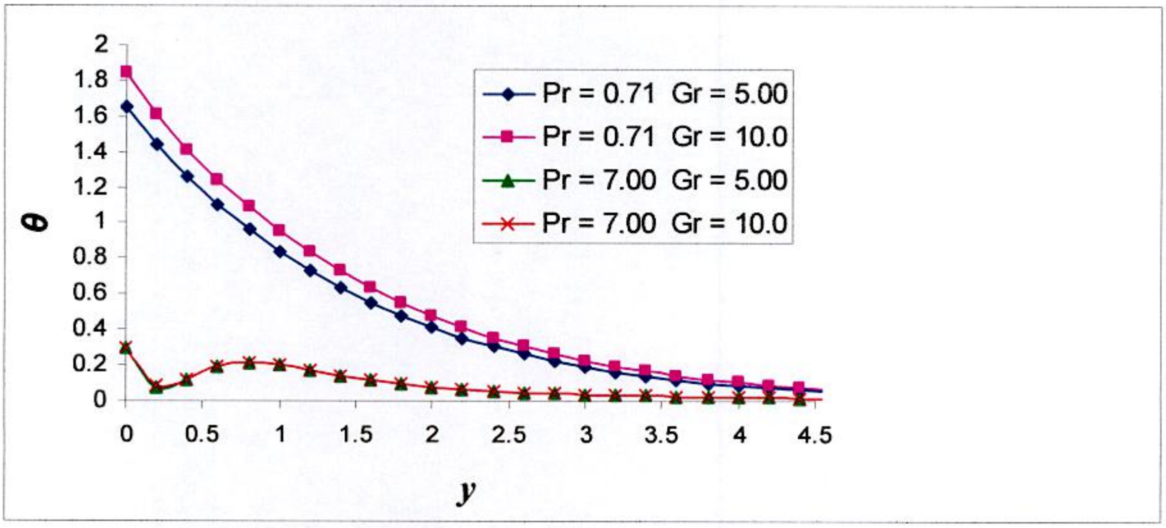

Figure 4.16 Comparison of the variation of temperature profiles for different Prandtl number (P_r) and Grashof number (G_r) with $S_c = 0.30$, $D_f = 0.05$ and $M = 6.0$.

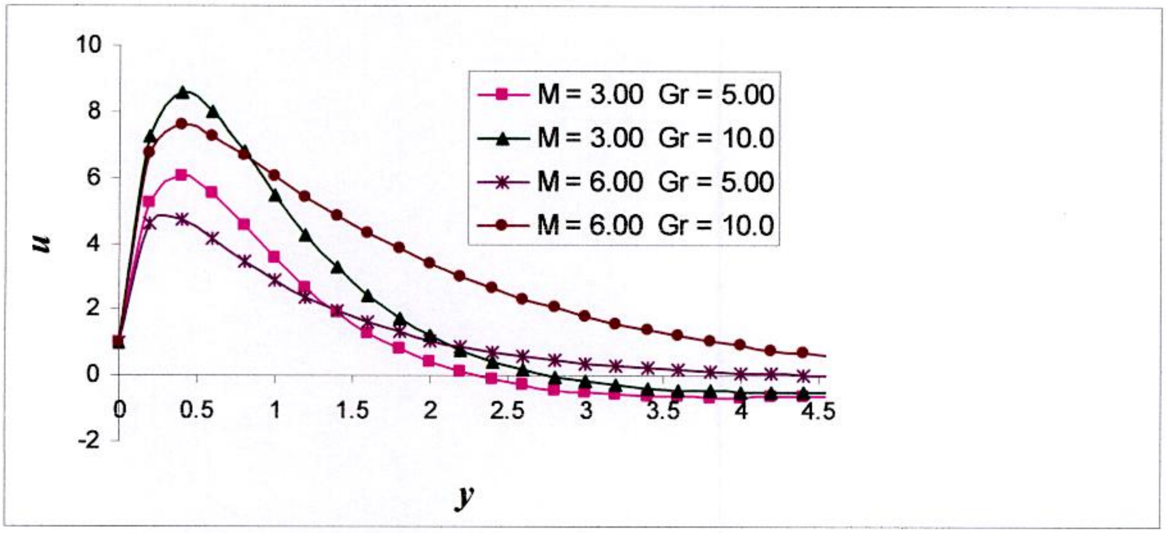

Figure 4.17 Variation of velocity profiles for different Magnetic parameter M and Grashof number (G_r) with $P_r = 0.71$, $S_c = 0.30$ and $D_f = 0.05$.

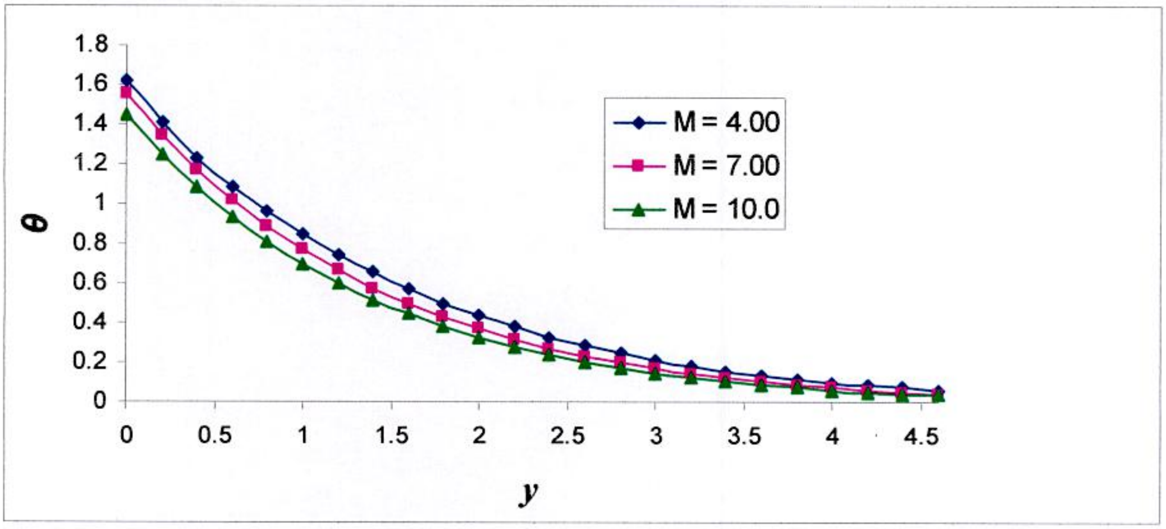

Figure 4.18 Variation of temperature profiles for different values of Magnetic parameter M with $P_r = 0.71$, $S_c = 0.30$, $G_r = 3.0$ and $D_f = 0.05$.

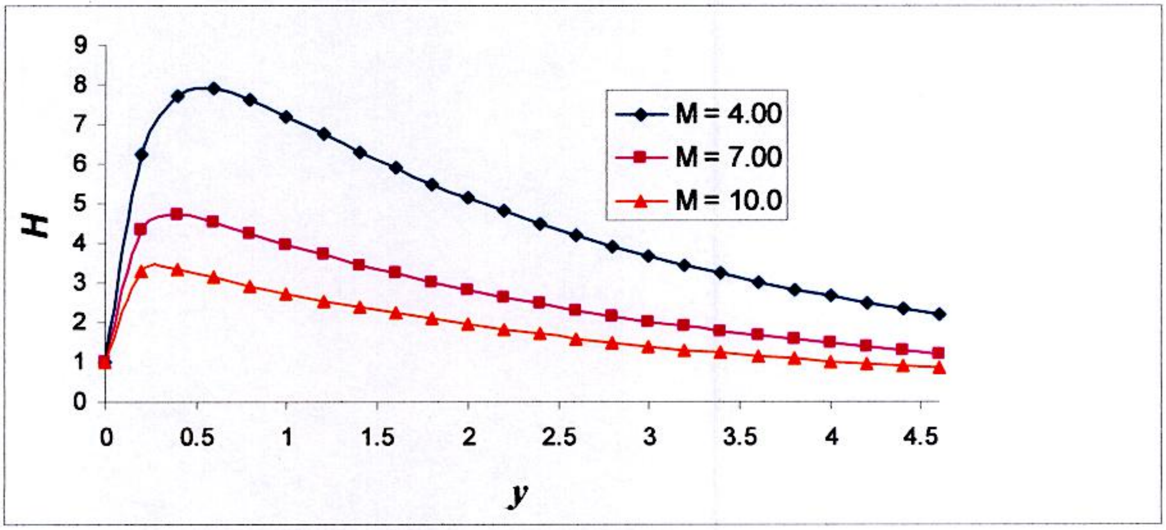

Figure 4.19 Variation of magnetic fields for different values of Magnetic parameter M with $P_r = 0.71$, $S_c = 0.30$, $G_r = 3.0$ and $D_f = 0.05$.

CHAPTER 5

Concluding Remarks

In this paper we have studied Dufour or diffusion-thermo effect on the laminar mixed free-force convection flow and heat transfer of viscous incompressible electrically conducting fluid above a vertical porous plate under the action of a transverse applied magnetic field. The transformed system of nonlinear, coupled, ordinary differential equations governing the problem were solved numerically by using perturbation technique. The influences of various establish parameters on the velocity and temperature profiles as well as induced magnetic fields for the first order approximation are exhibited in the present analysis. From the numerical investigation it was observed that the Dufour number has a considerable effect on some exceptional types of fluids considered. It was also found that the dimensionless Prandtl number (P_r), Grashof number (G_r), Schmedit number (S_c) and magnetic parameter (M) have an appreciable influence in the study of flow and heat transfer process. Therefore, it can be confirmly predicted for fluid with medium molecular weight the Dufour effect can play an important role on the effects of velocity, temperature and induced magnetic field, so that this effect should be taken into account with other useful parameters associated. Furthermore, it is necessary to study the Soret (thermo-diffusion) effect for the problem in order to get more useful results.

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